The Annals of Applied Probability
1992, Vol. 2, No. 1, 87–112

A DUALITY METHOD FOR OPTIMAL CONSUMPTION AND INVESTMENT UNDER SHORT-SELLING PROHIBITION. I. GENERAL MARKET COEFFICIENTS[1]

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A continuous-time, consumption–investment problem on a finite horizon is considered for an agent seeking to maximize expected utility from consumption plus expected utility from terminal wealth. The agent is prohibited from selling stocks short, so the usual martingale methods for solving this problem do not directly apply. A dual problem is posed and solved, and the solution to the dual problem provides information about the existence and nature of the solution to the original problem.

1. Introduction. This is the first of two papers which treat a consumption–investment decision problem for a single agent, endowed with some initial wealth, who can consume the wealth at some rate $C(t)$ and invest it in any of $d + 1$ available assets. The agent is attempting to maximize a linear combination of two quantities, namely:

1. $E\int_0^T U_1(t, C(t))\,dt$, the total expected discounted *utility from consumption* over the time interval $[0, T]$;
2. $EU_2(X(T))$, the expected *utility from terminal wealth*.

The $d + 1$ *assets* or *securities* available to the agent are very general. One of them is a *bond*, a security whose instantaneous rate of return may fluctuate (possibly randomly), but which is otherwise riskless. The other assets are *stocks*, risky securities whose prices have randomly fluctuating mean rates of return $b_i(t)$ and dispersion coefficients $\sigma_{ij}(t)$. Section 2 provides a careful exposition of these matters. The stock prices are driven by independent Wiener processes; these represent the sources of uncertainty in the market model, which we assume to be *complete* in the sense of Harrison and Pliska (1981, 1983) and Bensoussan (1984).

In our context, completeness amounts to nondegeneracy of the "diffusion" matrix $a(t) = \sigma(t)\sigma^T(t)$, as imposed in condition (2.3). This condition guarantees, roughly speaking, that there are exactly as many stocks as there are sources of uncertainty in the market model. It also enables us to construct a new probability measure under which the stock prices, discounted at the rate $r(t)$ of the bond, become a local martingale; this fact is of great importance in the modern theory of financial economics, and we refer the reader to Harrison and Pliska (1981, 1983) for a fuller account of its ramifications.

Received January 1990; revised February 1991.
[1]This work was supported by NSF Grants DMS-87-02537 and DMS-90-02588.
AMS 1980 *subject classifications*. Primary 93E20; secondary 60G44, 90A16, 49B60.
Key words and phrases. Portfolio and consumption processes, utility functions, stochastic control, martingale representation theorems, duality.

The processes $r(t)$, $b_i(t)$ and $\sigma_{ij}(t)$, $1 \le i, j \le d$, will be collectively referred to as the *coefficients of the market model*. In a companion paper, Xu and Shreve (1992), we examine the case where r, b_i and σ_{ij}, $1 \le i, j \le d$, are constants. The companion paper is nearly independent of the present paper, although it does refer to the present paper for notation and interpretation. In both papers, we assume that our agent is a "small investor," in that his decisions do not influence the asset prices, which are treated as exogenous.

Single-agent consumption–portfolio problems have been investigated by a number of authors. A significant plateau was reached by Merton (1969, 1971), who found closed-form solutions to the Hamilton–Jacobi–Bellman equation for a constant-coefficient model with power utility functions. Karatzas, Lehoczky, Sethi and Shreve (1986) generalized this work to allow general utility functions. More recently, Cox and Huang (1989), Pliska (1986) and Karatzas, Lehoczky and Shreve (1987) used martingale methods to study the problem with nonconstant market coefficients. Using the Girsanov theorem to change to a probability measure under which all the stock prices discounted by the bond rate become martingales, these authors found a simple expression for the optimal consumption process. The fact that every martingale relative to a Brownian filtration can be represented as a stochastic integral with respect to the underlying Brownian motion played a key role in the proof that this consumption process can be *financed*, that is, that there is a corresponding portfolio process which, together with the consumption process, results in a nonnegative wealth process. However, the portfolio process which is obtained by this method may require short-selling of the stocks. This paper and Xu and Shreve (1992) examine the model in which such short-selling is prohibited.

The approach of this paper is to define a *dual problem* for the original consumption–portfolio problem, hereafter referred to as the *primal problem*. Rockafellar and Wets (1976) have developed such a duality theory for discrete-time stochastic control, and Bismut (1973) has studied the continuous-time case. This approach has been used to get necessary conditions for optimal control processes, for example, Rockafellar and Wets (1978) and Frank (1984). Bismut (1975) applied this theory to stochastic growth problems and asset allocation problems. Our work is in the spirit of Bismut (1975), but our problem is complicated by the presence of a short-selling constraint and our analysis is enhanced by the martingale representation theorem. Pagès (1989) has studied a complete market problem with an endowment stream and with the constraint that wealth be nonnegative at all times. He provides conditions sufficient for the existence of an optimal solution, which he characterizes via duality. For the problem at hand, we do not know how to directly establish existence of an optimal solution, but we can establish existence of an optimal dual control process under fairly general conditions, and then use complementary slackness to obtain existence in the original (primal) problem and to characterize the optimal consumption and portfolio processes in that problem.

Our dual problem is defined in Section 3, and the relations between the two problems are developed in Section 4. Section 5 proves the existence of the optimal dual and primal control processes.

The approach of Xu and Shreve (1992) is to attack directly the Hamilton–Jacobi–Bellman equation associated with the consumption–investment problem. The solution to this equation was discovered by specializing the results of the present paper, but once the solution is known, the verification of the solution can proceed without reference to this paper.

Both this paper and Xu and Shreve (1992) are derived from the first author's Ph.D. dissertation [Xu (1990)]. The duality method presented in this paper is also useful in the study of optimal consumption and investment in incomplete markets; we refer the reader to Karatzas, Lehoczky, Shreve and Xu (1991) for an analysis of the maximization of the utility of terminal wealth in an incomplete market. He and Pearson (1989) have developed a closely related approach for both the incomplete market problem and the complete market problem with short-selling prohibition.

2. Formulation of the primal problem. In this section we formulate the problem of optimal consumption and investment when short-selling of the stocks is prohibited. We follow the notation of Karatzas and Shreve (1987), Section 5.8, which can be consulted for a more detailed formulation of this model. See also the survey by Karatzas (1989).

2.1. *Assets.* To model uncertainty, we will consider our problem on a probability space $(\Omega, \mathscr{F}, P)$. We assume that the σ-field $\mathscr{F}$ is rich enough to support a d-dimensional Brownian motion $\{w(t), \mathscr{F}(t); 0 \le t \le T\}$, where T is a fixed finite horizon and $\{\mathscr{F}(t)\}$ is the augmentation by null sets of the filtration generated by w. There are $d+1$ assets being traded continuously on the finite horizon $[0, T]$. One of them is a *bond*, whose price $p_0(t)$ at time t evolves according to the differential equation

$$dp_0(t) = r(t) p_0(t)\, dt, \qquad 0 \le t \le T. \tag{2.1}$$

The remaining d assets are *stocks*, and their prices are modeled by the stochastic differential equations

$$dp_i(t) = p_i(t)\left[b_i(t)\, dt + \sum_{j=1}^{d} \sigma_{ij}(t)\, dw^{(j)}(t)\right], \tag{2.2}$$
$$0 \le t \le T, i = 1, \dots, d.$$

The *interest rate process* $r(\cdot)$ as well as the vector process $b(\cdot) = (b_1(\cdot), \dots, b_d(\cdot))^T$ of *mean rates of return* and the $d \times d$ matrix *volatility process* $\sigma(\cdot) = (\sigma_{ij}(\cdot))$ are assumed to be $\{\mathscr{F}(t)\}$-progressively measurable and uniformly bounded. We introduce the *covariance process* $a(\cdot) \triangleq \sigma(\cdot)\sigma^T(\cdot)$ and assume the strong nondegeneracy condition

$$\xi^T a(t) \xi \ge \kappa_0 \|\xi\|^2, \qquad \forall\, \xi \in \mathbb{R}^d, \forall\, t \in [0, T], \text{ a.s.}, \tag{2.3}$$

for some $\kappa_0 > 0$. This implies that there is a constant κ_1 such that [see, e.g., Karatzas and Shreve (1987), Problem 5.8.1, with solution on page 393]

$$\max\Big\{\|(\sigma^T(t))^{-1}\xi\|, \|(\sigma(t))^{-1}\xi\|\Big\} \le \kappa_1\|\xi\|, \qquad \forall\, \xi \in \mathbb{R}^d, \forall\, t \in [0,T], \text{a.s.}, \tag{2.4}$$

$$\min\Big\{\|(\sigma^T(t))^{-1}\xi\|, \|(\sigma(t))^{-1}\xi\|\Big\} \ge \frac{1}{\kappa_1}\|\xi\|, \qquad \forall\, \xi \in \mathbb{R}^d, \forall\, t \in [0,T], \text{a.s.} \tag{2.5}$$

For specificity, we assume that $p_i(0) = 1$, $i = 0, \dots, d$. The solution to (2.1) with this initial condition is $p_0(t) = \exp(\int_0^t r(s)\,ds)$. We define for future reference a *discount process*

$$\beta(t) \triangleq \frac{1}{p_0(t)} = \exp\left(-\int_0^t r(s)\,ds\right), \qquad 0 \le t \le T. \tag{2.6}$$

2.2. *Portfolio and consumption processes.*

DEFINITION 2.1. A *portfolio process* $\pi(\cdot) = (\pi_1(\cdot), \dots, \pi_d(\cdot))^T$ is a measurable, $\{\mathscr{F}(t)\}$-adapted, $\mathbb{R}^d$-valued process satisfying $\int_0^T \|\pi(t)\|^2\,dt < \infty$ a.s. A *consumption process* $C(\cdot)$ is a measurable, $\{\mathscr{F}(t)\}$-adapted, $dt \times dP$-almost everywhere nonnegative process satisfying $\int_0^T C(t)\,dt < \infty$ a.s.

In Definition 2.1 we regard $\pi_i(t)$ as the value of stock i held by an agent at time t and we regard $C(t)$ as the rate of the agent's consumption at time t. If $X(t)$ denotes the wealth of the agent at time t, then the amount of money invested in the bond is $X(t) - \mathbf{1}^T\pi(t)$, where $\mathbf{1}$ denotes the d-dimensional vector of ones. In view of (2.1) and (2.2), the agent's wealth must evolve according to

$$\begin{aligned} dX(t) &= (r(t)X(t) - C(t))\,dt + \pi^T(t)(b(t) - r(t)\mathbf{1})\,dt \\ &\quad + \pi^T(t)\sigma(t)\,dw(t), \qquad 0 \le t \le T, \end{aligned} \tag{2.7}$$

whose solution is given by

$$\begin{aligned} \beta(t)X(t) &= x + \int_0^t \beta(s)\big[-C(s) + \pi^T(s)(b(s) - r(s)\mathbf{1})\big]\,ds \\ &\quad + \int_0^t \beta(s)\pi^T(s)\sigma(s)\,dw(s), \end{aligned} \tag{2.8}$$

where $x \ge 0$ denotes the agent's initial wealth.

DEFINITION 2.2. Let an initial wealth $x \ge 0$ and a consumption–portfolio process pair (C, π) be given. We say that (C, π) is *admissible* for x if for $i = 1, \dots, d$, we have

$$\pi_i(t) \ge 0, \qquad dt \times dP\text{-a.e.}, \tag{2.9}$$

and the wealth process $X(\cdot)$ defined by (2.8) satisfies

$$X(t) \geq 0, \qquad 0 \leq t \leq T, \text{ a.s.} \tag{2.10}$$

The set of all consumption–portfolio process pairs which are admissible for x will be denoted by $A(x)$.

Condition (2.9) rules out short-selling of stocks. However, $X(t) - \mathbf{1}^T\pi(t)$ is allowed to become negative, that is, borrowing from the bond is permitted.

Following the notation of Karatzas and Shreve (1987), Section 5.8, we define the *relative risk process*

$$\theta(t) \triangleq (\sigma(t))^{-1}[b(t) - r(t)\mathbf{1}], \qquad 0 \leq t \leq T. \tag{2.11}$$

We then introduce the martingale

$$Z(t) \triangleq \exp\left\{-\int_0^t \theta^T(s)\,dw(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2\,ds\right\}, \qquad 0 \leq t \leq T, \tag{2.12}$$

and the new probability measure $\tilde{P}$ defined by

$$\tilde{P}(A) \triangleq E[Z(T)1_A], \qquad \forall A \in \mathscr{F}, \tag{2.13}$$

and the drifted Brownian motion

$$\tilde{w}(t) \triangleq w(t) + \int_0^t \theta(s)\,ds, \qquad 0 \leq t \leq T. \tag{2.14}$$

According to Girsanov's theorem, $\tilde{w}$ is a standard Brownian motion under $\tilde{P}$. In terms of $\tilde{w}$, we may rewrite (2.8) as

$$\beta(t)X(t) + \int_0^t \beta(s)C(s)\,ds = x + \int_0^t \beta(s)\pi^T(s)\sigma(s)\,d\tilde{w}(s). \tag{2.15}$$

If (C, π) is admissible, then the left-hand side of (2.15) is nonnegative and the right-hand side is a local $\{\mathscr{F}(t)\}$-martingale under $\tilde{P}$. But Fatou's lemma shows that any nonnegative local martingale is a supermartingale, and the supermartingale property in (2.15) yields

$$\begin{aligned} &E\left[\beta(T)Z(T)X(T) + \int_0^T \beta(t)Z(t)C(t)\,dt\right] \\ &\quad = \tilde{E}\left[\beta(T)X(T) + \int_0^T \beta(t)C(t)\,dt\right] \leq x. \end{aligned} \tag{2.16}$$

We have obtained the following necessary condition for admissibility.

PROPOSITION 2.3. *If* $(C, \pi) \in A(x)$ *and* $X(\cdot)$ *is the corresponding wealth process, then* (2.16) *is satisfied.*

2.3. *Utility functions.*

DEFINITION 2.4. A *utility function* U is a strictly increasing, strictly concave, twice continuously differentiable, real-valued function defined on $[0, \infty)$ which satisfies

$$U(0) = 0, \tag{2.17}$$

$$U'(0) \triangleq \lim_{x \downarrow 0} U'(x) = \infty, \qquad U'(\infty) \triangleq \lim_{x \to \infty} U'(x) = 0, \tag{2.18}$$

$$0 \le U(x) \le \kappa_1(1 + x^{\rho_0}) \qquad \forall\, x \ge 0, \tag{2.19}$$

for some constants $\kappa_1 > 0$, $0 < \rho_0 < 1$.

Condition (2.17) can be replaced by the assumption that $U(0) > -\infty$; we assume (2.17) only for notational convenience. However, our model does not include utility functions such as log, for which $U(0) = -\infty$. Condition (2.18) ensures that the strictly decreasing, C^1 function U' maps $(0, \infty)$ *onto* $(0, \infty)$, and hence has a strictly decreasing, C^1 inverse I: $(0, \infty) \to (0, \infty)$, that is,

$$\begin{aligned} U'(I(y)) &= y, \qquad \forall\, y > 0, \\ I(U'(x)) &= x, \qquad \forall\, x > 0. \end{aligned} \tag{2.20}$$

We define

$$I(0) \triangleq \lim_{y \downarrow 0} I(y) = \infty, \tag{2.21}$$

the last equality resulting from $U'(\infty) = 0$.

Throughout the remainder of the paper, we will have a *terminal wealth utility function* U_2: $[0, \infty) \to \mathbb{R}$ satisfying the properties in Definition 2.4 and with I_2 denoting the inverse of U_2', and we will have a *consumption utility function* U_1: $[0, T] \times [0, \infty) \to \mathbb{R}$ which is (jointly) Borel measurable. For every $t \in [0, T]$, $U_1(t, \cdot\,)$ is assumed to satisfy Definition 2.4 with κ_1 and ρ_0 independent of t. We denote by $U_1'(t, x)$ the derivative of U_1 with respect to its second variable, and we denote by $I_1(t, \cdot\,)$ the inverse of $U_1'(t, \cdot\,)$.

2.4. *The value function.* For $x \ge 0$ and $(C, \pi) \in A(x)$, we define the *expected utility* of (C, π) as

$$J(x, C, \pi) \triangleq E\int_0^T U_1(t, C(t))\, dt + EU_2(X(T)), \tag{2.22}$$

where $X(\cdot)$ is given by (2.8) [or equivalently, (2.15)]. The *primal value function* is

$$V(x) \triangleq \sup\{J(x, C, \pi) | (C, \pi) \in A(x)\}, \qquad \forall\, x \ge 0. \tag{2.23}$$

An *optimal consumption–portfolio process pair* is one which attains the supremum in (2.23). Because of the strict concavity of $U_1(t, \cdot\,)$ and U_2, if such a pair exists, the consumption process component $C(\cdot)$ and the corresponding terminal wealth $X(T)$ are uniquely determined [see Xu (1990), Theorem 2.4.5].

Our goal is to characterize V, to obtain conditions under which an optimal consumption–portfolio process pair exists and to characterize this pair. We begin with the following description of V.

PROPOSITION 2.5. *The primal value function* $V: [0, \infty) \to [0, \infty)$ *is a continuous, nondecreasing concave function.*

PROOF. We first obtain an upper bound on $V(x)$. With ρ_0 as in (2.19), choose $q_1 \in [1, 1/\rho_0)$ and define $q_2 = 1 - q_1\rho_0 \in (0, 1)$. Given $(C, \pi) \in A(x)$ and the corresponding wealth process $X(\cdot)$ of (2.15), we use (2.19), the inequality $(a + b)^{q_1} \le 2^{q_1}(a^{q_1} + b^{q_1})\ \forall\, a, b \ge 0$, and the boundedness of $r(\cdot)$ to write

$$E\int_0^T [U_1(t, C(t))]^{q_1}\, dt \le (2\kappa_1)^{q_1} E\int_0^T \left[1 + (C(t))^{q_1\rho_0}\right] dt$$
$$\le (2\kappa_1)^{q_1}\left[T + e^{Tq_1\rho_0 \max|r(\cdot)|} E\int_0^T [\beta(t)C(t)]^{q_1\rho_0}\, dt\right].$$

Hölder's inequality and Proposition 2.3 imply

$$E\int_0^T [\beta(t)C(t)]^{q_1\rho_0}\, dt = E\int_0^T Z^{-q_1\rho_0}(t)[\beta(t)Z(t)C(t)]^{q_1\rho_0}\, dt$$
$$\le \left(E\int_0^T Z^{-q_1\rho_0/q_2}(t)\, dt\right)^{q_2}\left(E\int_0^T \beta(t)Z(t)C(t)\, dt\right)^{q_1\rho_0}$$
$$\le \left(\int_0^T EZ^{-q_1\rho_0/q_2}(t)\, dt\right)^{q_2} x^{q_1\rho_0}.$$

Because θ appearing in (2.12) is bounded, $EZ^{-q_1\rho_0/q_2}(t)$ is bounded uniformly in $t \in [0, T]$. Therefore, for some constant $\kappa(q_1)$ independent of x, C and π, we have

$$E\int_0^T [U_1(t, C(t))]^{q_1}\, dt \le \kappa(q_1)(1 + x^{q_1\rho_0}). \tag{2.24}$$

A similar estimation applied to $E([U_2(X(T))]^{q_1})$ results in the inequality

$$E\left([U_2(X(T))]^{q_1}\right) \le \kappa(q_1)(1 + x^{q_1\rho_0}). \tag{2.25}$$

Setting $q_1 = 1$ in (2.24) and (2.25), we obtain an upper bound on $J(x, C, \pi)$ which is independent of C and π; the finiteness of $V(x)$ follows.

Since the sets $A(x)$ increase with x, V must be nondecreasing. To prove concavity, note that for $x_1, x_2 \ge 0$, $\lambda \in (0, 1)$, $(C_1, \pi_1) \in A(x_1)$ and $(C_2, \pi_2) \in A(x_2)$, the linearity of the wealth equation (2.7) implies that $(\lambda C_1 + (1 - \lambda)C_2, \lambda\pi_1 + (1 - \lambda)\pi_2) \in A(\lambda x_1 + (1 - \lambda)x_2)$. The concavity of $U_1(t, \cdot\,)$ and U_2 allows us to conclude that

$$\lambda J(x_1, C_1, \pi_1) + (1 - \lambda)J(x_2, C_2, \pi_2)$$
$$\le J(\lambda x_1 + (1 - \lambda)x_2, \lambda C_1 + (1 - \lambda)C_2, \lambda\pi_1 + (1 - \lambda)\pi_2)$$
$$\le V(\lambda x_1 + (1 - \lambda)x_2).$$

Maximize the left-hand side of this inequality over $(C_1, \pi_1) \in A(x_1)$ and $(C_2, \pi_2) \in A(x_2)$ to obtain the concavity of V.

The concavity of V implies its continuity on $(0, \infty)$. Now $V(0) = 0$ [recall (2.17)], so to establish the continuity of V at 0, it suffices to show

$$\lim_{x \downarrow 0} V(x) \le 0. \tag{2.26}$$

For every $\varepsilon \in (0,1)$, choose $(C_\varepsilon, \pi_\varepsilon) \in A(\varepsilon)$ such that

$$V(\varepsilon) \le J(\varepsilon, C_\varepsilon, \pi_\varepsilon) + \varepsilon. \tag{2.27}$$

Let $X_\varepsilon(\cdot)$ be the associated wealth process. Inequality (2.16) implies

$$E\beta(T)Z(T)X_\varepsilon(T) \le \varepsilon, \qquad E\int_0^T \beta(t)Z(t)C_\varepsilon(t)\,dt \le \varepsilon.$$

Because L^1 convergence implies convergence almost everywhere along a subsequence, we can choose $\{\varepsilon_n\}_{n=1}^\infty$ such that $\varepsilon_n \downarrow 0$, $\beta(T)Z(T)X_{\varepsilon_n}(T) \to 0$, P-a.e., and $\beta(\cdot)Z(\cdot)C_{\varepsilon_n}(\cdot) \to 0$, $dt \times dP$-a.e. But (2.25) with $q_1 > 1$ implies that $\{U_2(X(T))\}_{n=1}^\infty$ is uniformly P-integrable, and (2.24) with $q_1 > 1$ implies that $\{U_1(\cdot C_{\varepsilon_n}(\cdot))\}_{n=1}^\infty$ is uniformly $dt \times dP$-integrable. Therefore, $\lim_{n\to\infty} J(\varepsilon_n, C_{\varepsilon_n}, \pi_{\varepsilon_n}) = 0$, and (2.26) follows from (2.27). □

3. Formulation of the dual problem. In this section we introduce a stochastic control problem which is dual to the problem of Section 2. We define the dual value function and establish its basic properties. The relationship between the dual problem of this section and the primal problem of Section 2 will be explored in Sections 4 and 5.

3.1. *Concave–convex conjugate function pairs.*

DEFINITION 3.1. Let U be a utility function (Definition 2.4). The *convex conjugate* of the concave function U is the convex function $\tilde{U}$ defined by

$$\tilde{U}(y) \triangleq \sup_{x \ge 0}\{U(x) - xy\}, \qquad \forall\, y > 0. \tag{3.1}$$

It is an easy exercise to verify that

$$\tilde{U}(0) \triangleq \lim_{y \downarrow 0} \tilde{U}(y) = U(\infty), \qquad \tilde{U}(\infty) \triangleq \lim_{y\to\infty} \tilde{U}(y) = U(0) = 0. \tag{3.2}$$

From (2.20) we have

$$\tilde{U}(y) = U(I(y)) - yI(y), \qquad \forall\, y > 0, \tag{3.3}$$

so

$$\tilde{U}'(y) = -I(y), \qquad \tilde{U}''(y) = -I'(y) > 0, \qquad \forall\, y > 0. \tag{3.4}$$

In particular, $\tilde{U}$ is a strictly decreasing, strictly convex, C^2 function. Equation

(3.1) implies

$$U(x) \le \tilde{U}(y) + xy, \qquad \forall\, x \ge 0, \forall\, y > 0, \tag{3.5}$$

and equality holds if and only if $x = I(y)$, or equivalently, $y = U'(x)$. It follows that

$$U(x) = \inf_{y>0}\{\tilde{U}(y) + xy\} = \tilde{U}(U'(x)) + xU'(x), \qquad \forall\, x > 0. \tag{3.6}$$

Finally, (3.1) and (2.19) imply

$$0 \le \tilde{U}(y) \le \sup_{x>0}\{\kappa_1(1 + x^{\rho_0}) - xy\} \le \kappa_2(1 + y^{-\alpha}), \qquad \forall\, y > 0, \tag{3.7}$$

where κ_2 is a positive constant and $\alpha = \rho_0/(1 - \rho_0)$.

Associated with the utility functions U_1 and U_2 introduced in Section 2.3, we have the convex conjugate functions $\tilde{U}_1, \tilde{U}_2$ defined by

$$\tilde{U}_1(t, y) = \sup_{x>0}\{U_1(t, x) - xy\}, \qquad \forall\, t \in [0, T], \forall\, y > 0, \tag{3.8}$$

$$\tilde{U}_2(y) = \sup_{x>0}\{U_2(x) - xy\}, \qquad \forall\, y > 0. \tag{3.9}$$

We define these functions at $y = 0$ and $y = \infty$ as in (3.2). We denote by $\tilde{U}_1'(t, y)$ the derivative of $\tilde{U}_1$ with respect to its second argument.

3.2. *Dual control processes.*

DEFINITION 3.2. A *dual control process* is a measurable, $\{\mathscr{F}(t)\}$-adapted, $\mathbb{R}^d$-valued process $\tilde{\pi}(\cdot) = (\tilde{\pi}_1(\cdot), \dots, \tilde{\pi}_d(\cdot))^T$ which satisfies $E\int_0^T \|\tilde{\pi}(t)\|^2\, dt < \infty$ and

$$\tilde{\pi}_i(t) \ge 0, \qquad dt \times dP\text{-a.e.} \tag{3.10}$$

The set of all dual control processes will be denoted by $\tilde{A}$.

For $\tilde{\pi} \in \tilde{A}$, we define the nonnegative local martingale (hence supermartingale)

$$\begin{aligned} Z_{\tilde{\pi}}(t) \triangleq \exp\bigg\{ &- \int_0^t [\theta(s) + \sigma^{-1}(s)\tilde{\pi}(s)]^T\, dw(s) \\ &- \frac{1}{2}\int_0^t \|\theta(s) + \sigma^{-1}(s)\tilde{\pi}(s)\|^2\, ds \bigg\}, \qquad 0 \le t \le T. \end{aligned} \tag{3.11}$$

LEMMA 3.3. *The set of processes* $\mathscr{D} \triangleq \{Z_{\tilde{\pi}}(\cdot) | \tilde{\pi} \in \tilde{A}\}$ *is convex.*

PROOF. For every $\lambda > 0$, $\mu > 0$ with $\lambda + \mu = 1$, and for every $\tilde{\pi}_1, \tilde{\pi}_2 \in \tilde{A}$, define $\xi = \lambda Z_{\tilde{\pi}_1} + \mu Z_{\tilde{\pi}_2}$, $\tilde{\pi} = (1/\xi)(\lambda\tilde{\pi}_1 Z_{\tilde{\pi}_1} + \mu\tilde{\pi}_2 Z_{\tilde{\pi}_2})$. Then $\tilde{\pi} \in \tilde{A}$, $\xi(0) = 1$, and

$$\begin{aligned} d\xi(t) &= \lambda\, dZ_{\tilde{\pi}_1}(t) + \mu\, dZ_{\tilde{\pi}_2}(t) \\ &= -\lambda Z_{\tilde{\pi}_1}(t)\big[\theta(t) + \sigma^{-1}(t)\tilde{\pi}_1(t)\big]^T dw(t) \\ &\quad - \mu Z_{\tilde{\pi}_2}(t)\big[\theta(t) + \sigma^{-1}(t)\tilde{\pi}_2(t)\big]^T dw(t) \\ &= -\xi(t)\big[\theta(t) + \sigma^{-1}(t)\tilde{\pi}(t)\big]\, dw(t). \end{aligned}$$

Therefore, $\xi = Z_{\tilde{\pi}} \in \mathscr{Z}$. □

3.3. *The dual control problem.* Let $\tilde{U}_1$ and $\tilde{U}_2$ be defined by (3.8) and (3.9). For $y \geq 0$ and $\tilde{\pi} \in \tilde{A}$, define the *dual objective function*

$$\tilde{J}(y, \tilde{\pi}) \triangleq E\int_0^T \tilde{U}_1(t, y\beta(t)Z_{\tilde{\pi}}(t))\, dt + E\tilde{U}_2(y\beta(T)Z_{\tilde{\pi}}(T)). \tag{3.12}$$

The dual problem is to minimize $\tilde{J}(y, \tilde{\pi})$ over $\tilde{A}$ for fixed y. The *dual value function* $\tilde{V}$ is defined by

$$\tilde{V}(y) \triangleq \inf\{\tilde{J}(y, \tilde{\pi}) | \tilde{\pi} \in \tilde{A}\}, \qquad \forall\, y \geq 0. \tag{3.13}$$

An *optimal process for the dual problem with initial condition* y is a process $\tilde{\pi}_y \in \tilde{A}$ which attains the infimum in (3.13). Because of the strict convexity of $\tilde{U}_1(t, \cdot\,)$ and $\tilde{U}_2$, if such a process exists, it must be unique [see Xu (1990), Theorem 3.3.1].

THEOREM 3.4. *Restricted to* $(0, \infty)$, *the dual value function* $\tilde{V}$ *is finite, nonnegative, continuous, nonincreasing and convex. Moreover,*

$$\tilde{V}(0) \triangleq \int_0^T \tilde{U}_1(t, 0)\, dt + \tilde{U}_2(0) = \lim_{y \downarrow 0} \tilde{V}(y), \tag{3.14}$$

but $\tilde{V}(0)$ *may be infinite. If* $\tilde{V}(0)$ *is finite, then*

$$\tilde{V}'(0) \triangleq \lim_{y \downarrow 0} \frac{\tilde{V}(y) - \tilde{V}(0)}{y} = -\infty. \tag{3.15}$$

PROOF. Because $\tilde{U}_1(t, \cdot\,)$ and $\tilde{U}_2$ are nonincreasing and nonnegative, $\tilde{V}$ is also. Let $\tilde{0}$ denote the identically zero dual control process, and note that $Z_{\tilde{0}}$ is the martingale Z defined by (2.12). Inequality (3.7) implies that for every $y > 0$,

$$\begin{aligned} \tilde{V}(y) \leq \tilde{J}(y, \tilde{0}) \leq E\int_0^T \kappa_2\big[1 + (y\beta(t)Z(t))^{-\alpha}\big]\, dt \\ + E\kappa_2\big[1 + (y\beta(T)Z(T))^{-\alpha}\big]. \end{aligned}$$

Because $\beta(\cdot)$ and $\theta(\cdot)$ are uniformly bounded, the above expectations are finite, so $0 \leq \tilde{V}(y) < \infty$, $\forall\, y > 0$.

We now prove convexity of $\tilde{V}$. For $y_1, y_2 > 0$, $\lambda, \mu > 0$ such that $\lambda + \mu = 1$, and $\tilde{\pi}_1, \tilde{\pi}_2 \in \tilde{A}$, by Lemma 3.3 there exists $\tilde{\pi} \in \tilde{A}$ such that

$$Z_{\tilde{\pi}} = \frac{1}{\lambda y_1 + \mu y_2}\left(\lambda y_1 Z_{\tilde{\pi}_1} + \mu y_2 Z_{\tilde{\pi}_2}\right).$$

Therefore,

$$\begin{aligned}
\tilde{V}(\lambda y_1 + \mu y_2) &\le \tilde{J}(\lambda y_1 + \mu y_2, \tilde{\pi}) \\
&= E\int_0^T \tilde{U}_1\big(t, \beta(t)\big(\lambda y_1 Z_{\tilde{\pi}_1}(t) + \mu y_2 Z_{\tilde{\pi}_2}(t)\big)\big)\,dt \\
&\quad + E\tilde{U}_2\big(\beta(T)\big(\lambda y_1 Z_{\tilde{\pi}_1}(T) + \mu y_2 Z_{\tilde{\pi}_2}(T)\big)\big) \\
&\le E\int_0^T \big[\lambda \tilde{U}_1\big(t, y_1\beta(t)Z_{\tilde{\pi}_1}(t)\big) + \mu\tilde{U}_1\big(t, y_2\beta(t)Z_{\tilde{\pi}_2}(t)\big)\big]\,dt \\
&\quad + E\big[\lambda\tilde{U}_2\big(y_1\beta(T)Z_{\tilde{\pi}_1}(T)\big) + \mu\tilde{U}_2\big(y_2\beta(T)\tilde{Z}_{\tilde{\pi}_2}(T)\big)\big] \\
&= \lambda\tilde{J}(y_1, \tilde{\pi}_1) + \mu\tilde{J}(y_2, \tilde{\pi}_2).
\end{aligned}$$

Minimization of the right-hand side of this inequality over $\tilde{\pi}_1, \tilde{\pi}_2 \in \tilde{A}$ yields the convexity of $\tilde{V}$. The continuity of $\tilde{V}$ on $(0, \infty)$ follows from its convexity.

The monotonicity of $\tilde{V}$ implies $\tilde{V}(0) \ge \lim_{y\downarrow 0}\tilde{V}(y)$. For the reverse inequality, let κ be an upper bound on $\beta(\cdot)$. The monotonicity of $\tilde{U}_1(t, \cdot\,)$ and $\tilde{U}_2$, Jensen's inequality and the supermartingale property imply that for $y > 0$, $\tilde{\pi} \in \tilde{A}$,

$$\begin{aligned}
\tilde{J}(y, \tilde{\pi}) &\ge E\int_0^T \tilde{U}_1(t, y\kappa Z_{\tilde{\pi}}(t))\,dt + E\tilde{U}_2(y\kappa Z_{\tilde{\pi}}(T)) \\
&\ge \int_0^T \tilde{U}_2(t, y\kappa EZ_{\tilde{\pi}}(t))\,dt + \tilde{U}_2(y\kappa EZ_{\tilde{\pi}}(T)) \\
&\ge \int_0^T \tilde{U}_1(t, y\kappa)\,dt + \tilde{U}_2(y\kappa).
\end{aligned}$$

Therefore,

$$\tilde{V}(y) \ge \int_0^T \tilde{U}_1(t, y\kappa)\,dt + \tilde{U}_2(y\kappa),$$

and the monotone convergence theorem implies

$$\lim_{y\downarrow 0}\tilde{V}(y) \ge \int_0^T \tilde{U}_1(t, 0)\,dt + \tilde{U}_2(0) = \tilde{V}(0).$$

If $\tilde{V}(0) < \infty$, then

$$
\begin{aligned}
-\tilde{V}'(0) &= \lim_{y\downarrow 0} \frac{\tilde{V}(0) - \tilde{V}(y)}{y} \geq \liminf_{y\downarrow 0} \frac{\tilde{V}(0) - \tilde{J}(y,\tilde{0})}{y} \\
&= \liminf_{y\downarrow 0} \left\{ \frac{1}{y} E \int_0^T \left[\tilde{U}_1(t,0) - \tilde{U}_1(t, y\beta(t)Z(t)) \right] dt \right. \\
&\qquad \left. + \frac{1}{y} E\left[\tilde{U}_2(0) - \tilde{U}_2(y\beta(T)Z(T)) \right] \right\} \\
&\geq \liminf_{y\downarrow 0} \frac{1}{y} E\left[\tilde{U}_2(0) - \tilde{U}_2(y\beta(T)Z(T)) \right].
\end{aligned}
$$

The convexity of $\tilde{U}_2$, (3.4), (2.21) and the monotone convergence theorem imply

$$
\begin{aligned}
&\liminf_{y\downarrow 0} \frac{1}{y} E\left[\tilde{U}_2(0) - \tilde{U}_2(y\beta(T)Z(T)) \right] \\
&\qquad \geq \lim_{y\downarrow 0} E\left[\beta(T)Z(T) I_2(y\beta(T)Z(T)) \right] = \infty. \qquad \square
\end{aligned}
$$

COROLLARY 3.5. *For every $x > 0$, there exists $y_x > 0$ such that*

$$
\tilde{V}(y_x) + xy_x = \inf_{y>0} \{ \tilde{V}(y) + xy \}. \tag{3.16}
$$

PROOF. Define $f\colon (0,\infty) \to \mathbb{R}$ by $f(y) = \tilde{V}(y) + xy$. Note that f is continuous and $\lim_{y\to\infty} f(y) = \infty$. If $\tilde{V}(0) = \infty$, then $\lim_{y\downarrow 0} f(y) = \infty$, and f attains its minimum on $(0,\infty)$. If $\tilde{V}(0) < \infty$, then f has a continuous extension to $[0,\infty)$ and must attain its minimum at some $y_x \in [0,\infty)$. According to (3.15), $f'(0) = -\infty$, so y_x must be positive. $\square$

4. Relations between the primal and dual problems.

In this section we show that, for any dual control process $\tilde{\pi}$, the objective function $\tilde{J}(\cdot, \tilde{\pi})$ in the dual problem provides a bound on the value function V for the primal problem. Moreover, the existence of an optimal dual control process $\tilde{\pi}$ implies the existence of an optimal consumption–portfolio process pair (C,π), and π is related to $\tilde{\pi}$ by the complementarity condition (4.4) below.

4.1. *Weak duality.*

WEAK DUALITY THEOREM 4.1. *For every $x \geq 0$, $y > 0$, $(C,\pi) \in A(x)$ and $\tilde{\pi} \in \tilde{A}$, the inequality*

$$
J(x, C, \pi) \leq \tilde{J}(y, \tilde{\pi}) + xy \tag{4.1}
$$

holds. Equality holds in (4.1) *if and only if*

$$C(t) = I_1(t, y\beta(t)Z_{\tilde{\pi}}(t)), \qquad dt \times dP\text{-}a.e., \tag{4.2}$$

$$X(T) = I_2(y\beta(T)Z_{\tilde{\pi}}(T)), \quad a.s., \tag{4.3}$$

$$\pi^T(t)\tilde{\pi}(t) = 0, \qquad dt \times dP\text{-}a.e., \tag{4.4}$$

$$E\int_0^T Z_{\tilde{\pi}}(t)\beta(t)C(t)\,dt + EZ_{\tilde{\pi}}(T)\beta(T)X(T) = x, \tag{4.5}$$

where $X(\cdot)$ *is the wealth process associated with* x, $C(\cdot)$ *and* $\pi(\cdot)$ [*see* (2.8)], *and* $Z_{\tilde{\pi}}(\cdot)$ *is given by* (3.11).

Proof. From (2.8), (3.11), (2.11) and Itô's rule, we have

$$\begin{aligned} &d(Z_{\tilde{\pi}}(t)\beta(t)X(t)) \\ &\quad = -Z_{\tilde{\pi}}(t)\beta(t)C(t)\,dt - Z_{\tilde{\pi}}(t)\beta(t)\pi^T(t)\tilde{\pi}(t)\,dt \\ &\qquad + Z_{\tilde{\pi}}(t)\beta(t)\Big[\pi^T(t)\sigma(t) - X(t)\big(\theta(t) + \sigma^{-1}(t)\tilde{\pi}(t)\big)^T\Big]\,dw(t), \end{aligned}$$

so

$$\begin{aligned} &Z_{\tilde{\pi}}(t)\beta(t)X(t) + \int_0^t Z_{\tilde{\pi}}(s)\beta(s)C(s)\,ds \\ &\qquad + \int_0^t Z_{\tilde{\pi}}(s)\beta(s)\pi^T(s)\tilde{\pi}(s)\,ds, \qquad 0 \le t \le T, \end{aligned}$$

is a nonnegative local martingale, hence a supermartingale. This supermartingale has initial condition x, so

$$\begin{aligned} &EZ_{\tilde{\pi}}(T)\beta(T)X(T) + E\int_0^T Z_{\tilde{\pi}}(s)\beta(s)C(s)\,ds \\ &\qquad + E\int_0^T Z_{\tilde{\pi}}(s)\beta(s)\pi^T(s)\tilde{\pi}(s)\,ds \le x. \end{aligned} \tag{4.6}$$

From (3.5) we have

$$U_1(t, C(t)) \le \tilde{U}_1(t, y\beta(t)Z_{\tilde{\pi}}(t)) + y\beta(t)Z_{\tilde{\pi}}(t)C(t), \qquad dt \times dP\text{-a.e.},$$

$$U_2(X(T)) \le \tilde{U}_2(y\beta(T)Z_{\tilde{\pi}}(T)) + y\beta(T)Z_{\tilde{\pi}}(T)X(T), \quad \text{a.s.},$$

and equality holds if and only if (4.2) and (4.3) hold. Therefore,

$$\begin{aligned} J(x, C, \pi) &\le \tilde{J}(y, \tilde{\pi}) + y\bigg\{E\int_0^T \beta(t)Z_{\tilde{\pi}}(t)C(t)\,dt \\ &\qquad\qquad + E\beta(T)Z_{\tilde{\pi}}(T)X(T)\bigg\} \\ &\le \tilde{J}(y, \tilde{\pi}) + yx \end{aligned} \tag{4.7}$$

because of (4.6) and the fact that $\pi^T(t)\tilde{\pi}(t) \ge 0$, $0 \le t \le T$, a.s. Equality holds in (4.7) if and only if (4.2)–(4.5) hold. □

COROLLARY 4.2. *For every* $x \geq 0$ *and* $y > 0$,

$$V(x) \leq \tilde{V}(y) + xy. \tag{4.8}$$

If $(C, \pi_y) \in A(x)$ *and* $\tilde{\pi}_y \in \tilde{A}$ *satisfy* (4.2)–(4.5), *then they are optimal in their respective problems, that is,*

$$V(x) = J(x, C, \pi_y), \qquad \tilde{V}(y) = \tilde{J}(y, \tilde{\pi}_y). \tag{4.9}$$

REMARK 4.3. Corollary 4.2 implies that

$$\tilde{V}(y) \geq \sup_{x \leq 0} \{V(x) - xy\}, \qquad \forall\, y > 0, \tag{4.10}$$

that is, $\tilde{V}$ dominates the convex conjugate of V. We provide conditions in Corollary 4.9 and Remark 5.7 under which the reverse inequality holds.

4.2. *Strong duality.* In order to construct pairs $(C, \pi) \in A(x)$ and $\tilde{\pi} \in \tilde{A}$ which are related by the duality conditions (4.2)–(4.5), we begin with $y > 0$ and $\tilde{\pi} \in \tilde{A}$. We can define $C(\cdot)$ by (4.2) and x by (4.5) [with $X(T)$ given by (4.3)], and we must then ask whether there is a portfolio process $\pi \in A(x)$ satisfying (4.4) such that the wealth process $X(\cdot)$ associated with x, $C(\cdot)$ and $\pi(\cdot)$ satisfies (4.3). We first construct a portfolio process π such that (4.3) is satisfied, but π may take negative values and so may fail to be admissible. We subsequently show that if $\tilde{\pi}$ is optimal, then π is indeed admissible and (4.4) holds.

LEMMA 4.4. *Let* $y > 0$ *and* $\tilde{\pi} \in \tilde{A}$ *be given. Define* $C(\cdot)$ *by* (4.2) *and assume the finiteness of*

$$x \triangleq E\int_0^T Z_{\tilde{\pi}}(t)\beta(t)C(t)\,dt + E\left[Z_{\tilde{\pi}}(T)\beta(T)I_2(y\beta(T)Z_{\tilde{\pi}}(T))\right]. \tag{4.11}$$

Then there exists a portfolio process $\pi(\cdot)$, *which may take negative values, and there exists a continuous, nonnegative process* $\tilde{X}(\cdot)$, *such that*

$$\tilde{X}(0) = x, \qquad \tilde{X}(T) = I_2(y\beta(T)Z_{\tilde{\pi}}(T)), \tag{4.12}$$

$$\begin{aligned} d\tilde{X}(t) &= \left(r(t)\tilde{X}(t) - C(t)\right)dt \\ &\quad + \pi^T(t)(b(t) - r(t)\mathbf{1} + \tilde{\pi}(t))\,dt \\ &\quad + \pi^T(t)\sigma(t)\,dw(t), \qquad 0 \leq t \leq T. \end{aligned} \tag{4.13}$$

PROOF. Define

$$D \triangleq \int_0^T Z_{\tilde{\pi}}(t)\beta(t)C(t)\,dt + Z_{\tilde{\pi}}(T)\beta(T)I_2(y\beta(T)Z_{\tilde{\pi}}(T)),$$

so $x = ED$. We may assume that P-a.e. path of the martingale $B(t) \triangleq E(D|\mathscr{F}(t))$ is right-continuous [Karatzas and Shreve (1987), Theorem 1.3.13], and so B has a representation as

$$B(t) = x + \int_0^t Y^T(s)\,dw(s), \qquad 0 \leq t \leq T,$$

where $Y(\cdot)$ is an $\mathbb{R}^d$-valued, $\{\mathscr{F}(t)\}$-progressively measurable process satisfying $\int_0^T \|Y(t)\|^2\,dt < \infty$ a.s. [use Karatzas and Shreve (1987), Theorem 3.4.15 and a localization argument]. In particular, B is actually continuous. Define

$$
\begin{aligned}
&\xi(t) = B(t) - \int_0^t \beta(s)C(s)Z_{\tilde{\pi}}(s)\,ds,\\
(4.14)\qquad &\tilde{X}(t) = \xi(t)[\beta(t)Z_{\tilde{\pi}}(t)]^{-1},\\
&\pi(t) = \tilde{X}(t)(\sigma^T(t))^{-1}\left[\theta(t) + \sigma^{-1}(t)\tilde{\pi}(t) + \frac{1}{\xi(t)}Y(t)\right].
\end{aligned}
$$

Then $\tilde{X}(0) = x$, $\tilde{X}(T) = I_2(y\beta(T)Z_{\tilde{\pi}}(T))$. To verify (4.13), we observe that

$$
\begin{aligned}
d\xi(t) &= Y^T(t)\,dw(t) - \beta(t)C(t)Z_{\tilde{\pi}}(t)\,dt,\\
d[\beta(t)Z_{\tilde{\pi}}(t)]^{-1} &= [\beta(t)Z_{\tilde{\pi}}(t)]^{-1}\left[r(t) + \|\theta(t) + \sigma^{-1}(t)\tilde{\pi}(t)\|^2\right]dt\\
&\quad + [\beta(t)Z_{\tilde{\pi}}(t)]^{-1}[\theta(t) + \sigma^{-1}(t)\tilde{\pi}(t)]^T\,dw(t).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
d\tilde{X}(t) &= \tilde{X}(t)\left[r(t) + \|\theta(t) + \sigma^{-1}(t)\tilde{\pi}(t)\|^2\right]dt\\
&\quad + \tilde{X}(t)\left[\theta(t) + \sigma^{-1}(t)\tilde{\pi}(t)\right]^T dw(t)\\
&\quad + \frac{\tilde{X}(t)}{\xi(t)}Y^T(t)\,dw(t) - C(t)\,dt\\
&\quad + \frac{\tilde{X}(t)}{\xi(t)}Y^T(t)\left[\theta(t) + \sigma^{-1}(t)\tilde{\pi}(t)\right]dt\\
&= \left(r(t)\tilde{X}(t) - C(t)\right)dt + \pi^T(t)\sigma(t)\left[\theta(t) + \sigma^{-1}(t)\tilde{\pi}(t)\right]dt\\
&\quad + \pi^T(t)\sigma(t)\,dw(t),
\end{aligned}
$$

which agrees with (4.13). □

REMARK 4.5. Note that (4.13) differs from the wealth equation (2.8) because of the term $\pi^T(t)\tilde{\pi}(t)\,dt$ in (4.13); when the complementary slackness condition (4.4) holds, the two equations agree. The solution to (4.12) and (4.13) satisfies

$$
(4.15)\qquad \beta(t)\tilde{X}(t) = M(t) - \int_0^t \beta(s)C(s)\,ds, \qquad 0 \le t \le T,
$$

where

$$
(4.16)\qquad M(t) \triangleq x + \int_0^t \beta(s)\pi^T(s)\sigma(s)\,dw_{\tilde{\pi}}(s), \qquad 0 \le t \le T,
$$

$$
(4.17)\qquad w_{\tilde{\pi}}(t) \triangleq w(t) + \int_0^t (\theta(s) + \sigma^{-1}(s)\tilde{\pi}(s))\,ds, \qquad 0 \le t \le T.
$$

From the definitions in the proof of Lemma 4.4, we also have the useful formula

$$Z_{\tilde{\pi}}(\tau)\beta(\tau)\tilde{X}(\tau) = E\left[\int_\tau^T Z_{\tilde{\pi}}(s)\beta(s)C(s)\,ds \mid \mathscr{F}(\tau)\right] + E\left[Z_{\tilde{\pi}}(T)\beta(T)\tilde{X}(T) \mid \mathscr{F}(\tau)\right], \tag{4.18}$$

for any $\{\mathscr{F}(t)\}$-stopping time τ taking values in $[0, T]$.

Let $y > 0$ be given and assume the dual problem with initial condition y has an optimal solution $\tilde{\pi}_y$, that is,

$$\tilde{J}(y, \tilde{\pi}_y) = \tilde{V}(y). \tag{4.19}$$

In the remainder of this section, we show that the corresponding portfolio π given by Lemma 4.4 is optimal in the primal problem with initial wealth x given by (4.11) when $\tilde{\pi}_y$ is substituted for $\tilde{\pi}$. In order to obtain this result, we define

$$\begin{aligned} g_y(\lambda) &\triangleq \tilde{J}(\lambda y, \tilde{\pi}_y) \\ &= E\int_0^T \tilde{U}_1\big(t, \lambda y\beta(t)Z_{\tilde{\pi}_y}(t)\big)\,dt \\ &\quad + E\tilde{U}_2\big(\lambda y\beta(T)Z_{\tilde{\pi}_y}(T)\big), \quad \forall\, \lambda > 0, \end{aligned} \tag{4.20}$$

and we need to assume

$$\exists\, \delta_y \in (0,1) \text{ such that } g_y(\lambda) < \infty, \qquad \forall\, \lambda \in (1 - \delta_y, 1 + \delta_y). \tag{4.21}$$

A sufficient condition for (4.21) is that for some $\alpha \in (0, 1)$, $\gamma \in (1, \infty)$,

$$\alpha U_1'(t,x) \geq U_1'(t, \gamma x), \qquad \alpha U_2'(x) \geq U_2'(\gamma x), \qquad \forall\, t \in [0, T],\ x > 0; \tag{4.22}$$

see Karatzas, Lehoczky, Shreve and Xu (1989), Lemma 11.5.

LEMMA 4.6. *Let $y > 0$ be given, assume $\tilde{\pi}_y \in \tilde{A}$ satisfies* (4.19) *and assume* (4.21). *Then g_y is differentiable at* 1 *and*

$$\begin{aligned} g_y'(1) = &-E\int_0^T y\beta(t)Z_{\tilde{\pi}_y}(t)I_1\big(t, y\beta(t)Z_{\tilde{\pi}_y}(t)\big)\,dt \\ &- E\big[y\beta(T)Z_{\tilde{\pi}_y}(T)I_2\big(y\beta(T)Z_{\tilde{\pi}_y}(T)\big)\big]. \end{aligned} \tag{4.23}$$

PROOF. Because of the convexity of $\tilde{U}_2$, we have for $\lambda \in (1 - \delta_y/2, 1) \cup (1, \infty)$,

$$\begin{aligned} &\frac{1}{|\lambda - 1|}\left|\tilde{U}_2\big(\lambda y\beta(T)Z_{\tilde{\pi}_y}(T)\big) - \tilde{U}_2\big(y\beta(T)Z_{\tilde{\pi}_y}(T)\big)\right| \\ &\quad \leq \frac{2}{\delta_y}\left|\tilde{U}_2\left(\left(1 - \frac{1}{2}\delta_y\right)y\beta(T)Z_{\tilde{\pi}_y}(T)\right) - \tilde{U}_2\big(y\beta(T)Z_{\tilde{\pi}_y}(T)\big)\right|. \end{aligned}$$

The right-hand side is integrable, so the dominated convergence theorem and (3.4) imply

$$\frac{\partial}{\partial\lambda} E\tilde{U}_2\big(\lambda y\beta(T)Z_{\tilde{\pi}_y}(T)\big)\Big|_{\lambda=1} = -E\big[y\beta(T)Z_{\tilde{\pi}_y}(T)I_2\big(y\beta(T)Z_{\tilde{\pi}_y}(T)\big)\big].$$

A similar analysis applies to $\tilde{U}_1$, and we thereby obtain (4.23). □

Let $\tilde{\pi}_y \in \tilde{A}$ satisfy (4.19) and let $\tilde{\pi}$ be another process in $\tilde{A}$. For any $\varepsilon \in [0,1]$, the "perturbed" process $\tilde{\pi}_\varepsilon \triangleq \tilde{\pi}_y + \varepsilon(\tilde{\pi} - \tilde{\pi}_y)$ is also in $\tilde{A}$, so we can study the sensitivity of $\tilde{J}(y, \tilde{\pi}_\varepsilon)$ to variations in ε. In order to carry out this program, we introduce some notation. Define

$$N(t) \triangleq \int_0^t \big[\sigma^{-1}(s)\big(\tilde{\pi}(s) - \tilde{\pi}_y(s)\big)\big]^T dw_{\tilde{\pi}_y}(s), \qquad 0 \le t \le T, \tag{4.24}$$

where $w_{\tilde{\pi}_y}$ is defined by (4.17). Corresponding to $\tilde{\pi}_y$, let π_y be the portfolio process constructed in Lemma 4.4 and let $C(\cdot)$ and $\tilde{X}(\cdot)$ be given by (4.2), (4.12) and (4.13) when π and $\tilde{\pi}$ are replaced by π_y and $\tilde{\pi}_y$, respectively. For each positive integer n, define the stopping time

$$\begin{aligned}\tau_n \triangleq T \wedge \inf\Big\{ t \in [0,T] \,\Big|\, &|N(t)| + |\tilde{X}(t)| + |Z_{\tilde{\pi}_y}(t)| \\ &+ \int_0^t \|\theta(s) + \sigma^{-1}(s)\tilde{\pi}_y(s)\|^2\,ds \\ &+ \int_0^t \beta(s)C(s)\,ds + \int_0^t \|\sigma^{-1}(s)\big(\tilde{\pi}(s) - \tilde{\pi}_y(s)\big)\|^2\,ds \\ &+ \int_0^t \|\sigma^T(s)\pi_y(s)\|^2\,ds \ge n \Big\},\end{aligned} \tag{4.25}$$

and note that

$$\tau_n \uparrow T \quad \text{as } n \to \infty. \tag{4.26}$$

Set $\pi_\varepsilon^n(t) \triangleq \pi_\varepsilon(t)1_{\{t \le \tau_n\}}$, $0 \le t \le T$.

Lemma 4.7. *Assume* (4.21). *Then*

$$\begin{aligned} yE\int_0^{\tau_n} Z_{\tilde{\pi}_y}(t)\beta(t)\pi_y^T(t)\big(\tilde{\pi}(t) - \tilde{\pi}_y(t)\big)\,dt \\ = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\big[\tilde{J}(y,\tilde{\pi}_\varepsilon^n) - \tilde{J}(y,\tilde{\pi}_y)\big] \ge 0.\end{aligned} \tag{4.27}$$

Proof. Because $\tilde{\pi}_y$ satisfies (4.19), the inequality

$$\liminf_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\big[\tilde{J}(y,\tilde{\pi}_\varepsilon^n) - \tilde{J}(y,\tilde{\pi}_y)\big] \ge 0$$

holds. As for the equality in (4.27), direct computation reveals

$$Z_{\tilde{\pi}_\varepsilon^n}(t) = Z_{\tilde{\pi}_y}(t)\exp\left\{-\varepsilon N(t\wedge\tau_n) - \tfrac{1}{2}\varepsilon^2\int_0^{t\wedge\tau_n}\left\|\sigma^{-1}(s)\big(\tilde{\pi}(s)-\tilde{\pi}_y(s)\big)\right\|^2 ds\right\}. \tag{4.28}$$

From the definition of τ_n, we have

$$e^{-2n\varepsilon}Z_{\tilde{\pi}_y}(t) \le Z_{\tilde{\pi}_\varepsilon^n}(t) \le e^{n\varepsilon}Z_{\tilde{\pi}_y}(t), \qquad \forall\, n\ge 1,\ \varepsilon\in[0,1],\ t\in[0,T].$$

Choose $\varepsilon_0 \in (0,1]$ such that $1-e^{-2n\varepsilon} \le \frac{1}{2}\delta_y$ for all $\varepsilon\in(0,\varepsilon_0)$. If $\varepsilon\in(0,\varepsilon_0)$ and $Z_{\tilde{\pi}_\varepsilon^n}(t) \ne Z_{\tilde{\pi}_y}(t)$, then the convexity of $\tilde{U}_2$ implies

$$\begin{aligned}
&\frac{1}{\varepsilon}\left|\tilde{U}_2\big(y\beta(T)Z_{\tilde{\pi}_\varepsilon^n}(T)\big) - \tilde{U}_2\big(y\beta(T)Z_{\tilde{\pi}_y}(T)\big)\right| \\
&\quad\le \frac{|Z_{\tilde{\pi}_\varepsilon^n}(T)Z_{\tilde{\pi}_y}^{-1}(T)-1|}{\varepsilon}\,\frac{\left|\tilde{U}_2\big(y\beta(T)Z_{\tilde{\pi}_\varepsilon^n}(T)\big) - \tilde{U}_2\big(y\beta(T)Z_{\tilde{\pi}_y}(T)\big)\right|}{|Z_{\tilde{\pi}_\varepsilon^n}(T)Z_{\tilde{\pi}_y}^{-1}(T)-1|} \\
&\quad\le \frac{1}{\varepsilon}\max\{1-e^{-2n\varepsilon}, e^{n\varepsilon}-1\} \\
&\qquad\times\frac{\left|\tilde{U}_2\big(y\beta(T)(1-\frac{1}{2}\delta_y)Z_{\tilde{\pi}_y}(T)\big) - \tilde{U}_2\big(y\beta(T)Z_{\tilde{\pi}_y}(T)\big)\right|}{\frac{1}{2}\delta_y}.
\end{aligned}$$

If $Z_{\tilde{\pi}_\varepsilon^n}(t) = Z_{\tilde{\pi}_y}(t)$, the first expression in the above string of inequalities is still dominated by the last expression. The last expression is the product of a bounded function of $\varepsilon\in(0,\varepsilon_0]$ and an integrable random variable, because of assumption (4.21). By the dominated convergence theorem, (4.28) and (3.4), we have

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\tfrac{1}{\varepsilon}\left[\tilde{U}_2\big(y\beta(T)Z_{\tilde{\pi}_\varepsilon^n}(T)\big) - \tilde{U}_2\big(y\beta(T)Z_{\tilde{\pi}_y}(T)\big)\right] \\
&\quad= E\frac{\partial}{\partial\varepsilon}\tilde{U}_2\left(y\beta(T)Z_{\tilde{\pi}_y}(T)\exp\left\{-\varepsilon N(\tau_n) - \frac{1}{2}\varepsilon^2\int_0^{\tau_n}\left\|\sigma^{-1}(s)\big(\tilde{\pi}(s)-\tilde{\pi}_y(s)\big)\right\|^2 ds\right\}\right)\Bigg|_{\varepsilon=0} \qquad (4.29)\\
&\quad= E\left[y\beta(T)Z_{\tilde{\pi}_y}(T)I_2\big(y\beta(T)Z_{\tilde{\pi}_y}(T)\big)N(\tau_n)\right] \\
&\quad= E\left[y\beta(T)Z_{\tilde{\pi}_y}(T)\tilde{X}(T)N(\tau_n)\right].
\end{aligned}$$

A similar analysis for U_1 results in the formula

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}E\left[\int_0^T\tilde{U}_1\big(t,y\beta(t)Z_{\tilde{\pi}_\varepsilon^n}(t)\big)\,dt - \int_0^T\tilde{U}_1\big(t,y\beta(t)Z_{\tilde{\pi}_y}(t)\big)\,dt\right] \\
&\quad= E\left[\int_0^T y\beta(t)Z_{\tilde{\pi}_y}(t)C(t)N(t\wedge\tau_n)\,dt\right].
\end{aligned} \tag{4.30}$$

Summing (4.29) and (4.30), we obtain

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\left[\tilde{J}(y,\tilde{\pi}_\varepsilon^n)-\tilde{J}(y,\tilde{\pi}_y)\right]$$

$$(4.31)\qquad = yE\left[\int_0^T Z_{\tilde{\pi}_y}(s)\beta(s)C(s)N(s\wedge\tau_n)\,ds + Z_{\tilde{\pi}_y}(T)\beta(T)\tilde{X}(T)N(\tau_n)\right].$$

It remains to show that the right-hand side of (4.31) agrees with the left-hand side of (4.27). Note first that (4.18) with $\tilde{\pi}_y$ replacing $\tilde{\pi}$ implies

$$E\left[\int_0^T Z_{\tilde{\pi}_y}(s)\beta(s)C(s)N(s\wedge\tau_n)\,ds + Z_{\tilde{\pi}_y}(T)\beta(T)\tilde{X}(T)N(\tau_n)\right]$$

$$(4.32)\qquad \begin{aligned} &= E\int_0^{\tau_n} Z_{\tilde{\pi}_y}(s)\beta(s)C(s)N(s)\,ds \\ &\quad + E\left\{N(\tau_n)E\left[\int_{\tau_n}^T Z_{\tilde{\pi}_y}(s)\beta(s)C(s)\,ds + Z_{\tilde{\pi}_y}(T)\beta(T)\tilde{X}(T)|\mathscr{F}(\tau_n)\right]\right\} \\ &= E\left[\int_0^{\tau_n} Z_{\tilde{\pi}_y}(t)\beta(t)C(t)N(t)\,dt + Z_{\tilde{\pi}_y}(\tau_n)\beta(\tau_n)\tilde{X}(\tau_n)N(\tau_n)\right],\end{aligned}$$

so it suffices to prove that this last expression equals

$$E\int_0^{\tau_n} Z_{\tilde{\pi}_y}(t)\beta(t)\pi_y^T(t)\big(\tilde{\pi}(t)-\tilde{\pi}_y(t)\big)\,dt.$$

Since $\int_0^{\tau_n}\|\theta(s)+\sigma^{-1}(s)\tilde{\pi}_y(s)\|^2\,ds \le n$ a.s., the Novikov condition [see, e.g., Karatzas and Shreve, Corollary 3.5.13] implies that $Z_{\tilde{\pi}_y}(t\wedge\tau_n)$ is an $\{\mathscr{F}(t)\}$-martingale. Define a new probability measure P_n on $\mathscr{F}$ by $P_n(A)\triangleq E[1_A Z_{\tilde{\pi}_y}(\tau_n)]$, $\forall\, A\in\mathscr{F}$. Girsanov's theorem implies that, under P_n, the process $w_{\tilde{\pi}_y}(t\wedge\tau_n)$ is a standard Brownian motion stopped at time τ_n. According to Remark 4.5, $d(\beta(t)\tilde{X}(t)) = dM(t)-\beta(t)C(t)\,dt$, where $dM(t)=\beta(t)\pi_y^T(t)\sigma(t)\,dw_{\tilde{\pi}_y}(t)$. Therefore,

$$(4.33)\qquad \begin{aligned} d\big(\beta(t)\tilde{X}(t)N(t)\big) &= \beta(t)\tilde{X}(t)\,dN(t)+N(t)\,dM(t) \\ &\quad - N(t)\beta(t)C(t)\,dt \\ &\quad + \beta(t)\pi_y^T(t)\big(\tilde{\pi}(t)-\tilde{\pi}_y(t)\big)\,dt.\end{aligned}$$

Integrating (4.33) and taking expectation under P_n, with respect to which $N(t\wedge\tau_n)$ and $M(t\wedge\tau_n)$ are martingales, we obtain

$$(4.34)\qquad \begin{aligned} &E\left[Z_{\tilde{\pi}_y}(\tau_n)\beta(\tau_n)\tilde{X}(\tau_n)N(\tau_n)+\int_0^{\tau_n} Z_{\tilde{\pi}_y}(t)\beta(t)C(t)N(t)\,dt\right] \\ &\quad = E\int_0^{\tau_n} Z_{\tilde{\pi}_y}(t)\beta(t)\pi_y^T(t)\big(\tilde{\pi}(t)-\tilde{\pi}_y(t)\big)\,dt.\end{aligned}$$

Equation (4.27) follows from (4.31), (4.32) and (4.34). □

STRONG DUALITY THEOREM 4.8. *Let $y > 0$ be given and let $\tilde{\pi}_y \in \tilde{A}$ be optimal for the dual problem with initial condition y. Assume that (4.21) holds. With $\tilde{\pi}_y$ replacing $\tilde{\pi}$, let $C(\cdot)$ be given by (4.2), x by (4.11), and let π_y be the portfolio process whose existence is guaranteed by Lemma 4.4. Then $(C, \pi_y) \in A(x)$ and*

$$\pi_y^T(t)\tilde{\pi}(t) = 0, \qquad dt \times dP\text{-a.e.} \tag{4.35}$$

In particular, the pair (C, π_y) is optimal in the primal problem with initial wealth x, that is, (4.9) holds.

PROOF. According to Corollary 4.2 and Lemma 4.4, we need only to verify that

$$\pi_y(t) \geq 0, \qquad dt \times dP\text{-a.e.}, \tag{4.36}$$

and that (4.35) holds. Define $\tilde{\pi} = (\tilde{\pi}_1, \ldots, \tilde{\pi}_d) \in \tilde{A}$ by

$$\tilde{\pi}_j(t) \triangleq (\tilde{\pi}_y)_j(t) - \frac{(\pi_y)_j(t)}{1 + \|\pi_y(t)\|^2} 1_{\{(\pi_y)_j(t) < 0\}}, \qquad 0 \leq t \leq T, j = 1, \ldots, d.$$

Lemma 4.7 implies

$$-E\int_0^{\tau_n} Z_{\tilde{\pi}_y}(t)\beta(t)\frac{1}{1 + \|\pi_y(t)\|^2}\sum_{j=1}^d (\pi_y)_j^2(t) 1_{\{(\pi_y)_j(t) < 0\}}\, dt \geq 0,$$

from which we conclude that $(\pi_y)_j \geq 0$, $dt \times dP$-a.e. on the set $\{(t, \omega) | 0 \leq t \leq \tau_n(\omega)\}$. Because of (4.26), we have (4.36).

Now take $\tilde{\pi} = \frac{1}{2}\tilde{\pi}_y$ and apply Lemma 4.7 again to conclude

$$-E\int_0^{\tau_n} Z_{\tilde{\pi}_y}(t)\beta(t)\pi_y^T(t)\tilde{\pi}_y(t)\, dt \geq 0, \qquad n = 1, 2, \ldots. \tag{4.37}$$

Since $\pi_y(t) \geq 0$, $\tilde{\pi}_y(t) \geq 0$, $dt \times dP$-almost everywhere, (4.37) implies $\pi_y^T(t)\tilde{\pi}_y(t) = 0$, first on $\{(t, \omega) | 0 \leq t \leq \tau_n(\omega)\}$ and then on $[0, T] \times \Omega$, $dt \times dP$-almost everywhere. □

COROLLARY 4.9. *Under the assumptions of Theorem 4.8, $\tilde{V}(y) = \sup_{\xi \geq 0}\{V(\xi) - \xi y\}$.*

PROOF. With $\tilde{\pi}_y$, π_y and x as in Theorem 4.10, we have from the Weak Duality Theorem 4.1,

$$\tilde{V}(y) = \tilde{J}(y, \tilde{\pi}_y) = J(x, C, \pi_y) - xy \leq V(x) - xy \leq \sup_{\xi \geq 0}\{V(\xi) - \xi y\}.$$

The reverse inequality follows from Remark 4.3. □

The Strong Duality Theorem 4.8 begins with a dual variable $y > 0$ and an optimal dual process $\tilde{\pi}_y$, and then constructs an optimal consumption–portfolio process pair (C, π_y) for the primal problem with initial wealth x, where x

is defined in terms of y and $\tilde{\pi}_y$ by (4.11) with $\tilde{\pi}$ replaced by $\tilde{\pi}_y$. We now show how, beginning with x, to find the corresponding dual variable y which permits this construction.

THEOREM 4.10. *Assume that for every $y > 0$, there exists an optimal control process $\tilde{\pi}_y \in \tilde{A}$ for the dual problem with initial condition y. Assume further that* (4.21) *holds for every $y > 0$. For every $x > 0$, let $y_x > 0$ be a minimizer of $\tilde{V}(y) + xy$ (the existence of y_x is guaranteed by Corollary* 3.5). *Then* (4.11) *holds with $\tilde{\pi}$ replaced by $\tilde{\pi}_{y_x}$. In particular, the consumption–portfolio process (C, π_{y_x}) constructed in Theorem* 4.8 *is optimal for the primal problem with initial wealth x.*

PROOF. We are given that y_x satisfies (3.16), and must prove that

$$
\begin{aligned}
x &= E\int_0^T Z_{\tilde{\pi}_{y_x}}(t)\beta(t)I_1\big(t, y_x\beta(t)Z_{\tilde{\pi}_{y_x}}(t)\big)\,dt \\
&\quad + E\Big[Z_{\tilde{\pi}_{y_x}}(T)\beta(T)I_2\big(y\beta(T)Z_{\tilde{\pi}_{y_x}}(T)\big)\Big] \qquad (4.38)\\
&= -\frac{1}{y_x}g'_{y_x}(1),
\end{aligned}
$$

the last equality being a restatement of (4.23). We have

$$
\begin{aligned}
\inf_{\lambda>0}\left\{\tilde{J}(\lambda y_x, \tilde{\pi}_{y_x}) + \lambda x y_x\right\} &= \inf_{y>0}\left\{\tilde{J}(y, \tilde{\pi}_{y_x}) + xy\right\} \ge \inf_{y>0}\{\tilde{V}(y) + xy\} \\
&= \tilde{V}(y_x) + xy_x = \tilde{J}(y_x, \tilde{\pi}_{y_x}) + xy_x.
\end{aligned}
$$

Therefore, the function $\lambda \mapsto g_{y_x}(\lambda) + \lambda x y_x$ is minimized by $\lambda = 1$, and consequently, $g'_{y_x}(1) + xy_x = 0$. □

COROLLARY 4.11. *Under the hypotheses of Theorem* 4.10, *we have*

$$
V(x) = \min_{y>0}\{\tilde{V}(y) + xy\}, \qquad \forall\, x > 0. \tag{4.39}
$$

PROOF. Given $x > 0$, let y_x, $\tilde{\pi}_{y_x}$ and (C, π_{y_x}) be as in Theorem 4.10. These processes were constructed to satisfy (4.2)–(4.5), so (4.1) holds with equality. From (4.6) we have

$$
\begin{aligned}
V(x) &\le \min_{y>0}\{\tilde{V}(y) + xy\} = \tilde{V}(y_x) + xy_x \\
&= \tilde{J}(y_x, \tilde{\pi}_{y_x}) + xy_x = J(x, C, \pi) \le V(x).
\end{aligned}
$$
□

5. Existence of optimal dual processes. A key assumption in the Strong Duality Theorem of the previous section was the existence of an optimal dual process. In this section, we show that if

$$(5.1)\qquad -\frac{xU_2''(x)}{U_2'(x)} \le 1, \qquad -\frac{xU_1''(t,x)}{U_1'(t,x)} \le 1, \qquad \forall\, t \in [0,T],\ x > 0,$$

then for every $y > 0$, the dual problem with initial condition y has an optimal solution. The ratios appearing on the left-hand side of the inequalities in (5.1) are called the *Arrow–Pratt indices of relative risk aversion*.

LEMMA 5.1. *Let U: $[0,\infty) \to [0,\infty)$ be a utility function (Definition 2.4). Then*

$$(5.2)\qquad -\frac{xU''(x)}{U'(x)} \le 1, \qquad \forall\, x > 0,$$

if and only if the mapping from $\mathbb{R}$ to $[0,\infty)$ given by $s \mapsto \tilde{U}(e^s)$ is convex. In this case,

$$(5.3)\qquad U(\infty) = \tilde{U}(0) = \infty.$$

PROOF. From (2.20) and (3.3), we have

$$\frac{d}{ds}\tilde{U}(e^s) = U'(I(e^s))I'(e^s)e^s - e^s I(e^s) - e^{2s}I'(e^s) = -e^s I(e^s),$$

$$\frac{d^2}{ds^2}\tilde{U}(e^s) = -e^{2s}I'(e^s) - e^s I(e^s) = \frac{-e^s}{U''(I(e^s))}\frac{d}{dx}(xU'(x))\bigg|_{x=I(e^s)}.$$

Therefore, $\tilde{U}(e^s)$ is a convex function of s if and only if

$$(5.4)\qquad \frac{d}{dx}(xU'(x)) \ge 0, \qquad \forall\, x > 0.$$

But (5.4) is equivalent to (5.2). Moreover, (5.4) implies $U'(x) \ge U'(1)/x$, $\forall\, x \ge 1$, and integration of this inequality yields $U(\infty) - U(1) = \infty$. The remainder of (5.3) is a restatement of (3.2). □

Let H denote the set of all measurable, $\{\mathscr{F}(t)\}$-adapted, $\mathbb{R}^d$-valued processes $\tilde{\pi}$ satisfying $E\int_0^T \|\tilde{\pi}(t)\|^2\,dt < \infty$. We impose on H the inner product

$$(5.5)\qquad \langle \tilde{\pi}_1, \tilde{\pi}_2 \rangle \triangleq E\int_0^T \tilde{\pi}_1^T(t)\tilde{\pi}_2(t)\,dt, \qquad \forall\, \tilde{\pi}_1, \tilde{\pi}_2 \in H,$$

and we denote the associated norm by $|\tilde{\pi}| \triangleq \sqrt{\langle \tilde{\pi}, \tilde{\pi} \rangle}$, $\forall\, \tilde{\pi} \in H$. The set of dual control processes $\tilde{A}$ of Definition 3.2 is a closed convex set in the Hilbert space H. For every fixed $y > 0$, $\tilde{J}(y, \cdot\,)$ given by (3.12) is a possibly ∞-valued nonlinear functional on $\tilde{A}$. The finiteness of $\tilde{J}(y, \tilde{\pi})$ for at least some $\tilde{\pi} \in \tilde{A}$ follows from Theorem 3.4. For $\tilde{\pi} \in H \setminus \tilde{A}$, we define $\tilde{J}(y, \tilde{\pi}) = \infty$.

LEMMA 5.2. *For every $y > 0$, the extended real-valued functional $\tilde{J}(y, \cdot\,)$ is lower semicontinuous on H.*

PROOF. It suffices to show that if $\{\tilde{\pi}_n\}_{n=1}^{\infty}$ is a sequence in $\tilde{A}$ which converges in norm to $\tilde{\pi} \in \tilde{A}$, then

$$\tilde{J}(y, \tilde{\pi}) \le \liminf_{n\to\infty} \tilde{J}(y, \tilde{\pi}_n). \tag{5.6}$$

Define $y_n \triangleq \theta + \sigma^{-1}\tilde{\pi}_n$, $y \triangleq \theta + \sigma^{-1}\tilde{\pi}$, and note that $\lim_{n\to\infty} E\int_0^T \|y_n(t) - y(t)\|^2\,dt = \lim_{n\to\infty} |\, y_n - y\,|^2 = 0$,

$$\begin{aligned}\lim_{n\to\infty} E\int_0^T \big|\|y_n(t)\|^2 - \|y(t)\|^2\big|\,dt &= \lim_{n\to\infty} E\int_0^T \big|(y_n(t) - y(t))^T(y_n(t) + y(t))\big|\,dt\\ &\le \lim_{n\to\infty} |y_n - y|\cdot|y_n + y| = 0.\end{aligned}$$

It follows that

$$\begin{aligned}\lim_{n\to\infty} E\int_0^T \bigg|&\left[\int_0^t y_n(s)\,dw(s) + \frac{1}{2}\int_0^t \|y_n(s)\|^2\,ds\right]\\ &-\left[\int_0^t y(s)\,dw(s) + \frac{1}{2}\int_0^t \|y(s)\|^2\,ds\right]\bigg|\,dt = 0,\end{aligned} \tag{5.7}$$

$$\begin{aligned}\lim_{n\to\infty} E\bigg|&\left[\int_0^T y_n(t)\,dw(t) + \frac{1}{2}\int_0^T \|y_n(t)\|^2\,dt\right]\\ &-\left[\int_0^T y(t)\,dw(t) + \frac{1}{2}\int_0^T \|y(t)\|^2\,dt\right]\bigg| = 0.\end{aligned} \tag{5.8}$$

Because L^1 convergence implies convergence almost surely along a subsequence, there exists a subsequence, also denoted by $\{y_n\}_{n=1}^{\infty}$, along which the convergences in (5.7) and (5.8) are almost sure. Consequently, $\lim_{n\to\infty} Z_{\tilde{\pi}_n}(t) = Z_{\tilde{\pi}}(t)$, $dt \times P$ almost everywhere on $[0, T] \times \Omega$, and $\lim_{n\to\infty} Z_{\tilde{\pi}_n}(T) = Z_{\pi}(T)$, almost surely on Ω. Inequality (5.6) follows from Fatou's lemma and the nonnegativity of $\tilde{U}_1$ and $\tilde{U}_2$. □

LEMMA 5.3. *If U_1 and U_2 satisfy* (5.1), *then for every $y > 0$, $\tilde{J}(y, \cdot\,)$ is a convex, extended real-valued functional on H.*

PROOF. It suffices to prove convexity of $\tilde{J}(y, \cdot\,)$ on the convex set $\tilde{A}$. Let $\tilde{\pi}_1, \tilde{\pi}_2 \in \tilde{A}$ and $\lambda_1 > 0$, $\lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$ be given. The convexity of the Euclidean norm implies

$$Z_{\lambda_1\tilde{\pi}_1 + \lambda_2\tilde{\pi}_2}(t) \ge \big(Z_{\tilde{\pi}_1}(t)\big)^{\lambda_1}\big(Z_{\tilde{\pi}_2}(t)\big)^{\lambda_2}, \qquad 0 \le t \le T, \text{a.s.}$$

The monotonicity of $\tilde{U}_2$ and Lemma 5.1 imply

$$\begin{aligned}\tilde{U}_2\big(y\beta(T)Z_{\lambda_1\tilde{\pi}_1+\lambda_2\tilde{\pi}_2}(T)\big) &\le \tilde{U}_2\Big(y\beta(T)\big(Z_{\tilde{\pi}_1}(T)\big)^{\lambda_1}\big(Z_{\tilde{\pi}_2}(T)\big)^{\lambda_2}\Big)\\ &\le \lambda_1\tilde{U}_2\big(y\beta(T)Z_{\tilde{\pi}_1}(T)\big) + \lambda_2\tilde{U}_2\big(y\beta(T)Z_{\tilde{\pi}_2}(T)\big), \quad \text{a.s.}\end{aligned}$$

A similar inequality holds for $\tilde{U}_1$, and the convexity of $\tilde{J}(y, \cdot\,)$ follows. □

LEMMA 5.4. *If U_2 satisfies* (5.1), *then for every* $y > 0$, *we have*

$$\lim_{|\tilde{\pi}| \to \infty} \tilde{J}(y, \tilde{\pi}) = \infty. \tag{5.9}$$

PROOF. Let κ be a constant such that $\beta(t) \le \kappa$, $0 \le t \le T$, a.s. From the monotonicity of $\tilde{U}_2$, Lemma 5.1 and Jensen's inequality applied to the function $s \mapsto \tilde{U}_2(y\kappa e^s)$, we have for all $\tilde{\pi} \in \tilde{A}$,

$$\tilde{J}(y, \tilde{\pi}) \ge E\tilde{U}_2(y\kappa Z_{\tilde{\pi}}(T)) \ge \tilde{U}_2\left(y\kappa \exp\left(-\frac{1}{2}|\theta + \sigma^{-1}\tilde{\pi}|^2\right)\right).$$

The result follows from (2.5) and (5.3). □

DUAL EXISTENCE THEOREM 5.5. *Assume that the utility functions U_1 and U_2 satisfy* (5.1). *Then, for each* $y > 0$, *there exists an optimal solution* $\tilde{\pi}_y \in \tilde{A}$ *to the dual problem* (3.13) *with initial condition* y.

PROOF. This follows immediately from Lemmas 5.2, 5.3 and 5.4. See, for example, Ekeland and Temam (1976), Corollary 1.2.2. □

We now summarize the principal result of this work. Examples with explicit computations are provided in Xu and Shreve (1991).

COROLLARY 5.6. *Assume that U_1 and U_2 satisfy* (5.1), *and* (4.21) [*or* (4.22)] *is satisfied as well. Then, for every* $x > 0$, *the optimal consumption–investment problem has an optimal solution* (C, π). *Moreover, let* $y > 0$ *solve the equation*

$$xy + g_y'(1) = 0, \tag{5.10}$$

and let $\tilde{\pi}_y \in \tilde{A}$ *be the optimal solution for the dual problem with initial condition* y. *Then optimal consumption and wealth processes are given by*

$$C(t) = I_1\big(t, y\beta(t)Z_{\tilde{\pi}_y}(t)\big), \qquad 0 \le t \le T, \text{ a.s.}, \tag{5.11}$$

$$X(T) = I_2\big(y\beta(T)Z_{\tilde{\pi}_y}(T)\big), \quad \text{a.s.}, \tag{5.12}$$

$$X(t) = \xi(t)\big[\beta(t)Z_{\tilde{\pi}_y}(t)\big]^{-1}, \qquad 0 \le t \le T, \text{ a.s.}, \tag{5.13}$$

where

$$\xi(t) = B(t) - \int_0^t \beta(s)C(s)Z_{\tilde{\pi}_y}(s)\,ds, \qquad 0 \le t \le T, \text{ a.s.}, \tag{5.14}$$

and B is a continuous version of

$$B(t) = E\left[\int_0^T Z_{\tilde{\pi}_y}(t)\beta(t)C(t)\,dt + Z_{\tilde{\pi}_y}(T)\beta(T)X(T) | \mathscr{F}(t)\right], \tag{5.15}$$
$$0 \le t \le T, \text{ a.s.}$$

The process $B(\cdot)$ has a representation as

$$B(t) = x + \int_0^t Y^T(s)\,dw(s), \qquad 0 \le t \le T, \tag{5.16}$$

for some $\mathbb{R}^d$-valued, $\{\mathcal{F}(t)\}$-progressively measurable process satisfying $\int_0^T \|Y(t)\|^2\,dt < \infty$ a.s., and in terms of Y, the optimal portfolio process is

$$\pi(t) = X(t)(\sigma^T(t))^{-1}\left[\theta(t) + \sigma^{-1}(t)\tilde{\pi}_y(t) + \frac{1}{\xi(t)}Y(t)\right], \qquad 0 \le t \le T,\ a.s. \tag{5.17}$$

Proof. This corollary is a restatement of Theorem 4.10 which takes advantage of the Dual Existence Theorem 5.5 and the characterization (4.38) of the minimizer y_x of $\tilde{V}(y) + xy$. □

Remark 5.7. Under the assumptions of the Dual Existence Theorem 5.5, Corollary 4.9 implies that the dual value function $\tilde{V}$ is the convex conjugate of the primal value function V. When there is no utility for terminal wealth, that is, $U_2 \equiv 0$, the proof of the Dual Existence Theorem breaks down and we do not know if the conclusion of that theorem holds. However, the conclusion of Corollary 4.9 still holds, as can be proved by introducing an artificial utility for terminal wealth $U_2(x) = \varepsilon\sqrt{x}$, and then letting $\varepsilon \downarrow 0$. See Xu (1990), Theorem 5.2.1, for details.

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