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FLUCTUATIONS IN A NONLINEAR REACTION–DIFFUSION MODEL[1]

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A law of large numbers and a central limit theorem are proved for a locally interacting particle system. This system describes a chemical reaction with diffusion with linear creation and quadratic annihilation of particles. The deterministic limit is the solution of a nonlinear reaction–diffusion equation defined on an n-dimensional unit cube. The law of large numbers holds for any dimension n and arbitrary times, whereas the central limit theorem holds only for dimension $n \leq 3$ and on a certain bounded time interval (depending on the initial distribution and on the creation rate). A propagation of chaos expansion of the correlation functions is used.

1. Introduction. To our knowledge the first rigorous study of stochastic models of chemical reactions with diffusion was carried out by Arnold [1] and Arnold and Theodosopulu [2]. In that model the "reactor" was represented by a finite interval which was split up into N cells where any particle in a given cell could react with any other particle from this cell and the cells were linked by diffusion. Under a high-density assumption (number of particles in each cell $\gg N$), it was shown in [2] that the stochastic particle system converges to the solution of a certain reaction–diffusion equation [law of large numbers (LLN)]. In Kotelenez [10] a corresponding central limit theorem (CLT) was proved under the assumption that the reaction is linear. Both Kotelenez [11] and Blount [3] are generalizations, respectively improvements, of the linear model studied in [10]. In Kotelenez [13, 14] CLT's were proved also for nonlinear reactions with diffusion under high-density assumptions. In particular, it was proved in [14] that the density necessarily has to be high both for the LLN and the CLT if the reaction rates in the stochastic model are the macroscopic ones given in the reaction–diffusion equation (cf. also our remarks in Section 5). In the terminology of interacting particle systems, the high-density assumption means that the interaction is not local. It should be mentioned that all stochastic models discussed so far were defined on a grid (following Arnold). The stochastic models in Dittrich [6, 7] were defined on the space of point measures on a bounded interval (the "reactor"). By taking rates

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different from the macroscopic ones [cf. (5.3)], Dittrich proved in [6] an LLN and in [7] a CLT under the assumption that the density is low, respectively the interaction is local. The assumptions in both [6] and [7] are that there is only killing and that the reactor is one dimensional. (Inasmuch as there is no creation in [6] and [7], the models herein are related to that considered in Lang and Xanh [16].) Moreover, in [7] the main tools were the correlation functions and the propagation of chaos, assuming that the stochastic system is initially Poisson distributed.

In this paper we essentially generalize Dittrich's model [7] to an n-dimensional cube where the reaction scheme contains a (linear) creation term. The result depends on the verification of the propagation of chaos (which we could not verify for creation or annihilation terms of order higher than 2). Moreover, we have made full use of the functional analytic methods developed in [10]–[14], in particular, of convolution integrals using "variation of constants." The advantage of this approach is, for example, that the uniqueness of the Ornstein–Uhlenbeck limit in the CLT is obtained for free as a consequence of the uniqueness of the limits of the coefficients of the rescaled stochastic model [cf. (2.17) and Theorem 4.2]. We obtain the LLN for any dimension n and on arbitrary bounded time intervals. The CLT, however, we can prove only for dimension $n \le 3$ and on time intervals which are bounded by a constant depending on the intensity of the initial (Poisson) distribution and the creation rate (cf. Theorem 3.1; the corresponding result was obtained by Dittrich [7]). The restriction on the dimension comes both in the verification of the propagation of chaos (Theorem 3.1) and its most important application in the proof of the CLT [Theorem 4.2(ii)] and in a speed of convergence result, where the solution of the reaction–diffusion equation is compared to the solution of an integro-differential equation associated with the correlation function (Theorem 2.1).

The paper is organized as follows. In Section 2 the deterministic and the stochastic models are described, and the main result, Theorem 2.1, is presented. In Section 3 the correlation function is analyzed and the propagation of chaos is proved. Section 4 contains the LLN (Theorem 4.1) and the CLT (Theorem 4.2). We only prove the CLT since the proof of the LLN is a simple consequence of the other proof. In Section 5 we compare this result with the result of Kotelenez [14].

As far as notation is concerned we use the same symbols c, $c(\hat{t})$, $c(\hat{t}, \alpha)$ and so on, for finite constants (depending on $\hat{t}$, resp. on $\hat{t}$ and α, etc.) which may be different at various steps in the proof of some statement. Our spatial domain is the unit cube, denoted by S. When integrating over S we will not specify this, whereas the integration over time intervals will be specified.

2. The deterministic and the stochastic models. Set $S := \{r = (r_1, \dots, r_n) \in \mathbb{R}^n\colon 0 \le r_i \le 1,\ i = 1, \dots, n\}$. Let $\mathring{\Delta}$ be the n-dimensional Laplacian, $D > 0$ a diffusion constant, $a \in \mathbb{R}$ and $b < 0$. Then the concentration of one particle with reflection at the boundary, linear creation (if $a > 0$) and quadratic annihilation is given by the solution of the following PDE:

$$(2.1)\qquad \begin{aligned} &\frac{\partial}{\partial t}X(t,r) = D\mathring{\Delta}X(t,r) + aX(t,r) + bX^2(t,r),\\ &\partial_i X(t,r) = 0 \quad \text{if } r_i \in \{0,1\},\ i = 1,\dots,n,\ X_0(r) \geq 0, \end{aligned}$$

where $\partial_i = \partial/\partial r_i$.

We briefly describe some properties of the solution of (2.1) and for the details refer to Kotelenez [12, 14].

Let

$$(H_0, \langle\,\cdot\,,\,\cdot\,\rangle_0) = (L_2(S), \langle\,\cdot\,,\,\cdot\,\rangle_0)$$

be the separable Hilbert space of square-integrable real-valued functions on S, equipped with the scalar product $\langle \varphi, \psi\rangle_0 := \int \varphi(r)\psi(r)\,dr$, where $dr = dr_1,\dots,dr_n$. Moreover, let C^k be the functions from S into $\mathbb{R}$ which are k times continuously differentiable in all variables. For $\varphi \in C^k$ set

$$|||\varphi|||_k := \max_{0\leq |l| \leq k} |||\partial^l \varphi|||.$$

Here $|l| = l_1 + \cdots + l_n$ for $l = (l_1,\dots,l_n)$, $\partial^l := \partial^{l_1},\dots,\partial^{l_n}$, $\partial^{l_i} := \partial^{l_i}/\partial r_i^{l_i}$ and

$$|||\psi||| = \sup_{q\in S} |\psi(q)|,$$

for those $\psi \in H_0$, where this definition makes sense. Now we denote by Δ the closure of $\mathring{\Delta}$ with respect to the homogeneous Neumann boundary conditions in (2.1) and by $C^\infty_{D\Delta}$ a core for $D\Delta$ whose elements are infinitely often differentiable. Let $(B_k, |||\cdot|||_k)$ be the closure of $C^\infty_{D\Delta}$ in C^k with respect to $|||\cdot|||_k$. We make the following hypothesis on $X_0(r)$.

H.1(k).

(i) $0 \leq X_0(r) \leq \rho$ *for all* $r \in S$, *where* ρ *is some number such that* $ax + bx^2 < 0$ *for all* $x \geq \rho$.
(ii) $X_0 \in B_k$.

H.1(k) implies the existence of a unique global mild solution X of (2.1) such that:

(i) $0 \leq X(t,r) \leq \rho$ for all $(t,r) \in [0,\infty) \times S$,
(ii) $X \in C([0,\infty); B_k)$,

where $C([0,\infty); B)$ are the B-valued continuous functions on $[0,\infty)$ with B being a Banach space.

We will assume H.1(k) for some fixed $k > n/2 + 1$ throughout the paper without explicitly stating it in the theorems.

Now we introduce the stochastic model following Dittrich [6, 7].

Let $\varepsilon > 0$, $E_N := \{\sum_{i=1}^N \varepsilon\delta_{r_i},\ r_i \in S\}$, where $\varepsilon\delta_{r_i}$ are Dirac measures with weight ε, and set

$$E := \bigcup_{N=0}^{\infty} E_N$$

(E_0 = {abstract point}). We can define a metric d_p on E (derived from the Prohorov metric) such that (E, d_p) is a locally compact separable metric space (cf. Ethier and Kurtz [9], page 408, Problem 6). Furthermore, let $R^\varepsilon\colon S \times S \to \mathbb{R}_+$ be continuous and symmetric such that

$$\begin{aligned} &\int R^\varepsilon(q,r)\,dq = 1 \quad \text{for all } r \in S, \\ &c_1 R^\varepsilon(r,q) \le G(\varepsilon^{2/n}, r, q) \le c_2 R^\varepsilon(r,q), \qquad 0 < c_1 < c_2 < \infty, \end{aligned} \tag{2.2}$$

where $G(t,r,q)$ is the fundamental solution (or Green's function) to $\partial/\partial t\, \xi(t) = D\,\Delta\xi(t)$, that is,

$$G(t,r,q) := \prod_{i=1}^{n} G(t, r_i, q_i), \tag{2.3}$$

with

$$G(t, r_i, q_i) := (4tD\pi)^{-1/2} \sum_{k=-\infty}^{\infty} \left\{ \exp\left(\frac{-(r_i - q_i + 2k)^2}{4tD} \right) + \exp\left(\frac{-(r_i + q_i + 2k)^2}{4tD} \right) \right\}.$$

On a fixed probability space we define a Markov process X^ε with state space E as follows. If X^ε belongs to E_N the process consists of N independent S-valued Brownian motions $w_i(t)$ with generator $D\Delta$, that is, $X^\varepsilon(t) = \sum_{i=1}^N \varepsilon\delta_{w_i(t)}$. The transition from E_N to E_{N+1} occurs with rate a (if $a > 0$). Finally, any pair x^i, x^j disappears with intensity $-b\varepsilon R^\varepsilon(x^i, x^j)$ resulting in a jump of X^ε from E_N to E_{N-2}. Condition (2.2) implies that typically only particles at a distance less than or equal to ε disappear, that is, that the interaction (annihilation) is local.

Denote by γ_N the canonical mapping from S^N (the Nth Cartesian product of S with itself) to E_N and let $f \in C_b(E, \mathbb{R})$ (continuous bounded from E into $\mathbb{R}$) such that $f \circ \gamma_N$ is twice continuously differentiable and satisfies homogeneous Neumann boundary conditions on S^N. Denote the corresponding closed Laplacian [on $C(S^N, \mathbb{R})$] by Δ_N and the operator induced by $\gamma_N \gamma_N^* \Delta_N$. Then the generator is as follows:

$$\begin{aligned} (A^\varepsilon f)(\mathscr{X}) := \gamma_N^* D\,\Delta_N f(\mathscr{X}) &+ a \sum_{i=1}^{N} \left[f(\mathscr{X} + \varepsilon\delta_{x^i}) - f(\mathscr{X}) \right] \\ &+ b\frac{\varepsilon}{2} \sum_{k \ne j}^{N} \left[f(\mathscr{X}) - f(\mathscr{X}_{k,j}) \right] R^\varepsilon(x^k, x^j), \end{aligned} \tag{2.4}$$

where $\mathscr{X} = \sum_{i=1}^{N} \varepsilon\delta_{x^i} \in E_N$ and $\mathscr{X}_{k,j} = \sum_{i=1, \neq k, \neq j}^{N} \varepsilon\delta_{x^i} \in E_{N-2}$ (i.e., $\mathscr{X}_{k,j}$ is obtained from $\mathscr{X}$ by deleting the kth and the jth particles). Dynkin [8], Chapter 10, Section 6, and Ethier and Kurtz [9], Section 4.2, ensure existence and uniqueness (in distribution) of a Markov process $X^\varepsilon \in D([0,\infty); E)$ (the Skorohod space of E-valued cadlag functions) with generator A^ε.

Dynkin's formula yields

$$\begin{aligned}
&\langle X^\varepsilon(t), \varphi\rangle \\
(2.5)\quad &= \langle X_0^\varepsilon, \varphi\rangle + \int_0^t ds \left\{ \langle X^\varepsilon(s), (D\Delta + a)\varphi\rangle + b\varepsilon^2 \sum_{k \neq j}^{N_s^\varepsilon} \varphi(x_s^k) R^\varepsilon(x_s^k, x_s^j) \right\} \\
&\quad + \varepsilon^{1/2} \langle M^\varepsilon(t), \varphi\rangle,
\end{aligned}$$

where $\varphi \in B_2$, $\langle \cdot , \cdot \rangle$ is the dual pairing (extending $\langle \cdot , \cdot \rangle_0$), and N_s^ε is the number of particles at time s and x_s^k is the position of the kth particle at time s (x_s^j is correspondingly defined). $\langle M^\varepsilon(t), \varphi\rangle$ is a real-valued martingale whose Meyer process is given by

$$\begin{aligned}
&\langle\langle M^\varepsilon(t), \varphi\rangle\rangle = \int_0^t ds \left\{ \left\langle X^\varepsilon(s), \left[2D \sum_{p=1}^{n} (\partial_p \varphi)^2\right] + a\varphi^2 \right\rangle \right. \\
(2.6)\quad & \\
&\qquad \left. -2b\varepsilon^2 \sum_{k \neq l}^{N_s^\varepsilon} \left[\frac{\varphi(x_s^k) + \varphi(x_s^l)}{2} \right]^2 R^\varepsilon(x_s^k, x_s^l) \right\}.
\end{aligned}$$

In order to compare X^ε and X we need more state spaces. We use the setup by Kotelenez [10, 14].

Define for $\gamma \in \mathbb{R}_+$,

$$(H_\gamma, \langle \cdot , \cdot \rangle_\gamma) := \text{the closure of } C_{D\Delta}^\infty \text{ in } H_0 \text{ with respect to } \langle \cdot , \cdot \rangle_\gamma,$$

where $\langle \varphi, \psi \rangle_\gamma := \langle (I - D\Delta)^\gamma \varphi, \psi \rangle_0$, $\varphi, \psi \in C_{D\Delta}^\infty$. Here I is the identity operator on H_0 and $(I - D\Delta)^\gamma$ is the γth power of the positive self-adjoint operator $(I - D\Delta)$. $(H_\gamma, \langle \cdot , \cdot \rangle_\gamma)$ is a real separable Hilbert space. Set $\Phi := \bigcap_{\gamma \geq 0} H_\gamma$ and endow Φ with the locally convex topology defined by $\{|\varphi|_\gamma := \langle \varphi, \varphi \rangle_\gamma^{1/2}, \varphi \in \Phi, \gamma \in \mathbb{R}_+\}$. Let Φ' be the strong dual of Φ and $H_{-\gamma} \subset \Phi'$ the strong dual of H_γ. Identifying H_0 with H_0', we obtain the chain of (dense) continuous inclusions:

$$\Phi \subset H_\gamma \subset H_0 = H_0' \subset H_{-\gamma} \subset \Phi', \qquad \gamma \in \mathbb{R}_+, \tag{2.7}$$

where Φ is a nuclear Frechet space. Let $l = (l_1, \dots, l_n)$ be a multiindex, where $l_i \in \mathbb{N} \cup \{0\}$, and set for $r_i \in [0; 1]$,

$$\phi_{l_i}(r_i) := \begin{cases} \sqrt{2}\cos(l_i \pi r_i), & l_i \geq 1, \\ 1, & l_i = 0. \end{cases}$$

Then $\{\phi_l := \prod_{i=1}^n \phi_{l_i}\}$ is a complete orthonormal system (CONS) of eigenvectors of $D\Delta$ in H_0 with eigenvalues

$$-\mu_l := -D\left(\sum_{i=1}^n l_i^2 \pi^2\right).$$

Setting $\lambda_l := 1 + \mu_l$, we obtain

$$\phi_l^\gamma := \lambda_l^{-\gamma/2}\phi_l$$

is a CONS for H_γ, $\gamma \in \mathbb{R}$. As a consequence,

$$|\varphi|_\gamma^2 = \sum_l \lambda_l^\gamma \langle \varphi, \phi_l \rangle^2, \tag{2.8}$$

where $\varphi \in H_\gamma$, $\gamma \in \mathbb{R}$. Moreover, if $T(t)$ is the semigroup generated by $D\Delta + a$ on H_0, then

$$T(t)\varphi = \sum_l \exp((-\mu_l + a)t)\phi_l \langle \varphi, \phi_l \rangle_0. \tag{2.9}$$

A more general object than a semigroup is an evolution operator $L(t, s)$, $0 \le s \le t$ (strongly continuous, cf. Curtain and Pritchard [4]). If $L(t, s)$ is defined on a separable Hilbert space H, then $L(t, s)$ is by definition of contraction-type or

$$L(t, s) \in \mathscr{G}(1, \beta)$$

if there is a $\beta \ge 0$ such that

$$|L(t, s)|_{\mathscr{L}(H)} \le e^{\beta(t-s)},$$

where $|\cdot|_{\mathscr{L}(H)}$ is the usual bounded operator norm (here with respect to H). Note that $T(t)$ can be extended (and restricted) to any H_γ such that

$$T(t) \in \mathscr{G}(1, \beta) \quad \text{with } \beta = a \vee 0$$

on H_γ, where $\vee$ denotes "maximum." [We will not make any notational distinction between $T(t)$ on H_0 and $T(t)$ on H_γ, $\gamma \ne 0$.] Since $E \subset H_{-\gamma}$ for $\gamma > n/2$, $X^\varepsilon(t) \in H_{-\gamma}$, if $\gamma > n/2$, (2.6) and the separability of H_γ imply that the "weak" martingale $\langle M^\varepsilon(t), \varphi \rangle$ defines a strong $H_{-\gamma}$-valued martingale, where $\gamma > n/2 + 1$. Thus we can compare X^ε and X in $H_{-\gamma}$ for $\gamma > n/2 + 1$.

We now define another deterministic model (depending on ε) which is "closer" to (2.5) and compare it to (2.1). Define an operator B^ε on B_0 (= continuous functions from S into $\mathbb{R}$) by

$$B^\varepsilon(\varphi)(r) := \int \varphi(q) R^\varepsilon(r, q)\, dq,$$

and consider

$$\begin{aligned}\frac{\partial}{\partial t} f^\varepsilon(t, r) = {} & [(D\Delta + a) f^\varepsilon](t, r) \\ & + b f^\varepsilon(t, r) B^\varepsilon(f^\varepsilon)(t, r), \qquad f^\varepsilon(0, r) \ge 0,\ f^\varepsilon(0) \in B_0.\end{aligned} \tag{2.10}$$

The existence of a unique mild positive solution f^ε of (2.10) such that $f^\varepsilon \in C([0,\infty); B_0)$ can be shown by the same means as the existence of a unique X for (2.1) (cf. Kotelenez [12], Appendix).

We will now compare X with f^ε, first in sup-norm and then in distribution norm.

LEMMA 2.1. *For any $\hat{t}$ there is a finite constant $c(\hat{t})$ such that*

$$\sup_{0\le t\le \hat{t}} |||X(t) - f^\varepsilon(t)||| \le c(\hat{t})(|||X(0) - f^\varepsilon(0)||| + \varepsilon^{1/n}). \tag{2.11}$$

PROOF. (i) By "variation of constants" we obtain

$$X(t) = T(t)X_0 + \int_0^t T(t-s)bX^2(s)\,ds, \tag{2.12}$$

$$f^\varepsilon(t) = T(t)f_0^\varepsilon + \int_0^t T(t-s)bf^\varepsilon(s)B^\varepsilon(f^\varepsilon(s))\,ds, \tag{2.13}$$

whence

$$\begin{aligned} X(t) - f^\varepsilon(t) = T(t)[X_0 - f_0^\varepsilon] & \\ + b\int_0^t T(t-s)\{&X(s)(X(s) - B^\varepsilon(X(s))) \\ &+ [X(s) - f^\varepsilon(s)]B^\varepsilon(X(s)) + f^\varepsilon(s)B^\varepsilon[X(s) - f^\varepsilon(s)]\}\,ds. \end{aligned}$$

(ii) Let $\varphi \in C^1$. Then by elementary calculations,

$$\int |\varphi(r) - \varphi(q)| R^\varepsilon(r,q)\,dq \le c\,|||\varphi|||_1 \varepsilon^{1/n}, \tag{2.14}$$

whence

$$|X(s,r) - B^\varepsilon[X(s)](r)| \le c(\hat{t})\varepsilon^{1/n} \quad [\text{by H.1(k)}].$$

(iii) Consequently, $|T(t)|_{\mathscr{L}(B_0)} \le e^{at}$ [$T(t)$ considered as a bounded operator on B_0] implies that for $t \le \hat{t}$,

$$|||X(t) - f^\varepsilon(t)||| \le c(\hat{t})\left\{|||X_0 - f_0^\varepsilon||| + \varepsilon^{1/n} + \int_0^t |||X(s) - f^\varepsilon(s)|||\,ds\right\}.$$

By the Gronwall lemma we obtain (2.11). □

Unfortunately, the bound of $O(\varepsilon^{1/n})$ provided by (2.11) is not good enough for a central limit theorem scaling in dimensions greater than 1, where we have to consider $\varepsilon^{-1/2}(X^\varepsilon - X)$ [cf. (2.5) and (2.6)]. Moreover, this bound seems to be quite sharp by step (ii) of the previous proof. On the other hand, we can considerably improve on this bound by computing the difference $X - f^{\varepsilon}$ in $H_{-\alpha}$, $\alpha > n/2 + 1$. To this end we make the following assumption.

H.2. $|||X_0 - f_0^{\varepsilon}||| \le c\varepsilon^{2/n}$.

THEOREM 2.1. *Assume* H.2. *Then for any* $\alpha > n/2 + 1$ *and any* $\hat{t}$ *there is a finite constant* $c(\hat{t}, \alpha)$ *such that*

$$\sup_{0 \le t \le \hat{t}} |X(t) - f^{\varepsilon}(t)|_{-\alpha} \le c(\hat{t}, \alpha)\varepsilon^{2/n}. \tag{2.15}$$

PROOF. (i) Note that H.1(k) with $k > n/2 + 1$ implies that the multiplication operator defined by multiplication by $X(s, r)$ is extendible to a bounded operator on $H_{-\alpha}$ for $\alpha \in [0, k]$ such that for any $\hat{t} > 0$ there is a finite constant $c(\hat{t})$ and

$$\sup_{0 \le s \le t} |X(s)|_{\mathscr{L}(H_{-\alpha})} \le c(\hat{t}) \tag{2.16}$$

(Kotelenez [12]). Assume for this proof $\alpha \in (n/2 + 1, k]$.

(ii) $$\begin{aligned} &X^2(s) - f^{\varepsilon}(s)B^{\varepsilon}(f^{\varepsilon}(s)) \\ &\quad = X(s)[f^{\varepsilon}(s) - B^{\varepsilon}[f^{\varepsilon}(s)]] + 2X(s)[X(s) - f^{\varepsilon}(s)] \\ &\qquad + [X(s) - f^{\varepsilon}(s)]\{B^{\varepsilon}[f^{\varepsilon}(s) - X(s)] + B^{\varepsilon}[X(s)] - X(s)\}. \end{aligned}$$

(iii) Let ψ and φ be in H_0. Then by the properties of R^{ε},

$$\langle \psi - B^{\varepsilon}(\psi), \varphi \rangle_0 = \tfrac{1}{2}\int\int[\psi(r) - \psi(q)][\varphi(r) - \varphi(q)]R^{\varepsilon}(r, q)\, dq\, dr. \tag{$*$}$$

(iv) $$\begin{aligned} &|X(s)[f^{\varepsilon}(s) - B^{\varepsilon}[f^{\varepsilon}(s)]]|_{-\alpha}^2 \\ &\quad \le c(\hat{t})|f^{\varepsilon}(s) - B^{\varepsilon}[f^{\varepsilon}(s)]|_{-\alpha}^2 \quad [\text{by (2.16)}] \\ &\quad = c(\hat{t})\sum_l \lambda_l^{-\alpha}\langle f^{\varepsilon}(s) - B^{\varepsilon}[f^{\varepsilon}(s)], \phi_l\rangle_0^2 \quad [\text{by (2.8)}]. \end{aligned}$$

Furthermore,

$$\begin{aligned} &\langle f^{\varepsilon}(s) - B^{\varepsilon}[f^{\varepsilon}(s)], \phi_l\rangle_0 \\ &\quad = \tfrac{1}{2}\int\int[X(s,r) - X(s,q)][\phi_l(r) - \phi_l(q)]R^{\varepsilon}(r,q)\, dq\, dr \\ &\qquad + \tfrac{1}{2}\int\int[X(s,q) - f^{\varepsilon}(s,q) - X(s,r) + f^{\varepsilon}(s,r)] \\ &\qquad\qquad \times[\phi_l(r) - \phi_l(q)]R^{\varepsilon}(r,q)\, dq\, dr \\ &\quad =: A + B \end{aligned}$$

[by ($*$) and adding and subtracting quantities under the integral]. Similarly as in step (ii) of the proof of Lemma 2.1 we obtain

$$\left|\int\int [X(s,r) - X(s,q)][\phi_l(r) - \phi_l(q)] R^\varepsilon(r,q)\, dq\, dr\right|$$

$$\leq c(\hat{t}) \, |||\phi_l|||_1 \int \|r - q\|^2 G(\varepsilon^{2/n}, r, q)\, dq$$

$$\sim c(\hat{t}) \, |||\phi_l|||_1 \, \varepsilon^{2/n}.$$

Note that $|||\phi_l|||_1 \leq c\lambda_l^{1/2}$. Thus

$$|A| \leq c(\hat{t}) \lambda_l^{1/2} \varepsilon^{2/n}.$$

By (2.11), H.2 and (2.14),

$$|B| \leq c(\hat{t}) \lambda_l^{1/2} \varepsilon^{2/n}.$$

Hence

$$|X(s)[f^\varepsilon(s) - B^\varepsilon[f^\varepsilon(s)]]|_{-\alpha}^2 \leq c(\hat{t}) \sum_l \lambda_l^{-\alpha+1} \varepsilon^{4/n}.$$

(v) $|||\{X(s) - f^\varepsilon(s)\}\{B^\varepsilon[f^\varepsilon(s) - X(s)] + B^\varepsilon[X(s)] - X(s)\}||| \leq c(\hat{t})\varepsilon^{2/n}$ by (2.11), H.2 and (2.14).

(vi) $|2X(s)[X(s) - f^\varepsilon(s)]|_{-\alpha} \leq c(\hat{t})|X(s) - f^\varepsilon(s)|_{-\alpha}$ by (2.16).

(vii) Since $|T(t)|_{\mathscr{L}(H_{-\alpha})} \leq e^{at}$ and $|||\cdot||| \geq |\cdot|_{-\alpha}$, we obtain from (2.12) and (2.13), the previous steps and H.2,

$$|X(t) - f^\varepsilon(t)|_{-\alpha} \leq c(\hat{t}, \alpha)\varepsilon^{2/n} + c(\hat{t}) \int_0^t |X(s) - f^\varepsilon(s)|_{-\alpha}\, ds.$$

Hence Gronwall's lemma implies (2.15) for $\alpha \in (n/2 + 1, k]$. Since $|\cdot|_{-\gamma} \leq |\cdot|_{-\alpha}$ if $\gamma \geq \alpha$, (2.15) holds for all $\alpha > n/2 + 1$. □

We abbreviate

$$Y^\varepsilon = \varepsilon^{-1/2}(X^\varepsilon - X),$$

$$\bar{Y}^\varepsilon = \varepsilon^{-1/2}(X^\varepsilon - f^\varepsilon),$$

$$\xi^\varepsilon = \varepsilon^{-1/2}(f^\varepsilon - X),$$

that is,

$$Y^\varepsilon = \bar{Y}^\varepsilon + \xi^\varepsilon.$$

By Theorem 2.1, $|\xi^\varepsilon(t)|_{-\alpha} \to 0$ uniformly on compact intervals, provided $\alpha > n/2 + 1$, $n \leq 3$, and if H.2 holds. We will assume H.2 throughout the rest of the paper.

Let $U(t, s)$ be the evolution operator generator by $D\Delta + a + 2bX(s)$, where $2bX(s)$ acts as a multiplication operator. Since $T(t) \in \mathscr{G}(1/\beta)$ on $H_{-\alpha}$ for all α with $\beta = a \vee 0$ and $X(s)$ satisfies (2.16) for $\alpha \in [0, k]$, $k > n/2 + 1$, $U(t, s)$ is extendible to an evolution operator $\in \mathscr{G}(1/\beta)$ to $H_{-\alpha}$ for all $\alpha \in [0, k]$ with $\beta = \beta(t, \alpha)$ depending on t and α for $\alpha \neq 0$. We will use the same symbol for the extended evolution operators. From (2.5) and (2.10) we obtain by "variation of constants"

$$\bar{Y}^\varepsilon(t) = U(t,0)\bar{Y}_0^\varepsilon + \int_0^t U(t,s)\,dM^\varepsilon(s) + \int_0^t U(t,s)b\varepsilon^{-1/2}\{\eta^\varepsilon(s) + 2\zeta^\varepsilon(s)\}\,ds, \tag{2.17}$$

where

$$\begin{aligned}\langle \eta^\varepsilon(s), \varphi\rangle &= \varepsilon^2 \sum_{i\neq j}^{N_s^\varepsilon} R^\varepsilon(x_s^i, x_s^j)\varphi(x_s^i)\\ &\quad - 2\varepsilon \sum_{i=1}^{N_s^\varepsilon} \int f^\varepsilon(s,q)\left[\frac{\varphi(q) + \varphi(x_s^i)}{2}\right] R^\varepsilon(x_s^i, q)\,dq\\ &\quad + \int\int f^\varepsilon(s,q) f^\varepsilon(s,r) R^\varepsilon(q,r)\varphi(q)\,dq\,dr\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}&\langle \zeta^\varepsilon(s), \varphi\rangle\\ &\quad = \varepsilon \sum_{i=1}^{N_s^\varepsilon} \int f^\varepsilon(s,q)\left[\frac{\varphi(q) + \varphi(x_s^i)}{2}\right] R^\varepsilon(x_s^i, q)\,dq\\ &\qquad - \int\int f^\varepsilon(s,q) f^\varepsilon(s,r) R^\varepsilon(q,r)\varphi(q)\,dq\,dr\\ &\qquad + \int X(s,r) f^\varepsilon(s,r)\varphi(r)\,dr.\end{aligned} \tag{2.19}$$

Our aim is to show that $\eta^\varepsilon + 2\zeta^\varepsilon \to 0$ (suitably) and $M^\varepsilon \Rightarrow M$, where M is an $H_{-\alpha}$-valued Gaussian martingale ($\alpha > n/2 + 1$) whose covariance is obtained from (2.6) by substituting X for X^ε and letting $\varepsilon \to 0$, that is,

$$\begin{aligned}&E(\langle M(t), \varphi\rangle^2)\\ &\quad = \int_0^t ds \left\{\left\langle X(s), \left[2D\sum_{p=1}^n (\partial_p\varphi)^2\right] + a\varphi^2\right\rangle_0 - 2b\langle X^2(s), \varphi^2\rangle_0\right\}\end{aligned} \tag{2.20}$$

(cf. Kotelenez [14] and Dittrich [7]). Then the limit of $\bar{Y}^\varepsilon$, respectively of Y^ε if $n \le 3$, would be the generalized Ornstein–Uhlenbeck process Y which satisfies the following Langevin equation:

$$dY(t) = [D\Delta + a + 2bX(t)]Y(t)\,dt + dM(t) \tag{2.21}$$

and has an explicit representation (through "variation of constants" if $Y_0 \in H_{-\alpha}$ for $\alpha > n/2 + 1$)

$$Y(t) = U(t,0)Y_0 + \int_0^t U(t,s)\,dM(s). \tag{2.22}$$

It follows from Kotelenez [13, 14] that for all $\alpha > n/2 + 1$,

$$Y \in \begin{cases} C^{\mu}([0,\infty); H_{-\alpha}), & \text{a.s. for all } \mu < \frac{1}{2}, \\ C((0,\infty); H_{-\alpha+1}), & \text{a.s.,} \end{cases} \tag{2.23}$$

where $C^{\mu}([0,\infty); H_{-\alpha})$ are the $H_{-\alpha}$-valued Hölder continuous functions with Hölder exponent μ.

The most difficult part in the proof $\bar{Y}^{\varepsilon} \Rightarrow Y$ is showing that $\eta^{\varepsilon} \Rightarrow 0$. To this end we will now analyze the correlation functional and show that a propagation of chaos hypothesis is satisfied if X_0 is Poisson distributed.

3. Correlation function analysis. Let $\phi \in C(S^k, \mathbb{R})$, $k \geq 1$. Then the kth correlation function $F^{k,\varepsilon}(t, y_1, \dots, y_n)$ is defined by

$$\begin{aligned} \int \cdots \int \phi(y_1, \dots, y_k) F^{k,\varepsilon}(t, y_1, \dots, y_k)\, dy_1 \cdots dy_k \\ = E \sum_{\substack{i_1, \dots, i_k \\ \text{(diff)}}}^{N_t^{\varepsilon}} \varepsilon^k \phi(x_t^{i_1}, \dots, x_k^{i_k}), \end{aligned} \tag{3.1}$$

where $y_i \in S$ and "(diff)" under the sum sign means that the $i_1, \dots, i_k$ have to be all different from one another (cf. Dittrich [7] and Lang and Xanh [16]).

Let us now assume that on some interval $[0, \hat{t}]$ the correlation functions can be written for all $k \in \mathbb{N}$ as

$$\begin{aligned} &F^{k,\varepsilon}(t, x_1, \dots, x_k) \\ &\quad = \prod_{i=1}^{k} f^{\varepsilon}(t, x_i) \\ &\qquad - \varepsilon \left[\sum_{i<j}^{k} g^{\varepsilon}(t, x_i, x_j) \prod_{l \neq i, j}^{k} f^{\varepsilon}(t, x_l) + \sum_{i=1}^{k} h^{\varepsilon}(t, x_i) \prod_{j \neq i}^{k} f^{\varepsilon}(t, x_j) \right] \\ &\qquad + \varepsilon \Theta^{k,\varepsilon}(t, x_1, \dots, x_k), \end{aligned} \tag{3.2}$$

where g^{ε}, h^{ε} and $\Theta^{k,\varepsilon}$ are real-valued functions satisfying the conditions given below. Denoting by $|||\cdot|||$ also the sup-norm for functions from $S^k \to \mathbb{R}$, $k \in \mathbb{N}$ and setting

$$p(\varepsilon, n, t) := \begin{cases} (t + \varepsilon^2)^{1/2} - \varepsilon, & n = 1, \\ \ln(t + \varepsilon) - \ln(\varepsilon), & n = 2, \\ \varepsilon^{(2/n)-1} - (t + \varepsilon^{2/n})^{1-(n/2)}, & n > 2, \end{cases} \tag{3.3}$$

we assume on g^{ε} and h^{ε},

$$\sup_{\varepsilon > 0} \sup_{0 \leq t \leq \hat{t}} [p(\varepsilon, n, t)]^{-1} \{|||g^{\varepsilon}(t)||| + |||h^{\varepsilon}(t)|||\} < \infty. \tag{3.4}$$

Concerning $\Theta^{k,\varepsilon}$ we make two different propagation of chaos assumptions.

H.3(L). *Assume in addition to* (3.2) *and* (3.4) *that for all* $k \in \mathbb{N}$,

$$\varepsilon \sup_{0 \le t \le \hat{t}} |||\Theta^{k,\varepsilon}(t)||| \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

H.3(C). *Assume in addition to* (3.2) *and* (3.4) *that for all* $k \in \mathbb{N}$,

$$\sup_{0 \le t \le \hat{t}} |||\Theta^{k,\varepsilon}(t)||| \to 0 \quad \text{as } \varepsilon \Rightarrow 0.$$

H.3(C) obviously implies H.3(L) if the $\hat{t}$ is the same in both conditions. It turns out that H.3(L) implies the law of large numbers (LLN—Theorem 4.1) and H.3(C) implies the central limit theorem (CLT—Theorem 4.2). The assumptions on the initial conditions in the following theorem are by H.2 trivially satisfied if X_0^ε is Poisson distributed with intensity $f_0^\varepsilon(y)\,dy$ (cf. Neveu [18]).

THEOREM 3.1. *Assume that for any* $k \in \mathbb{N}$, $\sup_{x_1,\dots,x_k} |F^{k,\varepsilon}(0, x_1, \dots, x_k) - \prod_{i=1}^k X_0(x_i)| \to 0$ *as* $\varepsilon \to 0$. *Then* H.3(L) *holds for all* $\hat{t} < \infty$, *and if* $n \le 3$ *in addition to* $(*)$, *then* H.3(C) *holds for all* $\hat{t}$ *such that*

$$|b| \hat{t} e^{a\hat{t}} \sup_{\varepsilon} |||f_0^\varepsilon||| < 1.$$

PROOF. To simplify the notation we will drop the superscript "ε" whenever possible.

(i) If $H^k(x_1, \dots, x_k)$ is a sequence of functions we define

$$P^k H^{k+1}(x_1, \dots, x_k) := b \sum_{l=1}^k \int H^{k+1}(x_1, \dots, x_k, q) R(x_l, q)\, dq$$

and

$$Q^k(x_1, \dots, x_k) := b \sum_{i<j}^k R(x_i, x_j)$$

acting as a multiplication operator.

Set $A^k(t, x_1, \dots, x_k) := \prod_{i=1}^k f(t, x_i)$ and denote by $T_k(t)$ the semigroup generated by $D\Delta_k + ak$ and by $V_k(t)$ the semigroup generated by $D\Delta_k + ak + \varepsilon Q^k$ [both on $C(S^k, \mathbb{R})$]. Then Dynkin's formula applied to

$$l(s, \mathscr{X}^\varepsilon(s)) = \varepsilon^k \sum_{\substack{i_1, \dots, i_n \\ \text{(diff)}}}^{N_s^\varepsilon} T_k(t-s) \phi\left(x_s^{i_1}, \dots, x_s^{i_k}\right), \qquad 0 \le s \le t,$$

yields

$$(3.5) \quad F^k(t) = T_k(t) F^k(0) + \int_0^t T_k(t-s)\{P^k F^{k+1}(s) + \varepsilon Q^k F^k(s)\}\, ds,$$

which is equivalent to

$$F^k(t) = V_k(t)F^k(0) + \int_0^t V_k(t-s)P^kF^{k+1}(s)\,ds. \tag{3.6}$$

It can be easily verified that $A^k(t)$ satisfies

$$A^k(t) = T_k(t)A^k(0) + \int_0^t T_k(t-s)P^kA^{k+1}(s)\,ds. \tag{3.7}$$

(ii) We now define the functions $g^\varepsilon =: g$ and $h^\varepsilon =: h$ for the decomposition (3.2) as solutions of the following integral equations:

$$\begin{aligned} g(t,r,q) = \int_0^t ds\, T_2(t-s)\Big\{ & bg(s,r,q)\int f(s,z)[R(q,z)+R(r,z)]\,dz \\ & + bf(s,q)\int g(s,z,r)R(q,z)\,dz \\ & + bf(s,r)\int g(s,q,z)R(r,z)\,dz \\ & + f(s,q)f(s,r)R(q,r)\Big\}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} h(t,r) = \int_0^t ds\, T(t-s)\Big\{ & bh(s,r)\int f(s,z)R(r,z)\,dz \\ & + bf(s,r)\int h(s,z)R(r,z)\,dz \\ & + b\int g(s,r,z)R(r,z)\,dz\Big\}. \end{aligned} \tag{3.9}$$

The existence of a unique solution—first for (3.8) in $C(S^2, \mathbb{R})$ and then for (3.9) in B_0—is no problem in view of the linearity and the boundedness of $R(r,q)$. Let us now derive an estimate for g. Note that for any $t > 0$ there is a $c(t) < \infty$ such that

$$\sup_\varepsilon \sup_{0\le s\le t} |||\, f^\varepsilon(s)\,||| \le c(t), \qquad f^\varepsilon =: f$$

[by Lemma 2.1, H.2 and the boundedness of $|||\,X(s)\,|||$]. Hence for $s \le t$,

$$|||\, g(s)\,||| \le c(t)\left\{\int_0^s du\, |||\, g(u)\,||| + \left|\left|\left|\int_0^s du\, T_2(s-u)R(\cdot,\cdot)\right|\right|\right|\right\}.$$

From (2.3) we obtain

$$\left|\left|\left|\int_0^t\int\int G(t-s,q,u)G(t-s,r,v)R(u,v)\,du\,dv\,ds\right|\right|\right| \le cp(\varepsilon,n,t), \tag{3.10}$$

with $p(\varepsilon,n,t)$ defined in (3.3). Consequently, using the kernel representation of $T_2(t)$, (2.2) (3.10) and Gronwall's lemma imply that for any $t \ge 0$ there is a

$c(t) < \infty$ such that for any $\varepsilon > 0$,

$$\sup_{0 \le s \le t} ||| g^{\varepsilon}(s) ||| \le c(t) p(\varepsilon, n, t), \qquad g^{\varepsilon} =: g. \tag{3.11}$$

Using (3.11), we obtain by the same argument for any $t > 0$ a $c(t) < \infty$ such that for any $\varepsilon \ge 0$,

$$\sup_{0 \le s \le t} ||| h^{\varepsilon}(s) ||| \le c(t) p(\varepsilon, n, t), \qquad h^{\varepsilon} =: h, \tag{3.12}$$

that is, (3.4) holds on any bounded interval. Setting for $k \in \mathbb{N}$,

$$\begin{aligned} B^k(t, x_1, \dots, x_n) &:= A^k(t, x_1, \dots, x_n) \\ &\quad - \varepsilon \Bigg[\sum_{i<j}^{k} g(t, x_i, x_j) \prod_{l \ne i, j}^{k} f(t, x_l) \\ &\qquad + \sum_{i=1}^{k} h(t, x_i) \prod_{j \ne i}^{k} f(t, x_j) \Bigg], \end{aligned} \tag{3.13}$$

it can be verified by an elementary calculation that for any k,

$$B^k(t) = T_k(t) A^k(0) + \int_0^t T_k(t-s) \left[P^k B^{k+1}(s) + \varepsilon Q^k A^k(s) \right] ds. \tag{3.14}$$

Note that (3.14) is similar to (3.5). By adding and subtracting on the right-hand side of (3.14) $\int_0^t T_k(t-s) \varepsilon Q^k B^k(s)\, ds$, we obtain by "variation of constants"

$$\begin{aligned} B^k(t) &= V_k(t) A^k(0) + \int_0^t V_k(t-s) P^k B^{k+1}(s)\, ds \\ &\quad + \varepsilon^2 \int_0^t V_k(t-s) Q^k L^k(s)\, ds, \end{aligned} \tag{3.15}$$

where

$$L^k(s) := \varepsilon^{-1} \left[A^k(s) - B^k(s) \right].$$

(iii) Denoting

$$\hat{\Theta}^k(t) := F^k(t) - B^k(t),$$

we obtain from (3.5), (3.14) and $(*)$ for any $t > 0$, a $c(t) < \infty$ such that for any $k \in \mathbb{N}$ and $\varepsilon > 0$,

$$\sup_{0 \le s \le t} ||| \hat{\Theta}^{k, \varepsilon}(s) ||| \le c(t) k^2 e^{akt}, \qquad \hat{\Theta}^{k, \varepsilon} =: \hat{\Theta}^k. \tag{3.16}$$

Moreover, (3.6) and (3.15) imply for $0 \le s \le t$,

$$\begin{aligned} \hat{\Theta}^k(t) &= V_k(t-s) \hat{\Theta}^k(s) + \int_s^t V_k(t-u) P^k \hat{\Theta}^{k+1}(u)\, du \\ &\quad + \varepsilon^2 \int_s^t V_k(t-u) Q^k L^k(u)\, du. \end{aligned} \tag{3.17}$$

Using this equation for $\hat{\Theta}^{k+1}(u)$ on the right-hand side and repeating this procedure, we obtain for any $m \geq 0$,

$$\hat{\Theta}^k(t) = V_k(t-s)\hat{\Theta}^k(s) + \sum_{i=0}^{m} I(t,s,k,i,\varepsilon) + \varepsilon^2 \sum_{i=0}^{m} J(t,s,k,i,\varepsilon) + K(t,s,k,m,\varepsilon),$$

where

$$I(t,s,k,i,\varepsilon) := \int_s^t \cdots \int_s^{t_{i-1}} V_k(t-t_1)P^k \cdots P^{k+i-1}V_{k+i}(t_{i-1}-s) \times \hat{\Theta}^{k+i}(s)\, dt_i \cdots dt_1,$$

$$J(t,s,k,i,\varepsilon) := \int_s^t \cdots \int_s^{t_{i-1}} V_k(t-t_1)P^k \cdots P^{k+i-1}V_{k+i}(t_i-t_{i+1}) \times Q^{k+i}L^{k+i}(t_{i+1})\, dt_{i+1} \cdots dt_1,$$

$$K(t,s,k,m,\varepsilon) := \int_s^t \cdots \int_s^{t_m} V_k(t-t_1)P^k \cdots P^{k+m-1}V_{k+m}(t_m-t_{m+1}) \times P^{k+m}\hat{\Theta}^{k+m+1}(t_{m+1})\, dt_{m+1} \cdots dt_1,$$

$t_0 = t$ and $\Sigma_{i=1}^{0} = 0$.

By the second inequality in (2.2) there is for any $\varepsilon > 0$ and $c > 0$ a $j(c,\varepsilon)$ such that

$$(3.18) \qquad cl + \varepsilon Q^l < 0 \quad \text{for all } l \geq j(c,\varepsilon).$$

On the other hand, setting $\tilde{V}_l(t) := V_l(t)e^{clt}$ and denoting by $S_l(t)$ the semigroup generated by $D\Delta_l$, then with $\phi \in C(S^l, \mathbb{R})$ and $0 \leq s \leq t$,

$$(3.19) \qquad \tilde{V}_l(t)\phi = S_l(t)\phi + \int_0^t S_l(t-u)\left[al + \varepsilon Q^l + cl\right]\tilde{V}_l(u)\phi\, du.$$

The semigroups are positive, and assuming l large we get $[al + \varepsilon Q^l + cl] < 0$. Since $C(S^l, \mathbb{R})$ is a Banach lattice, this implies $|\tilde{V}_l(t)| \leq 1$, where for notational convenience we drop the reference space for the bounded operator norm [cf. Davies [5], Lemma 7.1—$\tilde{V}_l(t)$ is a bounded operator on $C(S^l, \mathbb{R})$]. Consequently [taking $c + a$ instead of c in (3.18)], there is for any $\varepsilon > 0$ and $c > 0$ an $i(c,\varepsilon)$ such that

$$(3.20) \qquad |V_{k+l}(t)| \leq \exp[-c(k+l)t] \quad \text{for all } l \geq i(c,\varepsilon).$$

Second, we obviously have for any j,

$$|P^j| \leq j|b|$$

[P^j is a bounded operator form $C(S^{j+1}, \mathbb{R})$ into $C(S^j, \mathbb{R})$].

For $t > 0$ choose $c > 0$ such that $\ln(c) > at$. Then by (3.16) and (3.20) (choosing m large),

$$
\begin{aligned}
&|||K(t,s,k,m,\varepsilon)||| \\
&\quad \le c(t)(k+m+1)^2 \int_0^t \cdots \int_0^{t_{i-1}} c(t,k,i) \\
&\qquad \times \left[\int_0^{t_i} \cdots \int_0^{t_m} e^{-c(k+i)(t_i - t_{i+1})} \cdots e^{-c(k+m)(t_m - t_{m+1})} \right. \\
&\qquad\qquad \left. \times e^{a(k+m+1)t_{m+1}} \, dt_{m+1} \cdots dt_{i+1} \right] dt_i \cdots dt_1 \prod_{l=1}^{m} (k+l) \\
&\quad \le c(t,k,i)(m-i)^2 \exp[(-\ln(c) + at)(m-i)] \to 0 \quad \text{as } m \to \infty.
\end{aligned}
$$

Therefore, for any $0 \le s \le t$ and any k,

$$
\hat{\Theta}^k(t) = V_k(t-s)\hat{\Theta}^k(s) + \sum_{i=1}^{\infty} I(t,s,k,i,\varepsilon) + \varepsilon^2 \sum_{i=0}^{\infty} J(t,s,k,i,\varepsilon). \tag{3.21}
$$

(iv) By (3.7) and (3.11)–(3.13) there is for any $t > 0$ a $c(t) < \infty$ such that for any $\varepsilon > 0$ and $j \in \mathbb{N}$,

$$
\sup_{0 \le s \le t} |||L^j(s)||| \le c(t) j^2 e^{ajt} \rho^j p(\varepsilon, n, t), \tag{3.22}
$$

where $\rho := \sup_\varepsilon |||f^\varepsilon(0)|||$. We will first estimate the innermost integral in $J(t, s, k, i, \varepsilon)$. From (3.22) we obtain

$$
\begin{aligned}
&\left|\left|\left| \int_s^u V_j(u-v) Q^j L^j(v)\, dv \right|\right|\right| \\
&\qquad \le c(u) j^2 p^j \left|\left|\left| \int_s^u V_j(u-v) e^{ajv} Q^j \, dv \right|\right|\right| p(\varepsilon, n, u) \\
&\qquad \le c(u) j^4 \rho^j e^{aju} \left|\left|\left| \int_s^u S_2(u-v) R(\cdot,\cdot)\, du \right|\right|\right| p(\varepsilon, n, u).
\end{aligned}
$$

[Similarly as in (3.19)—with $\tilde{V}_j(t) = V_j(t)e^{-ajt}$—we obtain $\tilde{V}_j(t)|Q^j| \le S_j(t)|Q^j|$. Here $|Q^j|$ is the function $(x_1, \dots, x_j) \to |Q^j(x_1, \dots, x_j)|$.]

Thus, using the kernel representation of $S_2(t)$ and (3.10),

$$
\left|\left|\left| \int_s^u V_j(u-v) Q^j L^j(v)\, dv \right|\right|\right| \le c(s) j^4 \rho^j e^{aju} [p(\varepsilon, n, u)]^2.
$$

By the estimate $|V_j(t)| \le e^{ajt}$ in the other integrals of $J(t, s, k, i, \varepsilon)$, we obtain now

$$
\begin{aligned}
|||J(t,s,k,i,\varepsilon)||| &\le c(t)(k+i)^{4+k} \rho^k [p(\varepsilon, n, t)]^2 \\
&\quad \times e^{a(k+1)t} \left(e^{at}(t-s)\rho|b| \right)^i.
\end{aligned} \tag{3.23}
$$

Finally, for any $t > 0$,

$$\varepsilon[p(\varepsilon, n, t)]^2 = O(\varepsilon^{1/3}) \quad \text{if } n \le 3,$$

$$\varepsilon^2[p(\varepsilon, n, t)]^2 = o(1) \qquad \text{for all } n.$$

(v) As in the second part of the last step, we obtain by (3.16)

$$\text{(3.24)} \qquad |||I(t, s, k, i, \varepsilon)||| \le c(s)(k + i)^{2+k} e^{akt} (e^{at}(t - s)|b|)^i.$$

Assume now that the first statement of Theorem 3.1 holds for all $s < t < \infty$ (it holds for $\hat{t} = 0$ by H.2). Then we choose s so close to $\hat{t}$ that $(e^{a\hat{t}}(\hat{t} - s)(\rho + 1)) < 1$, whence there is a $\delta > 0$ such that for all $t \in [s, \hat{t} + \delta]$ both series on the right-hand side of (3.21) are dominated by absolutely convergent series (independent of ε!). Thus the first statement of our theorem holds on $[0, t + \delta]$ by Lebesgue's dominated convergence theorem (applied to $\Sigma_i |||I(t, s, k, i, \varepsilon)|||$). These arguments and the estimates in step (iv) imply the second statement. □

4. Limit theorems. In this section α will be some arbitrary fixed real number greater than $n/2 + 1$, and $D([0, \hat{t}]; H_{-\alpha})$ denotes the Skorohod space of $H_{-\alpha}$-valued cadlag functions defined on $[0, \hat{t}]$ for some $\hat{t} > 0$. "$\Rightarrow$" denotes "weak convergence." Y_0 is an $H_{-\alpha}$-valued square-integrable random variable defined on the same probability space as M [given by (2.20)], and supposed to be independent of M.

THEOREM 4.1 (LLN). *Assume* $|X_0^\varepsilon - X_0|_{-\alpha} \to 0$ *stochastically in addition to* H.3(L) *for any* $\hat{t} > 0$, H.1(k) *with* $k > n/2 + 1$ *and* H.2. *Then for any* $\hat{t} > 0$ *and* $\delta > 0$,

$$\text{(4.1)} \qquad P\Big\{ \sup_{0 \le t \le \hat{t}} |X^\varepsilon(t) - X(t)|_{-\alpha} > \delta \Big\} \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

THEOREM 4.2 (CLT). *Assume* $Y_0^\varepsilon \Rightarrow Y_0$, *and* Y_0 *is independent of* M [*defined by* (2.20)] *in addition to* H.3(C) *for some* $\hat{t} > 0$, H.1(k) *with* $k > n/2 + 1$, H.2 *and* $n \le 3$. *Then*

$$\text{(4.2)} \qquad Y^\varepsilon \Rightarrow Y \quad \text{on } D([0, \hat{t}]; H_{-\alpha}), \qquad Y^\varepsilon = \varepsilon^{-1/2}(X^\varepsilon - X),$$

where Y *is the Ornstein–Uhlenbeck process given by* (2.21) *and* (2.22) *with* $Y(0) = Y_0$.

First, we will derive an inequality needed in the proofs of the theorems, whose proof follows from the inequality

$$(*) \qquad |a_n|^2 \le \frac{c^{3n-2}}{c - 1} \sum_{i=1}^{n} |a_i - b_i|^2,$$

for real numbers a_n, b_n with $a_0 = 0$, $|b_i| \le c|a_{i-1}|$ and $c > 1$, $n \in \mathbb{N}$. [(*) is shown by induction: (i) It holds for $n = 1$ since our assumption implies $b_0 = 0$. (ii) If it holds for $n - 1 \geq 1$, then $(a_n)^2 \leq (a_n - b_n)^2 + [1/(c-1)] \times (a_n - b_n)^2 + (c-1)b_n^2 + b_n^2 \leq (a_n - b_n)^2[c^{3n-2}/(c-1)] + c^3(a_{n-1})^2 \leq$ the right-hand side of $(*)$ by assumption.]

LEMMA 4.1. *Let Z be an $H_{-\alpha}$-valued square-integrable process, $t \leq \hat{t}$ and $m \in \mathbb{N}$. Set*

$$N(t) := \int_0^t U(t,u)Z(u)\,du, \quad p = t2^{-m}, \quad \tilde{N}(kp) := U((k+1)p, kp)N(kp)$$

$$\text{and} \quad \beta := \beta(\hat{t}, \alpha).$$

Then

$$E \max_{1 \leq k \leq 2^m} |N(kp)|_{-\alpha}^2 \leq c(\hat{t}) \cdot 2^m \sum_{k=1}^{2^m} E\big|N(kp) - \tilde{N}((k-1)p)\big|_{-\alpha}^2. \tag{4.3}$$

PROOF. We set $c := e^{\beta 2^{-m}(t+1)}$ with $\beta = \beta(\hat{t}, \alpha)$ from the definition of the bounded operator norm of $U(t,s)$ on $H_{-\alpha}$. Furthermore, we set $a_k := |N(kp)|_{-\alpha}$, $b_k := |\tilde{N}((k\text{-}1)p)|_{-\alpha}$, apply the triangle inequality to $|\cdot|_{-\alpha}$ and use the monotonicity of the right-hand side of $(*)$. Then we take

$$c(\hat{t}) := c(\hat{t}, \alpha) := \frac{e^{3\beta(t+1)}}{(t+1)\beta}.$$

PROOF OF THEOREM 4.2. Let us denote by $c(\varepsilon)$ finite constants such that $(p(\varepsilon, n, t))^{-1}c(\varepsilon)$ is bounded uniformly in ε on finite intervals [cf. (3.3)]. The first step of the proof is a direct generalization of Dittrich's proof in [7] to our setting. Nevertheless we will give the most important steps.

(i) Take a smooth φ, abbreviate

$$Q(x,y) := \frac{[\varphi(x) + \varphi(y)]}{2} R^{\varepsilon}(x,y),$$

$$Q(u,x,y) := \int\int dq\,dr\,Q(r,q)G(u,q,x)G(u,r,y)e^{2ua}$$

and define

$$\begin{aligned}\langle \eta^{\varepsilon}(s,u), \varphi\rangle := \Bigg[&\varepsilon^2 \sum_{i \neq j}^{N_s^{\varepsilon}} Q\big(u, x_s^i, x_s^j\big) - 2\varepsilon \sum_{i=1}^{N_s^{\varepsilon}} \int dq\, f^{\varepsilon}(s,q) Q\big(u, x_s^i, q\big) \\ &+ \int\int dq\,dr\, f^{\varepsilon}(s,q)\, f^{\varepsilon}(s,r) Q(u,q,r)\Bigg].\end{aligned} \tag{4.4}$$

Note that $\eta^{\varepsilon}(s,0) = \eta^{\varepsilon}(s)$. Applying Dynkin's formula to the functional. $\langle \eta^{\varepsilon}(s+u, t-s-u), \varphi\rangle$ of the Markov process $(X_{s+u}^{\varepsilon}, u)$, $0 \leq u \leq t-s$, we obtain

$$
\begin{aligned}
(4.5)\qquad \eta^\varepsilon(t) = \eta^\varepsilon(s, t-s) &+ \int_0^{t-s} du\, H^\varepsilon(X^\varepsilon(s+u), f^\varepsilon(s+u)) \\
&+ m^\varepsilon(s, t-s),
\end{aligned}
$$

with

$$
\begin{aligned}
&\langle H^\varepsilon(X^\varepsilon(s+u), f^\varepsilon(s+u)), \varphi\rangle \\
&\quad := -2b\varepsilon \sum_{i=1}^{N^\varepsilon_{s+u}} \int\int dq\, dr\, f^\varepsilon(s+u, q) f^\varepsilon(s+u, r) R^\varepsilon(q, r) \\
&\qquad\quad \times Q(t-s-u, X^i_{s+u}, r) \\
&\qquad + 2b \int\int\int dq\, dr\, dy\, f^\varepsilon(s+u, q) f^\varepsilon(s+u, r) f^\varepsilon(s+u, y) \\
&\qquad\quad \times R^\varepsilon(q, y) Q(t-s-u, q, r) \\
&\qquad + b\frac{\varepsilon^3}{2} \sum_{k \neq l}^{N^\varepsilon_{s+u}} \left(\sum_{\substack{i \neq j \\ \{i,j\} \cap \{k,l\} \neq \varnothing}}^{N^\varepsilon_{s+u}} Q(t-s-u, x^i_{s+u}, x^j_{s+u}) \right) R^\varepsilon(x^k_{s+u}, x^l_{s+u}) \\
&\qquad - 2b\varepsilon^2 \sum_{k \neq l}^{N^\varepsilon_{s+u}} \int dq\, f^\varepsilon(s+u, q) Q(t-s-u, x^k_{s+u}, q) R^\varepsilon(x^k_{s+u}, x^l_{s+u})
\end{aligned}
$$

and $m^\varepsilon(s,u)$ a family (in s) of $H_{-\alpha}$-valued square-integrable F_{s+u} martingales $[F_{s+u} := \sigma(X^\varepsilon_v, v \le s+u)]$ such that $m^\varepsilon(s,0) = 0$ and $\langle m^\varepsilon(s,u), \varphi\rangle$ is (s,u)-measurable.

By elementary calculations using (3.2), we obtain for $0 \le u, s, v \le t \le \hat{t}$,

$$
\begin{aligned}
(4.6)\qquad & E\left[\langle \eta^\varepsilon(u), U(t,u)\varphi\rangle^2\right] + E\langle H^\varepsilon(X^\varepsilon(s), f^\varepsilon(s)), U(t,v)\varphi\rangle_0^2 \\
&\quad \le c(\hat{t})\varepsilon c(\varepsilon) |||\varphi|||^2.
\end{aligned}
$$

Consequently, for $0 \le s \le t \le \hat{t}$,

$$
\begin{aligned}
&E\left(\int_s^t \langle \eta^\varepsilon(u), U(t,u)\varphi\rangle\, du\right)^2 \\
&\quad = 2E\left\{\int_s^t\int_s^v \Big[\langle \eta^\varepsilon(u), U(t,u)\varphi\rangle\langle \eta^\varepsilon(u, v-u), U(t,v)\varphi\rangle \right. \\
&\qquad + \langle \eta^\varepsilon(u), U(t,u)\varphi\rangle \int_0^{v-u} \langle H^\varepsilon(X^\varepsilon(u+w), f^\varepsilon(u+w)), U(t,v)\varphi\rangle\, dw \\
&\qquad \left. + \langle \eta^\varepsilon(u), U(t,u)\varphi\rangle\langle m^\varepsilon(u, v-u), U(t,v)\varphi\rangle_0\Big]\, du\, dv\right\} \\
&\quad \le c(\hat{t})\left\{\int_s^t\int_s^v E\langle \eta^\varepsilon(u), U(t,u)\varphi\rangle\langle \eta^\varepsilon(u, v-u), U(t,v)\varphi\rangle\, du\, dv \right. \\
&\qquad \left. + (t-s)^3 \varepsilon c(\varepsilon)\right\} |||\varphi|||^2
\end{aligned}
$$

by the properties of $m^\varepsilon(u, v-u)$ and (4.6). Another calculation using the propagation of chaos hypothesis shows that for $0 \le s \le s+u \le t \le \hat{t}$,

$$
\begin{aligned}
&E\langle \eta^\varepsilon(s), U(t,s)\varphi\rangle\langle \eta^\varepsilon(s,u), U(t,s+u)\varphi\rangle \\
&\quad= \int\!\!\int\!\!\int\!\!\int dz_1 \cdots dz_4 \\
&\qquad \times\Big\{\varepsilon\Theta^{4,\varepsilon}(s,z_1,\dots,z_4)\overline{Q}(z_1,z_2)\overline{Q}(u_1,z_1,z_2) \\
&\qquad\quad -2\varepsilon\Theta^{3,\varepsilon}(s,z_1,z_2,z_3)f^\varepsilon(s,z_4) \\
&\qquad\qquad \times\big[\overline{Q}(z_1,z_2)\overline{Q}(u_1,z_3,z_4)+\overline{Q}(u_1,z_1,z_2)\overline{Q}(z_3,z_4)\big] \\
&\qquad\quad +4\varepsilon\Theta^{2,\varepsilon}(s,z_1,z_2)f^\varepsilon(s,z_3)f^\varepsilon(s,z_4)\overline{Q}(z_1,z_2)\overline{Q}(u,z_3,z_4) \\
&\qquad\quad +\varepsilon\Theta^{2,\varepsilon}(s,z_1,z_2)f^\varepsilon(s,z_3)f^\varepsilon(s,z_4) \\
&\qquad\qquad \times\big[\overline{Q}(z_1,z_2)\overline{Q}(u,z_3,z_4)+\overline{Q}(u,z_1,z_2)\overline{Q}(z_3,z_4)\big] \\
&\qquad\quad -2\varepsilon\Theta^{1,\varepsilon}(s,z_1)f^\varepsilon(s,z_2)f^\varepsilon(s,z_3)f^\varepsilon(s,z_4) \\
&\qquad\qquad \times\big[\overline{Q}(z_1,z_2)\overline{Q}(u,z_3,z_4)+\overline{Q}(u,z_1,z_2)\overline{Q}(z_3,z_4)\big]\Big\} \\
&\qquad + 4\varepsilon\int\!\!\int\!\!\int dz_1\,dz_2\,dz_3 \\
&\qquad\quad \times\Big\{\varepsilon\overline{F}^{3,\varepsilon}(s,z_1,z_2,z_3)\overline{Q}(z_1,z_2)\overline{Q}(u,z_2,z_3) \\
&\qquad\quad -\varepsilon\overline{F}^{2,\varepsilon}(s,z_1,z_2)f^\varepsilon(s,z_3) \\
&\qquad\qquad \times\big[\overline{Q}(z_1,z_2)\overline{Q}(u,z_2,z_3)+\overline{Q}(u,z_1,z_2)\overline{Q}(z_2,z_3) \\
&\qquad\quad +\varepsilon\overline{F}^{1,\varepsilon}(s,z_1)f^\varepsilon(s,z_2)f^\varepsilon(s,z_3)\overline{Q}(z_1,z_2)\overline{Q}(u,z_1,z_3)\big]\Big\} \\
&\qquad + 2\varepsilon^2\int\!\!\int dz_1, dz_2\, F^{2,\varepsilon}(s,z_1,z_2)\overline{Q}(z_1,z_2)\overline{Q}(u,z_1,z_2),
\end{aligned}
$$

with

$$
\overline{Q}(x,y) := bR^\varepsilon(x,y)\left\{\frac{U(t,s)\varphi(x)+U(t,s)\varphi(y)}{2}\right\},
$$

$$
\begin{aligned}
\overline{Q}(u,x,y) &= \int\!\!\int dz_1\,dz_2\left\{\frac{U(t,s+u)\varphi(z_1)+U(t,s+u)\varphi(z_2)}{2}\right\}bR^\varepsilon(z_1,z_2) \\
&\quad\times G(u,z_1,x)G(u,z_2,y)e^{2au}
\end{aligned}
$$

and

$$
\varepsilon\overline{F}^{k,\varepsilon}(s,z_1,\dots,z_k) := F^{k,\varepsilon}(s,z_1,\dots,z_k) - \prod_{j=1}^{k} f^\varepsilon(s,z_j).
$$

Hence, for $0 \le s \le u \le v \le t \le \hat{t}$,

$$E\langle \eta^{\varepsilon}(u), U(t,u)\varphi\rangle\langle \eta^{\varepsilon}(u, v-u), U(t,v)\varphi\rangle \le c(\hat{t})\left[o(\varepsilon) + \varepsilon^2 O\left(\varepsilon^{2/n} + (v-u)^{-n/2}\right)\right] |||\varphi|||^2. \tag{4.7}$$

Let $p, q \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, $p < q$, and define the projection operator π_p^q on $H_{-\alpha}$ by

$$\pi_p^q \phi = \sum_{p \le |l| < q} \langle \phi, \phi_l^{-\alpha}\rangle_{-\alpha} \phi_l^{-\alpha}, \quad \text{where } \phi \in H_{-\alpha}.$$

(4.7) implies by (2.8) and $|||\phi_l|||^2 \le 2^n$,

$$\begin{aligned} &E\left|\pi_p^q \int_s^t U(t,u)\eta^{\varepsilon}(u)\,du\right|_{-\alpha}^2 \\ &\quad \le c(\hat{t},\alpha)\Big\{o(\varepsilon)(t-s)^2 + \left[\varepsilon^2(t-s)c(\varepsilon) \wedge \varepsilon(t-s)^2\right] \\ &\qquad + (t-s)^3 \varepsilon c(\varepsilon)\Big\} \sum_{p \le |l| < q} \lambda_l^{-\alpha}, \end{aligned} \tag{4.8}$$

where "$\wedge$" denotes minimum.

(ii) We apply inequality (4.3) with $Z := \varepsilon^{-1/2}\eta^{\varepsilon}$ and obtain for $\bar{t} \le \hat{t}$ [with $p = \bar{t}2^{-m}$, $s = (k-1)p$ and $t = kp$ and π_0^{∞} in (4.8)]:

$$\begin{aligned} &E\left|\int_0^{\bar{t}} U(\bar{t},u)\varepsilon^{-1/2}\eta^{\varepsilon}(u)\,du\right|_{-\alpha}^2 \\ &\quad \le c(\hat{t},\alpha)\bar{t}^2\{o(1) + c(\varepsilon)\varepsilon 2^m + c(\varepsilon)2^{-m}\} \to 0 \quad \text{as } \varepsilon \downarrow 0 \end{aligned} \tag{4.9}$$

and $m = m(\varepsilon) \sim -\ln(\varepsilon)/2\ln(2) \to \infty$. In particular,

$$N(t) := \int_0^t U(t,u)\varepsilon^{-1/2}\eta^{\varepsilon}(u)\,du$$

is compact on $H_{-\alpha}$ for any $t \le \hat{t}$.

We will now show that $N(\cdot)$ satisfies a "modulus of continuity" condition on $D([0,\hat{t}]; H_{-\alpha})$ given in Kurtz [15], Theorem 2.7. Define for $\delta > 0$ and $\mu \in (0,1)$,

$$\begin{aligned} \gamma_{\varepsilon}^t(\delta) &:= 2 \sup_{0 \le t \le \hat{t}} |(U(t+\delta,t) - I)N(t)|_{-\alpha}^2 \\ &\quad + \delta^{\mu} \sup_{0 \le s \le t \le \hat{t}} \left|\int_s^t [U(t,u)\varepsilon^{-1/2}\eta^{\varepsilon}(u)\,du]\right|_{-\alpha}^2 (t-s)^{-\mu} \\ &=: A_{\varepsilon}(\delta) + B_{\varepsilon}(\delta). \end{aligned}$$

Note that $\sup_{\varepsilon} E(\sup_{0 \le \delta \le \hat{t}} A_{\varepsilon}(\delta)) \le c(\hat{t}) < \infty$ by taking the monotone limit in

(4.3) and using (4.8) with $p = 0$ and $q = \infty$. Furthermore,

$$\begin{aligned} A_\varepsilon(\delta) &\le 4 \sup_{0\le t\le \hat t} \left|(U(t+\delta,t) - I)\pi_p^\infty N(t)\right|_{-\alpha}^2 \\ &\quad + 4 \sup_{0\le t\le \hat t} \left|(U(t+\delta,t) - I)\pi_0^p N(t)\right|_{-\alpha}^2 \\ &=: A_\varepsilon(\pi_p^\infty, \delta) + A_\varepsilon(\pi_0^p, \delta). \end{aligned}$$

Since $A_\varepsilon(\pi_0^\infty, \delta) = 2A_\varepsilon(\delta)$ we may by the above argument for arbitrary $\gamma > 0$ choose a $p = p(\gamma)$ so that

$$\sup_\varepsilon E \sup_{0\le s\le \hat t} A_\varepsilon(\pi_p^\infty, \delta) \le \frac{\gamma}{2},$$

by Dini's theorem and the monotone convergence theorem.

Similarly we obtain

$$\begin{aligned} \sup_\varepsilon E A_\varepsilon(\pi_0^p, \delta) &\le c(\hat t) \sum_{|l|\le p} \sup_{0\le t\le \hat t} \left|(U(t+\delta,t) - I)\phi_\lambda^{-\alpha}\right|_{-\alpha}^2 \\ &\to 0 \quad \text{as } \delta \downarrow 0, \end{aligned}$$

by the strong continuity of $U(t,s)$. Therefore there is a $\delta = \delta(\gamma) > 0$ so that $\sup_\varepsilon EA_\varepsilon(\delta) < \gamma$. Hence, $\sup_\varepsilon EA_\varepsilon(\delta) \to 0$, as $\delta \downarrow 0$. Estimating $B_\varepsilon(\delta)$, we get (using the notation of the statement of Lemma 4.1) for $j < k$,

$$\begin{aligned} &\left|N(kp) - \tilde N((j-1)p)\right|_{-\alpha}^2 ((k-j)p)^{-\mu} \\ &\quad \le e^{2\beta\hat t} \sum_{i=j}^{k-1} \left|N(ip) - \tilde N((i-1)p)\right|_{-\alpha}^2 ((k-j)p)^{-\mu} \\ &\qquad + 2e^{2\beta\hat t} \sum_{j\le i<l<k} \left|N(ip) - \tilde N((i-1)p)\right|_{-\alpha} \\ &\qquad\qquad \times \left|N(lp) - \tilde N((l-1)p)\right|_{-\alpha} ((k-j)p)^{-\mu} \\ &\quad \le c(\hat t)\left\{\sum_{i=1}^{2^m} \left|N(ip) - \tilde N((i-1)p)\right|_{-\alpha}^2 p^{-\mu}\right\} \\ &\qquad + c(\hat t)\left\{\sum_{1\le i<l<2^m} \left|N(ip) - \tilde N((i-1)p)\right|_{-\alpha}\right. \\ &\qquad\qquad \left. \times \left|N(lp) - \tilde N((l-1)p)\right|_{-\alpha} ((l-i)p)^{-\mu}\right\} \\ &\quad =: c(\hat t)\left|C_\varepsilon^m + D_\varepsilon^m\right|. \end{aligned}$$

The inequality (4.8) yields

$$EC_\varepsilon^m \le c(\hat t) 2^{m(\mu-1)}[1 + 2^{-m}c(\varepsilon)] \to 0 \quad \text{as } m \to \infty.$$

Again from (4.8) we obtain

$$\begin{aligned} ED_\varepsilon^m &\le c(\hat{t}) \sum_{i=1}^{2^m} 2^{-m} \sum_{l=i+1}^{2^m} 2^{-m}((l-i)2^{-m})^{-\mu}\{1 + 2^{-m}c(\varepsilon)\} \\ &\sim c(\hat{t}) \frac{1}{1-\mu}\{1 + 2^{-m}c(\varepsilon)\} \\ &\to c(\hat{t}) \frac{1}{1-\mu} \quad \text{as } m \to \infty. \end{aligned}$$

Consequently, by monotonicity

$$\begin{aligned} \sup_\varepsilon EB_\varepsilon(\delta) &= \sup_\varepsilon \delta^\mu 2 \lim_{m\to\infty} E \max_{1\le j<k\le 2^m} |N(kp) - \tilde{N}((j-1)p)|_{-\alpha}^2((k-j)p)^{-\mu} \\ &\le \delta^\mu c(\hat{t}) \frac{1}{1-\mu}. \end{aligned}$$

Altogether we obtain [setting $\eta^\varepsilon(u) = 0$ for $u > \hat{t}$]

$$\left| \int_0^{t+\delta} U(t+\delta, u)\varepsilon^{-1/2}\eta^\varepsilon(u)\, du - \int_0^t U(t,u)\varepsilon^{-1/2}\eta^\varepsilon(u)\, du \right|_{-\alpha}^2 \le \gamma_\varepsilon^{\hat{t}}(\delta) \tag{4.10}$$

and

$$\lim_{\delta\downarrow 0} \limsup_{\varepsilon\downarrow 0} E\gamma_\varepsilon^{\hat{t}}(\delta) = 0. \tag{4.11}$$

This implies by (4.9) that

$$\int_0^{\cdot} U(\cdot, u)\varepsilon^{-1/2}\eta^\varepsilon(u)\, du \Rightarrow 0 \quad \text{on } D([0,\hat{t}]; H_{-\alpha}). \tag{4.12}$$

(iii) We will now deal with the ζ^ε-term in (2.17). Let φ be smooth and denote $\psi(t,u,q) := U(t,u)\varphi(q)$. Then by (2.19),

$$\begin{aligned} &E\langle \zeta^\varepsilon(u), U(t,u)\varphi\rangle^2 \\ &= E\left\{\left\langle X^\varepsilon(u) - f^\varepsilon(u), \int f^\varepsilon(u,q) \frac{\psi(t,u,q) + \psi(t,u,\cdot)}{2} R^\varepsilon(q,\cdot)\, dq \right\rangle \right. \\ &\qquad \left. - \langle X^\varepsilon(u) - f^\varepsilon(u), X(u)\psi(t,u,\cdot)\rangle \right\}^2 \\ &= E\langle X^\varepsilon(u) - f^\varepsilon(u), l^\varepsilon(u)\rangle^2 \\ &\qquad \left[\text{with } l^\varepsilon(u) := \int f^\varepsilon(u,q) \frac{(\psi(t,u,q) + \psi(t,u,\cdot))}{2} R^\varepsilon(q,\cdot)\, dq \right. \\ &\qquad \left. - X(u)\psi(t,u,\cdot) \right] \\ &= \varepsilon \int\int \bar{F}^{2,\varepsilon}(u,r,q) l^\varepsilon(u,r) l^\varepsilon(u,q)\, dr\, dq \\ &\quad - 2\varepsilon\langle F^{1,\varepsilon}(u), l^\varepsilon(u)\rangle_0 \langle f^\varepsilon(u), l^\varepsilon(u)\rangle_0. \end{aligned}$$

By elementary calculations using the smoothness of X (which implies the smoothness of $\psi(t, u, \cdot)$; cf. Kotelenez [12]), we obtain for $\hat{t} > 0$ a $c(\hat{t}) < \infty$ such that for $u \le \hat{t}$,

$$|||\, l^{\varepsilon}(u)\, ||| \le c(\hat{t})\varepsilon^{1/n}\, |||\, \varphi\, |||_1.$$

Hence

$$E\langle X^{\varepsilon}(u) - f^{\varepsilon}(u), l^{\varepsilon}(u)\rangle^2 \le c(\hat{t})\varepsilon c(\varepsilon)\varepsilon^{2/n}\, |||\, \varphi\, |||_1^2,$$

and thus for $0 \le s \le t \le \hat{t}$,

$$E\left|\int_s^t b\varepsilon^{-1/2} 2U(t,u)\zeta^{\varepsilon}(u)\, du\right|_{-\alpha}^2 \le c(\hat{t}, \alpha)(t-s)^2 c(\varepsilon)\varepsilon^{2/n}. \tag{4.13}$$

Applying (4.3) with $Z(u) := 2b\varepsilon^{-1/2}\zeta^{\varepsilon}(u)$ and passing to the monotone limit yields for any $\delta > 0$,

$$P\left\{\sup_{0\le t\le \hat{t}}\left|\int_0^t b\varepsilon^{-1/2} 2U(t,u)\zeta^{\varepsilon}(u)\, du\right|_{-\alpha} \ge \delta\right\} \to 0 \quad \text{as } \varepsilon \downarrow 0. \tag{4.14}$$

(iv) $M^{\varepsilon} \Rightarrow M$. Comparing (2.6) and (2.20), it can be easily seen that for any $t \le \hat{t}$ and smooth φ,

$$E(\langle\langle M^{\varepsilon}(t), \varphi\rangle\rangle - \langle\langle M(t), \varphi\rangle\rangle)^2 = o(1)\, |||\, \varphi\, |||_1^2. \tag{4.15}$$

Moreover, we have

$$\begin{aligned}\sup_{0\le t\le \hat{t}} |\langle M^{\varepsilon}(t), \varphi\rangle - \langle M^{\varepsilon}(t-), \varphi\rangle| &= \varepsilon^{-1/2} \sup_{0\le t\le \hat{t}} |\langle X^{\varepsilon}(t), \varphi\rangle - \langle X^{\varepsilon}(t-), \varphi\rangle| \\ &\le 2\varepsilon^{1/2}\, |||\, \varphi\, ||| \to 0 \quad \text{as } \varepsilon \downarrow 0.\end{aligned}$$

This implies by Lipcer and Shiryayev [17] that

$$\langle M^{\varepsilon}(\cdot), \varphi\rangle \Rightarrow \langle M(\cdot), \varphi\rangle \quad \text{on } D([0, \hat{t}]; \mathbb{R}). \tag{4.16}$$

Denoting by π_k the projection of $H_{-\alpha}$ onto the space spanned by all $\phi_l^{-\alpha}$ such that $|l| \le k$, (4.16) implies

$$\pi_k M^{\varepsilon} \Rightarrow \pi_k M \quad \text{on } D([0, \hat{t}]; H_{-\alpha}). \tag{4.17}$$

On the other hand, we obtain from (2.6) and Doob's inequality with $\pi_k^{\perp} = I - \pi_k$,

$$\begin{aligned} E\sup_{t\le \hat{t}} |\pi_k^{\perp} M^{\varepsilon}(t)|_{-\alpha}^2 &\le c(\hat{t}) \sum_{l, |l|>k} \lambda^{-\alpha+1} \\ &\to 0 \quad (\text{uniformly in } \varepsilon) \text{ as } k \to \infty. \end{aligned} \tag{4.18}$$

From elementary manipulations with a suitable metric on $D([0, \hat{t}]; H_{-\alpha})$ (cf. Kotelenez [10]), we obtain from (4.17) and (4.18) that

$$M^{\varepsilon} \Rightarrow M \quad \text{on } D([0, \hat{t}]; H_{-\alpha}). \tag{4.19}$$

Hence by Kotelenez [11] (cf. also [10] and [14]),

$$(4.20)\qquad \int_0^{\cdot} U(\cdot, s)\, dM^{\varepsilon}(s) \Rightarrow \int_0^{\cdot} U(\cdot, s)\, dM(s) \quad \text{on } D([0, \hat{t}]; H_{-\alpha}).$$

(v) Note that our assumption on Y_0 and (4.19) imply that Y_0^{ε} and M^{ε} are asymptotically independent (the proof is the same as that of Lemma 4.12 of Blount [3]). Now (4.2) is obtained from (4.12), (4.14), (4.20) and Theorem 2.1 by using the metric d_p on E. □

The proof of Theorem 4.1 easily follows from the proof of Theorem 4.2.

5. Remarks. The CLT implies (at least for some finite time) that (in distribution)

$$(5.1)\qquad X^{\varepsilon} = X + \varepsilon^{1/2} Y + o(\varepsilon^{1/2}),$$

where X^{ε} formally satisfies

$$(5.2)\qquad dX^{\varepsilon} = [(D\Delta + a)X^{\varepsilon} + bX^{\varepsilon}(X^{\varepsilon} - 1)R^{\varepsilon}]\, dt + \varepsilon^{1/2}\, dM^{\varepsilon}$$

[cf. (2.5) where R^{ε} is appropriately defined as an operator]. Since $R^{\varepsilon}(q, r)$ is close to the delta function, (5.2) is (roughly speaking) a perturbation of the macroscopic equation (2.1) by a semimartingale with a small martingale part. As mentioned in the introduction, (2.1) (with a more general polynomial interaction) was obtained in Kotelenez [14] as the limit of a system of particles $X_{v,N}$ (defined on a grid) where the interaction was not local. In that model N is the number of grid points ($\sim 1/\varepsilon$) and v is proportional to the number of particles interacting with one another. The evolution equation for $X_{v,N}$ is

$$(5.3)\qquad dX_{v,N} = \big[[D\Delta_{(N)} + a]X_{v,N} + bX_{v,N}^2\big]\, dt + (vN)^{-1/2}\, dM_{v,N},$$

where $M_{v,N}$ is a martingale similar to M^{ε} and $\Delta_{(N)}$ is the discrete Laplacian. The assumptions in [14] were: $v \to \infty$ as $N \to \infty$ for the LLN, and $v/N \to \infty$ as $N \to \infty$ for the CLT [they were also proved to be necessary for polynomials of second degree, i.e., for our macroscopic equation (2.1)]. Hence a perturbation of (2.1) by a small martingale is possible if the interaction is nonlocal.

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REFERENCES

[1] ARNOLD, L. (1981). Mathematical models of chemical reactions. In *Stochastic Systems* (M. Hazewinkel and J. Willams, eds.). Reidel, Dordrecht.

[2] ARNOLD, L. and THEODOSOPULU, M. (1980). Deterministic limit of the stochastic model of chemical reactions with diffusion. *Adv. in Appl. Probab.* **12** 367–379.

[3] BLOUNT, D. J. (1991). Comparison of stochastic and deterministic models of a linear chemical reaction with diffusion *Ann. Probab.* **19** 1440–1462.

[4] CURTAIN, R. F. and PRITCHARD, A. J. (1978). Infinite dimensional linear systems theory. In *Lecture Notes in Control and Inform. Sci.* **8**. Springer, Berlin.
[5] DAVIES, E. B. (1980). *One-Parameter Semigroups*. Academic, London.
[6] DITTRICH, P. (1986). A stochastic model of a chemical reaction with diffusion. Preprint, Akademie der Wissenschaften der DDR, Berlin.
[7] DITTRICH, P. (1988). A stochastic particle system: Fluctuations around a nonlinear reaction–diffusion equation. *Stochastic Process. Appl.* **30** 149–164.
[8] DYNKIN, E. B. (1965). *Markov Processes* **1**. Springer, Berlin.
[9] ETHIER, N. E. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence*. Wiley, New York.
[10] KOTELENEZ, P. (1986). Law of large numbers and central limit theorem for linear chemical reactions with diffusion. *Ann. Probab.* **14** 173–193.
[11] KOTELENEZ, P. (1986). Linear parabolic differential equations as limits of space–time jump Markov processes. *J. Math. Anal. Appl.* **116** 42–76.
[12] KOTELENEZ, P. (1986). Report 146, Forschungschwerpunkt Dynamische Systeme, Universität Bremen.
[13] KOTELENEZ, P. (1987). Fluctuations near homogeneous states of chemical reactions with diffusion. *Adv. in Appl. Probab.* **19** 352–370.
[14] KOTELENEZ, P. (1988). High density limit theorems for nonlinear chemical reactions with diffusion. *Probab. Theory Related Fields* **78** 11–37.
[15] KURTZ, T. (1981). *Approximation of Population Processes*. SIAM, Philadelphia.
[16] LANG, R. and XANH, N. X. (1980). Smolychowski's theory of coagulation in colloids holds rigorously in the Boltzmann–Grad–Limit. *Z. Wahrsch. Verw. Gebiete* **54** 227–280.
[17] LIPCER, R. S. and SHIRYAYEV, A. N. (1980). A functional central limit theorem for semimartingales. *Teor. Verojatnost. i Primenen.* **25** 683–703. (English translation in *Theory Probab. Appl.*)
[18] NEVEU, J. (1977). Processus ponctuels. *Ecole d'Eté de Probabilités de Saint-Flour VI, 1976. Lecture Notes in Math.* **598** 250–447. Springer, Berlin.

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