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THE TAIL OF THE CONVOLUTION OF DENSITIES AND ITS APPLICATION TO A MODEL OF HIV-LATENCY TIME[1]

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Let $p(x)$ and $q(x)$ be density functions and let $(p * q)(x)$ be their convolution. Define

$$w(x) = -(d/dx)\log q(x) \quad \text{and} \quad v(x) = -(d/dx)\log p(x).$$

Under the hypothesis of the regular oscillation of the functions w and v, the asymptotic form of $(p * q)(x)$, for $x \to \infty$, is obtained. The results are applied to a model previously introduced by the author for the estimation of the distribution of HIV latency time.

1. Introduction and summary. Let $p(x)$ and $q(x)$ be probability density functions and $(p * q)(x)$ their convolution. The focus of this paper is the determination of the asymptotic form of $(p * q)(x)$ for $x \to \infty$ or $x \to b = \sup$ (support of $p * q$) on the basis of the asymptotic forms of $p(x)$ and $q(x)$. In most of this paper p and q are assumed to be of different orders of magnitude for $x \to \infty$ and q will have the role of the density with the heavier tail. The key tools for densities p and q with unbounded support are the functions $v(x) = -(d/dx)\log p(x)$ and $w(x) = -(d/dx)\log q(x)$, which are closely related to the hazard functions used in extreme value theory. The tail of q obviously dominates the tail of p whenever the reverse holds for their corresponding functions w and v. Throughout this paper it is assumed that $v(x)$ and $w(x)$ are nonnegative for all sufficiently large x. In particular, it follows that the corresponding densities are nonincreasing for such x.

Theorem 3.1 states that if $\limsup w(x) < \liminf v(x)$ for $x \to \infty$, then

$$\int_{-\infty}^{\infty} p(x-t)q(t)\,dt \sim q(x)\int_{-\infty}^{\infty} e^{tw(x)}p(t)\,dt.$$

This represents an extension of a corresponding result of Breiman (1965) and Cline (1986), stated in terms of distributions instead of densities for the case $w(x) \to c \geq 0$. Theorem 3.2 furnishes a general result under the condition $w(x)/v(x) \to 0$, which allows for even the cases $w(x) \to \infty$ or $v(x) \to 0$. It

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implies, in particular, that

$$\log \int_{-\infty}^{\infty} p(x-t) q(t)\, dt \sim \log q(x)$$

for $x \to \infty$ (Corollary 3.3). The latter result is of a form similar to the Tauberian theorems of exponential type for Laplace transforms even though the convolution integral is of a more general form than such a transform. [See Bingham, Goldie and Teugels (1987), pages 247–258.]

Theorems 4.1 and 4.2 deal with the case where $p(x)$ has its support bounded above, so that the function v is not involved. The function $w(x)$ corresponding to q and the behavior of p at the upper endpoint of its support determine the asymptotic form of the convolution.

The results of Sections 3 and 4 are applied in Section 5 to the following problem. Let X and T be independent random variables with densities $p(x)$ and $q(t)$, respectively. The conditional density of T, given $X + T = x$, is by elementary considerations, given by

$$\frac{p(x-t) q(t)}{\int_{-\infty}^{\infty} p(x-s) q(s)\, ds}. \tag{1.1}$$

In the application in Section 6, one is interested in the limiting form of this conditional density for $x \to \infty$. The theorems on the tail of the convolution are used to estimate the denominator in the ratio.

The main hypothesis employed here is that the functions v and w are of regular oscillation. This concept was introduced by the author [Berman (1982)] and applied in the context of extreme value theory for one-dimensional diffusion processes [Berman (1982, 1983, 1988)]. A positive continuous function $f(x)$, $x \geq 0$, is of regular oscillation if $\lim_{u, v \to \infty,\ u/v \to 1} f(u)/f(v) = 1$. The concept of regular oscillation is developed further in Section 2. It has similarities to certain recent extensions of concepts in the theory of regular variation, as presented, for example, by Bingham, Goldie and Teugels (1987). It is used as the basic hypothesis in the main results in this paper. Furthermore, it is used to obtain an apparently new result of independent interest in extreme value theory (Lemma 2.3).

This study was motivated by an analysis of a family of density functions arising in a model proposed for the latency time for HIV infection, that is, the time between the moment of infection and the discovery of the presence of the virus in a blood test [Berman (1990)]. It is a consequence of the analysis that the conditional density of the latency time, given that the logarithm of the T4-count in the blood is x, is of the form

$$\frac{p(x+t) q(t)}{\int_{0}^{\infty} p(x+s) q(s)\, ds},$$

where p and q are densities. The denominator may be expressed as a convolution integral $\int_{-\infty}^{0} p(x-s)q(-s)\,ds$. Our interest in the forms of the convolution for large and small x led to the current study. The application is given in Section 6.

Theorems on the form of the tail of the convolution are of interest in the study of the domains of attraction of extreme value limit distributions. Such domains are characterized by specific conditions on the asymptotic behavior of the distribution function near the supremum of its support. A general problem that has attracted attention in recent years is the effect of a convolution operator on the properties of the distribution tail which characterize its membership in a particular domain of attraction. A central role in the study of this problem has been that of the class of exponential tail distributions, that is, distribution functions F satisfying

$$\lim_{x\to\infty} \frac{1-F(x-u)}{1-F(x)} = e^{u\alpha} \tag{1.2}$$

for some $\alpha > 0$ and every u.

On the one hand, all of this recent theory is presented in the context of tails of distribution functions; and, on the other hand, the current paper is done for tails of density functions, which is a more restricted setting. The reason for the latter setting is that the object of our investigation is the limiting form of the ratio (1.1), which is given in terms of density functions. The logical relation of this work on densities to the other work on distributions is that the former is more general than previous work done in the context of densities, while the work on distributions does not require the existence of densities, so that it has an inherent edge of generality. To demonstrate this relation, consider the condition for a density function f corresponding to (1.2):

$$\lim_{x\to\infty} \frac{f(x-u)}{f(x)} = e^{u\alpha}. \tag{1.3}$$

If f satisfies (1.3), then, by l'Hôpital's rule, F satisfies (1.2), so that the latter represents a more general class of distributions. The novelty of the current work is the use of densities f much more general than those satisfying (1.3). Putting $\alpha(x) = -(d/dx)\log f(x)$, we find

$$\frac{f(x-u)}{f(x)} = \exp\left[-\int_{x-u}^{x} \alpha(y)\,dy\right].$$

Under the hypothesis of the regular oscillation of α, we have

$$\int_{x-u}^{x} \alpha(y)\,dy \sim u\alpha(x)$$

for $x \to \infty$, so that the relation (1.3) assumes the more general form

$$-\log\frac{f(x-u)}{f(x)} \sim u\alpha(x).$$

Related work on the tail of a convolution of distributions and domains of extreme value distributions has been done by Cline (1986), Goldie and Resnick (1988), Kluppelberg (1988) and Leslie (1989). References to other work in this area may be found in these papers. An earlier result is that of Breiman (1965). The author is grateful to the referee for calling these publications to his attention.

2. Regular oscillation and tail behavior of functions. Let $f(x)$ be a positive, continuously differentiable function and put

$$h(x) = -\frac{d}{dx}\log f(x); \tag{2.1}$$

then, for every x and y,

$$\frac{f(x)}{f(y)} = \exp\left(-\int_y^x h(t)\,dt\right). \tag{2.2}$$

LEMMA 2.1. *If the function f above is a density function, and, for some $c > 0$,*

$$\liminf_{t\to\infty} h(t) > c, \tag{2.3}$$

then

$$\int_{-\infty}^{\infty} e^{tc} f(t)\,dt < \infty. \tag{2.4}$$

PROOF. It suffices to show that for some $b > 0$,

$$\int_b^{\infty} e^{tc} f(t)\,dt < \infty.$$

The latter follows from (2.2) and (2.3): For sufficiently large b and some $c' > c$, $f(x) = f(b)\exp(-\int_b^x h(t)\,dt) \le f(b)e^{-c'(x-b)}$ for $x \ge b$. □

We recall the definition of regular oscillation. A positive continuous function $g(t)$ is regularly oscillating if

$$\lim_{u,u'\to\infty,\, u/u'\to 1} g(u)/g(u') = 1. \tag{2.5}$$

[See Berman (1982).]

LEMMA 2.2. *If $g(x)$ is regularly oscillating, then so is the function $G(x) = \int_b^x g(t)\,dt$ for fixed b.*

PROOF. It suffices to show that

$$\lim_{u,v\to\infty,\, u/v\to 1} \frac{|G(u) - G(v)|}{G(v)} = 0. \tag{2.6}$$

The ratio in (2.6) is representable as

$$\frac{\left|\int_1^{v/u} g(su)\,ds\right|}{\int_{b/u}^1 g(su)\,ds}. \tag{2.7}$$

By regular oscillation,

$$\left|\int_1^{v/u} g(su)\,ds\right| \sim g(u)\left|\frac{v}{u} - 1\right|;$$

furthermore, there exists θ, $0 < \theta < 1$, such that

$$\liminf_{u\to\infty} \frac{g(su)}{g(u)} > \frac{1}{2}$$

for all $\theta \le s \le 1$. Therefore, by Fatou's Lemma,

$$\begin{aligned}\liminf_{u\to\infty} \int_{b/u}^1 \frac{g(su)}{g(u)}\,ds &\ge \liminf_{u\to\infty} \int_\theta^1 \frac{g(su)}{g(u)}\,ds \\ &\ge \int_\theta^1 \liminf_{u\to\infty} \frac{g(su)}{g(u)}\,ds \ge \frac{1}{2}(1-\theta).\end{aligned}$$

It follows that the lim sup of (2.7) is at most equal to that of $2|v/u - 1|/(1-\theta)$, which is equal to 0 under the limit operation in (2.6). □

The following is closely related to classical results in extreme value limit theory. [See de Haan (1970).]

LEMMA 2.3. *Let $h(x)$ be related to the function $f(x)$ by (2.1). If $h(x)$ is regularly oscillating and*

$$\lim_{x\to\infty} xh(x) = \infty, \tag{2.8}$$

then, for every x,

$$\lim_{t\to\infty} \frac{f(t + (x/h(t)))}{f(t)} = e^{-x}. \tag{2.9}$$

PROOF. By (2.2):

$$\frac{f(t + (x/h(t)))}{f(t)} = \exp\left\{-\int_t^{t+x/h(t)} h(s)\,ds\right\}.$$

For $x \ge 0$, this has an upper bound

$$\exp\left\{-\frac{x}{h(t)} \inf\left[h(s) : t \le s \le t\left(1 + \frac{x}{th(t)}\right)\right]\right\}$$

and a lower bound

$$\exp\left\{-\frac{x}{h(t)}\sup\left[h(s): t \le s \le t\left(1+\frac{x}{th(t)}\right)\right]\right\}.$$

Under (2.8) and the assumption of the regular oscillation of $h(t)$, these two bounds have the common limit e^{-x}. The argument for $x < 0$ is similar. □

REMARK. Lemma 2.3 provides an apparently new result in extreme value theory, namely, a sufficient condition for a distribution function $F(x)$ to be in the domain of attraction of the extreme value limiting distribution $\Lambda(x) = \exp(-e^{-x})$. Let F have a density function $F'(x)$ which is positive for all sufficiently large x and define $h(x) = F'(x)/(1 - F(x))$. It follows that for some x_0 sufficiently large,

$$1 - F(x) = (1 - F(x_0))\exp\left(-\int_{x_0}^{x} h(y)\,dy\right)$$

for all $x > x_0$. As a consequence of the statement of Lemma 2.3, with $1 - F$ in the role of f, if h is regularly oscillating and satisfies (2.8), then, for all x,

$$\frac{1 - F(t + (x/h(t)))}{1 - F(t)} \to e^{-x},$$

which is a necessary and sufficient condition for the membership of F in the domain of attraction of Λ [de Haan (1970)]. This implication is different from the classical result of von Mises on the sufficiency of the condition $(d/dx)(1/h(x)) \to 0$ for $x \to \infty$.

LEMMA 2.4. *If h is of regular oscillation, then for every $k > 0$,*

$$0 < \liminf_{x\to\infty}\frac{h(kx)}{h(x)} \le \limsup_{x\to\infty}\frac{h(kx)}{h(x)} < \infty. \tag{2.10}$$

PROOF. It suffices to prove the last inequality and only in the case $k > 1$. For arbitrary $\delta > 0$, there is a finite set of real numbers $1 < k_1 < k_2 < \cdots < k_m$, with $k_m < k$, such that the ratios $k/k_m, k_m/k_{m-1}, \dots, k_2/k_1$ and $k_1/1$ are all less than $1 + \delta$. For arbitrary $\varepsilon > 0$, there exists, by the definition (2.5) of regular oscillation, a number $\delta > 0$ sufficiently small so that

$$\limsup_{x\to\infty}\frac{h(x(1+\bar\delta))}{h(x)} < 1 + \varepsilon$$

for $0 < \bar\delta < \delta$. Now choose $k_1 < \cdots < k_m$ to satisfy the condition stated above for this particular δ and write

$$\frac{h(kx)}{h(x)} = \frac{h(kx)}{h(k_m x)}\cdot\frac{h(k_m x)}{h(k_{m-1}x)}\cdots\frac{h(k_2 x)}{h(k_1 x)}\cdot\frac{h(k_1 x)}{h(x)}.$$

It follows that the lim sup in (2.10) is at most equal to $(1 + \varepsilon)^{m+1}$. □

REMARK. The relations (2.10) imply that a function of regular oscillation is 0-regularly varying in the sense of Avakumovic (1936) and Karamata (1936). Conversely, a function which is extended regularly varying [see Matuszewska (1965)] is also of regular oscillation. A complete discussion of 0-regular and extended regular variation appears in Bingham, Goldie and Teugels (1987).

LEMMA 2.5. *Let f_1 and f_2 be positive, continuously differentiable functions and put*

$$h_i(x) = -\frac{d}{dx}\log f_i(x), \qquad i = 1,2. \tag{2.11}$$

If

$$\limsup_{x\to\infty} h_1(x) < \liminf_{x\to\infty} h_2(x), \tag{2.12}$$

then there exists $\varepsilon > 0$ such that

$$\lim_{x\to\infty}\frac{f_2(x(1-\varepsilon'))}{f_1(x)} = 0 \tag{2.13}$$

for all ε', $0 < \varepsilon' < \varepsilon$.

PROOF. Let b_1 and b_2 be numbers satisfying

$$\limsup_{x\to\infty} h_1(x) < b_1 < b_2 < \liminf_{x\to\infty} h_2(x);$$

then there exists x_0 such that $h_1(x) < b_1 < b_2 < h_2(x)$ for all $x \ge x_0$. It follows that for $\varepsilon' > 0$ and $x \ge x_0(1-\varepsilon')^{-1}$,

$$\begin{aligned}\frac{f_2(x(1-\varepsilon'))}{f_1(x)} &= \frac{f_2(x_0)}{f_1(x_0)}\exp\left\{\int_{x_0}^{x} h_1(t)\,dt - \int_{x_0}^{x(1-\varepsilon')} h_2(t)\,dt\right\}\\ &\le \frac{f_2(x_0)}{f_1(x_0)}\exp\{b_1(x-x_0) - b_2(x(1-\varepsilon')-x_0)\}.\end{aligned}$$

This converges to 0 if ε' is chosen so small that $b_1 < b_2(1-\varepsilon')$. The latter choice is possible because $b_1 < b_2$. □

LEMMA 2.6. *Let f_i and h_i, $i = 1,2$, be defined as in Lemma 2.5. Assume that $xh_i(x) \to \infty$ for $x \to \infty$; that h_i is regularly oscillating, $i = 1,2$; and that*

$$\lim_{x\to\infty} h_1(x)/h_2(x) = 0. \tag{2.14}$$

For each x sufficiently large, define

$$u(x) = f_2^{-1}(f_1(x)), \tag{2.15}$$

where the inverse is well defined because $xh_i(x) \to \infty$ implies that $f_i(x)$ is ultimately decreasing. Then

$$\lim_{x\to\infty} u(x) = \infty \tag{2.16}$$

and

$$\lim_{x\to\infty} u(x)/x = 0. \tag{2.17}$$

PROOF. The monotonicity of f_1 and f_2 implies that $u(x)$ is nondecreasing; hence, the limit (2.16) exists. If the limit is denoted as u_0, then, as a consequence of (2.14), $f_2(u_0) = \lim_{x\to\infty} f_1(x) = 0$. Since $f_2(x) > 0$ for all x, it is necessary that $u_0 = \infty$, which proves (2.16).

In order to prove (2.17), let us assume the contrary, namely, that for some d, $0 < d < 1$, $\liminf_{x\to\infty} u(x)/x > d$. It follows that

$$\begin{aligned}\liminf_{x\to\infty}\frac{\log f_1(x)}{\log f_2(x)} &= \liminf_{x\to\infty}\frac{\log f_2(xu(x)/x)}{\log f_2(x)}\\ &\geq \liminf_{x\to\infty}\frac{\log f_2(dx)}{\log f_2(x)}.\end{aligned} \tag{2.18}$$

By Lemma 2.2, $-\log f_2(x)$ is regularly oscillating; hence, by Lemma 2.4, the last member of (2.18) is positive. But this leads to a contradiction because the assumption (2.14) and the representation (2.2) imply $\log f_1(x)/\log f_2(x) \to 0$. □

3. The tail of the convolution: Unbounded supports for the densities. In the following, $p(x)$ and $q(x)$ will represent density functions which are positive and continuously differentiable for all sufficiently large x. We define, for such x,

$$v(x) = -\frac{d}{dx}\log p(x), \qquad w(x) = -\frac{d}{dx}\log q(x). \tag{3.1}$$

Our aim is to estimate the convolution integral

$$\int_{-\infty}^{\infty} p(x-t)q(t)\,dt \tag{3.2}$$

for $x \to \infty$. The integral may be written as the sum

$$\left(\int_{-\infty}^{0} + \int_{0}^{x} + \int_{x}^{\infty}\right)p(x-t)q(t)\,dt. \tag{3.3}$$

By the definition (3.1) of $v(x)$, the first integral in (3.3) is equal to

$$p(x)\int_0^{\infty}\exp\left(-\int_x^{x+t} v(s)\,ds\right)q(-t)\,dt. \tag{3.4}$$

By a simple change of variable, the second integral in (3.3) is equal to

$$\int_0^x q(x-t)p(t)\,dt. \tag{3.5}$$

Finally, the third integral is representable as

$$q(x)\int_0^\infty \exp\left(-\int_x^{x+t} w(s)\,ds\right)p(-t)\,dt. \tag{3.6}$$

THEOREM 3.1. *Suppose that $v(x)$ and $w(x)$, defined in* (3.1), *are regularly oscillating, and that*

$$\limsup_{t\to\infty} w(t) < \limsup_{t\to\infty} v(t); \tag{3.7}$$

then

$$\int_{-\infty}^\infty p(x-t)q(t)\,dt \sim q(x)\int_{-\infty}^\infty e^{tw(x)}p(t)\,dt \tag{3.8}$$

for $x \to \infty$, where

$$\int_{-\infty}^\infty e^{tw(x)}p(t)\,dt \tag{3.9}$$

is bounded for all large values of x.

PROOF. As a consequence of (3.7) there exists a number c such that

$$w(t) < c < v(t) \tag{3.10}$$

for all sufficiently large t.

The portion (3.4) of the integral (3.2) is obviously of the order $p(x)$ which, by Lemma 2.5 and the assumption (3.7), is of order smaller than $q(x)$. The portion (3.5) may, for arbitrary $\varepsilon > 0$, be written as

$$\int_0^{x(1-\varepsilon)} q(x-t)p(t)\,dt + \int_{x(1-\varepsilon)}^x q(x-t)p(t)\,dt. \tag{3.11}$$

The assumption (3.7) implies that $p(x)$ is ultimately decreasing, so that the second term in (3.11) is at most equal to $p(x(1-\varepsilon))$, which, by Lemma 2.5, is of order less than $q(x)$. The first term in (3.11) is representable as

$$q(x)\int_0^{x(1-\varepsilon)} \exp\left[\int_{x-t}^x w(s)\,ds\right]p(t)\,dt. \tag{3.12}$$

The integral in (3.12) has a lim inf at least equal to $\int_0^\infty p(t)\,dt$, and a lim sup at most equal to

$$\int_0^\infty e^{tc}p(t)\,dt,$$

which, by (3.10) and Lemma 2.1, is finite.

Next we show that the integral in (3.12) may be replaced by

$$\int_0^{x(1-\varepsilon)} e^{tw(x)}p(t)\,dt \tag{3.13}$$

before passing to the limit $x \to \infty$. Indeed, the absolute difference between the two integrals is at most

$$\int_0^{x(1-\varepsilon)} e^{tw(x)} p(t) \left| \exp\left[\int_{x-t}^{x} (w(s) - w(x))\, ds \right] - 1 \right| dt. \tag{3.14}$$

By the assumed boundedness and regular oscillation of w,

$$\int_{x-t}^{x} (w(s) - w(x))\, ds \to 0$$

for $x \to \infty$ and t fixed. Furthermore, the integrand in (3.14) is dominated by the integrable function $2e^{tc}p(t)$ for all large x. Thus, the integrand in (3.14) converges to 0 for all $t > 0$ and is dominated by an integrable function, so that the integral (3.14) also converges to 0. It follows that the integral (3.13) may be substituted for the integral in (3.12), so that the latter is asymptotically equal to

$$q(x) \int_0^{x(1-\varepsilon)} e^{tw(x)} p(t)\, dt.$$

This, in turn, is asymptotically equal to

$$q(x) \int_0^{\infty} e^{tw(x)} p(t)\, dt \tag{3.15}$$

because, for large x,

$$\int_{x(1-\varepsilon)}^{\infty} e^{tw(x)} p(t)\, dt \le \int_{x(1-\varepsilon)}^{\infty} e^{tc} p(t)\, dt,$$

and the latter, by Lemma 2.1, converges to 0 for $x \to \infty$.

Finally, we consider the portion (3.6). An argument similar to that for (3.15) shows that (3.6) is asymptotically equal to

$$q(x) \int_{-\infty}^{0} e^{tw(x)} p(t)\, dt. \tag{3.16}$$

Indeed, the integral in (3.6) is equal to

$$\int_{-\infty}^{0} \exp\left(\int_{x-t}^{x} w(s)\, ds \right) p(t)\, dt,$$

and

$$\left| \int_{-\infty}^{0} \left[\exp\left(\int_{x-t}^{x} w(s)\, ds \right) - \exp(tw(x)) \right] p(t)\, dt \right| \to 0.$$

The proof is completed by noting that the right-hand member of (3.8) is the sum of (3.15) and (3.16). The proof above also shows that the lim sup of (3.9) is at most equal to $\int_{-\infty}^{\infty} e^{tc} p(t)\, dt$. □

THEOREM 3.2. *For sufficiently large x define, as in Lemma 2.6,*

$$u(x) = p^{-1}(q(x)). \tag{3.17}$$

Suppose that w and v are of regular oscillation and that

$$\lim_{x\to\infty} w(x)/v(x) = 0 \tag{3.18}$$

and

$$\lim_{x\to\infty} u(x)w(x) = \infty; \tag{3.19}$$

then

$$\limsup_{x\to\infty} \int_{-\infty}^{\infty} \frac{p(x-t)q(t)\,dt}{q(x-u(x))} \le \int_0^{\infty} p(t)\,dt \tag{3.20}$$

and

$$\liminf_{x\to\infty} \frac{\int_{-\infty}^{\infty} p(x-t)q(t)\,dt}{q(x)} \ge \int_0^{\infty} p(t)\,dt. \tag{3.21}$$

PROOF. The assumptions (3.18) and (3.19) imply the conditions $xw(x) \to \infty$ and $xv(x) \to \infty$ in the hypothesis of Lemma 2.6. Indeed, (3.18) itself implies $p(x)/q(x) \to 0$, which, in turn, implies $u(x) \le x$ for all large x. Then $xw(x) \to \infty$ is a consequence of (3.19) and $xv(x) \to \infty$ then follows from (3.18).

We apply Lemma 2.6 with q, p, w and v in the roles of f_1, f_2, h_1 and h_2, respectively. As in the proof of Theorem 3.1, we estimate the integrals (3.4), (3.5) and (3.6). The first of these, as in the latter proof, is $O(p(x))$, which, under (3.18), is $o(q(x))$; hence, it may be neglected in estimating the left-hand members of (3.20) and (3.21).

Now we estimate the integral (3.6) and show that it is $o(q(x - u(x)))$, so that it may also be ignored in establishing (3.20) as well as (3.21). For the proof, it suffices to show that

$$\lim_{x\to\infty} q(x)/q(x-u(x)) = 0. \tag{3.22}$$

Indeed the ratio $q(x)/q(x-u)$ is representable as

$$\exp\left[-\int_{x-u}^{x} w(s)\,ds\right].$$

The latter is dominated by

$$\exp\left[-u(x)\min\left(w(s): x\left(1-\frac{u(x)}{x}\right) \le s \le x\right)\right],$$

which, by (2.17) and the regular variation of w, is dominated for large x by $e^{-u(x)w(x)/2}$, which, under (3.19), converges to 0 for $x \to \infty$.

The integral (3.5) is equal to the sum

$$\int_0^{u(x)} q(x-t)p(t)\,dt + \int_{u(x)}^{x} q(x-t)p(t)\,dt. \tag{3.23}$$

Under (3.19), q is ultimately decreasing, so the first term in (3.23) is at least equal to

$$q(x)\int_0^{u(x)} p(t)\,dt \sim q(x)\int_0^{\infty} p(t)\,dt$$

for $x \to \infty$ [see (2.16)]. This completes the proof of (3.21). Similarly, the first term in (3.23) is at most equal to

$$q(x-u(x))\int_0^{\infty} p(t)\,dt. \tag{3.24}$$

To complete the proof we show that the second term in (3.23) is $o(q(x-u(x)))$. The ratio of that term to $q(x-u)$ is, by the ultimate monotonicity of p, at most

$$\frac{p(u)\int_u^x q(x-t)\,dt}{q(x-u)}$$

which, by (3.17), is at most equal to $q(x)/q(x-u)$, which, by (3.22), converges to 0. □

COROLLARY 3.3. *Under the conditions of Theorem* 3.2, *for* $x \to \infty$,

$$\log \int_{-\infty}^{\infty} p(x-t)q(t)\,dt \sim \log q(x). \tag{3.25}$$

PROOF. By Lemma 2.2, the regular oscillation of $w(t)$ implies the regular oscillation of $-\log q(x)$ for $x \to \infty$. Therefore, by (2.17),

$$\begin{aligned} -\log q(x-u(x)) &= -\log q(x[1-u(x)/x]) \\ &\sim -\log q(x). \end{aligned}$$

From this, it follows by elementary reasoning concerning (3.20) and (3.21) that (3.25) holds. □

4. The convolution tail when the support of p is bounded above. Suppose that $p(x)$ has support on $(-\infty, b]$. For simplicity we take $b = 0$.

THEOREM 4.1. *Suppose* $q(x) > 0$ *for all large* x *and* $p(x) = 0$ *for all* $x > 0$. *If* $w(x)$ *is regularly oscillating and is bounded for* $x \to \infty$, *then*

$$\int_{-\infty}^{0} q(x-t)p(t)\,dt \sim q(x)\int_{-\infty}^{0} e^{tw(x)}p(t)\,dt. \tag{4.1}$$

PROOF. The proof is essentially the same as that of Theorem 3.1. The proof of the latter depended on the condition (3.7) to ensure the boundedness of w on the interval of convergence of the moment generating function of the density p. Here the boundedness of w is explicitly assumed and the moment generating function converges everywhere because the support of p is bounded above. □

THEOREM 4.2. *Suppose that* $p(x) = 0$ *for all* $x > 0$, *and, for some* $\varepsilon > 0$, $p(x) > 0$ *for* $-\varepsilon < x < 0$. *If* $w(x)$ *is ultimately monotonic and regularly oscillating and* $w(x) \to \infty$ *for* $x \to \infty$, *then the convolution* $\int_{-\infty}^{0} q(x-t)p(t)\,dt$ *has the lower asymptotic value*

$$q(x)\int_{-\infty}^{0} e^{tw(x+\varepsilon)}p(t)\,dt \tag{4.2}$$

and the upper asymptotic value

$$q(x)\int_{-\infty}^{0} e^{tw(x)}p(t)\,dt \tag{4.3}$$

for $x \to \infty$.

PROOF. The convolution may be expressed as

$$q(x)\int_{-\infty}^{0} \exp\left(-\int_{x}^{x-t} w(s)\,ds\right)p(t)\,dt. \tag{4.4}$$

We claim: For $\varepsilon > 0$,

$$\lim_{x\to\infty} \frac{\int_{-\infty}^{-\varepsilon} \exp\left(-\int_{x}^{x-t} w(s)\,ds\right)p(t)\,dt}{\int_{-\varepsilon}^{0} \exp\left(-\int_{x}^{x-t} w(s)\,ds\right)p(t)\,dt} = 0. \tag{4.5}$$

For the proof, we note that the numerator is at most equal to

$$\int_{-\infty}^{-\varepsilon} e^{tw(x)}p(t)\,dt \le e^{-\varepsilon w(x)}.$$

The denominator is at least equal to

$$\int_{-\varepsilon}^{0} e^{tw(x+\varepsilon)}p(t)\,dt.$$

Therefore, the ratio of the denominator to the numerator is at least equal to

$$\int_{-\varepsilon}^{0} e^{tw(x+\varepsilon)+\varepsilon w(x)}p(t)\,dt$$
$$= \int_{-\varepsilon}^{0} \exp\left\{\varepsilon w(x)\left[1 + \frac{tw(x+\varepsilon)}{\varepsilon w(x)}\right]\right\}p(t)\,dt.$$

This converges to ∞ for $x \to \infty$ because $w(x+\varepsilon) \sim w(x)$ for $x \to \infty$ and so the expression $1 + tw(x+\varepsilon)/[\varepsilon w(x)]$ is ultimately positive on a fixed subset of $[-\varepsilon, 0]$ of positive measure. This confirms (4.5). It follows that the convolution (4.4) is asymptotically equal to

$$q(x)\int_{-\varepsilon}^{0} \exp\left(-\int_{x}^{x-t} w(s)\,ds\right)p(t)\,dt. \tag{4.6}$$

It follows from the monotonicity of w and from (4.6) that the convolution has the lower and upper asymptotic bounds

$$q(x)\int_{-\varepsilon}^{0} \exp(tw(x+\varepsilon))p(t)\,dt$$

and

$$q(x)\int_{-\varepsilon}^{0} \exp(tw(x))p(t)\,dt,$$

respectively. These are asymptotically equal to (4.2) and (4.3), respectively. Indeed, this is trivially true if $p(t) = 0$ a.e. $t < -\varepsilon$. On the other hand, if $p(t) > 0$ on a subset of $(-\infty, -\varepsilon)$ of positive Lebesgue measure, then the assertion is still true by virtue of the relations

$$\frac{\int_{-\infty}^{-\varepsilon} e^{tw}p(t)\,dt}{\int_{-\varepsilon}^{0} e^{tw}p(t)\,dt} \le \frac{\int_{-\infty}^{-\varepsilon} e^{tw}p(t)\,dt}{e^{-w\varepsilon}\int_{-\varepsilon}^{0} p(t)\,dt}$$

$$= \frac{\int_{-\infty}^{-\varepsilon} e^{w(t+\varepsilon)}p(t)\,dt}{\int_{-\varepsilon}^{0} p(t)\,dt} \to 0$$

for $w \to \infty$. □

Under specified conditions on the behavior of $p(x)$ near $x = 0$, more exact estimates of the convolution follow.

Corollary 4.1. *If, in addition to the conditions in the hypothesis of Theorem 4.2, it is assumed that $p(x)$ is continuous from below at $x = 0$, then*

$$\int_{-\infty}^{0} q(x-t)p(t)\,dt \sim \frac{q(x)}{w(x)}p(0-) \tag{4.7}$$

for $x \to \infty$.

Proof. The lower value (4.2) is equal to

$$\frac{q(x)}{w(x+\varepsilon)}\int_{-\infty}^{0} e^{t}p\left(\frac{t}{w(x+\varepsilon)}\right)dt \sim \frac{q(x)}{w(x+\varepsilon)}p(0-)\int_{-\infty}^{0} e^{t}\,dt,$$

which, by the regular oscillation of w, is asymptotically equal to the right-hand member of (4.7). The argument for the upper value (4.3) is similar. □

Corollary 4.2. *If, in addition to the conditions in the hypothesis of Theorem 4.2, it is assumed that $p(0) = 0$ and that $p(-t)$ is of regular* *variation of index* $\alpha \geq 0$ *for* $t \to 0+$, *then*

$$\int_{-\infty}^{0} q(x-t)p(t)\,dt \sim \frac{q(x)}{w(x)} p\left(-\frac{1}{w(x)}\right)\Gamma(\alpha+1) \tag{4.8}$$

for $x \to \infty$.

Proof. In the course of proving Theorem 4.2, we showed that, for appropriate $\varepsilon > 0$,

$$q(x)\int_{-\varepsilon}^{0} \exp(tw(x+\varepsilon))p(t)\,dt$$

was a lower asymptotic bound for the left-hand member of (4.8). Division of this bound by $q(x)p(-1/w(x))/w(x)$ and a change of the variable of integration yield

$$\int_{-\varepsilon w(x)}^{0} \exp\left(t\frac{w(x+\varepsilon)}{w(x)}\right)\frac{p(t/w(x))}{p(-1/w(x))}\,dt.$$

By the regular oscillation of w and the regular variation of p, the integrand converges for all $t < 0$ to $e^{t}(-t)^{\alpha}$, whose integral over this domain is $\Gamma(\alpha+1)$. Passage to the limit under the sign of integration is permitted by the use of the Karamata representation of $p(t)$ for $-\varepsilon \leq t < 0$. This proves that the right-hand member of (4.8) is a lower asymptotic estimate of the left-hand member. A similar argument shows that it is also an upper asymptotic estimate. □

For the sake of completeness, we close this section with a simple result for the convolution tail when both supports are bounded.

Proposition 4.1. *If* $p(x) = q(x) = 0$ *for* $x > 0$ *and* $p(-x)$ *and* $q(-x)$ *are regularly varying with indices* $\alpha > 0$ *and* $\beta > 0$, *respectively, for* $x \to 0$, *then*

$$\int_{-x}^{0} p(-x-t)q(t)\,dt \sim xp(-x)q(-x)B(\alpha,\beta)$$

for $x \downarrow 0$, *where* B *is the Beta function.*

The proof is based on the Karamata representation.

For $\alpha = \beta$ this result appears in Davis and Resnick (1991).

5. Application to the conditional distribution of a random variable, given the sum of itself and another random variable. Let X and T be independent random variables with densities $p(x)$ and $q(t)$, respectively; then $X + T$ and T have the joint density $p(x-t)q(t)$ and so the conditional density of T, given $X + T = x$ is

$$\frac{p(x-t)q(t)}{\int_{-\infty}^{\infty} p(x-s)q(s)\,ds}. \tag{5.1}$$

In this section we investigate the limiting conditional distribution of T, with appropriate normalization by functions of x for $x \to \infty$. The results on the tail of the convolution are applied to the denominator in (5.1). Our first result here is based on Theorem 3.1.

THEOREM 5.1. *Under the assumptions of Theorem* 3.1, *the conditional density of* $T - x$, *at the point* t, *given* $X + T = x$, *is, for* $x \to \infty$, *approximately equal to*

$$\frac{p(-t)e^{-tw(x)}}{\int_{-\infty}^{\infty} e^{sw(x)}p(s)\,ds}, \tag{5.2}$$

in the sense that the difference between (5.2) *and the conditional density converges to* 0 *for each* t.

PROOF. By (5.1), the conditional density of $T - x$ is

$$\frac{p(-t)q(t+x)}{\int_{-\infty}^{\infty} p(x-s)q(s)\,ds},$$

which is the ratio of

$$\frac{p(-t)q(t+x)}{q(x)} \tag{5.3}$$

to

$$\frac{\int_{-\infty}^{\infty} p(x-s)q(s)\,ds}{q(x)}. \tag{5.4}$$

By (2.2), the ratio (5.3) is representable as

$$p(-t)\exp\left(-\int_x^{x+t} w(s)\,ds\right),$$

which, by the regular oscillation and boundedness of w, differs by a negligible amount from $p(-t)e^{-tw(x)}$ for $x \to \infty$. By Theorem 3.1, the ratio (5.4) differs negligibly from

$$\int_{-\infty}^{\infty} p(s)e^{sw(x)}\,ds$$

for large x. □

Now we consider the limiting conditional density under conditions not restricted to the case where the tail of q dominates that of p. Although we were unable to find a linear normalization leading to a limiting form like (5.3), we found one leading to what may be described as limiting Lebesgue measure on the line. We first state a preliminary analytic result:

LEMMA 5.1. *Let $v(x)$, $x \geq 0$, and $w(x)$, $x \geq 0$, be continuous functions such that $v(x) \to \infty$, $w(x) \to \infty$ for $x \to \infty$. For any x such that $v(x) > \max(v(0), w(0))$, $w(x) > \max(v(0), w(0))$, there exists t, $0 < t < x$, such that*

$$v(x-t) = w(t). \tag{5.5}$$

Similarly, if v and w are continuous positive functions for $x \geq 0$ and $v(x) \to 0$ and $w(x) \to 0$ for $x \to \infty$, then, for any x such that $v(x) < \min(v(0), w(0))$, $w(x) < \min(v(0), w(0))$, there exists t, $0 < t < x$, such that (5.5) *holds. In either case, if $\tau(x)$ is the smallest t satisfying* (5.5), *then*

$$\tau(x) \to \infty \quad \text{and} \quad x - \tau(x) \to \infty \tag{5.6}$$

for $x \to \infty$.

PROOF. Since $v(x) > w(0)$ and $v(0) < w(x)$, there must exist t, $0 < t < x$, for which (5.4) holds. The result for $v(x) < w(0)$ and $v(0) > w(x)$ is similarly proved. Since $\tau(x)$ satisfies $v(x - \tau(x)) = w(\tau(x))$, the boundedness of $\tau(x)$ is inconsistent with the assumptions $v(x) \to \infty$ for $x \to \infty$ and the boundedness of w on compact sets, and is also inconsistent with the assumptions that $v(x) \to 0$ and $w(x)$ is positive and continuous (hence bounded away from 0 on compact sets). A similar argument shows that $\tau(x) \to \infty$ if and only if $x - \tau(x) \to \infty$. □

In the following theorem, the tail of one density does not necessarily dominate the tail of the other.

THEOREM 5.2. *Let $v(x)$ and $w(x)$ be of regular oscillation for $x \geq 0$ and satisfy either*

$$v(x) \to \infty \quad \text{and} \quad w(x) \to \infty \tag{5.7}$$

or

$$v(x) \to 0, \qquad w(x) \to 0 \quad \text{and} \quad xv(x) \to \infty, \qquad xw(x) \to \infty \tag{5.8}$$

for $x \to \infty$. If $\tau(x)$ is defined as the smallest solution t of (5.5), *then for all t_1, t_2, t_3, t_4 with $t_1 < t_2$ and $t_3 < t_4$,*

$$\lim_{x \to \infty} \frac{P(w(\tau(x))(T - \tau(x)) \in [t_1, t_2] \mid X + T = x)}{P(w(\tau(x))(T - \tau(x)) \in [t_3, t_4] \mid X + T = x)} = \frac{t_2 - t_1}{t_4 - t_3}. \tag{5.9}$$

PROOF. The conditional density of

$$w(\tau)(T-\tau), \tag{5.10}$$

where $\tau = \tau(x)$, at the point t is, by (5.1),

$$\frac{p(x-\tau-(t/w(\tau)))q(\tau+(t/w(\tau)))}{w(\tau)\int_{-\infty}^{\infty}p(x-s)q(s)\,ds}.$$

Therefore, the ratio in the left-hand member of (5.9) is equal to

$$\frac{\int_{t_1}^{t_2}p(x-\tau-(t/w(\tau)))q(\tau+(t/w(\tau)))\,dt}{\int_{t_3}^{t_4}p(x-\tau-(t/w(\tau)))q(\tau+(t/w(\tau)))\,dt}. \tag{5.11}$$

For the proof of (5.9), it suffices to show that

$$\lim_{x\to\infty}\int_{t_1}^{t_2}\frac{p(x-\tau-(t/w(\tau)))q(\tau+(t/w(\tau)))}{p(x-\tau)q(\tau)}\,dt = t_2 - t_1. \tag{5.12}$$

Since, by (5.5), $w(\tau) = v(x-\tau)$, we have

$$\frac{p(x-\tau-(t/w(\tau)))}{p(x-\tau)} = \exp\left\{\int_{x-\tau-t/(v(x-\tau))}^{x-\tau}v(s)\,ds\right\}.$$

The right-hand member above is a bounded monotonic function on $t_1 \le t \le t_2$ and by (5.6) and the proof of Lemma 2.3, converges everywhere to e^t. Similarly, we have

$$\frac{q(\tau+(t/w(\tau)))}{q(\tau)} = \exp\left\{-\int_{\tau}^{\tau+t/w(\tau)}w(s)\,ds\right\} \to e^{-t}.$$

Therefore, the integrand in (5.12) is uniformly bounded and has the limit 1, so that (5.12) holds. □

Theorems 4.1 and 4.2 and Corollaries 4.1 and 4.2 have analogous implications for the limit of the conditional density similar to Theorem 5.1. However, here we consider the case where q, the density of T, has support on the negative axis, so that p, the density of X, has the heavier tail. Therefore, in applying the results of Section 4 to the problem of the conditional density of T, we have to switch the role of p and q in estimating the convolution in formula (5.1). The normalization of T is $-v(x)T$ instead of $T - x$. We do not explicitly state and prove all the relevant results, but furnish an application of Corollary 4.1 as an illustration.

THEOREM 5.3. *If the conditions for the densities p and q of Corollary 4.1 are satisfied by the densities q and p of T and X, respectively, and if $q(0-) > 0$, then the conditional density of $-v(x)T$, given $X + T = x$, converges for $x \to \infty$, to the standard exponential.*

PROOF. By formula (5.1), the conditional density of $-vT$ is

$$\frac{p(x + t/v)q(-t/v)1/v}{\int_{-\infty}^{0} p(x - s)q(s)\, ds}.$$

By Corollary 4.1, with the roles of p and q reversed, the ratio is asymptotically equal to

$$\frac{p(x + t/v)}{p(x)} \frac{q(-t/v)}{q(0-)},$$

which, by Lemma 2.3, converges to e^{-t}. □

There is a similar result under the conditions of Corollary 4.2: The conditional density of $-v(x)T$ converges to the gamma density with parameter $\alpha + 1$.

6. Application to the T4-count in HIV-positive individuals. In Berman (1990), we introduced a stochastic model for the behavior of the T4-count in the blood of an HIV-infected individual and used it to show how to estimate the distribution of the time since the initial infection based on subsequent T4-counts. The model involves two distributions. The first is the prior distribution of the time between the moment of infection and the moment when the T4-count is subsequently observed. It is assumed to have a density function $q(t)$, $t > 0$. The second distribution is that of the logarithm of the T4-count at the moment of infection; this is taken to be identical with the distribution of the logarithm of the T4-count for HIV-negative individuals. The distribution is assumed to have a density function $p(t)$, $-\infty < t < \infty$. Then the distribution of the logarithm of the first T4-count for HIV-positive individuals is assumed to be the distribution of $X - \delta T$, where X and T are independent random variables having the densities p and q, respectively, and where $\delta > 0$ is a parameter representing the rate of decay of the log-T4-count after infection. Suppose, for simplicity, we take $\delta = 1$. Then the density of $X - T$ is $\int_0^\infty p(x + t)q(t)\, dt$; furthermore, the conditional density of T, given $X - T = x$, is

$$\frac{p(x + t)q(t)}{\int_0^\infty p(x + s)q(s)\, ds}, \qquad t > 0. \tag{6.1}$$

The latter is called the posterior density of the time since infection.

Berman (1990) assumed that p was a normal density and its mean and variance were estimated. Then the moments of T were estimated and these suggested that T was exponentially distributed. The resulting density for $X - T$ seemed to fit the observed histogram of log-T4-counts. The resulting posterior density (6.1) was a censored normal density, cut off on the left tail at the point $x + k$, where k is a constant depending on δ, the mean of the exponential density, and the mean and variance of the normal density. For large negative values of x, the censored normal density is obviously nearly identical with the uncensored density, so that for large negative x, the posterior density of $T + x$ has a normal limit with mean k. Let us show how the last result can be extended as a consequence of Theorem 3.1. Let p be the normal density which, as in Berman (1990), is taken to have mean 0 and variance 1. The function v is $v(x) = x$. Let the function $w(x)$ corresponding to the density $q(t)$ be nonnegative and bounded. By the symmetry of the normal density, the conditional density of $T + x$, given $X - T = x$ for $x \to -\infty$, is asymptotically the same as the conditional density of $T - x$, given $X + T = x$ for $x \to \infty$. By Theorem 5.1, this conditional density is approximately equal to the ratio (5.2). A direct computation with the normal density shows that the ratio (5.2) is equal to a normal density with mean $w(x)$ and variance 1. In the model of Berman (1990), $q(t)$ was the exponential density, where $w(x) \equiv c = (ET)^{-1}$. The result here is more general because $w(x)$ may vary with x and T is not necessarily exponentially distributed.

For large positive values of x, the normal density censored on the left at the point x is approximately exponential when suitably scaled. Indeed, if Y is a random variable with a standard normal distribution, then

$$P(x(Y - x) > y \mid Y > x) \to e^{-y} \text{ for } x \to \infty.$$

Thus the posterior density of T is, with suitable scaling, approximately exponential for large x. This may also be deduced as a special case of Theorem 5.3 with $-T$ in the place of T and $v(x) = x$. Since this theorem does not require a specific form for either density p or q, the resulting form of the posterior density holds for a general class of distributions containing the specific ones considered in Berman (1990).

The fact that the limiting form of the posterior density for $x \to -\infty$ is normal was shown to be a consequence of the assumed normality of the density p in Theorem 5.1. However, the conclusion of the theorem is still valid for a more general p and this leads to the general question of estimating p on the basis of the observed data described by Berman (1990), namely, sample values of $X - T$. The latter have the density $\int_0^\infty p(x + t)q(t)\,dt$. The precise mathematical question here is to what extent the latter integral determines $p(x)$. We will show how the values of the convolution for large x are related to those of $p(x)$. Transform the convolution to the form $\int_{-\infty}^0 p(x - t)q(-t)\,dt$ and then apply Corollary 4.1 with p and q interchanged:

$$\int_{-\infty}^{0} p(x - t)q(t)\,dt \sim q(0)p(x)/v(x). \tag{6.2}$$

The following identity holds for all positive and continuously differentiable functions p:

$$p(x) = \left[\frac{1}{p(a)} + \int_a^x \frac{d}{dy}\left(\frac{1}{p(y)}\right) dy\right]^{-1} \tag{6.3}$$

for arbitrary a and x. It follows from the definition of v that

$$\frac{d}{dx}\left(\frac{1}{p(x)}\right) = \frac{v(x)}{p(x)}. \tag{6.4}$$

Therefore, the integral appearing in (6.3) is, by (6.2) and (6.4), determined by the left-hand member of (6.2) for large x up to the constant multiple $q(0)$.

These remarks suggest the possibility of estimating the limiting posterior density of T in (5.2) on the basis of the estimated density of $X - T$ for large values of x.

7. A remark on the mode of the posterior density. If the conditional density of T, given $X + T = x$, has a critical point t, then the latter satisfies the equation $p'(x - t)q(t) = p(x - t)q'(t)$, or, equivalently, the equation

$$v(x - t) = w(t). \tag{7.1}$$

This is identical to (5.5). Thus, the solution $t = \tau(x)$ represents the mode of the conditional density if the solution is unique. The posterior density of T, given $X - T = x$, defined in (6.1), has a critical point which satisfies the corresponding equation

$$v(x + t) = -w(t). \tag{7.2}$$

In Berman (1990), p is the standard normal density, so that $v(x) = x$ and (7.2) implies that the mode t of the posterior density of T satisfies $t + w(t) = -x$. In particular, it follows that $t = -x + O(1)$ for $x \to -\infty$, if w is bounded.

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