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MATCHING RANDOM SUBSETS OF THE CUBE WITH A TIGHT CONTROL ON ONE COORDINATE

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Consider a measure μ on $[0,1]^2$, and $2n$ points $X_1, \dots, X_n, Y_1, \dots, Y_n$ that are independent and distributed according to μ. Consider $2n$ points $U_1, \dots, U_n, V_1, \dots, V_n$ that are independent and uniformly distributed on $[0,1]$. Then there exists a constant K (independent of μ) such that if $s \le \sqrt{n}/K$, with probability close to 1 we can find a one-to-one map π from $\{1, \dots, n\}$ to itself such that

$$\forall i \le n, \qquad |U_i - V_{\pi(i)}| \le \frac{K}{s},$$

$$\frac{1}{n}\sum_{i \le n} |X_i - Y_{\pi(i)}| \le K\left(\frac{s}{n}\right)^{1/2}.$$

1. Introduction. A matching of two sets of n points in a metric space is a one-to-one correspondence between these two sets. The existence of a matching for which the points that are matched are "close" (in various senses) is a way to measure the distance of the two sets. The topic of how well random sets of points can be matched is of considerable interest and depth [1, 9], and has many applications to the probabilistic analysis of certain algorithms, as is exemplified by the beautiful work of Shor [9]. There are many possible variations on these problems; only a few of them have been investigated. The theorem presented in the abstract was motivated by the work of the first-named author on a transportation problem [7]. The most natural case of this theorem is where $1/s$ [and thus also $(s/n)^{1/2}$] is of order $n^{-1/3}$, and when μ is uniform on $[0,1]^2$. However, in view of the application to [7], it is necessary to consider other values of s and general probability measures μ on $[0,1]^2$. When one considers general probability measures on $[0,1]^2$, it becomes clear that the special structure of $[0,1]^2$ plays a very little role. In order to clarify the proofs and to unify a whole family of results in one single theorem, we have decided to use a more abstract setting.

Consider a metric space (Z, d). The diameter $D(A)$ of a subset $A \subset Z$ is defined by

$$D(A) = \sup\{d(x,y);\, x, y \in A\}.$$

Given $\varepsilon > 0$, we will denote by $N(Z, \varepsilon)$ the minimum number of closed balls

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(for the distance d) in Z that can cover Z. Thus, $N(Z, \varepsilon) = 1$ for $\varepsilon \geq D(Z)$. The one single property of $[0, 1]^2$ that is relevant for the theorem presented in the abstract is that $N([0, 1]^2, \varepsilon) \leq K\varepsilon^{-2}$ for some constant K. This theorem is thus a special case of the following theorem.

THEOREM 1. *For each $\alpha > 1$, there exist constants K_1, K_2 depending on α only with the following property. Consider a metric space Z such that for some constant D we have $N(Z, \varepsilon) \leq (D/\varepsilon)^\alpha$ for all $\varepsilon > 0$. Consider a probability μ on Z, and n random points $X_1, \dots, X_n$ of Z that are independent and distributed according to μ. Consider points $U_1, \dots, U_n$ that are independent and uniformly distributed on $[0, 1]$. Consider an integer s such that $K_1 s^2 \leq n$, $s \geq n^{1-\alpha/2}$. Set $r = \lfloor n/s \rfloor$, $r' = n - rs$ (so that $r' \leq s \leq r$). Then we can find fixed points $x_1, \dots, x_r$ of Z such that with probability greater than or equal to $1 - K_1 \exp(-n \min(s^{-2}, r^{-2/\alpha})/K_2)$ one can find a one-to-one map $i \to (k(i), l(i))$ from $\{1, \dots, n\}$ to*

$$J = \{(k, l); 1 \leq k \leq r, 0 \leq l \leq s \text{ if } k \leq r', 0 \leq l \leq s - 1 \text{ if } r' < k \leq r\},$$

such that

$$\forall\, i \leq n, \qquad |U_i - l(i)/s| \leq \frac{K_1}{s}, \tag{1}$$

$$\frac{1}{n} \sum_{i \leq n} d(X_i, x_{k(i)}) \leq \frac{DK_2}{r^{1/\alpha}}. \tag{2}$$

COMMENTS.

1. The n points $(x_k, l/s)$ for $(k, l) \in J$ are the substitute for a "grid" of points evenly spread on $Z \times [0, 1]$.
2. The proof will show that the constant K_2 can be made arbitrarily small if one accepts large values of K_1.
3. This statement is easily seen to be equivalent to the statement where the random points U_i are replaced by the fixed points $u_i = i/n$.
4. An immediate consequence of this statement is a result about matching the points (X_i, U_i) with an independent copy (X_i', U_i'), in the spirit of the result mentioned in the abstract.
5. A case of special interest is when s is of order $n^{1/(1+\alpha)}$, so that r is of order $n^{\alpha/(1+\alpha)}$.
6. By homogeneity, we can assume $D = 1$.

As of today, there exist two rather different approaches to matching problems of the type studied here. One is the "transportation method" of [1]. The other could be called "the stochastic process method" and was used (implicitly) in [9], and in [8]. The hardest matching problems are in dimension 2. This the the case because all the terms of a certain series connected to the problem have the same order. This difficulty disappears in higher dimensions (our result is in "dimension $1 + \alpha$"). For that reason, it is possible to give proofs that do not use any of the delicate tools that are required in dimension

2. We have actually succeeded in writing the proofs in a rather elementary way (on the other hand, we have not hesitated to use powerful abstract principles to replace ad hoc uninspiring computations).

The paper is organized as follows. In Section 2 we perform several reductions of the problem. The heart of the proof lies in Section 3. There we show that the problem reduces to estimating the supremum of a certain Gaussian process, and we estimate this supremum. The proof is then completed in Section 4.

As our methods are not adapted to give sharp constants, we will not attempt to track the value of the constants involved. We will denote by K a number depending on α only, which may vary at each occurrence.

2. Reducing the problem. We first recall the following simple lemma.

LEMMA 1 [6]. *Consider a metric space (H,d), an integer q_0 such that $2^{-q_0} \geq D(H)$ and $q \geq q_0$. Then there exists an increasing sequence of partitions $\mathscr{P}_{q_0}, \dots, \mathscr{P}_q$ of H with the following properties:*

$$\operatorname{card} \mathscr{P}_l \leq N(H, 2^{-l}), \tag{3}$$

$$\forall A \in \mathscr{P}_l, \qquad D(A) \leq 2^{-l+2}. \tag{4}$$

We assume that the metric space Z satisfies $N(Z,\varepsilon) \leq \varepsilon^{-\alpha}$ for all $\alpha > 0$. We denote by δ_x the unit mass at x.

LEMMA 2. *Consider $r \geq 1$. Then we can find $x_1, \dots, x_r \in Z$ such that the measure $\nu = (1/r)\sum_{i \leq r} \delta_{x_i}$ has the following property. There exists a probability measure η on Z^2 such that if ψ, φ denote the projections from Z^2 to Z, we have $\mu = \psi(\eta)$, $\nu = \varphi(\eta)$, and $\int_\Omega d(\omega_1, \omega_2)\, d\eta((\omega_1, \omega_2)) \leq Kr^{-1/\alpha}$.*

COMMENT. A more intuitive way to express this is that ν can be obtained from μ by transporting the mass of μ by an average distance less than or equal to $Kr^{-1/\alpha}$, and that is what we will actually prove. The purpose of the more abstract formulation of Lemma 2 will be apparent later.

PROOF. Consider the largest integer q such that $2^{-\alpha q} \geq r$, so that

$$2^{-q} \leq 2r^{1/\alpha}. \tag{5}$$

It follows from Lemma 1 that we can find an increasing sequence $(\mathscr{P}_l)$ of partitions of Z such that for $0 \leq l \leq q$, we have

$$\operatorname{card} \mathscr{P}_l \leq 2^{l\alpha}, \tag{6}$$

$$\forall A \in \mathscr{P}_l, \qquad D(A) \leq 2^{-l+2}. \tag{7}$$

The basic procedure is as follows. Assume that we have a probability measure μ_l on Z such that

$$\forall A \in \mathscr{P}_l, \qquad r\mu_l(A) \in \mathbb{N}. \tag{8}$$

Then we show that by transporting a mass less than or equal to card $\mathscr{P}_l/r$ for a distance less than or equal to 2^{-l+2}, we can obtain a probability measure μ_{l+1} on Z such that

$$\forall A \in \mathscr{P}_{l+1}, \qquad r\mu_{l+1}(A) \in \mathbb{N}. \tag{9}$$

Consider $A \in \mathscr{P}_l$, and $B_1, \dots, B_p$, the elements of $\mathscr{P}_{l+1}$ that it contains. Clearly, we can find integers $n_1, \dots, n_p$ such that

$$\sum_{j \le p} n_j = r\mu_l(A), \tag{10}$$

while

$$\forall j \le p, \qquad |n_j - r\mu_l(B_j)| \le 1.$$

Set

$$J = \{j \le p;\, n_j \le r\mu_l(B_j)\}.$$

For each $j \in J$, we remove (in an arbitrary way) an amount of mass $\mu_l(B_j) - n_j/r$ from B_j. For $j \notin J$, we add an amount of mass $n_j/r - \mu_l(B_j)$ to B_j. This is possible without changing the mass of A by (10). The total amount of mass transported is less than or equal to j/r, and since A is of diameter less than or equal to 2^{-l+2}, it is transported over a distance less than or equal to 2^{-l+2}.

We repeat this operation inside each $A \in \mathscr{P}_l$; this concludes the basic step. We observe that the mass has been transported over an average distance

$$\frac{1}{r} 2^{-l+2} \operatorname{card} \mathscr{P}_l \le \frac{4}{r} 2^{(\alpha-1)l}.$$

We now apply the basic procedure inductively, starting with $l = 0$, $\mu_0 = \mu$. We thus go from μ to a measure μ_q that satisfies

$$\forall A \in \mathscr{P}_q, \qquad r\mu_q(A) \in \mathbb{N}$$

by transporting the mass over an average distance

$$\sum_{0 \le l \le q-1} \frac{4}{r} 2^{(\alpha-1)l} \le \frac{K}{r} 2^{(\alpha-1)q} \le K r^{-1/\alpha},$$

by (5) and since $\alpha > 1$.

As a last step, for each $A \in \mathscr{P}$, we replace the mass $\mu_q(A)$ by $\mu_q(A)\delta_{x(A)}$ for some arbitrary $x_A \in A$, thereby transporting the mass an average distance less than or equal to $2^{-q+2} \le K r^{-1/\alpha}$. This finishes the proof. □

PROPOSITION 1. *To prove Theorem* 1 *in full generality, it suffices to prove this theorem in the special case where* $\mu = \nu$ *and* $n = rs$ (ν *being constructed in Lemma* 2).

PROOF. We first show that we can replace μ by ν. It follows from Lemma 2 that one can define a couple X, Y of random variables valued in Z, such that

X is distributed as μ, while Y is distributed as ν, and that $E(d(X,Y)) \leq Kr^{-1/\alpha}$. Consider n such independent couples (X_i, Y_i). Then, by Hoeffding's inequality [3],

$$P\left(\frac{1}{n}\sum_{i\leq n} d(X_i, Y_i) \geq Kr^{-1/\alpha}\right) \leq 2\exp(-nr^{-2/\alpha}).$$

On the other hand, for all $x \in Z$, we have

$$d(x, Y_i) \leq d(x, X_i) + d(X_i, Y_i).$$

We next show that we can replace n by rs. Consider two maps π_1, π_2 from $\{1,\dots,rs\}$ to $\{1,\dots,r\}$ and $\{1,\dots,s\}$, respectively, and assume that the map $\pi = (\pi_1, \pi_2)$ is one to one and that $|U_i - \pi_2(i)/r| \leq K/r$. For $rs < i \leq n$, set $\pi_1(i) = i - rs$, and consider $\pi_2(i)$ such that $|U_i - \pi_2(i)/r| \leq 1/r$. We define $\pi_2'(i)$ as follows. If $i > rs$, $\pi_2'(i) = \pi_2(i)$. If $i \leq rs$, and if $\pi_1(i) > n - rs$, then $\pi_2'(i) = \pi_2(i)$. If $\pi_1(i) \leq n - rs$, let $i' = rs + \pi_1(i)$. If $\pi_2(i) \leq \pi_2(i')$, then $\pi_2'(i) = \pi_2(i)$. If $\pi_2(i) \geq \pi_2(i')$, then $\pi_2'(i) = \pi_1(i) + 1$. It is easy to see that $\pi = (\pi_1, \pi_2')$ is one to one from $\{1,\dots,n\}$ to J, and that $|U_i - \pi_2'(i)/r| \leq (K+1)/r$. Also,

$$\begin{aligned}\sum_{i\leq n} d\big(X_i, x_{\pi_1(i)}\big) &\leq \sum_{i\leq rs} d\big(X_i, x_{\pi_1(i)}\big) + (n - rs)\\ &\leq \sum_{i\leq rs} d\big(X_i, x_{\pi_1(i)}\big) + s.\end{aligned}$$

Now $s \leq nr^{-1/\alpha}$ since $sr^{1/\alpha} \leq sr \leq n$. □

We will work from now on with the measure ν instead of μ. Only the points $x_1,\dots,x_r$ of Z are relevant; thus, we can assume as well that $Z = \{1,\dots,r\}$ for simplicity of notation. This space will be provided with the distance d, which satisfies

$$N(Z, d, \varepsilon) \leq \varepsilon^{-\alpha} \tag{11}$$

(and is unrelated to the distance induced on Z by the distance on $\mathbb{R}$!).

If the variable U is uniform on $[0,1]$, the variable $\lceil sU \rceil$ is uniform on $1,\dots,s$. Thus, clearly, we have reduced the proof of Theorem 1 to that of the following statement.

PROPOSITION 2. *Consider integers r, s. Set $n = rs$, and assume that $s^2 \leq n$, $r^{2/\alpha} \leq n$. Consider points $(X_i)_{i\leq n}$ that are independent and uniformly distributed over the grid $G = \{1,\dots,r\} \times \{1,\dots,s\}$. Then, with probability greater than or equal to $1 - K\exp(-(n/K)\min(s^{-2}, r^{-2/\alpha}))$, we can find a one-to-one map $\pi = (\pi_1, \pi_2)$ from $\{1,\dots,n\}$ to G such that*

$$\forall\, i \leq n, \qquad |l(i) - \pi_2(i)| \leq 1, \tag{12}$$

$$\frac{1}{n}\sum_{i\leq n} d(k(i), \pi_1(i)) \leq Kr^{-1/\alpha}, \tag{13}$$

where we have set $X_i = (k(i), l(i))$.

Consider now fixed points $x_i = (k(i), l(i))$ of G. We must understand when there is a one-to-one map π for which (12) and (13) hold. For this, we will (in a standard way) use a "duality" argument, which, in the present case, can be stated as follows.

LEMMA 3. *Given points $(x_i)_{i \le n}$ in G, consider the quantity*

$$W = \min \sum_{i \le n} d(k(i), \pi_1(i)),$$

where the minimum is taken over all the possible choices of a one-to-one map $\pi = (\pi_1, \pi_2)$ for which (12) *holds. Then*

$$W \le \sup\Bigl(\sum_{i \le n} w_i - \sum_{(k,l) \in G} w(k,l)\Bigr),$$

where the sup *is taken over all the families $(w_i), (w(k,l))$ of numbers that satisfy*

$$|l' - l(i)| \le 1, \qquad k' \le r \Rightarrow w_i \le w(k', l') + d(k(i), k'). \tag{14}$$

To see this, let us observe that this is a special case of the "assignment problem" of minimizing $\sum_{i \le n} a_{i, \pi(i)}$ over all permutations of $\{1, \dots, n\}$ for a matrix $(a_{i,j})_{i,j}$ of nonnegative numbers. From the classical fact that the extreme points of the set of bistochastic matrices are permutation matrices, this is the same as minimizing $\sum_{i,j \le n} x_{i,j} a_{i,j}$ under the conditions $x_{i,j} \ge 1$, $\sum_{j \le n} x_{k,j} = 1 = \sum_{i \le n} x_{i,l}$ for all $k, l \le n$. The result then follows from the duality theorem of linear programming.

Consider the class $\mathscr{F}$ of functions on G that satisfy

$$\forall\, l \le s, \forall\, k, k' \le r, \qquad |f(k,l) - f(k',l)| \le d(k,k'), \tag{15}$$

$$\sum_{k \le r,\, l \le s} f(k,l) = 0. \tag{16}$$

We now obtain an important corollary of Lemma 3.

PROPOSITION 3.

$$W \le \sup_{f \in \mathscr{F}} \Bigl(\sum_{i \le n} f(x_i) - \tfrac{1}{2} \sum_{k \le r,\, l < s} |f(k, l+1) - f(k,l)|\Bigr). \tag{17}$$

PROOF. Consider numbers $w(k,l)$ for $(k,l) \in G$. Set

$$g(k,l) = \min(w(k',l') + d(k,k');\, |l - l'| \le 1,\, k' \le r).$$

We observe that if numbers w_i satisfy (14) we have $w_i \le g(x_i)$. Thus,

$$\sum_{i \le n} w_i - \sum_{(k,l) \in G} w(k,l)$$
$$\le \sum_{i \le n} g(x_i) - \sum_{(k,l) \in G} g(k,l) - \sum_{(k,l) \in G} (w(k,l) - g(k,l)).$$

It is easy to see from the definition of g that

$$\forall\, l \le s, \forall\, k, k' \le r, \qquad |g(k,l) - g(k',l)| \le d(k,k').$$

To conclude the proof, it suffices to show that

$$(18) \qquad \sum_{(k,l)\in G} (w(k,l) - g(k,l)) \ge \tfrac{1}{2} \sum_{k\le r,\, l<s} |g(k,l+1) - g(k,l)|.$$

[One then obtains (17) by considering the function

$$f = g - (1/n) \sum_{(k,l)\in G} g(k,l).]$$

If $g(k,l+1) \ge g(k,l)$, we note that $g(k,l+1) \le w(k,l)$ so that

$$g(k,l+1) - g(k,l) \le w(k,l) - g(k,l).$$

In the case $g(k,l+1) \le g(k,l)$, we obtain similarly

$$g(k,l) - g(k,l+1) \le w(k,l+1) - g(k,l+1).$$

Since $g(k,l) \le w(k,l)$, this gives

$$|g(k,l+1) - g(k,l)| \le (w(k,l) - g(k,l)) + (w(k,l+1) - g(k,l+1)),$$

from which (18) follows by summation. □

For $s > 0$, we consider the class

$$\mathscr{F}(S) = \left\{ f \in \mathscr{F};\ \sum_{k\le r,\, l<s} |f(k,l+1) - f(k,l)| \le S \right\}.$$

To prove Proposition 2, it suffices to prove the following fact.

Proposition 4. *One can find a constant K such that if* $S = Knr^{-1/\alpha}$, *then with probability at least* $1 - K\exp(-(n/K)\min(s^{-2}, r^{-2/\alpha}))$, *we have*

$$(19) \qquad \sup_{f\in\mathscr{F}(S)} \left| \sum_{i\le n} f(X_i) \right| \le S/2.$$

The proof of this will be the object of Sections 3 and 4.

Proof of Proposition 2. First, we show that the right-hand side of (17) is bounded by S. Indeed, consider $f \in \mathscr{F}$, and set

$$\Delta(f) = \sum_{k\le r,\, l<s} |f(k,l+1) - f(k,l)|.$$

If $\Delta(f) \le S$, then $f \in \mathscr{F}(S)$, so that $|\sum_{i\le n} f(X_i)| \le S/2$ by (19). Thus

$$\sum_{i\le n} f(X_i) - \Delta(f)/2 \le S.$$

If $\Delta(f) \ge S$, consider $g = Sf/\Delta(f)$. Thus, $\Delta(g) = S$, and, clearly, $g \in \mathscr{F}(S)$.

By (19), we have

$$\Big|\sum_{i\le n} g(X_i)\Big| \le S/2,$$

so that

$$\Big|\sum_{i\le n} f(X_i)\Big| \le \Delta(f)/2$$

and thus

$$\sum_{i\le n} f(X_i) - \Delta(f)/2 \le 0 \le S.$$

It then follows form Proposition 3 that the required matching exists. □

3. Gaussian processes. We first establish the linkage between the proof of Proposition 4 and Gaussian processes. Consider an i.i.d. sequence $(g_i)_{i\le n}$ of $N(0,1)$ random variables. We denote by E_g the conditional expectation at $X_1,\dots,X_n$ fixed. The core of the proof of Proposition 4 is the following proposition.

PROPOSITION 5. *There exists a constant* K_1 *such that if* $r^{2/\alpha-1} \le s \le r/K_1$, *then for* $S = K_1 n r^{-1/\alpha}$, *with probability at least* $1 - K_1\exp(-r/K_1)$, *we have*

$$E_g \sup_{f\in\mathscr{F}(S)} \Big|\sum_{i\le n} g_i f(X_i)\Big| \le S/16.$$

This will be established through several steps; but we first show how to deduce Proposition 4 from Proposition 5. Consider a sequence $(\varepsilon_i)_{i\le n}$ of independent Bernoulli random variables $(P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = 1/2)$ which is independent of the sequences g_i, X_i. Set

$$T_S = \sup_{f\in\mathscr{F}(S)} \Big|\sum_{i\le n} \varepsilon_i f(X_i)\Big|.$$

Then, by the "contraction principle" ([4], page 96, Lemma 4.5), if we denote by E_ε the conditional expectation given $(X_i)_{i\le n}, (g_i)_{i\le n}$, we get

$$E_\varepsilon(T_S) \le \sqrt{\frac{2}{\pi}}\, E_g \sup_{f\in\mathscr{F}(S)} \Big|\sum_{i\le n} g_i f(X_i)\Big| \le \frac{S}{16}. \tag{20}$$

LEMMA 4. *We have*

$$P_\varepsilon(T_S \ge ET_S + t) \le 2\exp\left(-\frac{t^2}{2n(2+S/r)}\right).$$

PROOF. This follows from the fact that for $f\in\mathscr{F}(S)$, we have $\max|f| \le 2 + S/r$ and by a straightforward use of martingale difference sequences and

Azuma's inequality, for example, as in [8]. Combining with (20), we have

$$P_\varepsilon\left(T_S \geq \frac{S}{8}\right) \leq 2\exp\left(-\frac{S^2}{2^7 n(2+S/r)^2}\right)$$
$$\leq 2\exp\left(-\frac{n}{K}\min\{s^{-2}, r^{-2/\alpha}\}\right).$$

We observe now that

$$r^{1-1/\alpha} = ns^{-1}r^{-1/\alpha} \geq n\min\{s^{-2}, r^{-2/\alpha}\}.$$

Thus, we have proved that

$$P\left(\sup_{f\in\mathscr{F}(S)}\left|\sum_{i\leq n}\varepsilon_i f(X_i)\right| \geq \frac{S}{8}\right) \leq Ks\exp\left(-\frac{n}{K}\min\{s^{-2}, r^{-2/\alpha}\}\right)$$
$$\leq K\exp\left(-\frac{n}{K}\min\{s^{-2}, r^{-2/\alpha}\}\right).$$

We now appeal to [2], Lemma 2.7b, to get

$$P\left(\sup_{f\in\mathscr{F}(S)}\left|\sum_{i\leq n} f(X_i)\right| \geq S/2\right) \leq K\exp\left(-\frac{n}{K}\min\{s^{-2}, r^{-2/\alpha}\}\right).$$

This concludes the proof. □

Consider again $Z = \{1,\dots,r\}$, provided with a distance d that satisfies (11). We will say that a function h on Z is *Lipschitz* if it satisfies

$$\forall\, k, k' \leq r, \qquad |h(k) - h(k')| \leq d(k,k').$$

Consider $\beta = (\beta_k)_{k\leq r}$, such that $\beta_k \geq 0$, $\sum_{k\leq r}\beta_k = 1$. Consider $C > 0$, and the class

$$\mathscr{S}(\beta, C) = \left\{h\colon Z \to \mathbb{R};\, h \text{ Lipschitz}, \sum_{k\leq r}\beta_k|h(k)| \leq C\right\}.$$

Consider a sequence $(g_k)_{k\leq r}$ of i.i.d. $N(0,1)$ random variables, and the random variable

$$\theta(\beta, C) = \sup_{h\in\mathscr{S}(\beta,C)}\left|\sum_{k\leq r}\beta_k^{1/2}g_k h(k)\right|. \tag{21}$$

An essential step in the proof of Proposition 4 is to obtain sharp estimates of $E\theta(\beta, C)$. This is the purpose of the next few pages. We consider $q_1 \geq 0$ which will be determined later. We appeal to Lemma 1 to get an increasing sequence of partitions $(\mathscr{P}_q)$ of Z, $0 \leq q \leq q_1$, that satisfy

$$\operatorname{card}\mathscr{P}_q \leq 2^{\alpha q}, \tag{22}$$

$$\forall\, A \in \mathscr{P}_q, \qquad \operatorname{diam} A \leq 2^{-q+2}. \tag{23}$$

We denote by $E_q(h)$ the conditional expectation of h with respect to the algebra $\mathscr{A}_q$ generated by $\mathscr{P}_q$; that is, for $A \in \mathscr{P}_q$, $E_q(h)$ is constant over A

with the value $(\sum_{k\in A}\beta_k)^{-1}\sum_{k\in A}\beta_k h(k)$. (Throughout the paper, we use the convention that $0/0 = 0$.)

We set $h_0 = E_0(h)$. We set $E_{q_1+1}(h) = h$, and for $0 < q \le q_1 + 1$, we set $h_q = E_q(h) - E_{q-1}(h)$. Thus,

$$h = \sum_{0\le q\le q_1+1} h_q$$

and

$$E(\theta(\beta, C)) \le \sum_{0\le q\le q_1+1} E\big(\theta_q(\beta, C)\big), \tag{24}$$

where

$$\theta_q(\beta, C) = \sup_{h\in\mathscr{G}(\beta,C)} \left|\sum_{k\le r} \beta_k^{1/2} g_k h_q(k)\right|. \tag{25}$$

We first take care of the term for $q = 0$. We observe that $h_0(k)$ is constant, equal to $\sum\beta_k h(k)$, and thus $|h_0(k)| \le C$, so that

$$\theta_0(\beta, C) \le C\left|\sum_{k\le r}\beta_k^{1/2} g_k\right|.$$

Since the variable $\sum_{k\le r}\beta_k^{1/2} g_k$ is $N(0,1)$, we have

$$E\theta_0(\beta, C) \le \sqrt{\frac{2}{\pi}}\, C. \tag{26}$$

We now observe an important property of h_q, for $q > 0$. For $A \in \mathscr{P}_{q-1}$, $k, l \in A$, we have

$$|h(k) - h(l)| \le d(k,l) \le D(A) \le 2^{-q+3}.$$

It follows easily (averaging over $l \in A$) that

$$|h(k) - E_{q-1}(h)(l)| \le 2^{-q+3}.$$

This implies that $|h_{q_1+1}(k)| \le 2^{-q_1+2}$, and (by averaging over $k \in B \in \mathscr{P}_q$, $B \subset A$) that for $q < q_1$, we have

$$|h_q(k)| \le 2^{-q+3}. \tag{27}$$

We now estimate $E\theta_{q_1+1}(\beta, C)$. Since $|h_{q_1+1}(k)| \le 2^{-q_1+2}$, we have

$$\left|\sum_{k\le r}\beta_k^{1/2} g_k h_{q_1+1}(k)\right| \le 2^{-q_1+2}\sum_{k\le r}\beta_k^{1/2}|g_k|,$$

so that

$$E\theta_{q_1+1}(\beta, C) \le K 2^{-q_1}\sum_{k\le r}\beta_k^{1/2} \le K 2^{-q_1} r^{1/2} \tag{28}$$

by Cauchy–Schwarz.

We now estimate $E\theta_q(\beta, C)$ for a fixed q, $0 \le q \le q_1$. For $A \in \mathscr{P}_q$, set $\beta_A = \sum_{k\in A}\beta_k$, and consider the random variable $g_A = (\sum_{k\in A}\beta_k^{1/2} g_k)$, so that g_A is $N(0, \beta_A^{1/2})$, and the variables $(g_A)_{A\in\mathscr{P}_q}$ are independent. For $h \in \mathscr{G}(\beta, C)$, consider the constant value h_A of h_q on A. Thus (with the convention $0/0 = 0$), by Cauchy–Schwarz,

$$\left(\sum_{k\le r}\beta_k^{1/2}g_k h_q(k)\right)^2 = \left(\sum_{A\in\mathscr{P}_q} g_A h_A\right)^2 \le \left(\sum_{A\in\mathscr{P}_q}\frac{g_A^2}{\beta_A}\right)\left(\sum_{A\in\mathscr{P}_q}\beta_A h_A^2\right). \tag{29}$$

We know by (27) that $|h_A| \le 2^{-q+3}$. On the other hand, since $\sum_{k\le r}\beta_k|h(k)| \le C$, we have $\sum_{k\le r}\beta_k|E_p(h)(k)| \le C$ for all p, and thus $\sum_{k\le r}\beta_k|h_q(k)| \le 2C$, which implies $\sum_{A\in\mathscr{P}_q}\beta_A|h_A| \le 2C$. Since $|h_A| \le 2^{-q+3}$, we have $\sum_{A\in\mathscr{P}_q}\beta_A h_A^2 \le C2^{-q+4}$. Since $\sum_{A\in\mathscr{P}_q}\beta_A = 1$, we also have $\sum_{A\in\mathscr{P}_q}\beta_A h_A^2 \le 2^{-2q+6}$. Finally,

$$\sum_{A\in\mathscr{P}_q}\beta_A h_A^2 \le K2^{-q}\min(C,2^{-q}).$$

Since $Eg_A^2/\beta_A \le 1$, we see from (29) and (22) that

$$E\theta_q(\beta,C) \le \left(E\theta_q(\beta,C)^2\right)^{1/2} \le K\min(2^{-q},\sqrt{C2^{-q}})(\operatorname{card}\mathscr{P}_q)^{1/2} \le K\min(2^{-q},\sqrt{C2^{-q}})2^{\alpha q/2}. \tag{30}$$

To obtain our estimate of $E(\theta(\beta,C))$, it suffices now to combine (24) with the estimates we obtained for each term, and in particular (30). Which of these dominates depends on the value of α.

Case $\alpha < 2$. Then $\alpha/2 - 1 < 0$, $\alpha/2 - 1/2 > 0$; thus, if q_2 is the largest integer for which $2^{-q_2} \ge C$, we see that

$$\sum_{q\in\mathbb{Z}}\min(2^{-q},\sqrt{C2^{-q}})2^{\alpha q/2} \le \sum_{q\le q_2}\sqrt{C}\,2^{(\alpha/2-1/2)q} + \sum_{q>q_2}2^{(\alpha/2-1)q} \le K\left(\sqrt{C}\,2^{(\alpha/2-1/2)q_2} + 2^{(\alpha/2-1)q_2}\right) \le KC^{1-\alpha/2}.$$

Thus, from (24), (26) and (28), letting $q_1 \to \infty$, we get

$$E\theta(\beta,C) \le K(C + C^{1-\alpha/2}). \tag{31}$$

Case $\alpha > 2$. Then $\alpha/2 - 1/2 > 0$ so that

$$\sum_{q\le q_1}2^{(\alpha/2-1)q} \le K2^{(\alpha/2-1)q_1}.$$

We take q_1 such that 2^{-q_1} is of order $r^{-1/\alpha}$ to get, using (28) and (26),

$$E\theta(\beta,C) \le K(C + r^{1/2-1/\alpha}). \tag{32}$$

Case $\alpha = 2$. We take q_1 as in the preceding case to get

$$E\theta(\beta,C) \le K\left(C + \max(1,\log C\sqrt{r})\right). \tag{33}$$

For simplicity of notation, we set

$$F(C) = \begin{cases} K(C + C^{1-\alpha/2}), & \text{for } \alpha < 2, \\ K(C + \max(1, \log C\sqrt{r})), & \text{for } \alpha = 2, \\ K(C + r^{1/2-1/\alpha}), & \text{for } \alpha > 2. \end{cases}$$

Thus, with this notation, we have $E\theta(\beta, C) \le F(C)$.

PROPOSITION 6. *For $t > 0$, we have*

$$P(\theta(\beta, C) \ge F(C) + t) \le \exp\left(-\frac{t^2}{K(C^2 + C)}\right).$$

PROOF. We know that $E\theta(\beta, C) \le F(C)$. We now appeal to the Maurey–Pisier deviation inequality [5], which in the present case implies

$$P(\theta(\beta, C) \ge E\theta(\beta, C) + t) \le 2\exp\left(-\frac{t^2}{K\sigma^2}\right),$$

where

$$\sigma^2 = \sup\left\{\sum_{k \le r} \beta_k h^2(k);\ h \in \mathscr{G}(\beta, C)\right\}.$$

Thus, $\sigma^2 \le C \max\{|h(k)|;\ k \le r\}$. Since $|h(k) - h(k')| \le d(k, k') \le 1$, we have

$$\max\{|h(k)|;\ k \le r\} \le \min\{|h(k)|;\ k \le r\} + 1 \le C + 1,$$

so that $\sigma^2 \le C^2 + C$. □

Consider now the grid $G = Z \times \{1, \dots, s\}$. Consider points $(x_i)_{i \le rs}$, $x_i = (k(i), l(i))$ of G and consider independent $N(0,1)$ r.v. $(g_i)_{i \le rs}$. We want to estimate

$$E \sum_{f \in \mathscr{F}(S)} \left|\sum_{i \le n} g_i f(x_i)\right|.$$

For $l \le s$, set $I_l = \{i \le n;\ l(i) = l\}$. We assume that for certain numbers M_1, M_2, the following occur, for all $l \le s$:

(H) For all Lipschitz functions h on Z, we have

$$\sum_{i \in I_l} |h(k(i))| \le M_1 \sum_{k \le r} |h(k)| + M_2.$$

(H′) $\forall\ l \le s$, $r/2 \le \operatorname{card} I_l \le 2r$.

These assumptions will be justified in Lemma 7, where the appropriate values of M_1 and M_2 will be determined.

Consider the smallest integer p such that $s \le 2^p$. Consider the sequence $Q_0, Q_1, \dots, Q_p$ of partitions of $\{1, \dots, s\}$ defined as follows: For $0 \le m \le p$, Q_m consists of the intervals $]\lfloor ls2^{-m}\rfloor, \lfloor (l+1)s2^{-m}\rfloor]$ for $0 \le l < 2^m$. We observe that this sequence of partitions increases. Also, a set $A \in Q_m$ satisfies

$$2^{p-m-1} \le \operatorname{card} A \le 2^{p-m}.$$

For a function f on G and $0 \le m \le p$, we write $E_m(f)$ for the function defined as follows; the value of $E_m(f)$ at (k, l) is the average of the values of $f_m(k, l)$ for $l \in A$, where A is the element of Q_m that contains l. Note that $E_p(f) = f$. We set $f_0 = E_0(f)$, $f_m = E_m(f) - E_{m-1}(f)$ for $1 \le m \le p$. Thus, $f = \sum_{0 \le m \le p} f_m$. This implies

$$(34) \qquad E \sup_{f \in \mathscr{F}(S)} \left| \sum_{i \le n} g_i f(x_i) \right| \le \sum_{0 \le m \le p} E \sup_{f \in \mathscr{F}(S)} \left| \sum_{i \le n} g_i f_m(x_i) \right|.$$

We will evaluate each term on the right (the one that dominates is the last one) and eventually bound the left-hand side of (34) in Proposition 7. But we first investigate the properties of f_m when the functions f and m are fixed, treating first the case $m > 0$ and then the case $m = 0$. For $A \in Q_m$, $k \le r$, we denote by $h_A(k)$ the common value that f_m takes on all the points (k, l) for $l \in A$. Observe the important fact that the function h_A is Lipschitz.

Lemma 5. *For* $m > 0$, *we have*

$$\sum_{A \in Q_m} \sum_{k \le r} |h_A(k)| \le S.$$

Proof. We denote by B the element of Q_{m-1} that contains A. If $B = A$, the definition of f_m shows that $h_A(k) = 0$. If $B \ne A$, then B contains another unique element A' of Q_m. The definition of f_m then shows that card $A h_A(k)$ + card $A' h_{A'}(k) = 0$ so that $h_A(k)$ and $h_{A'}(k)$ are of opposite signs and thus

$$(35) \qquad |h_A(k)| + |h_{A'}(k)| \le |h_A(k) - h_{A'}(k)|.$$

We note that we have

$$(36) \qquad |h_A(k) - h_{A'}(k)| \le \sum_{l \in B} |f_m(k, l+1) - f_m(k, l)|,$$

since one of the terms of this sum is $|h_A(k) - h_{A'}(k)|$ (e.g., if A is to the left of A', the term corresponding to the largest $l \in A$.). Using (35) and (36), we get by summation over all $B \in Q_{m-1}$ that

$$\sum_{A \in Q_m} |h_A(k)| \le \sum_{0 \le l < s} |f_m(k, l+1) - f_m(k, l)|.$$

It is a simple matter, left to the reader, to show that this latter sum is at most $\sum_{0 \le l < s} |f(k, l+1) - f(k, l)|$. The result then follows from summation over k. □

We fix $m > 0$. For $A \in Q_m$, $k \le r$, we get

$$I(A, k) = \{i \le n;\, k(i) = k,\, l(i) \in A\}.$$

So we have

$$\Big|\sum_{i\le n} g_i f_m(x_i)\Big| = \Big|\sum_{A\in Q_m}\sum_{k\le r}\sum_{i\in I(A,k)} g_i h_A(k)\Big|$$
$$\le \sum_{A\in Q_m}\Big|\sum_{k\le r}\sum_{i\in I(A,k)} g_i h_A(k)\Big|.$$

We set

$$b(A,k) = \operatorname{card} I(A,k), \qquad b(A) = \sum_{k\le r} b(A,k), \qquad \beta_{A,k} = \frac{b(A,k)}{b(A)}.$$

Thus, $\sum_{k\le r}\beta_{A,k} = 1$. By (H′) we have

$$r2^{p-m-2} \le \frac{r}{2}\operatorname{card} A \le b(A) \le 2r \operatorname{card} A \le r2^{p-m+2}. \tag{37}$$

We set $g_{A,k} = [1/\sqrt{b(A,k)}\,]\sum_{i\in I(A,k)} g_i$. Thus,

$$\Big|\sum_{k\le r}\sum_{i\in I(A,k)} g_i h_A(k)\Big| = \sqrt{b(A)}\,\Big|\sum_{k\le r}\sqrt{\beta_{A,k}}\,g_{A,k}h_A(k)\Big| \tag{38}$$

and the variables $(g_{A,k})_{k\le r}$ are independent standard normal. We set now $S_A = \sum_{k\le r}|h_A(k)|$. Thus, by Lemma 5, we have $\sum_{A\in Q_m} S_A \le S$. We appeal to (H) to get

$$\sum_{l\in A}\sum_{i\in I_l}|h_A(k(i))| \le \operatorname{card} A\Big(M_1\sum_{k\le r}|h_A(k)| + M_2\Big),$$

so that, since $\operatorname{card} A \le 2^{p-m}$,

$$\sum_{k\le r} b(A,k)|h_A(k)| \le 2^{p-m}(M_1 S_A + M_2)$$

and thus, by (37),

$$\sum_{k\le r}\beta_{A,k}|h_A(k)| \le \frac{4}{r}(M_1 S_A + M_2). \tag{39}$$

We set $C_A(f) = (4/r)(M_1 S_A + M_2)$. We set

$$\mathscr{G}(A,C) = \Big\{h;\, Z\to\mathbb{R},\ h \text{ Lipschitz},\ \sum_{k\le r}\beta_{A,k}|h(k)| \le C\Big\}$$

and we consider the variable

$$\eta(A,C) = \sup_{h\in\mathscr{G}(A,C)}\Big|\sum_{k\le r}\sqrt{\beta_{A,k}}\,g_{A,k}h(k)\Big|.$$

Thus,

$$\Big|\sum_{k\le r}\sum_{i\in I(A,k)} g_i h_A(k)\Big| \le \sqrt{b(A)}\,\eta(A, C_A(f))$$

by (38) and (39), and thus

$$\left|\sum_{i \le n} g_i f_m(x_i)\right| \le \sum_{A \in Q_m} \sqrt{b(A)}\,\eta(A, C_A(f)) \le 2\sqrt{r}\,2^{(p-m)/2} \sum_{A \in Q_m} \eta(A, C_A(f)). \tag{40}$$

We observe that the numbers $C_A(f)$ satisfy

$$\sum_{A \in Q_m} C_A(f) \le \frac{4}{r}(M_1 S + s M_2) := M.$$

Consider the collection $\mathscr{H}$ of families of integers $(l_A)_{A \in Q_m}$ that satisfy

$$M 2^{-m} \le 2^{l_A}, \qquad \sum_{A \in Q_m} 2^{l_A} \le 4M.$$

Clearly, we can find $(l_A) \in \mathscr{H}$ such that $C_A(f) \le 2^{l_A}$. We observe that $\eta(A, 2^l)$ is distributed like $\theta((\beta_{A,k})_{k \le r}, 2^l)$, so that by Proposition 6, we have

$$P\big(\eta(A, 2^l) \ge F(2^l) + Kt(2^l + 2^{l/2})\big) \le \exp(-t^2).$$

Consider the random variable

$$\xi = \max \frac{\eta(A, 2^l) - F(2^l)}{2^l + 2^{l/2}},$$

where the maximum is over all $A \in Q_m$, and all l for which $M 2^{-m} \le 2^l \le 4M$. There are at most $m + 2$ possible choices for l, so that

$$P(\xi \ge t) \le (m+2) 2^m \exp(-t^2) \le \exp Km \exp(-t^2).$$

Thus, by a simple computation $E\xi \le K\sqrt{m}$. On the other hand,

$$\eta(A, 2^l) \le F(2^l) + K\xi(2^l + 2^{l/2}).$$

Thus, for all families $(l_A) \in \mathscr{H}$, we have

$$\sum_{A \in Q_m} \eta(A, 2^{l_A}) \le \sum_{A \in Q_m} F(2^{l_A}) + K\xi\Big(\sum 2^{l_A} + 2^{l_A/2}\Big). \tag{41}$$

We now make the extra assumption that

$$S \ge n r^{-1/\alpha}. \tag{42}$$

Thus, since $M_1 \ge 1$, we have

$$2^{l_A} \ge \frac{4S}{rs} \ge r^{-1/\alpha}.$$

Setting $\mathscr{Q}_m = \operatorname{card} Q_m$ and using the concavity of $F(x)$ for $x \ge r^{-1/\alpha}$, we have

$$\sum_{A \in Q_m} F(2^{l_A}) \le \mathscr{Q}_m F\Big(\mathscr{Q}_m^{-1} \sum_{A \in Q_m} 2^{l_A}\Big) \le 2^m F(2^{-m+3} M).$$

Thus, we get from (41) that (using the concavity of $\sqrt{\cdot}$),

$$\sum_{A \in Q_m} \eta(A, 2^{l_A}) \le 2^m F(2^{-m+3}M) + K\xi(M + \sqrt{M}).$$

Thus, by (40), we have

$$\left| \sum_{i \le n} g_i f_m(x_i) \right| \le K\sqrt{n}\, 2^{m/2} F(2^{-m+3}M) + K\sqrt{r}\, 2^{(p-m)/2} \xi(M + \sqrt{M}).$$

The right-hand side is now independent of $f \in \mathscr{F}(S)$ so the inequality still holds if we replace the left-hand side by its supremum over $f \in \mathscr{F}(S)$. Taking expectation, we obtain, since $E\xi \le K\sqrt{m}$, that

$$E \sup_{f \in \mathscr{F}(S)} \left| \sum_{i \le n} g_i f_m(x_i) \right| \le K\sqrt{n}\, 2^{m/2} F(2^{-m+3}M) + K\sqrt{r}\, 2^{(p-m)/2} \sqrt{m}\,(M + \sqrt{M}). \tag{43}$$

We now turn to the case $m = 0$.

Lemma 6. *Consider $f \in \mathscr{F}(S)$. Then* $\max|f(k,l)| \le S/r + 2$.

Proof. We find k such that

$$\sum_{l < s} |f(k, l+1) - f(k,l)| \le \frac{S}{r}.$$

Thus, $|f(k,l') - f(k,l)| \le S/r$ for all $l, l' \le s$. Now, for $k', k'' \le r$, and $l', l'' \le s$, we have

$$\begin{aligned} |f(k',l') - f(k'',l'')| &\le |f(k',l') - f(k,l')| \\ &\quad + |f(k,l') - f(k,l'')| + |f(k,l'') - f(k'',l'')| \\ &\le 2 + \frac{S}{r}. \end{aligned}$$

The result follows since $\sum_{k \le r,\, l \le r} f(k,l) = 0$. □

Since the value of $f_0(k,l)$ does not depend on l, it should be clear that

$$E \sup_{f \in \mathscr{F}(S)} \left| \sum_{i \le n} g_i f_0(x_i) \right| \le \sqrt{n}\, E\left(\theta\left(\beta, 2 + \frac{S}{r}\right)\right),$$

where $\beta_k = r^{-1/2} \operatorname{card}\{i \le n;\, k(i) = k\}$, $\beta = (\beta_k)_{k \le r}$. Thus,

$$E \sup_{f \in \mathscr{F}(S)} \left| \sum_{i \le n} g_i f_0(x_i) \right| \le \sqrt{n}\, F\left(2 + \frac{S}{r}\right). \tag{44}$$

The bound for (34) is achieved by summing (43) and (44) and going back to the definition of F. We obtain the following proposition.

PROPOSITION 7. *Assume* $S \geq nr^{-1/\alpha}$ *and conditions* (H) *and* (H′) *and let* $M = 4/r(M_1 S + sM_2)$. *Then*

$$E \sup_{f \in \mathscr{F}(S)} \left| \sum_{i \leq n} g_i f(x_i) \right| \leq Ks\sqrt{r}\, F\left(\frac{8M}{s}\right) + K\sqrt{n}\,(M + \sqrt{M}) + K\sqrt{n}\, F\left(2 + \frac{S}{r}\right).$$

4. End of proof. The purpose of the next lemma is to verify that conditions (H) and (H′) are satisfied.

LEMMA 7. *Consider the random variables* $(X_i)_{i \leq n}$, $X_i = (k(i), l(i))$, *which are uniformly distributed over the grid* G. *Then, for some constant* K, *the following occurs with probability at least* $1 - s\exp(-r^{1-1/\alpha}/K)$. *For each Lipschitz function* h *on* Z, *and for each* $l \leq s$, *we have*

$$\sum_{i \in I_l} |h(k(i))| \leq K\left(\sum_{k \leq r} |h(k)| + r^{1-1/\alpha}\right), \tag{45}$$

where $I_l = \{i \leq n;\ l(i) = l\}$.

COMMENT. When (45) occurs, we can take $M_1 = K$, $M_2 = Kr^{1-1/\alpha}$, so that $M = 4(M_1 S + sM_2)/r \leq KS/r$ whenever $S \geq nr^{-1/\alpha}$.

PROOF. *Step* 1. Consider the smallest integer q_1 such that $2^{-q_1} \leq r^{-1/\alpha}$, and the sequence $(\mathscr{P}_q)_{q \leq q_1}$ of partitions of Z given by Lemma 1.

We observe that if V is a random variable such that $0 \leq V \leq 1$, we have (since $1 + x \leq e^x \leq 1 + 2x$ for $x \leq 1$)

$$E \exp V \leq E(1 + 2V) = 1 + 2EV \leq \exp 2EV. \tag{46}$$

Consider a subset A of G, and the random variable A_i given by $A_i = 1$ if $X_i \in A$ and $A_i = 0$ otherwise. Thus, $EA_i = n^{-1}\operatorname{card} A$. By (46) and independence,

$$E \exp \sum_{i \leq n} A_i \leq \exp 2 \operatorname{card} A$$

and by the Chebyshev inequality, we have $P(\sum_{i \leq n} A_i \geq t) \leq \exp(-t) + 2\operatorname{card} A$. In other words, we have

$$P\big(\operatorname{card}\{i \leq n;\ X_i \in A\} \geq t + 2\operatorname{card} A\big) \leq \exp(-t). \tag{47}$$

Step 2. Consider now a sequence $(b_q)_{0 \leq q \leq q_1}$, to be specified later. Consider the following event:

Ω_0: Given any $0 \leq q \leq q_1$, and any subset Y of $Z = \{1, \dots, r\}$ that is $\mathscr{P}_q$-measurable, and any $l \leq s$, then

$$\operatorname{card}\{i \leq n;\ X_i \in Y \times \{l\}\} \leq 2\operatorname{card} Y + b_q. \tag{48}$$

Thus, by (47), we have

$$P(\Omega_0) \geq 1 - \sum_{0 \leq q \leq q_1} s 2^{\operatorname{card} \mathscr{P}_q} \exp(-b_q). \tag{49}$$

Assume now that (48) occurs, and consider a Lipschitz function h on Z. We define a "stopping time" as follows. For $k \in Z$, $q(k)$ is the smallest integer $0 \leq q \leq q_1$ such that for some point k', which belongs to the same element of $\mathscr{P}_q$ as k, we have $|h(k')| \geq 2^{-q+3}$. [If no such q exists, we set $q(k) = q_1$.] Consider $Y_q = \{k \in Z;\ q(k) = q\}$. Then clearly, Y_q is $\mathscr{P}_q$-measurable. Since each set of $\mathscr{P}_q$ has diameter $\leq 2^{-q+2}$, when $q \leq q_1$, we have $h(k) \geq 2^{-q+2}$ on Y_q. Thus,

$$\sum_{0 \leq q \leq q_1} 2^{-q+2} \operatorname{card} Y_q \leq \sum_{k \leq r} |h(k)| + r 2^{-q_1}. \tag{50}$$

The definition of $q(k)$ shows that, if $q > 0$, we have $|h(l)| \leq 2^{-q+4}$ for each $l \in Y_q$. Suppose that $\|h\|_\infty =: \max_{k \in Z} |h(k)| \leq 2^4$. Then we have $|h(k)| \leq 2^4$ for each $k \in Y_0$. Thus,

$$h \leq \sum_{0 \leq q \leq q_1} 2^{-q+4} 1_{Y_q}$$

so that by (48),

$$\begin{aligned} \sum_{i \in I_l} h(k(i)) &\leq \sum_{0 \leq q \leq q_1} 2^{-q+4} \operatorname{card}\{i \leq n;\ X_i \in Y_q \times \{l\}\} \\ &\leq \sum_{0 \leq q \leq q_1} 2^{-q+5} (\operatorname{card} Y_q + b_q) \\ &\leq 8 \sum_{k \leq r} |h(k)| + \sum_{0 \leq q \leq q_1} 2^{-q+5} b_q. \end{aligned} \tag{51}$$

Suppose now that $\|h\|_\infty \geq 2^4$. Then, for all $k' \in Z$, we have $|h(k')| \geq \|h\|_\infty - 1 \geq \|h\|_\infty / 2$. Thus,

$$\begin{aligned} \sum_{i \in I_l} |h(k(i))| &\leq \|h\|_\infty \operatorname{card}\{i \leq n;\ l(i) = l\} \leq (2r + b_0) \|h\|_\infty \\ &\leq 2(2 + r^{-1} b_0) \sum_{k \leq r} |h(k)|. \end{aligned}$$

Step 3. Thus, to conclude the proof, it suffices to show that one can find the sequence (b_q) such that the following occur:

$$b_0 \leq Kr, \tag{52}$$

$$\sum_{0 \leq q \leq q_1} 2^{-q} b_q \leq K r^{1 - 1/\alpha}, \tag{53}$$

$$s \sum_{0 \leq q \leq q_1} 2^{2^{\alpha q}} \exp(-b_q) \leq Ks \exp(-r^{1 - 1/\alpha}). \tag{54}$$

For that purpose, we take

$$b_q = 2^{q/2} r^{1 - 1/\alpha} + 2^{\alpha q}.$$

Thus, $b_0 \le 1 + r^{1-1/\alpha}$, and (52) holds. To prove (53), one simply observes that, since 2^{-q_1} is of order $r^{-1/\alpha}$, we have

$$\sum_{0 \le q \le q_1} 2^{\alpha q} 2^{-q} \le K 2^{q_1(\alpha - 1)} \le K r^{1-1/\alpha}.$$

To prove (54), one simply notes that

$$2^{2^{\alpha q}} \exp(-b_q) \le \exp(-2^{q/2} r^{1-1/\alpha}). \qquad \square$$

We leave to the reader to check (using Hoeffding's inequality [3]) that (H′) is satisfied with probability at least $1 - K \exp(-r/K)$. If we combine Lemma 7 with Proposition 7, we have proved the following lemma.

LEMMA 8. *Consider $S \ge nr^{-1/\alpha}$. Denote by E_g the conditional expectation at $X_1, \dots, X_n$ fixed. Then, with probability at least $1 - k\exp(-r^{1-1/\alpha}/K)$, we have*

$$(55)\qquad \begin{aligned} E_g \sup_{f \in \mathscr{F}(S)} \left| \sum_{i \le n} g_i f(X_i) \right| &\le K s \sqrt{r} F\left(\frac{KS}{n}\right) + K\sqrt{n}\left(\frac{S}{r} + \sqrt{\frac{S}{r}}\right) \\ &\quad + K\sqrt{n} F\left(2 + \frac{S}{r}\right). \end{aligned}$$

Proposition 4 now follows by a tedious but straightforward computation.

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