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THE PRICING OF THE AMERICAN OPTION

BY RAVI MYNENI

Stanford University

This paper summarizes the essential results on the pricing of the American option.

1. Introduction. The valuation of contingent claims is prominent in the theory of modern finance. Typical claims such as call and put options are significant not only in theory but in real security markets. A call (put) option is a "right to buy (sell) a certain asset at a specified price until or at a future date." The situation can be colorfully imagined as a game where the reward is the payoff of the option and the option holder pays a fee (the option price) for playing the game. If the option specifies that the holder may exercise the right only at the given future date, the claim is termed *European*. The pricing of European puts and calls on stocks has an interesting history, beginning with Bachelier [2] in 1900. The theory only reached a satisfying level with the celebrated papers by Black and Scholes [9] and Merton [51], using certain notions of *hedging* and *arbitrage-free pricing*. These ideas were formalized and extended in Harrison and Kreps [33] and Harrison and Pliska [34] by applying the fundamental concepts of stochastic integrals and the Girsanov theorem in stochastic calculus.

A more common option, however, is one with exercise possible at any instant until the given future date. These options are termed *American*, and it is the added dimension which makes them more interesting and complex to evaluate. The earliest, and still one of the most penetrating, analysis on the pricing of the American option is by McKean [49]. There the problem of pricing the American option is transformed into a *Stefan* or *free boundary* problem. Solving the latter, McKean writes the American option price explicitly up to knowing a certain function (the *optimal stopping boundary*). This work was taken further by van Moerbeke [61], who studied properties of the optimal stopping boundary. Although the American option problem was treated as an optimal stopping problem by McKean and van Moerbeke, a financial justification using hedging arguments was given only later by Bensoussan [7] and Karatzas [39], [40].

From the theory of optimal stopping, it is well-known that the value process of the optimal stopping problem can be characterized as the smallest supermartingale majorant to the stopping reward. As such, the value process of the

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American option has a Riesz decomposition into martingale and potential processes. Recent work has identified the martingale, which values the reward at the terminal date, and the potential, which values the early exercise feature. Among finance theorists, this is sometimes known as the *early exercise premium* representation. This result can be derived as a simple consequence of McKean's free boundary formulation or as a somewhat involved, but probabilistic, result directly from the optimal stopping representation. The latter approach follows from the work of El Karoui [24] and El Karoui and Karatzas [25], [26].

In addition to the free boundary method, another major technique that can be applied to understand the option price as the value of the optimal stopping problem is that of *variational inequalities*. This method, as developed by Bensoussan and Lions [8], was applied by Jaillet, Lamberton and Lapeyre [36] (JLL hereafter) to look at the regularity of the price function and its numerical approximation. Variational inequalities are superior for understanding the discretization of the American option but lack the explicitness of the other methods.

The purpose of this paper is to review these methodologies used in the American option pricing problem. Here, we survey the existing literature, present the recent results mentioned previously and indicate a few unresolved issues. Some proofs have been omitted; the reader will find the many details and background material in the references.

To round off this overview of the American option, it must be mentioned that there is an abundance of numerical work on the subject. With the notable exception of the so-called binomial model, these algorithms contain few rigorous connections to the underlying theory. This being the case, references to this work have been relegated to the appendix.

2. The American option problem. The story begins with a model for the behavior of the primitives which, for illustrative purposes, we have not made as general as possible. All activity occurs on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$ supporting Brownian motion with a finite horizon date T. The filtration will be the canonical one augmented with the Q-null sets of $\mathcal{F}$ and $\mathcal{F}_T = \mathcal{F}$. (For details of this as well as the following stochastic concepts, consult Karatzas and Shreve [41].) As is standard to much of option pricing, our economy has the following desirable properties.

1. Ideal markets: continuous trading, infinitely divisible assets, no transaction costs, taxes, restrictions on short sales, and so on.
2. Agents (market participants) have homogeneous beliefs of asset prices and symmetric information. That is, Q and $\{\mathcal{F}_t\}$ are common to everyone.
3. Agents are nonsatiated. That is, everyone prefers more wealth to less.

There is a *savings account* representing the time value of money which appreciates at a deterministic constant rate:

$$d\beta_t = r\beta_t\, dt, \qquad t \in [0, T], \tag{1}$$

where $r \in \mathbb{R}^+$ is the *interest rate*. By convention, we set $\beta_0 = 1$.

To this we introduce a risky asset, a *stock*, whose price process is modeled as a geometric Brownian motion,

$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t, \qquad t \in [0,T], \tag{2}$$

with $S_0 > 0$, *appreciation* rate $\mu \in \mathbb{R}$ and *volatility* coefficient $\sigma \in \mathbb{R}^+$, and where W is a standard Brownian motion on the probability space.

A *trading strategy in* (β, S), or portfolio in the savings account and stock, is a progressively measurable pair of processes (ϕ_1, ϕ_2) such that $\int_0^T \phi_1^2(t)\beta_t^2\,dt < \infty$ and $\int_0^T \phi_2^2(t)S_t^2\,dt < \infty$, almost surely. The process ϕ_1 (ϕ_2) represents the amount held, or shorted, of the savings account (stock) in units. A short position in the savings account should be thought of as a loan. The idea of managing the savings account and stock to replicate an option is central to its arbitrage-free pricing. However, we need to specify which replications are allowable. A *consumption* process C is an adapted, continuous, nondecreasing process with $C_0 = 0$.

A *trading and consumption strategy in* (β, S) is a triple (ϕ_1, ϕ_2, C), where (ϕ_1, ϕ_2) is a trading strategy in (β, S), C is a consumption process and the *self-financing condition* is satisfied:

$$\begin{aligned}\phi_1(t)\beta_t + \phi_2(t)S_t &= \phi_1(0) + \phi_2(0)S_0 + \int_0^t \phi_1(u)\,d\beta_u \\ &\quad + \int_0^t \phi_2(u)\,dS_u - C_t, \qquad t \in [0,T] \text{ a.s.},\end{aligned} \tag{3}$$

with $C_0 = 0$.

The meaning of the equation is that starting with an initial wealth, all changes come from gains in stock appreciation and in interest from the savings account *less* the amount consumed.

A new probability measure is now introduced that is critical to everything that follows. Define the probability measure $\tilde{Q}$ that is equivalent to Q by the Radon-Nikodym derivative

$$\left.\frac{d\tilde{Q}}{dQ}\right|_{\mathcal{F}} = \exp\left\{-\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T - \left(\frac{\mu - r}{\sigma}\right)W_T\right\}.$$

By the Girsanov theorem, the measure $\tilde{Q}$ is the unique probability so that S/β is a martingale with respect to $\tilde{Q}$ and, moreover, it is easy to see that the stock price process must evolve according to

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t, \qquad t \in [0,T], \tag{4}$$

where

$$\tilde{W}_t \triangleq W_t + \frac{\mu - r}{\sigma}t, \qquad t \in [0,T],$$

is $\tilde{Q}$-standard Brownian motion. All of our considerations will be with respect to $\tilde{Q}$ and the price process (4).

To our trading strategies, we impose a mild restriction limiting the size and speed of the position in the stock:

$$\tilde{E}\left[\int_0^T \phi_2^2(t) S_t^2 \, dt\right] < \infty,$$

where $\tilde{E}$ denotes expectation under $\tilde{Q}$. These strategies are called *admissible* and the class of admissible trading and consumption strategies in (β, S) will be denoted by $\mathscr{A}$. The requirement of admissibility is placed to avoid pathologies like doubling (see [34]), which allows unbounded losses and creates arbitrage opportunities.

More precisely, there is an *arbitrage in* (β, S) if there is an admissible trading and consumption strategy, that is,

$$\exists\, (\phi_1, \phi_2, C) \in \mathscr{A},$$

such that, starting with an initial gain (negative cost),

$$\phi_1(0) + \phi_2(0) S_0 < 0,$$

there are no liabilities at date T, that is,

$$\phi_1(T)\beta_T + \phi_2(T) S_T \geq 0 \quad \text{a.s.}$$

Such opportunities represent the limitless creation of wealth through riskless profits and ought not to exist in well-functioning securities markets.

It is due to the fundamental work of Harrison and Kreps [33] and Harrison and Pliska [34] that, thanks to the existence of the "martingale measure" $\tilde{Q}$, there are no arbitages in (β, S). For further information, see, for example, Duffie [20]. Defining the *wealth process*,

$$X_t \triangleq \phi_1(t)\beta_t + \phi_2(t) S_t, \qquad t \in [0, T],$$

relation (3) transforms into the wealth equation:

$$(5) \quad X_t = X_0 + \int_0^t r X_u \, du + \int_0^t \sigma \phi_2(u) S_u \, d\tilde{W}_u - C_t, \qquad t \in [0, T] \text{ a.s.},$$

for $(\phi_1, \phi_2, C) \in \mathscr{A}$.

Now we take up the definition of an American claim and the issue of pricing it. A *reward* function ψ is a continuous, nonnegative function on $\mathbb{R}^+ \times [0, T]$.

An *American claim* or *option* on the stock with reward ψ and maturity T is a financial security that pays the stochastic amount $\psi(S_t, t)$ when exercised at time $t \in [0, T]$. To sell such a claim means to accept the obligation to pay the reward $\psi(S_t, t)$ to the buyer at any time $t \in [0, T]$. The horizon time T is the claim's *expiration date*.

IMPORTANT EXAMPLE. In the case of an American *call* option,

$$\psi(x, t) = (x - K)^+,$$

and in the case of an American *put* option,

$$\psi(x, t) = (K - x)^+,$$

where $K \in \mathbb{R}^+$ is the *exercise price*.

Since the exercise of the option must be based on the information accumulated to date and not on future prices of the stock, we require that an agent's

exercise policy be restricted to a *stopping time* of the filtration $\{\mathscr{F}_t\}$. Denote the value process of the American option by $(V_t)_{0 \le t \le T}$. Having introduced this asset into the market, we now need to extend the definitions of trading strategy and arbitrage. First, we introduce some notation.

NOTATION. A tilded operator or process will denote an operator or process having properties with respect to the measure $\tilde{Q}$. The set $\mathscr{T}_{t_1, t_2}$ denotes all stopping times taking values in $[t_1, t_2]$.

Given a stopping time $\tau \in \mathscr{T}_{0,T}$, a *buy-and-hold strategy in* V is a pair (ϕ_3, τ), where ϕ_3 is the process

$$\phi_3(t) \triangleq k 1_{[0,\tau]}(t), \qquad t \in [0, T],$$

for $k \in \mathbb{R}$. Let Π^+ (Π^-) denote the set of all buy-and-hold strategies in V with $k \ge 0$ ($k < 0$). To simplify notation, we also write ϕ to stand for the triple (ϕ_1, ϕ_2, ϕ_3).

A *trading strategy in* (β, S, V) is a collection (ϕ, τ), where (ϕ_1, ϕ_2) is a trading strategy in (β, S) and (ϕ_3, τ) is a buy-and-hold strategy in V, such that, on the interval $(\tau, T]$, we have

$$\phi_1(t) = \phi_1(\tau) + \phi_2(\tau) S_\tau / \beta_\tau + \phi_3(\tau) \psi(S_\tau, \tau) / \beta_\tau,$$
$$\phi_2(t) = 0.$$

This particular trading strategy requires that at time τ we liquidate the stock and option accounts and invest the proceeds in the savings account. Notice that we do not allow dynamic trading in the American option. Our limited definition nevertheless will suffice for pricing the option.

The notions of self-financing and admissibility are almost as before. The collection (ϕ, τ, C) is an admissible *trading and consumption strategy in* (β, S, V) if (ϕ, τ) is a trading strategy in (β, S, V), the admissibility condition is satisfied, C is a consumption process and the following self-financing condition holds:

$$\phi_1(t)\beta_t + \phi_2(t) S_t = \phi_1(0) + \phi_2(0) S_0 + \int_0^t \phi_1(u)\, d\beta_u + \int_0^t \phi_2(u)\, dS_u - C_t, \qquad t \in [0, \tau] \text{ a.s.},$$

$$\int_\tau^t dC_u = 0, \qquad t \in (\tau, T] \text{ a.s.}$$

Denote the class of admissible trading and consumption strategies in (β, S, V) by $\mathscr{A}'$.

There is an *arbitrage in* (β, S, V) if either

$$\exists\, (\phi_3, \tau) \in \Pi^+ \quad \Rightarrow \quad [\exists\, (\phi_1, \phi_2, C) \text{ s.t. } (\phi, \tau, C) \in \mathscr{A}']$$

or

$$\forall\, (\phi_3, \tau) \in \Pi^- \quad \Rightarrow \quad [\exists\, (\phi_1, \phi_2, C) \text{ s.t. } (\phi, \tau, C) \in \mathscr{A}'],$$

and we have

$$\phi_1(0) + \phi_2(0) S_0 + \phi_3(0) V_0 < 0,$$
$$\phi_1(T)\beta_T \ge 0 \quad \text{a.s.}$$

The interpretation is that by holding long an American option, it should not be possible to find an exercise policy that will yield riskless profits. Conversely, by selling an option, it should not be possible to make riskless profits for *every* selection of exercise policy by the buyer.

The American option problem is to determine the price at which this option might trade in the securities market that is consistent with no arbitrage. This will be accomplished in two steps: First, we show the existence of a wealth process that hedges, in a sense to be made precise, against the payoff of the option. Then, we show that the option price must be the initial value of the obtained wealth process in order to preclude arbitrage opportunities.

For concreteness, the results in the rest of the paper are presented in the context of the American put option. The American call on a stock without dividends is known to be equivalent to its European counterpart (see [50] and [61]).

3. Optimal stopping. The following existence result, shown by Bensoussan [7] and Karatzas [39], [40], provides the basis for pricing the American option.

LEMMA 3.1. *The process*

$$X_t \triangleq \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_{t,T}} \tilde{E}\left[e^{-r(\tau-t)}(K - S_\tau)^+ | \mathscr{F}_t\right], \qquad t \in [0,T], \tag{6}$$

is a wealth process. That is, there exists $(\phi_1, \phi_2, C) \in \mathscr{A}$ *corresponding to* (6).

PROOF. This particular proof is a slight variation of Karatzas [39], [40]. Denote by J the smallest supermartingale majorant to the discounted reward (the *Snell envelope*). Then, we know (see [29] and [23])

$$J_t = \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_{t,T}} \tilde{E}\left[e^{-r\tau}(K - S_\tau)^+ | \mathscr{F}_t\right], \qquad t \in [0,T] \text{ a.s.}$$

The process J, being a right continuous with left limits (RCLL) supermartingale, regular and of class D, has a Doob–Meyer decomposition,

$$J = \tilde{M} - \Lambda,$$

where $\tilde{M}$ is a square-integrable martingale (in fact, even bounded by Theorem 3.2) and Λ is a unique, predictable, continuous, nondecreasing process with $\Lambda_0 = 0$. So,

$$d\left(e^{rt}J_t\right) = re^{rt}J_t\,dt + e^{rt}\,d\tilde{M}_t - e^{rt}\,d\Lambda_t, \qquad t \in [0,T] \text{ a.s.}$$

By the integral form of martingale representation, we can further write

$$d\tilde{M}_t = \eta_t\,d\tilde{W}_t, \qquad t \in [0,T] \text{ a.s.},$$

for some progressively measurable process η with $E[\int_0^T \eta_t^2\,dt] < \infty$. Therefore, with the identifications $\phi_1(t) = J_t - \eta_t\sigma^{-1}$, $\phi_2(t) = e^{rt}\eta_t\sigma^{-1}S_t^{-1}$ and $C_t = \int_0^t e^{ru}\,d\Lambda_u$, it follows that $X = (e^{rt}J_t)_{0 \le t \le T}$ is a wealth process. □

REMARK 3.1. From the majorizing property of the Snell envelope, we observe that X defined by (6) hedges against the American put option's reward in the following sense:

$$X_t \geq (K - S_t)^+, \qquad t \in [0, T) \text{ a.s.},$$
$$X_T = (K - S_T)^+ \quad \text{a.s.}$$

REMARK 3.2. The optimal stopping time ρ_t for the interval $[t, T]$ is also known (see [29], [23] and [59]) to be the first instant J drops to the level of the discounted reward:

$$\rho_t = \inf\{u \in [t, T] | J_u = e^{-ru}(K - S_u)^+\} \quad \text{a.s.} \tag{7}$$

Moreover, the stopped process $(J_{u \wedge \rho_t})_{t \leq u \leq T}$ is a martingale (that is, Λ is constant on the interval $[t, \rho_t]$).

The previous technical result shows that the price of the American option cannot exceed X_0. We can also show the reverse inequality. The arguments used in the next result are at the heart of all option pricing.

THEOREM 3.1. *Let X be defined by* (6). *If V_0 is the initial value of the American put option, then*

$$V_0 = X_0 \tag{8}$$

is necessary for no arbitrage in (β, S, V).

PROOF. Suppose that the actual market price of this option were $V_0 > X_0$. Let (η, ξ) and c form the admissible investment and consumption process of (6). Consider the trading strategy in (β, S, V) given by an exercise policy $\tau \in \mathscr{T}_{0,T}$ selected by the buyer and the following asset positions and consumption plan:

$$\phi_1(t) = \begin{cases} \eta_t, & t \in [0, \tau], \\ \eta_\tau + \xi_\tau S_\tau/\beta_\tau - (K - S_\tau)^+/\beta_\tau, & t \in (\tau, T], \end{cases}$$
$$\phi_2(t) = \xi_t 1_{[0,\tau]}(t),$$
$$\phi_3(t) = -1_{[0,\tau]}(t),$$
$$C_t = c_{t \wedge \tau}.$$

Then from the hedging property, that is,

$$\eta_\tau \beta_\tau + \xi_\tau S_\tau \geq (K - S_\tau)^+ \quad \text{a.s.},$$

it follows that

$$\phi_1(T)\beta_T \geq 0 \quad \text{a.s.}$$

But, by construction,

$$\phi_1(0) + \phi_2(0)S_0 + \phi_3(0)V_0 = X_0 - V_0 < 0,$$

and therefore we have an arbitrage in (β, S, V).

Similarly, suppose that $V_0 < X_0$. Again, let (η, ξ) and c form the admissible investment and consumption process of (6). Consider the trading strategy in (β, S, V) given by the exercise policy ρ_0 (the optimal stopping time) and the following asset positions and consumption plan:

$$\phi_1(t) = \begin{cases} -\eta_t, & t \in [0, \rho_0], \\ -\eta_{\rho_0} - \xi_{\rho_0} S_{\rho_0}/\beta_{\rho_0} + (K - S_{\rho_0})^+/\beta_{\rho_0}, & t \in (\rho_0, T], \end{cases}$$

$$\phi_2(t) = -\xi_t 1_{[0,\rho_0]}(t),$$

$$\phi_3(t) = 1_{[0,\rho_0]}(t),$$

$$C_t = -c_{t \wedge \rho_0}.$$

But $c \equiv 0$ on $[0, \rho_0]$ by Remark 3.2, and

$$\eta_{\rho_0}\beta_{\rho_0} + \xi_{\rho_0} S_{\rho_0} = (K - S_{\rho_0})^+ \quad \text{a.s.},$$

so,

$$\phi_1(T)\beta_T \geq 0 \quad \text{a.s.}$$

Again, by construction,

$$\phi_1(0) + \phi_2(0) S_0 + \phi_3(0) V_0 = V_0 - X_0 < 0,$$

and there is an arbitrage in (β, S, V). □

REMARK 3.3. Let us note that, more generally, the value function

$$P(x,t) \triangleq \sup_{\tau \in \mathcal{T}_{t,T}} \tilde{E}_x\left[e^{-r(\tau - t)}(K - S_\tau)^+\right], \tag{9}$$

where $\tilde{E}_x$ denotes expectation conditioned on $S_t = x$, is the *arbitrage-free* price of the American put option at time $t \in [0, T]$. In the terminology of potential theory, P is the r-réduite of the put option's reward.

Although the solution is thus completely specified by (9) and (7), it is implicit; to have a useful form we will need to characterize it by analytical methods. For this purpose, we introduce two regions which partition the domain of the value function.

Let

$$\mathscr{C} \triangleq \{(x,t) \in \mathbb{R}^+ \times [0,T) | P(x,t) > (K - x)^+\},$$

and let its complement be

$$\mathscr{S} \triangleq \{(x,t) \in \mathbb{R}^+ \times [0,T) | P(x,t) = (K - x)^+\}.$$

Given that the American put option's reward is a continuous, convex and nonincreasing function of the state variable, the following properties for its value function in our model are straightforward.

PROPOSITION 3.1. *The American put value function P is continuous on $\mathbb{R}^+\times[0,T]$. The function $P(\cdot,t)$ is convex and nonincreasing on $\mathbb{R}^+$ for every $t\in[0,T]$. The function $P(x,\cdot)$ is nonincreasing on $[0,T]$ for every $x\in\mathbb{R}^+$.*

PROOF. See van Moerbeke [61] and JLL [36]. □

From these properties of the value function, it is clear that $\mathscr{S}$ ($\mathscr{C}$) is closed (open) and that the intervals $\mathscr{S}_t \triangleq \{x|(x,t)\in\mathscr{S}\}$ and $\mathscr{C}_t \triangleq \{x|(x,t)\in\mathscr{C}\}$ are connected for every $t\in[0,T)$. Therefore, the graph of $S_t^* \triangleq \sup\{x|x\in\mathscr{S}_t\}$, $t\in[0,T)$, is contained in $\mathscr{S}$ and the optimal stopping time implies that S_t^* provides the level at or below which optimal exercise occurs for every $t\in[0,T)$. It is now appropriate to call $\mathscr{C}$ the *continuation region*, $\mathscr{S}$ the *stopping region* and S^* the *optimal stopping boundary*.

By the Riesz decomposition theorem, we can split the supermartingale J into unique martingale and potential processes. The next theorem, an important result for the optimal stopping problem, develops the decomposition explicitly in terms of the stopping boundary.

THEOREM 3.2. *The Snell envelope of the American put option's reward has the decomposition*

$$J_t = \tilde{E}\left[e^{-rT}(K-S_T)^+|\mathscr{F}_t\right] + \tilde{E}\left[\int_t^T e^{-ru} rK 1_{\{S_u<S_u^*\}}\,du\,\middle|\,\mathscr{F}_t\right], \quad t\in[0,T] \quad a.s. \tag{10}$$

PROOF. The proof is adapted from El Karoui and Karatzas [25], [26] (see also [27]). If we introduce the optimal stopping time (7) for the interval $[t,T]$ as

$$\rho_t = \inf\{u\in[t,T)|S_u\le S_u^*\}\wedge T,$$

then, by the optimality of ρ_t, the Snell envelope is

$$J_t = \tilde{E}\left[e^{-r\rho_t}(K-S_{\rho_t})^+|\mathscr{F}_t\right], \qquad t\in[0,T].$$

Write

$$J_t = \tilde{E}\left[e^{-rT}(K-S_T)^+|\mathscr{F}_t\right] + \tilde{E}\left[e^{-r\rho_t}(K-S_{\rho_t})^+ - e^{-rT}(K-S_T)^+|\mathscr{F}_t\right], \quad t\in[0,T],$$

where the second conditional expectation (the potential) is the so-called *early exercise premium* process and notice that, from the generalized Itô rule for convex functions,

$$J_t = \tilde{E}\left[e^{-rT}(K-S_T)^+|\mathscr{F}_t\right] + \tilde{E}\left[\int_{\rho_t}^T e^{-ru} rK 1_{\{S_u<K\}}\,du - \int_{\rho_t}^T e^{-ru}\,dL_u^K(S)\,\middle|\,\mathscr{F}_t\right], \quad t\in[0,T] \text{ a.s.},$$

where $L_u^K(S)$ is the *local time* of S at level K in the interval $[0,u]$.

Define the *anticipating*, finite variation, right continuous process

$$D_t \triangleq \int_{\rho_0}^{\rho_t} e^{-ru} rK 1_{\{S_u < K\}}\, du - \int_{\rho_0}^{\rho_t} e^{-ru}\, dL_u^K(S), \qquad t \in [0,T],$$

so that,

$$J_t = \tilde{E}\big[e^{-rT}(K - S_T)^+ | \mathcal{F}_t\big] + \tilde{E}\big[D_T - D_t | \mathcal{F}_t\big], \qquad t \in [0,T].$$

From Dellacherie and Meyer ([18], 75.b), there exists a unique predictable process D^p such that

$$\tilde{E}\big[D_T^p - D_t^p | \mathcal{F}_t\big] = \tilde{E}\big[D_T - D_t | \mathcal{F}_t\big], \qquad t \in [0,T] \text{ a.s.} \tag{11}$$

and $D_0^p = 0$. The process D^p is the *dual predictable projection* of D. By the Doob–Meyer decomposition, we see that $\Lambda = D^p$ and so D^p is a nondecreasing process.

Now, separate the integrals into two pieces:

$$D = A + B,$$

where

$$A_t \triangleq \int_{\rho_0}^{\rho_t} e^{-ru} rK 1_{\{S_u < K\}} 1_{\{S_u \le S_u^*\}}\, du - \int_{\rho_0}^{\rho_t} e^{-ru} 1_{\{S_u \le S_u^*\}}\, dL_u^K(S), \qquad t \in [0,T],$$

$$B_t \triangleq \int_{\rho_0}^{\rho_t} e^{-ru} rK 1_{\{S_u < K\}} 1_{\{S_u > S_u^*\}}\, du - \int_{\rho_0}^{\rho_t} e^{-ru} 1_{\{S_u > S_u^*\}}\, dL_u^K(S), \qquad t \in [0,T].$$

Since $S_t^* < K$ for $t \in [0,T)$ and dL^K does not charge $\{S < K\}$,

$$\begin{aligned} A_t &= \int_{\rho_0}^{\rho_t} e^{-ru} rK 1_{\{S_u \le S_u^*\}}\, du \quad \text{a.s.} \\ &= \int_0^t e^{-ru} rK 1_{\{S_u < S_u^*\}}\, du \quad \text{a.s.}, \end{aligned}$$

and we have that A is a predictable process. Therefore, $A^p = A$ and easily can be seen to be nondecreasing. The dual predictable projection of B does not come as cheaply. The proof is made simpler if we assume continuity of S^*, as we shall, although it is really not necessary to assume this; see [26].

Let $\chi(\omega)$ denote the excursion intervals of the stock price process into the continuation region:

$$\chi(\omega) \triangleq \{t \in [\rho_0(\omega), T) | S_t(\omega) > S_t^*\}.$$

From the almost sure continuity of the price process and the continuity of the optimal stopping boundary, the random set χ will be the countable union of open sets almost surely. Choose $\varepsilon > 0$ and note that, for every choice, the number of excursions (N^ε) in χ with duration exceeding ε is finite. Label these intervals by (a_n, b_n) and put $N_t^\varepsilon \triangleq \sup\{1 \le n \le N^\varepsilon | a_n \le t\}$; see Figure 1.

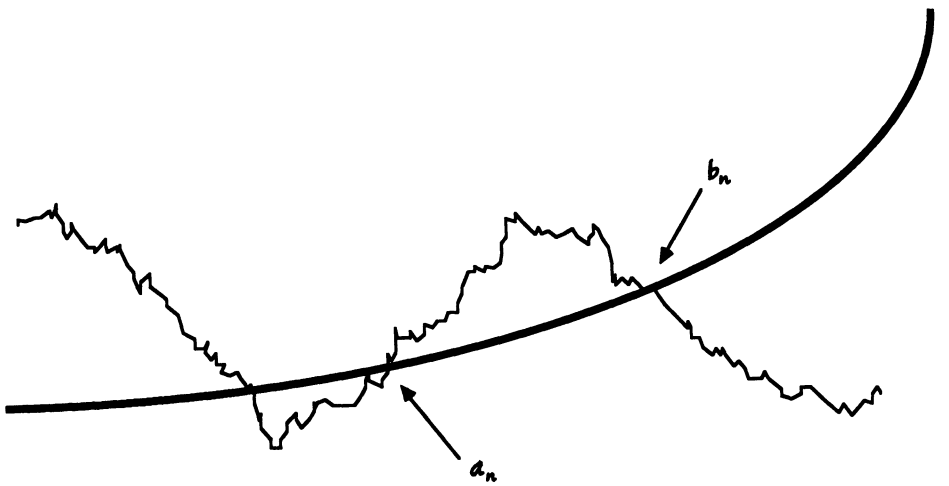

FIG. 1. *Excursions of the stock price process into the continuation region.*

By dominated convergence, the approximate process

$$B_t^\varepsilon \triangleq \sum_{n=1}^{N_t^\varepsilon} \left[\int_{a_n+\varepsilon}^{b_n} e^{-ru} rK 1_{\{S_u < K\}}\, du - \int_{a_n+\varepsilon}^{b_n} e^{-ru}\, dL_u^K(S) \right], \qquad t \in [0,T],$$

converges for every $t \in [0,T)$ to B_t in L^1 and for almost all $\omega \in \Omega$. Now, B^ε is constant off $\{t \in [0,T) | S_t \geq S_t^*\}$ and so the dual predictable projection $[B^\varepsilon]^p$ of B^ε will be constant away from there. It can also be shown (see [26] and [27]) that $[B^\varepsilon]^p$ is *nonincreasing* due to the martingale property of $(J_{u \wedge \rho_t})_{t \leq u \leq T}$. In the limit, the process B^p inherits both of these properties. Since $D^p = A^p + B^p$ is a *nondecreasing* process, the dual predictable projection of B is null. Therefore,

$$\tilde{E}[\, D_T^p - D_t^p | \mathscr{F}_t \,] = \tilde{E}\left[\int_t^T e^{-ru} rK 1_{\{S_u < S_u^*\}}\, du \,\middle|\, \mathscr{F}_t \right], \qquad t \in [0,T] \text{ a.s.} \quad \square$$

REMARK 3.4. In effect, the supermartingale property of the Snell envelope requires B^p to be a process with nondecreasing sample paths, whereas the minimality of the Snell envelope (as manifested in its martingale nature on the continuation region) requires B^p to be a process with nonincreasing sample paths. To satisfy both constraints, $B^p \equiv 0$ and the only contribution is from the interior of the stopping region.

REMARK 3.5. The relation (11) gives the dual predictable projection D^p the financial interpretation as an exact "hedge" against the (nonadapted) reward process D. To see this a bit more clearly, we can rewrite (11) as

$$^oD - D^p = \tilde{N},$$

where $\tilde{N}$ is a martingale and the process ${}^{o}D_t \triangleq \tilde{E}[D_t|\mathcal{F}_t]$, $t \in [0,T]$, is the *optional projection* of D. The conclusion is that the difference between the "observable" part of D and its dual predictable projection is a martingale begun at 0, and so has null value. Furthermore, the consumption process of Lemma 3.1 is identified as

$$C_t = \int_0^t rK1_{\{S_u < S_u^*\}}\,du, \qquad t \in [0,T] \text{ a.s.},$$

and C is *absolutely* continuous, nondecreasing, and constant off $\{t \in [0,T)|S_t < S_t^*\}$.

Closely related to the supermartingale property of J is the r-excessivity (Lemma 5.1) of the value function P. The Riesz decomposition for excessive functions is provided immediately by Theorem 3.2. The decomposition is also known as the early exercise premium representation.

COROLLARY 3.1. *The value function of the American put option has the representation*

$$P(x,t) = p(x,t) + e(x,t), \tag{12}$$

where

$$p(x,t) \triangleq \tilde{E}_x\left[e^{-r(T-t)}(K - S_T)^+\right],$$

$$e(x,t) \triangleq \tilde{E}_x\left[\int_t^T e^{-r(u-t)}rK1_{\{S_u < S_u^*\}}\,du\right],$$

with $S_t = x$.

The terms $p(x,t)$ and $e(x,t)$ are the European put option value and the early exercise premium, respectively. The European put option value measures the option's reward assuming it is realized on the horizon date T; it can be verified directly that the map $(x,t) \mapsto p(x,t)$ defines an r-harmonic function with respect to the space–time stock price process on $\mathbb{R}^+ \times [0,T)$. The early exercise premium accumulates the advantage of being able to stop at any time over the period of the option. Indeed, the term $e^{-r\Delta}rK$ has the interpretation as the discounted gain for exercising relative to continuing when the stock process belongs to the stopping region $\mathcal{S}$ over the instant $[u, u+\Delta]$. Imagining an asset with this payoff structure, the early exercise premium of an American option is the value of this asset.

REMARK 3.6. It is left as an exercise (without consulting [26] or [35]) to show that, from the representation formula, we have

$$P(\cdot,t) \in C^1(\mathbb{R}^+), \qquad t \in [0,T).$$

This characteristic property of the value function plays a crucial rôle in the next section, where we study the problem as a differential system.

REMARK 3.7. Had we effected the decomposition differently,

$$J_t = e^{-rt}(K - S_t)^+ + \tilde{E}\left[e^{-r\rho_t}(K - S_{\rho_t})^+ - e^{-rt}(K - S_t)^+ | \mathscr{F}_t\right], \quad t \in [0, T],$$

then we would define the conditional expectation as the *delayed exercise value* process, which, by the Itô rule, becomes

$$\tilde{E}\left[-\int_t^{\rho_t} e^{-ru} rK 1_{\{S_u < K\}}\, du + \int_t^{\rho_t} e^{-ru}\, dL_u^K(S) \middle| \mathscr{F}_t\right].$$

Carrying the computations through, we arrive at the delayed exercise value representation (see [13]):

$$P(x,t) = (K - x)^+ + \tilde{E}_x\left[\int_t^T e^{-r(u-t)}\, dL_u^K(S) - \int_t^T e^{-r(u-t)} rK 1_{\{S_u^* < S_u < K\}}\, du\right]. \tag{13}$$

The delayed exercise value references the gain for stopping from the current reward, whereas the early exercise premium references the gain for stopping from the terminal reward.

Thus far, the optimal stopping boundary is unknown. For its determination, we can apply the optimality property of this curve to generate a nonlinear integral equation [see equation (20)]. Unfortunately, there appears to be no hope of solving the integral equation explicitly. A more fruitful effort would be to investigate the behavior of the optimal stopping boundary in this parametrization.

4. Free boundary formulation. We would like now to formulate the optimal stopping problem as a particular free boundary problem for the value function. The relationship of optimal stopping to free boundary problems was discovered by Mikhalevich [51], Chernoff [14] and Lindley [47]. A little later, it was rediscovered and applied to the American option problem by McKean [49]. The free boundary formulation consists primarily of a partial differential equation and its Dirichlet conditions plus a Neumann condition to determine the unknown stopping boundary S^*. This approach, interesting in itself, will provide a "simple" way to derive the Riesz decomposition of the American option and will serve to clarify that result.

For later use, we need the following facts (here subscripts refer to partial derivatives with respect to that variable):

LEMMA 4.1. *The American put value function P is smooth on $\mathscr{C}$ and for every $t \in [0, T)$, $P_x(x,t) \in [-1, 0]$ for $x \in \mathscr{C}_t$. The optimal stopping boundary S^* is continuously differentiable and nondecreasing on $[0, T)$, and $S_T^* \triangleq \lim_{t \to T} S_t^* = K$.*

PROOF. See McKean [49] and van Moerbeke [61]. □

From the martingale property of $(J_{u \wedge \rho_t})_{t \le u \le T}$ and the smoothness of P on the continuation region follows a partial differential equation for the value function of the American put option (see [61], Lemma 5).

LEMMA 4.2. *The American put value function is r-harmonic to the space–time stock price process on the continuation region and therefore on* $\mathscr{C}$:

$$\mathscr{L}\left[e^{-rt}P(x,t)\right] = 0, \tag{14}$$

where

$$\mathscr{L} \triangleq \frac{\sigma^2}{2}x^2\frac{\partial^2}{\partial x^2} + rx\frac{\partial}{\partial x} + \frac{\partial}{\partial t}.$$

Now, associated with the partial differential equation are the Dirichlet and optimality conditions given by the following proposition.

PROPOSITION 4.1. *The American put value function satisfies*

$$\lim_{x \downarrow S_t^*} P(x,t) = K - S_t^*, \qquad t \in [0,T), \tag{15}$$

$$\lim_{t \to T} P(x,t) = (K-x)^+, \qquad x \in \mathbb{R}^+, \tag{16}$$

$$\lim_{x \to +\infty} P(x,t) = 0, \qquad t \in [0,T), \tag{17}$$

$$P(x,t) \ge (K-x)^+, \qquad (x,t) \in \mathbb{R}^+ \times [0,T). \tag{18}$$

PROOF. See also McKean [49] and Jacka [36]. The first two conditions are true thanks to the optimality of stopping at S^* and the continuity of the value function (Proposition 3.1). For the third condition, merely note that $P(x,t) \le \tilde{E}_x[e^{-r\tau^*}K]$ for $x > K$ and $t \in [0,T)$, where τ^* is the first hitting time to the level K and $\tilde{E}_x[e^{-r\tau^*}] \sim x^{-1}$ ([20], page 253), which uniformly tends to zero as x tends to infinity. The last condition is again the hedging property. □

From this information alone, we cannot determine the boundary S^*. One additional condition is needed to "close" the system. This is the condition which, in the theory of optimal stopping, is known as the *principle of "smooth fit."* Alternative derivations to this lemma can be found in Grigelionis and Shiryaev [33], Bather [6] and van Moerbeke [62].

LEMMA 4.3 (Smooth fit). P_x *is continuous a.e. across the stopping boundary* S^*. *That is, for almost every* $t \in [0,T]$,

$$\lim_{x \downarrow S_t^*} P_x(x,t) = -1. \tag{19}$$

PROOF. This proof is essentially from McKean [49]. The value function is r-excessive to the space–time stock price process (Lemma 5.1), so in the sense

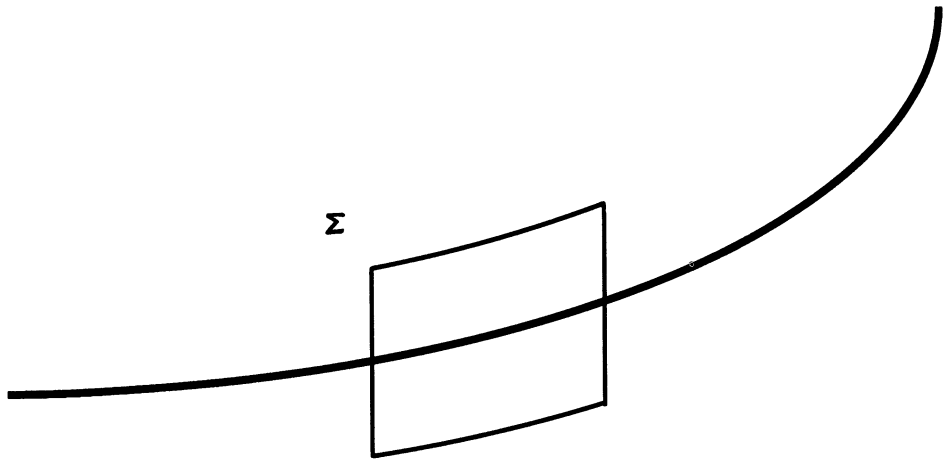

FIG. 2. *Integration across the optimal stopping boundary.*

of distributions,

$$\mathscr{L}\left[e^{-rt}P(x,t)\right] \leq 0, \qquad (x,t) \in \mathbb{R}^+ \times [0,T).$$

Introducing the new scale $\xi \triangleq \ln(x)$ and letting $\hat{P}(\xi,t) \triangleq P(\xi(x),t)$ gives

$$\frac{\sigma^2}{2}\hat{P}_{\xi\xi} \leq -\left(r - \frac{\sigma^2}{2}\right)\hat{P}_\xi - \hat{P}_t + r\hat{P}.$$

The arguments of $\hat{P}$ and its derivatives have been suppressed for notational ease.

Now integrate this expression on a region Σ of width 2ε over the log stopping boundary ξ^*, defined by $\xi^* \triangleq \ln(S^*)$, from t_1 to t_2 as shown in Figure 2. We have

$$\int_{t_1}^{t_2} \frac{\sigma^2}{2}\left[\hat{P}_\xi(\xi^* + \varepsilon, t) - \hat{P}_\xi(\xi^* - \varepsilon, t)\right] dt$$

$$\leq -\int_{t_1}^{t_2}\left(r - \frac{\sigma^2}{2}\right)\left[\hat{P}(\xi^* + \varepsilon, t) - \hat{P}(\xi^* - \varepsilon, t)\right] dt - \int_\Sigma \left[\hat{P}_t + r\hat{P}\right] d\xi\, dt.$$

Defining the horizontal strips of Σ, Σ_ξ, which start at time $t^-(\xi)$ and end at time $t^+(\xi)$, this becomes

$$\int_{t_1}^{t_2} \frac{\sigma^2}{2}\left[\hat{P}_\xi(\xi^* + \varepsilon, t) - \hat{P}_\xi(\xi^* - \varepsilon, t)\right] dt$$

$$\leq -\int_{t_1}^{t_2}\left(r - \frac{\sigma^2}{2}\right)\left[\hat{P}(\xi^* + \varepsilon, t) - \hat{P}(\xi^* - \varepsilon, t)\right] dt$$

$$-\int_{\Sigma_\xi}\left[\hat{P}(\xi, t^+) - \hat{P}(\xi, t^-)\right] d\xi + \int_\Sigma r\hat{P}\, d\xi\, dt.$$

As $\varepsilon \downarrow 0$, by dominated convergence and since $\hat{P}_\xi = -e^\xi$ on the stopping region $\mathscr{S}$, we have

$$\int_{t_1}^{t_2} \left[\lim_{\xi \downarrow \xi^*} \hat{P}_\xi + e^{\xi^*} \right] dt \leq 0.$$

Observing that the spatial derivative of P is bounded from below by Lemma 4.1,

$$\hat{P}_\xi \geq -e^\xi,$$

we conclude that the slope must exhibit "smooth fit" across the boundary ξ^*. □

REMARK 4.1. In words, the integration was performed to pick up any jumps across the boundary. Excessivity forces the jump to be nonpositive and minimality forces the jump to be nonnegative. Therefore, the jump size must be null. (Compare this with Remark 3.4.)

The results from Lemma 4.2, Proposition 4.1 and Lemma 4.3 constitute a free boundary or Stefan problem for the American put option. From this characterization, we can construct the value function and the optimal stopping boundary as the solution of an integral equation. McKean's original formula involved the stopping boundary in a complicated way. The following result recovers the Riesz decomposition or the early exercise premium representation as a candidate value function.

THEOREM 4.1. *The function $P = p + e$, where the European put value p and the early exercise premium e are given as in Corollary* 3.1, *and the boundary S^* determined by the integral equation*

$$P(S_u^*, u) = K - S_u^*, \qquad u \in [t, T), \tag{20}$$

along with $S_T^* = K$, solve the free boundary problem (14)–(19).

PROOF. Damien Lamberton (private communication) offers this proof; see also Jacka [36]. By Proposition 3.1 and Lemma 4.3, the function $(x, t) \mapsto P(x, t)$ is $C^{1,0}$ and piecewise $C^{2,1}$ on $\mathbb{R}^+ \times [0, T)$. Now, in general, the time derivative may suffer a discontinuity across the stopping boundary. But with the regularity we have for the boundary in Lemma 4.1, one can show that P_t is actually continuous across S^* and, hence, everywhere (see [60], Lemma 5, and [6]). It is also seen that the spatial derivative is absolutely continuous and, because of this, an extension of Itô's rule (for example, [44], Theorem 2.10.1) enables us to write

$$e^{-r(T-t)}P(S_T, T) = P(S_t, t) + \int_t^T e^{-r(u-t)}\sigma S_u P_x(S_u, u)\, d\tilde{W}_u + \int_t^T \mathscr{L}\left[e^{-r(u-t)}P(x, u)\right](S_u, u)\, du, \qquad t \in [0, T] \text{ a.s.}$$

By Lemma 4.2, $\mathscr{L}[e^{-r(u-t)}P(x,u)] = 0$ for $(x,u) \in \mathscr{C}$, and a simple calculation shows that

$$\mathscr{L}\left[e^{-r(u-t)}P(x,u)\right] = -e^{-r(u-t)}rK, \qquad (x,u) \in \mathscr{S}.$$

So,

$$e^{-r(T-t)}P(S_T,T) = P(S_t,t) + \int_t^T e^{-r(u-t)}\sigma S_u P_x(S_u,u)\,d\tilde{W}_u$$
$$- \int_t^T e^{-r(u-t)}rK1_{\{S_u < S_u^*\}}\,du, \qquad t \in [0,T] \text{ a.s.}$$

Because P_x is bounded, the stochastic integral is a martingale. Therefore, setting $P(x,T) = (K-x)^+$ and taking expectations, we have

$$P(x,t) = \tilde{E}_x\left[e^{-r(T-t)}(K-S_T)^+\right] + \tilde{E}_x\left[\int_t^T e^{-r(u-t)}rK1_{\{S_u < S_u^*\}}\,du\right].$$

The integral equation (20) for the boundary S^* follows from applying (15) with the boundary condition in Lemma 4.1. □

REMARK 4.2. Following the proof closely, we notice the rôle played by "smooth fit" in providing enough differentiability to avoid singular pieces such as Dirac measure (local time). In financial terms, "smooth fit" enables us to adjust *continuously* the hedging portfolio across S^* so that there are no "costs" for transitions through the boundary. This is why the early exercise premium takes on such a particularly simple form.

The uniqueness and regularity of the stopping boundary from this integral equation remain open. In a different setting, Jacka [35] and especially van Moerbeke [61] are able to address these issues in their framework.

Having now a candidate for the value function, we need to verify that our solution is in fact the American option value. The next theorem, whose proof is given by van Moerbeke [61], shows that the free boundary setup does capture faithfully the optimal stopping problem.

DEFINITION. A function $g \in C^{3,1}(\mathbb{R} \times [0,T))$ has *Tychonov growth* if g, g_t, g_x, g_{xx}, g_{xt} and g_{xxx} have growth at most $e^{o(x^2)}$ uniformly on compact sets as $|x|$ tends to infinity.

THEOREM 4.2. *Suppose a continuous function $(x,t) \mapsto f(x,t)$, defined on $\mathbb{R}^+ \times [0,T)$, and an open domain $\mathscr{D} \subseteq \mathbb{R}^+ \times [0,T)$ with continuously differentiable boundary b and such that $f \in C^{3,1}$ on $\mathscr{D}$ with $g(x,t) \triangleq f(e^x,t)$ having Tychonov growth, satisfy the following conditions*:

$$\mathscr{L}\left[e^{-rt}f(x,t)\right] = 0 \quad \text{on } \mathscr{D},$$
$$f(x,T) = (K-x)^+, \qquad x \in \mathbb{R}^+,$$
$$f(x,t) > (K-x)^+ \quad \text{on } \mathscr{D} \quad \text{and} \quad f(x,t) = (K-x)^+ \quad \text{on } \mathscr{D}^c,$$
$$\lim_{x \downarrow b_t} f_x(x,t) = -1, \qquad t \in [0,T).$$

Then f is the American put value function P, $\mathscr{D}$ is the continuation region $\mathscr{C}$ and b is the optimal stopping boundary S^*.

Assuming the integral equation (20) has a continuously differentiable solution, it is easily checked that our previous result in Theorem 4.1 satisfies the conditions of this theorem. Now we turn to the variational formulation of the problem as a way of avoiding the need for the properties of the boundary.

5. Variational inequalities. This section provides the variational inequality approach to the American option. The formulation in terms of variational inequalities allows us to treat the domain of the option as an entire region. An advantage of this approach over the free boundary method is the lack of need to introduce the stopping boundary S^* and to treat its regularity properties as well as the uniqueness of the solution to the integral equation (20). This advantage is valuable, in particular, when studying the stopping of multidimensional (diffusion) processes. On the other hand, this leads to a somewhat less explicit characterization for the option value.

On the domain, we need an extension of the harmonic property of the value function.

LEMMA 5.1. *The American put value function P is r-excessive to the space–time stock price process and therefore on $\mathbb{R}^+\times[0,T]$,*

$$\mathscr{L}\left[e^{-rt}P(x,t)\right] \leq 0, \tag{21}$$

in the sense of Schwartz distributions.

PROOF. The proof follows Dynkin and Yushkevich [22] in showing that, for every $t \in [0,T]$,

$$P(x,0) \geq \tilde{E}_x\left[e^{-rt}P(S_t,t)\right],$$

which implies (21) since any excessive function is the increasing limit of a sequence of infinitely differentiable excessive functions (for example, [54]). Choose $\varepsilon > 0$ and a stopping time τ_ε from the set,

$$\left\{\tau \in \mathscr{T}_{t,T} \middle| \tilde{E}\left[e^{-r(\tau-t)}(K-S_\tau)^+ | S_t\right] \geq P(S_t,t) - \varepsilon\right\}.$$

This set is necessarily nonempty for all $t \in [0,T)$. Then, letting $\tilde{E}_x$ denote expectation conditional on $S_0 = x$,

$$\begin{aligned}\tilde{E}_x\left[e^{-r\tau_\varepsilon}(K-S_{\tau_\varepsilon})^+\right] &= \tilde{E}_x\left[e^{-rt}\tilde{E}\left[e^{-r(\tau_\varepsilon-t)}(K-S_{\tau_\varepsilon})^+ | S_t\right]\right] \\ &\geq \tilde{E}_x\left[e^{-rt}P(S_t,t)\right] - \varepsilon e^{-rt}.\end{aligned}$$

But, for any stopping time τ,

$$P(x,0) \geq \tilde{E}_x\left[e^{-r\tau}(K-S_\tau)^+\right],$$

so,

$$P(x,0) \geq \tilde{E}_x\left[e^{-rt}P(S_t,t)\right] - \varepsilon e^{-rt}.$$

Letting $\varepsilon \downarrow 0$ yields the desired result. □

Furthermore, from the hedging and continuity properties of the value function, we have the following proposition.

PROPOSITION 5.1. *The American put value function satisfies the following conditions*:

$$P(x,t) \geq (K-x)^+, \qquad (x,t) \in \mathbb{R}^+ \times [0,T), \tag{22}$$

$$\lim_{t \to T} P(x,t) = (K-x)^+, \qquad x \in \mathbb{R}^+. \tag{23}$$

The condition expressing the minimality of the value function, which was the "smooth fit" relation in the free boundary method, is now given by the following lemma.

LEMMA 5.2. *The American put value function satisfies the following partial differential equation on* $\mathbb{R}^+ \times [0,T]$:

$$\left(\mathscr{L}\left[e^{-rt}P(x,t)\right]\right)\left((K-x)^+ - P(x,t)\right) = 0. \tag{24}$$

PROOF. We merely indicate that, from the free boundary formulation, the value function is r-harmonic on the continuation region $\mathscr{C}$ (Lemma 4.2) and is equal to its exercise value $(K-x)^+$ on the stopping region $\mathscr{S}$. □

REMARK 5.1. The "smooth fit" condition (19) can be derived from the results (21) and (24). (See [36], Corollary 3.7.)

The following theorem due to JLL [36], based on the work of Bensoussan and Lions [8], shows that the preceding system of inequalities and equalities determines the value of the American put option. For the technical conditions, we need the following definition.

DEFINITION. The space $H^{m,\lambda}$ is the set of measurable, real-valued functions f on $\mathbb{R}$ whose distributional derivatives of order up to and including m belong to $L^2(\mathbb{R}, e^{-\lambda|u|}\,dx)$ for some positive λ. This space is given the norm

$$\|f\| \triangleq \left[\sum_{i \leq m} \int_{\mathbb{R}} \left|\partial^i f(x)\right|^2 e^{-\lambda|x|}\,dx\right]^{1/2}.$$

The space $L^2([0,T]; H^{m,\lambda})$ is the set of measurable functions $g\colon [0,T] \to H^{m,\lambda}$ such that $\int_{[0,T]} \|g(t)\|^2\,dt < \infty$.

THEOREM 5.1. *Suppose a continuous function* $(x,t) \mapsto f(x,t)$, *defined on* $\mathbb{R}^+ \times [0,T]$, *such that* $f(e^x,t) \in L^2([0,T]; H^{2,\lambda})$ *and* $f_t(e^x,t) \in L^2([0,T]; H^{0,\lambda})$,

satisfies the following system on $\mathbb{R}^+ \times [0,T]$:

$$\mathcal{L}\left[e^{-rt}f(x,t)\right] \leq 0,$$
$$f(x,t) \geq (K-x)^+,$$
$$f(x,T) = (K-x)^+,$$
$$\left(\mathcal{L}\left[e^{-rt}f(x,t)\right]\right)\left(f(x,t) - (K-x)^+\right) = 0.$$

Then f is unique and is the American put value function P.

The existence of the solution is proved in Bensoussan and Lions [8], but, of course, it is not known explicitly. Despite this, the system provides a useful characterization of the option value from which one can derive many properties of the value function. Moreover, the above setup lends itself to an important algorithm for pricing the American option. All of the details can be found in JLL [36].

APPENDIX

This section contains notes and additional references.

Section 1. For an introduction to option theory, see Cox and Rubinstein [19] and Duffie [20]. Chapter 1 of Bensoussan and Lions [8], and Zabczyk [62] provide a nice discussion of the problem of optimal stopping and its characterizations as a free boundary problem and in terms of variational inequalities. Serlet [57] is a promising exposition of American option results.

Section 2. The modern framework for arbitrage-free pricing of securities is due to Harrison and Kreps [33] and Harrison and Pliska [34]. Our approach is a reconciliation of Karatzas [39], [40] to Harrison and Pliska. General semimartingale models require greater care; see Stricker [58].

Section 3. The result establishing the solution to the American option problem in terms of optimal stopping is due to Bensoussan [7] and Karatzas [39], [40]. These papers develop the more general model. Extensions to a semimartingale framework can be found in Schweizer [55] and an application to a model of stochastic interest rates in Amin [1]. The sufficiency for no arbitrage in Theorem 3.1 has not been treated in the literature. What might be more interesting is the generalization to discrete dividends. El Karoui [23] and Shiryayev [57] are authoritative on the subject of optimal stopping.

Section 4. McKean [49] and van Moerbeke [61] are standard references in this section. Other good sources for the mathematics of the free boundary approach are Grigelionis and Shiryaev [32], Bather [6] and Kotlow [43]. An extension of the "smooth fit" condition to the problem of stopping more general processes can be found in El Karoui [24] and Chitashvili [15]. The

equivalence of McKean's original solution to the early exercise premium representation is demonstrated in Carr, Jarrow and Myneni [13], and El Karoui et al. [28] and Jamshidian [37] provide applications for bond options. In the *perpetual* case ($T = +\infty$), the value function for our model is known explicitly (see McKean [49]). Not surprisingly, it is also in this highly symmetrical instance that the first passage time density for the optimal stopping boundary (a level) can be computed.

Section 5. For the theory of variational inequalities, see Bensoussan and Lions [8]. Extensive connections to the free boundary problem can be found in Friedman [30]. The application of variational inequalities to American options is due entirely to JLL [36].

Numerical methods. The most intuitive, and perhaps the most widely used, numerical approach for determining the American option value is through dynamic programming. A particular discretization of this technique is commonly known as the binomial or Cox, Ross and Rubinstein [16] model in the finance literature. Here, one recursively solves the Bellman equation $P_i = \max\{(K - S_i)^+, e^{-r\Delta}\tilde{E}[P_{i+1}|\mathcal{F}_i]\}$ with $P_T = (K - S_T)^+$, using a binomial lattice of stock prices. Similar in spirit to this is Parkinson [53]. The convergence of the discrete rule for dynamic programming is discussed in Kushner [45] and, more recently, Lamberton and Pagès [46].

In addition, there are two common numerical recipes in use: the Brennan and Schwartz [12] method and the Geske and Johnson [31] method. The Brennan and Schwartz algorithm turns out to be justifiable for the American put option, and only in this case, through the use of variational inequalities (see JLL [36]). As yet, there is no rigorous formulation of the Geske and Johnson algorithm.

Finally, various approximations to the value function or the optimal stopping boundary have been proposed by Johnson [38], MacMillan [48], Blomeyer [10], Omberg [52], Barone-Adesi and Whaley [4], [5], Kim [42] and Barone-Adesi and Elliott [3].

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GRADUATE SCHOOL OF BUSINESS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305