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THE RESOLVENT OF A DEGENERATE DIFFUSION ON THE PLANE, WITH APPLICATION TO PARTIALLY OBSERVED STOCHASTIC CONTROL

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We compute the resolvent of the degenerate, two-dimensional diffusion process introduced by Beneš, Karatzas and Rishel in the study of a stochastic control problem with partial observations. The explicit nature of our computations allows us to show that this diffusion can be constructed uniquely (in the sense of the probability law) starting at any point on the plane, including the origin, and to solve explicitly the control problem of Beneš, Karatzas and Rishel for very general cost functions. Our derivation combines probabilistic techniques, with use of the so-called "principle of smooth fit."

1. Introduction and summary. In this paper we consider the problem of optimal control for the scalar process

$$X_t^u = y + \int_0^t z u_s \, ds + B_t, \qquad 0 \le t < \infty. \tag{1.1}$$

Here z is a random drift parameter (observable only through X^u) with symmetric distribution, B is a Brownian motion independent of z and u is a control process with values in $[-1, 1]$. The paper [2] initiated the study of control problems for this model.

In the *completely observable case*, in which z is a known real constant $\bar{z}$, this problem goes back to Beneš [1], who showed that the control law

$$\bar{u}_t = -\operatorname{sgn}(\bar{z} X_t^{\bar{u}}) \tag{1.2}$$

minimizes any cost functional of the form $E[k(|X_T^u|)]$, where $k: [0, \infty) \to [0, \infty)$ is nondecreasing. This result was later established by different methods in [8], [4], [3] and [9], Section 6.5. In the *partially observed setting* of (1.1), it is natural to guess from (1.2) that, according to the so-called "separation" or "certainty-equivalence" principle, the law

$$u_t^* = -\operatorname{sgn}(\hat{z}_t^{u^*} X_t^{u^*}) \tag{1.3}$$

will be optimal for a wide variety of cost functionals, provided that we take for $\hat{z}_t^{u^*}$ the *least-mean-square estimate of z based on the observations $X_s^{u^*}$, $s \le t$.*

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An attempt to study this conjecture immediately faces a difficulty, which forces us to refine the formulation of the problem. If we formally substitute the control (1.3) into (1.1) and then derive the stochastic differential equation of filtering for $\hat{z}_t^{u^*}$, we find that the resulting equation *does not admit a strong* (i.e., $\{\mathscr{F}_t^{X^{u^*}}\} \triangleq \{\sigma(X_s^{u^*}, s \le t)\}$-adapted) *solution*. The paper [2] overcomes this difficulty by introducing a weak formulation. The processes X^u, for different u, are replaced by a fixed, canonical process Y. A fixed filtration $\{\mathscr{F}_t\}$, with respect to which Y is adapted, is then given, and the new class $\mathscr{U}$ of admissible controls consists of those which are adapted to $\{\mathscr{F}_t\}$ and take values in $[-1, 1]$. One then "solves" (1.1) by constructing a measure P^u (with corresponding expectation operator E^u), under which the process $Y_t - y - z\int_0^t u_s\, ds$ is a Brownian motion. The objective is to minimize a cost functional calculated with respect to E^u, and the conjectured optimal control suggested by (1.3) becomes

$$u_t^* = -\operatorname{sgn}(\hat{z}_t Y_t), \tag{1.4}$$

where $\hat{z}_t \triangleq E^{u^*}[z|\mathscr{F}_t]$. In [2] it was shown that if $\{\mathscr{F}_t\}$ is large enough (in particular, strictly larger than $\{\mathscr{F}_t^Y\} \triangleq \{\sigma(Y_s), s \le t\}$) and $\mathscr{U}_s$ denotes the subclass of $\mathscr{U}$ that contains the "strict-sense" (i.e., $\{\mathscr{F}_t^Y\}$-adapted) controls, then the following hold:

(i) The choice (1.4) is consistent—in the sense that the filtering equation for $\hat{z}_t$ admits an $\{\mathscr{F}_t\}$-adapted solution (cf. Section 6 for a detailed and rigorous formulation).

(ii) The resulting control $u^* \in \mathscr{U}$ minimizes the particular, infinite-horizon cost functional

$$J_1(u) \triangleq E^u \int_0^\infty e^{-\alpha t} Y_t^2\, dt \tag{1.5}$$

over the class $\mathscr{U}$.

(iii) u^* cannot be in $\mathscr{U}_s$.

(iv) No control in $\mathscr{U}_s$ can be optimal—despite the fact that the infima of (1.5) over these two classes $\mathscr{U}$ and $\mathscr{U}_s$ are the same. Such a situation had been envisaged in the pioneering work of Fleming and Pardoux [6].

The purpose of our paper is to analyze u^* and the corresponding optimal process in greater depth, with a view toward improved optimality results. In particular, we show that u^* minimizes any cost functional of the form

$$J_c(u) \triangleq E^u \int_0^\infty e^{-\alpha t} c(Y_t)\, dt, \tag{1.6}$$

assuming only that c is an even, convex function admitting at most exponential growth. The main result behind our proof is an explicit and tractable formula of $J_c(u^*)$, for any running cost function $c(\cdot)$, which follows from a more general calculation presented in Theorem 3.4. This theorem is the chief technical contribution of the paper, and deals with the two-dimensional diffusion process (Y, Z), solution of the stochastic differential equation

$$\begin{aligned} dY_t &= dW_t, & Y_0 &= y, \\ dZ_t &= -\operatorname{sgn}(Y_t Z_t)\, dW_t, & Z_0 &= \xi, \end{aligned} \tag{1.7}$$

on the plane. Here $\operatorname{sgn}(x) := 1_{(0,\infty)}(x) - 1_{(-\infty,0]}(x)$ is the usual signum function, and W is a standard, one-dimensional Brownian motion with respect to a filtration $\{\mathscr{F}_t\}$ large enough to support a solution of (1.7). Equation (1.7) is directly relevant to the control law $u_t^* = -\operatorname{sgn}(\hat{z}_t Y_t)$; namely, in the Bernoulli case $P[z = \theta] = 1 - P[z = -\theta] = \rho$ with $0 < \theta < \infty$, $0 < \rho < 1$, we have $\hat{z}_t = \theta \tanh(\theta \xi_t^*)$ and

$$(1.8) \qquad d\xi_t^* = -\operatorname{sgn}(\xi_t^* Y_t)\, dY_t, \qquad \xi_0 = \frac{1}{\theta} \tanh^{-1}(2\rho - 1);$$

see Section 6. Since we may use an equivalent probability measure under which the process Y becomes Brownian motion (thanks to the Girsanov theorem), the analyses of (1.7) and (1.8) are equivalent.

Our main result is *an explicit representation for the resolvent operator associated to solutions* $(Y^{y,\xi}, Z^{y,\xi})$ *of* (1.7), *for initial conditions* $(y,\xi) \in \mathscr{R}^2 \setminus \{\mathbf{0}\}$. That is, we compute explicitly

$$(1.9) \qquad V^g(y,\xi) = E\int_0^\infty e^{-\lambda t} g\big(Y_t^{y,\xi}, Z_t^{y,\xi}\big)\, dt$$

for any bounded, Borel-measurable function $g\colon \mathscr{R}^2 \to \mathscr{R}$ (cf. Theorem 3.4). By examining the resulting formula, we are then able to deduce further useful information about the solutions of (1.7). It turns out, in particular, that $V^g(y,\xi)$ extends continuously to the origin; this allows one to extend the Markov semigroup associated with $((Y_t^{y,\xi}, Z_t^{y,\xi}),\ 0 \le t < \infty,\ (y,\xi) \in \mathscr{R}^2 \setminus \{\mathbf{0}\})$ and restricted to functions defined on the punctured plane, to a Markovian semigroup of the Feller type acting on continuous functions defined on $\mathscr{R}^2$. This methodology allows us to conclude that *there is a weak solution of* (1.7) *starting at* $(y,\xi) = \mathbf{0}$, *and this solution is unique in the sense of the probability law* (cf. Proposition 3.7 and Theorem 8.1). Furthermore, we are able to obtain more refined results on the differentiability of V^g, and on how this property depends on the differentiability of g itself. We then return to the optimal control problem of (1.6) and obtain improved results (Theorem 7.1 and Remark 7.2); we show that the solution found in [2] is still optimal, if we require only that the cost function $c(\cdot)$ of (1.6) be even, convex and satisfy an exponential growth condition, and we also allow the initial condition y in (1.7) to be equal to 0.

The paper is organized as follows. In Section 2 we present the construction of the process $(Y_t^{y,\xi}, Z_t^{y,\xi})$ for $(y,\xi) \neq \mathbf{0}$. We repeat this construction here, because in its course we shall develop an important result (Proposition 2.4), crucial for further developments. In Section 3 we first state a result of the Feynman–Kac type (Proposition 3.4) that lays out conditions under which a solution $Q(y,\xi)$ of the equation

$$(1.10) \qquad \tfrac{1}{2}\big[Q_{yy} + Q_{\xi\xi}\big] - \operatorname{sgn}(\xi y) Q_{y\xi} + g(y,\xi) = \lambda Q$$

must, in fact, coincide with $V^g(y,\xi)$; then the explicit representation of V^g is stated in Theorem 3.4. Section 4 is devoted to the proof of Theorem 3.4. The idea is first to derive by probabilistic arguments a simple formula that expresses $V^g(y,\xi)$ in terms of g and of the values of $V^g(y,\xi)$ along the axes; and then to invoke a heuristic *smooth-fit principle* which postulates, as an Ansatz, that V^g be continuously differentiable across the axes. This is in the end justified because it leads to a candidate for V^g, by determining V^g along the axes. The candidate satisfies (1.10), and hence must actually represent V^g. Section 5 is a preliminary to the treatment of the stochastic control problem and shows under general conditions that, when $g(y,\xi) = c(y)\cosh(\theta\xi)$ and $\alpha = \lambda - \theta^2/2 > 0$, the function V^g of (1.9) satisfies the nonlinear partial differential equation

$$\tag{1.11} \tfrac{1}{2}V_{yy} + \min_{|u|\le 1}\left[uV_{y\xi} + \tfrac{1}{2}u^2\left(V_{\xi\xi} - \theta^2 V\right)\right] + c(y)\cosh(\theta\xi) = \alpha V$$

of the Hamilton–Jacobi–Bellman (HJB) type. Section 6 defines the optimal control problem of (1.4) and (1.6) precisely, and singles out a candidate optimal control. In Section 7 we show that this control law is indeed optimal, using a verification argument for (1.11). Section 8 develops the semigroup properties of $(Y^{y,\xi}, Z^{y,\xi})$, and proves the existence of a weak solution to (1.7) for $(y,\xi) = (0,0)$.

Many of the arguments require heavy calculation. To emphasize the essential points in Sections 4 and 5, we have relegated some of the proofs to an Appendix (Section 9). Readers interested primarily in the control problem of Sections 6 and 7 may wish, on first reading, to skip the proofs of Propositions 2.4 and 3.2 and Corollary 3.6, and to glance quickly through the "analysis" Section 4.

2. A degenerate, two-dimensional diffusion process. The following two-dimensional diffusion process (Y, Z) was introduced and studied in [2].

PROBLEM 2.1. *To find a complete probability space* $(\Omega, \mathscr{F}, P)$, *a filtration* $\{\mathscr{F}_t\}$ *of sub-σ-fields of* $\mathscr{F}$ *which satisfy the usual conditions, as well as two continuous and* $\{\mathscr{F}_t\}$*-adapted processes* Y, Z *on this space, such that*:

(i) $Y = \{Y_t, \mathscr{F}_t, 0 \le t < \infty\}$ *is a standard, one-dimensional Brownian motion process with* $Y_0 = y \in \mathscr{R}$ (*i.e., a continuous, square-integrable* $\{\mathscr{F}_t\}$*-martingale with* $Y_0 = y$ *and* $E[(Y_t - Y_s)^2 | \mathscr{F}_s] = t - s$ *a.s., for* $0 \le s \le t < \infty$);

(ii) *the equation*

$$\tag{2.1} Z_t = \xi - \int_0^t \operatorname{sgn}(Y_s Z_s)\, dY_s, \qquad 0 \le t < \infty,$$

is satisfied almost surely, for an arbitrary but fixed initial condition $(Y_0, Z_0) = (y, \xi) \in \mathscr{R}^2$.

In other words, one seeks a *weak solution* for the degenerate, two-dimensional stochastic differential equation:

$$\tag{2.2} \begin{aligned} dY &= dW, & Y_0 &= y,\\ dZ &= -\operatorname{sgn}(YZ)\, dW, & Z_0 &= \xi, \end{aligned}$$

where W is a standard, real-valued Brownian motion, for any given (y, ξ) in

$\mathscr{R}^2$. It was shown in [2] that Problem 2.1 admits a solution for any given $(y, \xi) \in \mathscr{R}^2 \setminus \{\mathbf{0}\}$, and this solution is unique in the sense of the probability law; for completeness and later usage, we repeat this construction here (Theorem 2.3 below). *The existence and uniqueness-in-law of a solution for* $(y, \xi) = \mathbf{0}$ *are established in Section* 8, *proof of Proposition* 3.7.

REMARK 2.2. We employ throughout the convention

$$\operatorname{sgn}(x) = \begin{cases} -1, & x \leq 0, \\ 1, & x > 0. \end{cases} \tag{2.3}$$

It should be observed that, for any solution of Problem 2.1, the process Z is an $\{\mathscr{F}_t\}$-Brownian motion starting at $Z_0 \equiv \xi$ [because it is an $\{\mathscr{F}_t\}$-local martingale with continuous paths and quadratic variation equal to $\langle Z \rangle_t = t$, as is easily checked from (2.1)].

THEOREM 2.3 [2]. *For any given* $(y, \xi) \neq 0$, *there is a solution to Problem* 2.1; *this solution is unique in the sense of the probability law.*

PROOF. Without loss of generality, we shall take $y \neq 0$, $\xi = 0$.

(a) *Existence*: Start with a complete probability space $(\Omega, \mathscr{F}, P)$ and a standard Brownian motion B on it ($B_0 = 0$), and let $\{\mathscr{F}_t\}$ be the P-augmentation of the filtration $\mathscr{F}_t^B = \sigma(B(s), 0 \leq s \leq t)$, $t \in [0, \infty)$. We define the continuous, $\{\mathscr{F}_t\}$-adapted processes

$$\begin{aligned} Z_1(t) &\triangleq B(t), \\ Y_1(t) &\triangleq y - \operatorname{sgn}(y) \int_0^t \operatorname{sgn}(Z_1(s))\, dB(s) \\ &= y - \int_0^t \operatorname{sgn}(Y_1(s) Z_1(s))\, dB(s) \end{aligned} \tag{2.4}$$

for $0 < t \leq \tau_1$, where $\tau_1 \triangleq \inf\{t \geq 0, Y_1(t) = 0\}$ is a stopping time of $\{\mathscr{F}_t\}$, with values in $(0, \infty)$. From the Tanaka formula for Brownian local time (cf. [9], page 205) and (2.4), we obtain

$$|Y_1(t)| + |Z_1(t)| = |y| + L^B(t), \qquad 0 \leq t \leq \tau_1, \tag{2.5}$$

where $L^B(\cdot)$ is the local time at the origin for the Brownian motion process B. In particular, we have from (2.5),

$$|Z_1(\tau_1)| = |y| + L^B(\tau_1) \geq |y| > 0, \quad \text{a.s.} \tag{2.5$'$}$$

This allows us to continue, by defining

$$\begin{aligned} Y_2(t) &\triangleq B(t) - B(\tau_1), \\ Z_2(t) &\triangleq Z_1(\tau_1) - \operatorname{sgn}(Z_1(\tau_1)) \int_{\tau_1}^t \operatorname{sgn}(Y_2(s))\, dB(s) \\ &= Z_2(\tau_1) - \int_{\tau_1}^t \operatorname{sgn}(Y_2(s) Z_2(s))\, dY_2(s) \end{aligned} \tag{2.6}$$

for $\tau_1 \le t < \tau_2$, where $\tau_2 \triangleq \inf\{t \ge \tau_1,\ Z_2(t) = 0\}$ is a stopping time of $\{\mathscr{F}_t\}$ with $P[\tau_1 < \tau_2 < \infty] = 1$. Using the Tanaka formula again, as well as (2.6) and (2.5′), we obtain

$$\begin{aligned} |Y_2(t)| + |Z_2(t)| &= |Z_1(\tau_1)| + L^{Y_2}(t) - L^{Y_2}(\tau_1) \\ &= |y| + L^{Y_2}(t), \qquad \tau_1 \le t \le \tau_2, \end{aligned} \tag{2.7}$$

and in particular

$$|Y_2(\tau_2)| = |Z_1(\tau_1)| + L^{Y_2}(\tau_2) - L^{Y_2}(\tau_1) \ge |Z_1(\tau_1)|, \tag{2.7′}$$

almost surely. Notice also from (2.5′) and (2.7′) that both τ_1 and $\tau_2 - \tau_1$ are stochastically larger than the first passage time of $|B|$ to the level $|y|$.

Continuing this construction, we create a strictly increasing sequence $\{\tau_m\}_{m=1}^{\infty}$ of stopping times, with each $\tau_m - \tau_{m-1}$ stochastically larger than the first passage time of $|B|$ to the level $|y|$ (and thus with $\lim_{m\to\infty} \tau_m = \infty$ a.s.), such that for every $n \ge 0$ we have almost surely:

(i) On $[\tau_{2n}, \tau_{2n+1})$: $Y_{2n}(\tau_{2n}) \ne 0$, $Z_{2n}(\tau_{2n}) = 0$,

$$Z_{2n+1}(t) = B(t) - B(\tau_{2n+1}),$$

$$\begin{aligned} Y_{2n+1}(t) &= Y_{2n}(\tau_{2n}) - \operatorname{sgn}(Y_{2n}(\tau_{2n}))\int_{\tau_{2n}}^{t} \operatorname{sgn}(Z_{2n+1}(s))\, dB(s) \\ &= Y_{2n+1}(\tau_{2n}) - \int_{\tau_{2n}}^{t} \operatorname{sgn}(Y_{2n+1}(s) Z_{2n+1}(s))\, dZ_{2n+1}(s), \end{aligned} \tag{2.8}$$

$$\tau_{2n+1} = \inf\{t \ge \tau_{2n},\ Y_{2n+1}(t) = 0\}, \tag{2.9}$$

$$|Y_{2n+1}(t)| + |Z_{2n+1}(t)| = |Y_{2n}(\tau_{2n})| + L^{Z_{2n+1}}(t) - L^{Z_{2n+1}}(\tau_{2n}), \qquad \tau_{2n} \le t \le \tau_{2n+1}. \tag{2.10}$$

(ii) On $[\tau_{2n+1}, \tau_{2n+2})$: $Z_{2n+1}(\tau_{2n+1}) \ne 0$, $Y_{2n+1}(\tau_{2n+1}) = 0$,

$$Y_{2n+2}(t) = B(t) - B(\tau_{2n+1}),$$

$$\begin{aligned} Z_{2n+2}(t) &= Z_{2n+1}(\tau_{2n+1}) \\ &\quad - \operatorname{sgn}(Z_{2n+1}(\tau_{2n+1}))\int_{\tau_{2n+1}}^{t} \operatorname{sgn}(Y_{2n+2}(s))\, dB(s) \\ &= Z_{2n+2}(\tau_{2n+1}) \\ &\quad - \int_{\tau_{2n+1}}^{t} \operatorname{sgn}(Z_{2n+2}(s) Y_{2n+2}(s))\, dY_{2n+2}(s), \end{aligned} \tag{2.11}$$

$$\tau_{2n+2} = \inf\{t \ge \tau_{2n+1},\ Z_{2n+2}(t) = 0\}, \tag{2.12}$$

$$\begin{aligned} &|Y_{2n+2}(t)| + |Z_{2n+2}(t)| \\ &\quad = |Z_{2n+1}(\tau_{2n+1})| + L^{Y_{2n+2}}(t) - L^{Y_{2n+2}}(\tau_{2n+1}), \qquad \tau_{2n+1} \le t \le \tau_{2n+2}. \end{aligned} \tag{2.13}$$

It is now straightforward to see that the $\{\mathscr{F}_t\}$-adapted processes Y and Z, defined consistently on the entirety of $[0, \infty)$ by

$$(Y(t), Z(t)) \triangleq (Y_m(t), Z_m(t)) \quad \text{for } \tau_{m-1} \le t < \tau_m, \tag{2.14}$$

are Brownian motions which satisfy (2.1), as well as

$$|Y(t)| + |Z(t)| = |y| + |\xi| + L^Y(t) + L^Z(t), \qquad 0 \le t < \infty. \tag{2.15}$$

(b) *Uniqueness*. [The following argument is due to L. C. G. Rogers (personal communication).] Let $(\Omega, \mathscr{F}, P), \{\mathscr{F}_t\}, (Y, Z)$ be a solution of Problem 2.1. Then from (2.1) we obtain

$$\begin{aligned} Z_t &= \int_0^t \operatorname{sgn}(Y_s)\, dW_s, \\ Y_t &= y - \int_0^t \operatorname{sgn}(Z_s)\, dW_s \end{aligned} \tag{2.16}$$

in terms of the $\{\mathscr{F}_t\}$-Brownian motion process

$$W_t \triangleq -\int_0^t \operatorname{sgn}(Z_s)\, dY_s.$$

Uniqueness-in-law for the pair (Y, Z) will follow, as soon as we have shown that *pathwise uniqueness holds for the stochastic equation* (2.16).

In order to do this, consider another pair of process (Y', Z') satisfying (2.16) on the same probability space $(\Omega, \mathscr{F}, P), \{\mathscr{F}_t\}$ and with respect to the same Brownian motion W, that is,

$$\begin{aligned} Z'_t &= \int_0^t \operatorname{sgn}(Y'_s)\, dW_s, \\ Y'_t &= y - \int_0^t \operatorname{sgn}(Z'_s)\, dW_s, \end{aligned} \tag{2.16$'$}$$

and let $\rho_1 \triangleq \inf\{t \ge 0,\ Y_t = 0 \text{ or } Y'_t = 0\}$. This stopping time is almost surely positive and finite, and the processes Y, Y' are of the same sign on $[0, \rho_1]$; therefore, $Z \equiv Z'$ on $[0, \rho_1]$, and this in turn implies $Y \equiv Y'$ on $[0, \rho_1]$. On the other hand, from Tanaka's formula we obtain as before

$$|Z_{\rho_1}| = |Y_{\rho_1}| + |Z_{\rho_1}| = |y| + L^Z(\rho_1) \ge |y|, \quad \text{a.s.}$$

Similarly, the stopping time $\rho_2 \triangleq \inf\{t \ge \rho_1,\ Z_t = 0 \text{ or } Z'_t = 0\}$ satisfies $P[\rho_1 < \rho_2 < \infty] = 1$, and it can be seen as before that

(i) $$(Y, Z) = (Y', Z') \quad \text{on } [\rho_1, \rho_2],$$

(ii) $$|Y_{\rho_2}| = |Z_{\rho_1}| + \left[L^Y(\rho_2) - L^Y(\rho_1)\right]$$

hold almost surely; in particular, both ρ_1 and $\rho_2 - \rho_1$ are stochastically larger than the first passage time of a Brownian motion to $\pm |y|$. Continuing this way, we construct a strictly increasing sequence $\{\rho_m\}_{m=1}^\infty$ of $\{\mathscr{F}_t\}$-stopping times with $\lim_{m\to\infty} \rho_m = \infty$ (because $\rho_m - \rho_{m-1}$ is stochastically larger than the

Brownian first passage time to $\pm|y|$, for every $m \geq 2$), and such that $(Y, Z) \equiv (Y', Z')$ on $[0, \rho_m]$, $\forall\ m \geq 1$. We conclude that the pairs (Y, Z) and (Y', Z') are indistinguishable. □

It is also shown in [2] that for the stopping time $\tau \equiv \tau_1$ as in (2.9), we have

$$\text{(2.17)} \qquad \operatorname{sgn}(Z_\tau) \text{ is independent of } \mathscr{F}_\tau^{Y},$$

and that

$$\text{(2.18)} \qquad \begin{array}{l}\text{the processes } Z, \operatorname{sgn}(Z) \text{ are not adapted to the}\\ \text{filtration } \{\mathscr{F}_t^{Y}\},\ \mathscr{F}_t^{Y} = \sigma(Y_s,\ 0 \leq s \leq t),\end{array}$$

for any solution $(\Omega, \mathscr{F}, P), \{\mathscr{F}_t\}, (Y, Z)$ of Problem 2.1. In other words, (2.1) *does not admit a strong solution.*

PROPOSITION 2.4. *For the process* $(Y^{y,\xi}, Z^{y,\xi})$ *of Problem* 2.1 *with* $(y, \xi) \neq \mathbf{0}$, *introduce the sequence of stopping times*

$$\text{(2.19)} \qquad \begin{aligned}\sigma_k \equiv \sigma_k(y, \xi) &\triangleq \inf\left\{t \geq 0,\ |Y_t^{y,\xi}| + |Z_t^{y,\xi}| \geq |y| + |\xi| + k\right\}\\ &= \inf\left\{t \geq 0,\ L^{Y^{y,\xi}}(t) + L^{Z^{y,\xi}}(t) \geq k\right\}\end{aligned}$$

for $k \in \mathbb{N}$ [*recall* (2.15)]. *Then for every* $\lambda \in (0, \infty)$ *and* $0 < \varepsilon < \sqrt{2\lambda}$, *there exists a positive constant* $M_\varepsilon < \infty$ *such that*

$$\text{(2.20)} \qquad Ee^{-\lambda\sigma_k} \leq M_\varepsilon e^{-(\sqrt{2\lambda} - \varepsilon)k} \quad \textit{for all } k \in \mathbb{N}.$$

PROOF. Let us recall the construction of Theorem 2.3 (Existence), and the fact that for a standard Brownian motion B, the stopping time $\tau_k \triangleq \inf\{t \geq 0, L^B(t) \geq k\}$ has moment-generating function

$$\text{(2.21)} \qquad Ee^{-\lambda\tau_k} = e^{-k\sqrt{2\lambda}}, \qquad k > 0$$

[cf. [9], Theorem 3.6.17 and formula (2.8.6)].

Set $f_k \triangleq Ee^{-\lambda\sigma_k}$, $k \in \mathbb{N}$, and observe from the strong Markov property:

$$\text{(2.22)} \qquad \begin{aligned} f_{k+1} &= E\left[e^{-\lambda\sigma_k} E\left\{e^{-\lambda(\sigma_{k+1} - \sigma_k)} | \mathscr{F}_{\sigma_k}\right\}\right] = E\left[e^{-\lambda\sigma_k} \tilde{f}_1\right],\\ \tilde{f}_1 &= E\left[e^{-\lambda\sigma_1(\tilde{y}, \tilde{\xi})}\right]\Big|_{(\tilde{y}, \tilde{\xi}) = (Y_{\sigma_k}^{y,\xi}, Z_{\sigma_k}^{y,\xi})}.\end{aligned}$$

Note that $(Y_{\sigma_k}^{y,\xi}, Z_{\sigma_k}^{y,\xi})$ takes values in the set $\mathscr{S}_l = \{(0, l), (0, -l), (l, 0), (-l, 0)\}$ with $l = k + |y| + |\xi|$. Let us consider the event

$$\text{(2.23)} \qquad A \triangleq \left\{\left(Y^{\tilde{y},\tilde{\xi}}, Z^{\tilde{y},\tilde{\xi}}\right) \text{ undergoes an excursion from one axis to another, during } \left(0, \sigma_1(\tilde{y}, \tilde{\xi})\right)\right\}.$$

From (2.21) we have

$$\text{(2.24)} \qquad E\left[e^{-\lambda\sigma_1(\tilde{y}, \tilde{\xi})} 1_{A^c}\right] \leq e^{-\sqrt{2\lambda}},$$

because, on the event A^c, $\sigma_1(\tilde{y}, \tilde{\xi})$ is just the time it takes for the local time of a standard Brownian motion (either $Y^{\tilde{y},\tilde{\xi}}$ or $Z^{\tilde{y},\tilde{\xi}}$) to increase by one unit.

On the other hand, with $(\tilde{y}, \tilde{\xi}) \in \mathscr{S}_l$, for the event A to occur, a standard Brownian motion (either $Y^{\tilde{y},\tilde{\xi}}$ or $Z^{\tilde{y},\tilde{\xi}}$) must undergo an excursion of size l *before* its local time grows by one unit:

$$P(A) = P[|B| \text{ hits } l, \text{before } L^B \text{ hits } 1].$$

Now the pairs $(|B|, L^B)$ and $(M - B, M)$, with $M(t) = \max_{0 \le s \le t} B(s)$, are equivalent in law (e.g., [9], Theorem 3.6.17); furthermore, for $M - B$ to hit l before M hits 1, B must hit $1 - l$ before it hits 1. But this event has probability $1/l$; thus,

$$P(A) \le \frac{1}{l}. \tag{2.25}$$

From (2.24) and (2.25),

$$E\left[e^{-\lambda\sigma_1(\tilde{y},\tilde{\xi})}\right] \le e^{-\sqrt{2\lambda}} + \frac{1}{l}, \qquad (\tilde{y}, \tilde{\xi}) \in \mathscr{S}_l,$$

and from this and (2.22) we deduce

$$f_{k+1} \le f_k\left[e^{-\sqrt{2\lambda}} + \frac{1}{k + |y| + |\xi|}\right] \le f_k\left(e^{-\sqrt{2\lambda}} + \frac{1}{k}\right),$$

as well as

$$Ee^{-\lambda\sigma_k} = f_k \le f_1 e^{-k\sqrt{2\lambda}} \prod_{j=1}^{k}\left(1 + \frac{e^{\sqrt{2\lambda}}}{j}\right) = f_1 g_k e^{-k\sqrt{2\lambda}}, \qquad k \in \mathbb{N}.$$

It is easy to see that

$$g_k \triangleq \prod_{j=1}^{k}\left(1 + \frac{e^{\sqrt{2\lambda}}}{j}\right)$$

satisfies $\lim_{k\to\infty} g_k e^{-\varepsilon k} = 0$ for any $\varepsilon > 0$, and the conclusion (2.20) follows. □

3. The resolvent of $(Y_t^{y,\xi}, Z_t^{y,\xi})_{t \ge 0}$. We would like now to compute explicitly the resolvent function

$$V^g(y, \xi) \triangleq E\int_0^\infty e^{-\lambda t} g(Y_t^{y,\xi}, Z_t^{y,\xi})\, dt, \qquad (y, \xi) \in \mathscr{R}^2 \setminus \{\mathbf{0}\}, \tag{3.1}$$

for the unique (in the sense of probability law) solution $(Y_t^{y,\xi}, Z_t^{y,\xi})_{t\ge 0}$ of Problem 2.1, corresponding to the initial condition $(Y_0^{y,\xi}, Z_0^{y,\xi}) = (y, \xi)$ and a given function g. We want our formula to be valid for a class of functions g sufficiently large, both to use the resolvent V^g in order to characterize $(Y_t^{y,\xi}, Z_t^{y,\xi})_{t\ge 0}$, and to include functions of the form

$$g(y, \xi) = c(y)\cosh(\theta\xi) \tag{3.2}$$

(where c is sufficiently smooth and satisfies suitable growth conditions) that arise in the optimal control problem treated in Sections 6 and 7. Bounded functions g suffice for characterizing $(Y_t^{y,\xi}, Z_t^{y,\xi})_{t \geq 0}$ via the resolvent. However, to handle (3.2) we shall have to consider g satisfying a growth condition of the type

$$|g(y,\xi)| \leq Ke^{\theta_1|y|+\theta_2|\xi|}, \qquad \forall (y,\xi) \in \mathscr{R}^2, \tag{3.3}$$

for some $K > 0$, $\theta_1 \geq 0$, $\theta_2 \geq 0$.

LEMMA 3.1. *Let g satisfy* (3.3). *Then, for every $\lambda > \frac{1}{2}(\theta_1 + \theta_2)^2$, the function $V^g(y,\xi)$ is well defined and finite, and there exists a real number $K_\lambda > 0$ such that*

$$|V^g(y,\xi)| \leq K_\lambda e^{\theta_1|y|+\theta_2|\xi|}, \qquad \forall (y,\xi) \in \mathscr{R}^2. \tag{3.4}$$

PROOF. Observe that $(Y_t^{y,\xi} - y)_{t \geq 0}$ and $(Z_t^{y,\xi} - \xi)_{t \geq 0}$ are Brownian motions. Moreover, for a Brownian motion B, we have

$$Ee^{\lambda|B_t|} \leq 2Ee^{\lambda B_t} = 2e^{\lambda^2 t/2}, \qquad \forall \lambda \in \mathscr{R}.$$

Thus, by Hölder's inequality and (3.3),

$$E\left|g\left(Y_t^{y,\xi}, Z_t^{y,\xi}\right)\right| \leq Ke^{\theta_1|y|+\theta_2|\xi|}\left(Ee^{p_1\theta_1|B_t|}\right)^{1/p_1}\left(Ee^{p_2\theta_2|B_t|}\right)^{1/p_2}$$
$$= 2Ke^{\theta_1|y|+\theta_2|\xi|}e^{(\theta_1^2 p_1+\theta_2^2 p_2)t/2}$$

for $p_1 > 1$, $p_2 > 1$ such that $1/p_1 + 1/p_2 = 1$. Since

$$\min\left\{\theta_1^2 p_1 + \theta_2^2 p_2;\, p_1 > 1,\, p_2 > 1,\, \frac{1}{p_1} + \frac{1}{p_2} = 1\right\} = (\theta_1 + \theta_2)^2,$$

we see that V^g is well defined and finite for $\lambda > \frac{1}{2}(\theta_1 + \theta_2)^2$, and (3.4) holds. □

Our approach to computing $V^g(y,\xi)$ will be to exhibit an explicit solution of the *resolvent equation* formally associated with (3.1):

$$\tfrac{1}{2}\left[Q_{yy} + Q_{\xi\xi}\right] - \operatorname{sgn}(y\xi)Q_{y\xi} + g = \lambda Q \quad \text{in } \mathscr{R}^2. \tag{3.5}$$

We make precise the connection between (3.1) and (3.5) in Proposition 3.2.

The discontinuity of the coefficient $\operatorname{sgn}(y\xi)$ across the axes makes it necessary to define carefully a suitable domain of functions for the operator

$$L = \frac{1}{2}\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial \xi^2}\right] - \operatorname{sgn}(y\xi)\frac{\partial^2}{\partial \xi\, \partial y}. \tag{3.6}$$

Let Γ_1, Γ_2, Γ_3 and Γ_4 denote, respectively, the first through fourth open quadrants of $\mathscr{R}^2$. We shall say that a function $V\colon \mathscr{R}^2 \to \mathscr{R}$ *is of class* $\mathscr{D}$, if

$$\begin{aligned} &V \in C^1(\mathscr{R}^2), \\ &V \in C^2(\overline{\Gamma}_i), \qquad \text{for each } i = 1,\dots,4. \end{aligned} \tag{3.7}$$

For each $V \in \mathscr{D}$, $LV(y,\xi)$ is unambiguously defined in every open quadrant. We define $LV(y,\xi)$ on the axes by continuation from the quadrant corresponding to the convention $\operatorname{sgn}(0) = -1$. That is, for $y = 0$, $\xi > 0$,

$$[LV](0,\xi) \triangleq \tfrac{1}{2}\left[V_{yy}(0-,\xi) + V_{\xi\xi}(0-,\xi)\right] + V_{y\xi}(0-,\xi);$$

for $y = 0$, $\xi < 0$,

$$[LV](0,\xi) \triangleq \tfrac{1}{2}\big[V_{yy}(0+,\xi) + V_{\xi\xi}(0+,\xi)\big] + V_{y\xi}(0+,\xi);$$

and so on. Finally, we define by convention $LV(0,0) \triangleq LV(0,0+)$. With those definitions, (3.5) has an unambiguous meaning on $\mathscr{R}^2$, for every $Q \in \mathscr{D}$.

PROPOSITION 3.2. *Let g: $\mathscr{R}^2 \to \mathscr{R}$ be a locally bounded, Borel-measurable function, and assume that $\lambda > 0$ is large enough, so that $V^{|g|}(y,\xi)$ exists for all $(y,\xi) \in \mathscr{R}^2 \setminus \{\mathbf{0}\}$. Assume that $Q \in \mathscr{D}$ obeys (3.5) and*

$$|Q(y,\xi)| \le Ke^{\theta(|y|+|\xi|)}, \qquad \forall\,(y,\xi) \in \mathscr{R}^2, \tag{3.8}$$

for some $0 < K < \infty$ and $0 \le \theta < \sqrt{2\lambda}$. Then

$$Q(y,\xi) = V^g(y,\xi) \triangleq E\int_0^\infty e^{-\lambda t} g\big(Y_t^{y,\xi}, Z_t^{y,\xi}\big)\,dt \tag{3.9}$$

for all $(y,\xi) \in \mathscr{R}^2 \setminus \{\mathbf{0}\}$.

PROOF. The well-known argument behind (3.9) is to apply Itô's rule, and we follow this procedure. However, we must deal with the fact that Q is not necessarily of class C^2 across the axes. We shall smooth out Q and take limits, using the fact that the time spent by the process $(Y_t^{y,\xi}, Z_t^{y,\xi})_{t\ge 0}$ on the axes has Lebesgue measure 0.

Accordingly, let p be a compactly supported, symmetric, nonnegative C^∞-function on $\mathscr{R}^2$ such that $\int_{\mathscr{R}^2} p(y,\xi)\,dy\,d\xi = 1$, and introduce the functions $\phi_n(y,\xi) \triangleq n^2 p(y/n, \xi/n)$ and $Q_n(y,\xi) \triangleq [\phi_n * Q](y,\xi)$. Here, $*$ denotes convolution. Because $Q \in \mathscr{D}$, it remains true that

$$\partial^\alpha Q_n(y,\xi) = \phi_n * \partial^\alpha Q(y,\xi),$$

for any multi-index $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ with $|\boldsymbol{\alpha}| \triangleq \alpha_1 + \alpha_2 \le 2$. Consequently,

$$\sup_n \sup_{(y,\xi)\in K} |\partial^\alpha Q_n(y,\xi)| < \infty, \tag{3.10}$$

for any bounded $K \subset \mathscr{R}^2$ and any multi-index $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ with $|\boldsymbol{\alpha}| \le 2$. Also,

$$\begin{aligned}
&\lim_{n\to\infty} \partial^\alpha Q_n(y,\xi) = \partial^\alpha Q(y,\xi), \qquad \forall\,(y,\xi) \in \mathscr{R}^2 \qquad \text{if } |\alpha| \le 1,\\
&\lim_{n\to\infty} \partial^\alpha Q_n(y,\xi) = \partial^\alpha Q(y,\xi), \qquad \forall\,(y,\xi) \in \bigcup_{j=1}^4 \Gamma_j \quad \text{if } |\alpha| = 2.
\end{aligned} \tag{3.11}$$

For every integer $k \ge 1$, recall the stopping time $\sigma_k = \inf\{t \ge 0,\ |Y_t^{y,\xi}| + |Z_t^{y,\xi}| \ge |y| + |\xi| + k\}$ of (2.19), as well as the upper bound (2.20) for its moment-generating function. Application of Itô's rule to $e^{-\lambda t} Q_n(Y_t^{y,\xi}, Z_t^{y,\xi})$ yields

$$\begin{aligned}
&e^{-\lambda t} Q_n\big(Y_t^{y,\xi}, Z_t^{y,\xi}\big) + \int_0^t e^{-\lambda s}\big\{\lambda Q_n\big(Y_s^{y,\xi}, Z_s^{y,\xi}\big) - [LQ_n]\big(Y_s^{y,\xi}, Z_s^{y,\xi}\big)\big\}\,ds\\
&\quad = Q_n(y,\xi) + \int_0^t e^{-\lambda s}\left\{\frac{\partial Q_n}{\partial y}\big(Y_s^{y,\xi}, Z_s^{y,\xi}\big)\,dY_s^{y,\xi} + \frac{\partial Q_n}{\partial \xi}\big(Y_s^{y,\xi}, Z_s^{y,\xi}\big)\,dZ_s^{y,\xi}\right\}.
\end{aligned}$$

Consequently, for every $t > 0$,

$$Q_n(y,\xi) = E\left[e^{-\lambda(t\wedge\sigma_k)}Q_n\left(Y^{y,\xi}_{t\wedge\sigma_k}, Z^{y,\xi}_{t\wedge\sigma_k}\right)\right] + E\int_0^{t\wedge\sigma_k} e^{-\lambda s}\left\{\lambda Q_n\left(Y^{y,\xi}_s, Z^{y,\xi}_s\right) - [LQ_n]\left(Y^{y,\xi}_s, Z^{y,\xi}_s\right)\right\} ds. \tag{3.12}$$

Now we can take limits in (3.12) as $n \to \infty$, using dominated convergence [valid by (3.10) and (3.11)] and the fact that the time spent by the process $(Y^{y,\xi}, Z^{y,\xi})$ on the axes has zero Lebesgue measure. After this, we take limits in $t \to \infty$ and appeal to the uniform boundedness of Q and g on bounded sets, in order to obtain

$$Q(y,\xi) = E\int_0^{\sigma_k} e^{-\lambda s} g\left(Y^{y,\xi}_s, Z^{y,\xi}_s\right) ds + E\left[e^{-\lambda\sigma_k}Q\left(Y^{y,\xi}_{\sigma_k}, Z^{y,\xi}_{\sigma_k}\right)\right]. \tag{3.13}$$

Letting $k \to \infty$ in (3.13), we obtain (3.9), since $E\int_0^\infty e^{-\lambda s}|g(Y^{y,\xi}_s, Z^{y,\xi}_s)|\,ds < \infty$ and, by (3.8), (2.19) and Proposition 2.4, we have for $0 < \varepsilon < \sqrt{2\lambda} - \theta$,

$$\begin{aligned}\left|Ee^{-\lambda\sigma_k}Q\left(Y_{\sigma_k}, Z_{\sigma_k}\right)\right| &\le Ee^{-\lambda\sigma_k + \theta(|Y^{y,\xi}_{\sigma_k}| + |Z^{y,\xi}_{\sigma_k}|)} \\ &\le M_\varepsilon e^{\theta(|y|+|\xi|+k)}e^{-k(\sqrt{2\lambda}-\varepsilon)} \xrightarrow[k\to\infty]{} 0.\end{aligned}$$

□

REMARK 3.3. The bound (3.8) is consistent with the bounds (3.4) and (3.3), from Lemma 3.1. Because Q belongs to $\mathscr{D}$ and obeys (3.5), we have $g \in C(\overline{\Gamma}_i)$, $1 \le i \le 4$, automatically.

We shall present now an explicit solution to (3.5), for suitable functions g. To state the result, it will be convenient to introduce the following notation:

$$F^g(s) \triangleq 2\int_0^s \sinh(u\sqrt{2\lambda})g(s-u,u)\,du, \tag{3.14}$$

$$G^g(s) \triangleq 2\int_0^s \sinh(u\sqrt{2\lambda})g(u,s-u)\,du, \tag{3.15}$$

$$M^g(s) \triangleq \int_s^\infty \frac{\left\{F^g(u)\left[\cosh(u\sqrt{2\lambda})\cosh(s\sqrt{2\lambda}) - 1\right] + G^g(u)\left[\cosh(u\sqrt{2\lambda}) - \cosh(s\sqrt{2\lambda})\right]\right\}}{\sinh^3(u\sqrt{2\lambda})}\,du, \tag{3.16}$$

$$N^g(s) \triangleq \int_s^\infty \frac{\left\{G^g(u)\left[\cosh(u\sqrt{2\lambda})\cosh(s\sqrt{2\lambda}) - 1\right] + F^g(u)\left[\cosh(u\sqrt{2\lambda}) - \cosh(s\sqrt{2\lambda})\right]\right\}}{\sinh^3(u\sqrt{2\lambda})}\,du, \tag{3.17}$$

$$J^g(y,\xi) \triangleq \sqrt{\frac{2}{\lambda}}\left[\frac{\sinh(y\sqrt{2\lambda})}{\sinh((y+\xi)\sqrt{2\lambda})}\int_0^\xi \sinh(u\sqrt{2\lambda})g(y+\xi-u,u)\,du + \frac{\sinh(\xi\sqrt{2\lambda})}{\sinh((y+\xi)\sqrt{2\lambda})}\int_0^y \sinh(u\sqrt{2\lambda})g(u,y+\xi-u)\,du\right]. \tag{3.18}$$

THEOREM 3.4. *Consider a function $g \in \mathscr{D}$ which satisfies the growth condition* (3.3).

(i) *For every $\lambda > \frac{1}{2}(\theta_1 + \theta_2)^2$, there exists a unique function $Q \in \mathscr{D}$ which satisfies* (3.5) *and the growth condition* (3.8) *with $\theta = \theta_1 \vee \theta_2$.*

(ii) *This function agrees with V^g of* (3.1) *on $\mathscr{R}^2 \setminus \{\mathbf{0}\}$, as follows directly from* (i) *and Proposition* 3.2.

(iii) *If g is even in both variables, that is, if $g \equiv g_0$ where*

$$g_0(y,\xi) \triangleq \tfrac{1}{4}[g(y,\xi) + g(-y,\xi) + g(y,-\xi) + g(-y,-\xi)],$$

then $Q \equiv Q_0$ is given as

$$\begin{aligned} Q_0(y,\xi) \triangleq J^{g_0}(y,\xi) &+ \frac{\sinh(\xi\sqrt{2\lambda})}{\sinh((y+\xi)\sqrt{2\lambda})} M^{g_0}(y+\xi) \\ &+ \frac{\sinh(y\sqrt{2\lambda})}{\sinh((y+\xi)\sqrt{2\lambda})} N^{g_0}(y+\xi). \end{aligned} \tag{3.19}$$

This function is of class $C^2(\mathscr{R}^2)$ and even in both variables.

(iv) *More generally, we have*

$$Q = Q_0 + Q_1 + Q_2 + Q_3, \tag{3.20}$$

where for $y \geq 0$, $\xi \geq 0$,

$$Q_1(y,\xi) \triangleq J^{g_1}(y,\xi) + \sinh(\xi\sqrt{2\lambda}) \int_{y+\xi}^{\infty} \frac{F^{g_1}(u)}{\sinh^2(u\sqrt{2\lambda})}\, du, \tag{3.21}$$

$$Q_2(y,\xi) \triangleq J^{g_2}(y,\xi) + \sinh(\xi\sqrt{2\lambda}) \int_{y+\xi}^{\infty} \frac{G^{g_2}(u)}{\sinh^2(u\sqrt{2\lambda})}\, du, \tag{3.22}$$

$$Q_3(y,\xi) \triangleq J^{g_3}(y,\xi), \tag{3.23}$$

and

$$\begin{aligned} g_1(y,\xi) &= \tfrac{1}{4}[g(y,\xi) + g(-y,\xi) - g(y,-\xi) - g(-y,-\xi)], \\ g_2(y,\xi) &= \tfrac{1}{4}[g(y,\xi) - g(-y,\xi) + g(y,-\xi) - g(-y,-\xi)], \\ g_3(y,\xi) &= \tfrac{1}{4}[g(y,\xi) - g(-y,\xi) - g(y,-\xi) + g(-y,-\xi)]; \end{aligned}$$

the functions $Q_0, \ldots, Q_3$ have the following symmetry properties:

$Q_0(y,\xi) = Q_0(y,-\xi) = Q_0(-y,\xi) = Q_0(-y,-\xi)$: *even in both y and ξ*,

$Q_1(y,\xi) = Q_1(-y,\xi)$, $Q_1(y,\xi) = -Q_1(y,-\xi)$: *even in y, odd in ξ*,

$Q_2(y,\xi) = -Q_2(-y,\xi)$, $Q_2(y,\xi) = Q_2(y,-\xi)$: *odd in y, even in ξ*,

$Q_3(y,\xi) = -Q_3(-y,\xi) = -Q_3(y,-\xi) = Q_3(-y,-\xi)$: *odd in both y and ξ*.

(*The same symmetry properties are satisfied, respectively, by g_0, g_1, g_2 and g_3.*)

REMARK 3.5. (i) The assumption (3.3) implies that for $s \geq 0$ and $\lambda > \frac{1}{2}(\theta_1 + \theta_2)^2$, we have

$$|F^g(s)| \leq K\int_0^s e^{u\sqrt{2\lambda}}e^{\theta_2 u}e^{\theta_1(s-u)}\,du \tag{3.24}$$
$$= \frac{K}{\theta_2 - \theta_1 + \sqrt{2\lambda}}\left[e^{(\sqrt{2\lambda}+\theta_2)s} - e^{\theta_1 s}\right].$$

Similarly,

$$|G^g(s)| \leq \frac{K}{\theta_1 - \theta_2 + \sqrt{2\lambda}}\left[e^{(\sqrt{2\lambda}+\theta_1)s} - e^{\theta_2 s}\right]. \tag{3.25}$$

It follows that there is a real constant $K > 0$, such that

$$\frac{|F^g(s)| + |G^g(s)|}{\sinh^2(s\sqrt{2\lambda})} \leq Ke^{-s\sqrt{2\lambda}}\left(e^{\theta_1 s} + e^{\theta_2 s}\right), \qquad \forall\, s \geq 1, \tag{3.26}$$

and hence $M^g(\cdot)$ and $N^g(\cdot)$ are well defined on $(0,\infty)$ since $\lambda > \frac{1}{2}(\theta_1 + \theta_2)^2$, by assumption. In Section 4 we shall see that these functions are, in fact, of class $C^2[0,\infty)$.

(ii) By Theorem 3.4 and Lemma 3.1, Q inherits the growth condition (3.3) from g. In fact, it is a straightforward consequence of (3.24) and (3.25) that the growth rate of (3.3) is inherited by the different components of Q in (3.19) and (3.21)–(3.23); if g satisfies (3.3), then, for suitable $\tilde{K} \in (0,\infty)$: $|J^g(y,\xi)| \leq \tilde{K}e^{\theta_1|y|+\theta_2|\xi|}$, $\forall\,(y,\xi) \in \mathscr{R}^2$, $[\sinh(\xi\sqrt{2\lambda})/\sinh((y+\xi)\sqrt{2\lambda})]M^g(y+\xi) \leq \tilde{K}e^{\theta_2|\xi|}$, and so on.

According to Proposition 3.2, in order to prove Theorem 3.4 it suffices to verify that the function Q defined by (3.19)–(3.23) is of class $\mathscr{D}$, and satisfies (3.5) as well as the growth condition (3.8). This could be checked analytically, directly from the expressions (3.19)–(3.23). We shall prefer, however, to give the derivation described in Section 4, which *computes V^{g_i} subject to the Ansatz that V^{g_i} be a C^1 function, for $i = 0,\dots,3$, and then shows that the resulting function is, in fact, a solution of* (3.5) *in the class $\mathscr{D}$.* In connection with the decomposition $g = \sum_{i=0}^3 g_i$, the Ansatz then determines $V^g = \sum_{i=0}^3 V^{g_i}$.

This derivation will clarify the probabilistic significance of the expression for Q and will also provide, in the context of the problem treated in Sections 6 and 7, a further example of the use of the *smooth-fit principle* in stochastic optimal control.

Theorem 3.4 allows us to compute explicitly the resolvent operator on the space of bounded measurable functions, as the following result indicates.

COROLLARY 3.6. *Let g: $\mathscr{R}^2 \to \mathscr{R}$ be any bounded, Borel-measurable function. Then the function $Q \equiv Q^g$ is well defined by* (3.19)–(3.23), *and*

$$V^g(y,\xi) \triangleq E\int_0^\infty e^{-\lambda t}g\left(Y_t^{y,\xi}, Z_t^{y,\xi}\right)dt = Q^g(y,\xi), \qquad \forall\,(y,\xi) \neq \mathbf{0}, \tag{3.27}$$

for any given $\lambda > 0$. Moreover, Q^g is continuous at the origin, and hence $V^g(y, \xi)$ extends continuously to the origin, for any bounded, Borel-measurable g.

PROOF. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of bounded, Borel-measurable, real-valued functions on $\mathscr{R}^2$. We say that $f\colon \mathscr{R}^2 \to \mathscr{R}$ is the *bounded, pointwise limit* of this sequence, and write $\text{bp-lim}_{n\to\infty} f_n = f$, if $\sup_{n\geq 1}\|f_n\| < \infty$ and $\lim_{n\to\infty} f_n(y,\xi) = f(y,\xi)$, $\forall\ (y,\xi) \in \mathscr{R}^2$ (here and in the sequel, $\|f\| = \sup_{(y,\xi)\in\mathscr{R}^2}|f(y,\xi)|$). It is clear from (3.1) and (3.19)–(3.23) that, if $\text{bp-lim}_{n\to\infty} g_n = g$, then $\text{bp-lim}_{n\to\infty} V^{g_n} = V^g$ and $\text{bp-lim}_{n\to\infty} Q^{g_n} = Q^g$. Let

$$\bar{C}(\mathscr{R}^2) \triangleq \{\phi \in C(\mathscr{R}^2), \|\phi\| < \infty\}. \tag{3.28}$$

The set $\bar{C}(\mathscr{R}^2) \cap C^2(\mathscr{R}^2)$ is bp-dense in $\bar{C}(\mathscr{R}^2)$, which in turn is bp-dense in the space of bounded, Borel-measurable functions on $\mathscr{R}^2$ (cf. [5], Chapter 3, Proposition 4.2). By Theorem 3.4, V^g and Q^g coincide on $\bar{C}(\mathscr{R}^2) \cap C^2(\mathscr{R}^2)$; hence, they are equal for any bounded, Borel-measurable $g\colon \mathscr{R}^2 \to \mathscr{R}$.

The continuity of Q^g at the origin can be proved directly from (3.19)–(3.23). The essential observation is that, if g is bounded, there exists a positive constant $K < \infty$ such that

$$|F^g(u)| \leq Ku^2, \qquad |G^g(u)| \leq Ku^2 \quad \text{for } 0 \leq u \leq 1. \tag{3.29}$$

Therefore,

$$\begin{aligned} M^g(0) &= N^g(0) \\ &= \int_0^{\infty} [F^g(u) + G^g(u)]\big(\cosh(u\sqrt{2\lambda}) - 1\big)\sinh^{-3}(\sqrt{2\lambda}\,u)\,du \end{aligned} \tag{3.30}$$

exists, and, by (3.16),

$$\begin{aligned} |M^g(s) - M^g(0)| &\leq \int_0^s [F^g(u) + G^g(u)]\,\frac{\cosh(u\sqrt{2\lambda}) - 1}{\sinh^3(u\sqrt{2\lambda})}\,du \\ &\quad + \big[\cosh(s\sqrt{2\lambda}) - 1\big]\int_s^{\infty} \frac{F^g(u)\cosh(u\sqrt{2\lambda}) - G^g(u)}{\sinh^3(u\sqrt{2\lambda})}\,du \\ &\leq \tilde{K}[s^2 + s^2 \ln s], \qquad \forall\, 0 < s \leq 1, \end{aligned} \tag{3.31}$$

for some $\tilde{K} < \infty$. It follows that M^g is continuous at 0, and a similar argument proves that the same is true for N^g. Thus,

$$\begin{aligned} \lim_{y\downarrow 0,\, \xi\downarrow 0} \sinh\big((y+\xi)\sqrt{2\lambda}\big)^{-1}\big[&\sinh(y\sqrt{2\lambda})N^g(y+\xi) \\ &+ \sinh(\xi\sqrt{2\lambda})M^g(y+\xi)\big] = M^g(0). \end{aligned}$$

Since an inequality like (3.29) proves $\lim_{y\downarrow 0,\,\xi\downarrow 0} J^g(y,\xi) = 0$ for the function of (3.18), we derive easily the continuity of $Q^g(y,\xi)$ at $\mathbf{0}$ from (3.19)–(3.23). □

In Section 8 we shall use the fact that $V^g(y, \xi)$ extends continuously to the origin, to prove the following result.

PROPOSITION 3.7. *Problem* 2.1 *admits a solution for* $(y, \xi) = \mathbf{0}$, *which is unique in the sense of the probability law.*

4. Analysis. Consider a function $g \in \mathscr{D}$ satisfying the growth condition (3.3), and fix $\lambda > \frac{1}{2}(\theta_1 + \theta_2)^2$. Recall the decomposition $g = g_0 + g_1 + g_2 + g_3$ of g into functions with various even and odd symmetries, as in Theorem 3.4. It suffices to prove Theorem 3.4 for each g_i, $0 \le i \le 3$, separately.

Thus, we begin by assuming that g *is even in both* y *and* ξ, that is, $g \equiv g_0$. It is easy to see that such a g belongs to $C^2(\mathscr{R}^2)$. Our candidate for Q is then $V^0 \triangleq V^{g_0}$, which inherits the symmetry of g, that is,

$$V^0(y, \xi) = V^0(-y, \xi) = V^0(y, -\xi) = V^0(-y, -\xi) \tag{4.1}$$

holds, and satisfies the growth condition

$$|V^0(y, \xi)| \le Ke^{\theta_1|y| + \theta_2|\xi|}, \qquad \forall\, (y, \xi) \in \mathscr{R}^2 \setminus \{\mathbf{0}\}, \tag{4.2}$$

for a suitable real constant $K > 0$, by Lemma 3.1.

Let us start our analysis by studying the function V^0 in the positive quadrant.

PROPOSITION 4.1. *For* $(y, \xi) \in \bar{\Gamma}_1 \setminus \{\mathbf{0}\}$ *we have*

$$\begin{aligned} &\sinh((y + \xi)\sqrt{2\lambda})V^0(y, \xi) \\ &\quad = \sqrt{\frac{2}{\lambda}}\, \sinh(y\sqrt{2\lambda}) \int_0^{\xi} \sinh(u\sqrt{2\lambda}) g(y + \xi - u, u)\, du \\ &\qquad + \sqrt{\frac{2}{\lambda}}\, \sinh(\xi\sqrt{2\lambda}) \int_0^{y} \sinh(u\sqrt{2\lambda}) g(u, y + \xi - u)\, du \\ &\qquad + \sinh(\xi\sqrt{2\lambda}) \tilde{M}(y + \xi) + \sinh(y\sqrt{2\lambda}) \tilde{N}(y + \xi), \end{aligned} \tag{4.3}$$

where

$$\tilde{M}(s) \triangleq V^0(0, s), \qquad \tilde{N}(s) \triangleq V^0(s, 0), \qquad 0 < s < \infty. \tag{4.4}$$

PROOF. First, let us take $(y, \xi) \in \Gamma_1$, and consider the stopping time

$$\tau = \inf\{t \ge 0,\ Z_t^{y, \xi} \notin (0, y + \xi)\}. \tag{4.5}$$

Obviously,

$$Y_t^{y, \xi} = y + \xi - Z_t^{y, \xi} \quad \text{for } 0 \le t \le \tau, \tag{4.6}$$

and we can write the function $V^0 = V^{g_0}$ of (3.1) as

$$V^0(y, \xi) = E\int_0^{\tau} e^{-\lambda t} g(\beta - Z_t^{y, \xi}, Z_t^{y, \xi})\, dt + E\int_{\tau}^{\infty} e^{-\lambda t} g(Y_t^{y, \xi}, Z_t^{y, \xi})\, dt, \tag{4.7}$$

where we have put $y + \xi = \beta$. The first expectation on the right-hand side of (4.7) equals

$$E\int_0^\infty e^{-\lambda t} g\big(\beta - W^\beta_{\xi,t}, W^\beta_{\xi,t}\big)\, dt, \tag{4.8}$$

where $(W^\beta_{\xi,t})_{t\ge 0}$ is the Brownian motion on $[0,\beta]$ which starts at ξ and is *killed* when it hits the endpoints. The expression of (4.8) is precisely the resolvent $(\lambda I - \mathscr{A}^\beta)^{-1} f$, where $\mathscr{A}^\beta$ is the generator of $(W^\beta_{\xi,t})_{t\ge 0}$ and $f(x) = g(\beta - x, x)$, $0 \le x \le \beta$. Since $\mathscr{A}^\beta = \frac{1}{2}(d^2/dx^2)$ on the domain $D(A^\beta) = \{f \in C[0,\beta] \cap C^2[0,\beta],\ f(0) = f(\beta) = 0\}$, if $\phi \in C[0,\beta]$, the function $r(\xi) = [(\lambda I - \mathscr{A}^\beta)^{-1}\phi](\xi)$ is the solution of $\lambda r(\xi) - \frac{1}{2} r''(\xi) = \phi(\xi)$, $r(0) = r(\beta) = 0$. It is well known then that

$$\Big[(\lambda I - \mathscr{A}^\beta)^{-1}\phi\Big](\xi) = \int_0^\beta G_\beta(\xi, u)\phi(u)\, du, \tag{4.9}$$

where G_β is the Green's function

$$G_\beta(\xi, u) \triangleq \begin{cases} \sqrt{\dfrac{2}{\lambda}}\, \dfrac{\sinh(\xi\sqrt{2\lambda})\sinh((\beta - u)\sqrt{2\lambda})}{\sinh(\beta\sqrt{2\lambda})}, & u \ge \xi, \\[2ex] \sqrt{\dfrac{2}{\lambda}}\, \dfrac{\sinh(u\sqrt{2\lambda})\sinh((\beta - \xi)\sqrt{2\lambda})}{\sinh(\beta\sqrt{2\lambda})}, & u < \xi. \end{cases}$$

Thus, the expression of (4.8), with $\beta = y + \xi$, becomes

$$\sqrt{\frac{2}{\lambda}}\, \frac{1}{\sinh((y+\xi)\sqrt{2\lambda})} \Bigg[\sinh(y\sqrt{2\lambda})\int_0^\xi \sinh(u\sqrt{2\lambda})\, g(y + \xi - u, u)\, du$$
$$+ \sinh(\xi\sqrt{2\lambda})\int_\xi^{y+\xi} \sinh((y + \xi - u)\sqrt{2\lambda})\, g(y + \xi - u, u)\, du\Bigg] = J^g(y, \xi),$$

where $J^g(y, \xi)$ is defined as in (3.18). This way, we obtain the first two terms on the right-hand side of the expression (4.3).

On the other hand, conditioning on $\mathscr{F}^{Y,Z}_\tau$ and using the fact that (Y, Z) of (2.1) is a strong Markov process (e.g., [9], page 322), we obtain for the second expectation of (4.7):

$$\begin{aligned}
E&\int_\tau^\infty e^{-\lambda t} g\big(Y^{y,\xi}_t, Z^{y,\xi}_t\big)\, dt \\
&= E\bigg[e^{-\lambda\tau}\int_0^\infty e^{-\lambda t} g\big(Y^{y,\xi}_{\tau+t}, Z^{y,\xi}_{\tau+t}\big)\, dt\bigg] \\
&= E\bigg[e^{-\lambda\tau} E\bigg\{\int_0^\infty e^{-\lambda t} g\big(Y^{y,\xi}_{\tau+t}, Z^{y,\xi}_{\tau+t}\big)\, dt \,\bigg|\, \mathscr{F}^{Y,Z}_\tau\bigg\}\bigg] \\
&= E\bigg[e^{-\lambda\tau}\bigg\{E\int_0^\infty e^{-\lambda t} g\big(Y^{\eta,z}_t, Z^{\eta,z}_t\big)\, dt\bigg\}\bigg|_{\eta = Y^{y,\xi}_\tau,\, z = Z^{y,\xi}_\tau}\bigg] \\
&= E\big[e^{-\lambda\tau} V^0\big(Y^{y,\xi}_\tau, Z^{y,\xi}_\tau\big)\big] \\
&= E\big[e^{-\lambda\tau} 1_{\{Z^{y,\xi}_\tau = y+\xi\}}\big] V^0(0, y + \xi) + E\big[e^{-\lambda\tau} 1_{\{Z^{y,\xi}_\tau = 0\}}\big] V^0(y + \xi, 0).
\end{aligned}$$

The last two terms on the right-hand side of (4.3) follow now from the definitions of (4.4), and from the computations (e.g., [9], page 100):

$$E\left[e^{-\lambda\tau}1_{\{Z_\tau^{y,\xi}=y+\xi\}}\right] = \frac{\sinh(\xi\sqrt{2\lambda})}{\sinh((y+\xi)\sqrt{2\lambda})},$$

$$E\left[e^{-\lambda\tau}1_{\{Z_\tau^{y,\xi}=0\}}\right] = \frac{\sinh(y\sqrt{2\lambda})}{\sinh((y+\xi)\sqrt{2\lambda})}.$$

It is quite straightforward that (4.3) remains valid if either one of y or ξ is equal to 0, as long as $(y, \xi) \neq \mathbf{0}$. □

REMARK 4.2. In the notation of the preceding proof, we have

$$\begin{aligned} &\frac{1}{2}\frac{d^2}{dx^2}(\lambda I - \mathscr{A}^b)^{-1} f(x) \\ (4.10)\qquad &= \mathscr{A}^b(\lambda I - \mathscr{A}^b)^{-1} f(x) \\ &= -f(x) + \lambda(\lambda I - \mathscr{A}^b)^{-1} f(x), \qquad 0 < x < b, \end{aligned}$$

$$\begin{aligned} V^0(b-x,x) &= (\lambda I - \mathscr{A}^b)^{-1} f(x) + \frac{\sinh(x\sqrt{2\lambda})}{\sinh(b\sqrt{2\lambda})}\tilde{M}(b) \\ (4.11)\qquad &\quad + \frac{\sinh((b-x)\sqrt{2\lambda})}{\sinh(b\sqrt{2\lambda})}\tilde{N}(b), \qquad 0<x<b, \end{aligned}$$

for every $b \in (0, \infty)$. Here again, $f(x) = g(b-x, x)$, $0 \le x \le b$.

Now, in order to prove Theorem 3.4 for a function $g \equiv g_0$ (which is even in y and ξ), we must show $V^0(y, \xi) = Q_0(y, \xi)$, where Q_0 is defined in (3.19). Notice that the first two terms of (4.3) correspond to $J^g(y, \xi)$ as defined in (3.18). Hence, to complete the argument, it remains to identify $\tilde{M}(s)$ with $M^g(s)$ and $\tilde{N}(s)$ with $N^g(s)$.

We shall proceed as follows: We take, as working assumptions,

$$(4.12)\qquad \tilde{M} \text{ and } \tilde{N} \text{ are of class } C^2 \text{ on } (0, \infty)$$

and

$$(4.13)\qquad V^0 \text{ is of class } C^1 \text{ on } \mathscr{R}^2 \setminus \{\mathbf{0}\}.$$

We shall show that (4.12) and (4.13) lead to $\tilde{M} \equiv M^g$, $\tilde{N} \equiv N^g$ (Proposition 4.4), and by virtue of (4.3) and (3.19) to $V^0 \equiv Q_0$. Always under (4.12), we shall see that V^0 is of class C^2 on $\bigcup_{i=1}^4 \Gamma_i$ and satisfies (3.5) there (Proposition 4.3); while (4.13) implies then that V^0 is of class $C^2(\overline{\Gamma}_i)$ for every $1 \le i \le 4$, and $V^0 \in \mathscr{D}$. In fact, since V^0 is even in both y and ξ, $V^0 \in C^2(\mathscr{R}^2)$.

In the opposite direction (synthesis), we shall show that Q_0 of (3.19) is of class $\mathscr{D}$ and satisfies (3.5) as well as a growth condition of the type (3.8) (Proposition 4.5 and the remarks preceding it). From Proposition 3.2 we shall conclude then $Q_0 \equiv V^0$, thus *vindicating the Ansätze* (4.12) and (4.13) and the above analysis.

PROPOSITION 4.3. *Under the Ansatz* (4.12), *the function* V^0 *is of class* C^2 *in each open quadrant* Γ_i, $1 \le i \le 4$, *and satisfies there the equation*

$$\tfrac{1}{2}\left[V^0_{yy}(y,\xi) + V^0_{\xi\xi}(y,\xi)\right] - V^0_{y\xi}(y,\xi) + g(y,\xi) = \lambda V^0(y,\xi). \tag{4.14}$$

PROOF. For any fixed $b \in (0,\infty)$, we have

$$\frac{1}{2}\frac{d^2}{dx^2}V^0(b-x,x) = \left[\frac{1}{2}\left(V^0_{yy} + V^0_{\xi\xi}\right) - V^0_{y\xi}\right](b-x,x). \tag{4.15}$$

But from (4.11) and (4.10),

$$\begin{aligned}
&\frac{1}{2}\frac{d^2}{dx^2}V^0(b-x,x)\\
&\quad= \frac{1}{2}\frac{d^2}{dx^2}(\lambda I - \mathscr{A}^b)^{-1}f(x)\\
&\qquad + \lambda\left[\frac{\sinh(x\sqrt{2\lambda})}{\sinh(b\sqrt{2\lambda})}M(b) + \frac{\sinh((b-x)\sqrt{2\lambda})}{\sinh(b\sqrt{2\lambda})}N(b)\right]\\
&\quad= -f(x) + \lambda\Bigg[(\lambda I - \mathscr{A}^b)^{-1}f(x) + \frac{\sinh(x\sqrt{2\lambda})}{\sinh(b\sqrt{2\lambda})}M(b)\\
&\qquad\qquad + \frac{\sinh((b-x)\sqrt{2\lambda})}{\sinh(b\sqrt{2\lambda})}N(b)\Bigg]\\
&\quad= -f(x) + \lambda V^0(b-x,x) = \lambda V^0(b-x,x) - g(b-x,x).
\end{aligned} \tag{4.16}$$

Equation (4.14) follows, upon setting $x = \xi$, $b = y + \xi$ and equating terms in (4.14) and (4.15). The required smoothness of the function V^0 follows directly from its representation (4.3) and from the corresponding smoothness of the functions $\tilde{N}$ and $\tilde{M}$ [assumption (4.12)]. □

We *proceed now with the derivation of* $\tilde{M} = M^g$ *and* $\tilde{N} = N^g$ *from* (4.12) *and* (4.13). Note first that, from the symmetry relation $V^0(y,\xi) = V^0(y,-\xi)$ of (4.1), V^0_y is automatically continuous across the half-axis ($y > 0$, $\xi = 0$); in order for V^0_ξ also to be continuous, we must have

$$V^0_\xi(y,0+) = 0, \qquad 0 < y < \infty. \tag{4.17(i)}$$

Similar considerations across the half-axis ($y = 0$, $\xi > 0$) lead to

$$V^0_y(0+,\xi) = 0, \qquad 0 < \xi < \infty. \tag{4.17(ii)}$$

Let us work in the quadrant Γ_1 where, by virtue of (4.3), the partial derivatives V_y and V_ξ are given, respectively, by

$$\begin{aligned}
&\sinh((y+\xi)\sqrt{2\lambda})V_y^0(y,\xi) + \sqrt{2\lambda}\cosh((y+\xi)\sqrt{2\lambda})V^0(y,\xi)\\
&\quad = \sqrt{2\lambda}\cosh(y\sqrt{2\lambda})\tilde{N}(y+\xi) + \sinh(\xi\sqrt{2\lambda})\tilde{M}'(y+\xi)\\
&\qquad + \sinh(y\sqrt{2\lambda})\tilde{N}'(y+\xi)\\
&\qquad + 2\cosh(y\sqrt{2\lambda})\int_0^\xi g(y+\xi-u,u)\sinh(u\sqrt{2\lambda})\,du\\
&\qquad + \sqrt{\frac{2}{\lambda}}\sinh(y\sqrt{2\lambda})\sinh(\xi\sqrt{2\lambda})g(y,\xi)\\
&\qquad + \sqrt{\frac{2}{\lambda}}\bigg[\sinh(y\sqrt{2\lambda})\int_0^\xi g_y(y+\xi-u,u)\sinh(u\sqrt{2\lambda})\,du\\
&\qquad\qquad + \sinh(\xi\sqrt{2\lambda})\int_0^y g_\xi(u,y+\xi-u)\sinh(u\sqrt{2\lambda})\,du\bigg],
\end{aligned}\tag{4.18}$$

$$\begin{aligned}
&\sinh((y+\xi)\sqrt{2\lambda})V_\xi^0(y,\xi) + \sqrt{2\lambda}\cosh((y+\xi)\sqrt{2\lambda})V^0(y,\xi)\\
&\quad = \sqrt{2\lambda}\cosh(\xi\sqrt{2\lambda})\tilde{M}(y+\xi) + \sinh(\xi\sqrt{2\lambda})\tilde{M}'(y+\xi)\\
&\qquad + \sinh(y\sqrt{2\lambda})\tilde{N}'(y+\xi)\\
&\qquad + 2\cosh(\xi\sqrt{2\lambda})\int_0^y \sinh(u\sqrt{2\lambda})g(u,y+\xi-u)\,du\\
&\qquad + \sqrt{\frac{2}{\lambda}}\sinh(y\sqrt{2\lambda})\sinh(\xi\sqrt{2\lambda})g(y,\xi)\\
&\qquad + \sqrt{\frac{2}{\lambda}}\bigg[\sinh(y\sqrt{2\lambda})\int_0^\xi g_y(y+\xi-u,u)\sinh(u\sqrt{2\lambda})\,du\\
&\qquad\qquad + \sinh(\xi\sqrt{2\lambda})\int_0^y g_\xi(u,y+\xi-u)\sinh(u\sqrt{2\lambda})\,du\bigg].
\end{aligned}\tag{4.19}$$

Letting $y \downarrow 0$ in (4.18), we obtain

$$\begin{aligned}
V_y^0(0+,\xi) &= \tilde{M}'(\xi) - \sqrt{2\lambda}\coth(\xi\sqrt{2\lambda})\tilde{M}(\xi) + \frac{\tilde{N}(\xi)\sqrt{2\lambda}}{\sinh(\xi\sqrt{2\lambda})}\\
&\quad + 2\int_0^\xi \frac{\sinh(u\sqrt{2\lambda})}{\sinh(\xi\sqrt{2\lambda})}g(\xi-u,u)\,du, \qquad 0<\xi<\infty.
\end{aligned}\tag{4.20}$$

The requirement (4.17)(ii) thus amounts to the differential equation

$$\sinh(s\sqrt{2\lambda})\tilde{M}'(s) = \sqrt{2\lambda}\left[\cosh(s\sqrt{2\lambda})\tilde{M}(s) - \tilde{N}(s)\right] - F^g(s), \qquad 0 < s < \infty,\tag{4.21}$$

where $F^g(\cdot)$ is the function of (3.14).

On the other hand, letting $\xi \downarrow 0$ in (4.19), we obtain

$$V_\xi^0(y+,0) = \tilde{N}'(y) - \sqrt{2\lambda}\coth(y\sqrt{2\lambda})\tilde{N}(y) + \frac{\sqrt{2\lambda}}{\sinh(y\sqrt{2\lambda})}\tilde{M}(y) + 2\int_0^y \frac{\sinh((y-u)\sqrt{2\lambda})}{\sinh(y\sqrt{2\lambda})} g(y-u,u)\,du, \qquad 0<y<\infty.$$

Then the requirement (4.17)(i) amounts to the differential equation

$$\sinh(s\sqrt{2\lambda})\tilde{N}'(s) = \sqrt{2\lambda}\left[\cosh(s\sqrt{2\lambda})\tilde{N}(s) - \tilde{M}(s)\right] - G^g(s), \qquad 0<s<\infty, \tag{4.22}$$

where $G^g(\cdot)$ is the function of (3.15).

In addition to (4.21) and (4.22), the growth constraint (4.2) implies the boundary conditions at infinity,

$$\limsup_{s\to\infty} e^{-\theta_2 s}\tilde{M}(s) < \infty, \tag{4.23}$$

$$\limsup_{s\to\infty} e^{-\theta_1 s}\tilde{N}(s) < \infty. \tag{4.24}$$

PROPOSITION 4.4. *The unique solution to* (4.21)–(4.24) *is given by the functions* $\tilde{M}(s) = M^g(s)$ *and* $\tilde{N}(s) = N^g(s)$ *defined in* (3.16) *and* (3.17). *In particular, the Ansätze* (4.12) *and* (4.13) *lead to* $V^0 \equiv Q_0$.

PROOF. Setting $L_\pm(s) \triangleq (\tilde{M}(s) \pm \tilde{N}(s))/\sinh(s\sqrt{2\lambda})$, (4.21) and (4.22) can be written equivalently as

$$\begin{aligned}\left(L_+(s)\tanh\left(\frac{s}{2}\sqrt{2\lambda}\right)\right)' &= -\frac{F^g(s)+G^g(s)}{2\sinh(s\sqrt{2\lambda})\cosh^2\left(\frac{s}{2}\sqrt{2\lambda}\right)},\\ \left(L_-(s)\coth\left(\frac{s}{2}\sqrt{2\lambda}\right)\right)' &= -\frac{F^g(s)-G^g(s)}{2\sinh(s\sqrt{2\lambda})\sinh^2\left(\frac{s}{2}\sqrt{2\lambda}\right)}.\end{aligned} \tag{4.25}$$

But from (4.23) and (4.24) and $\sqrt{2\lambda} > \theta_1 + \theta_2$, we have $\lim_{s\to\infty} L_\pm(s) = 0$. From this and (4.25) and (4.26), we deduce

$$\begin{aligned}\tilde{M}(s) + \tilde{N}(s) &= \cosh^2\left(\frac{s}{2}\sqrt{2\lambda}\right)\int_s^\infty \frac{F^g(u)+G^g(u)}{\sinh(u\sqrt{2\lambda})\cosh^2\left(\frac{u}{2}\sqrt{2\lambda}\right)}\,du,\\ \tilde{M}(s) - \tilde{N}(s) &= \sinh^2\left(\frac{s}{2}\sqrt{2\lambda}\right)\int_s^\infty \frac{F^g(u)-G^g(u)}{\sinh(u\sqrt{2\lambda})\sinh^2\left(\frac{u}{2}\sqrt{2\lambda}\right)}\,du,\end{aligned} \tag{4.26}$$

and thence the expressions of (3.16) and (3.17) for $\tilde{M}$ and $\tilde{N}$, after some algebra. □

Assumptions (4.12) and (4.13) have led us, therefore, to identify Q_0 as the appropriate candidate for V^0.

In the opposite direction (synthesis), it is easily seen by inspection that M^g and N^g are of class $C^2(0,\infty)$. Hence, reasoning as in Proposition 4.3, we see that Q_0 is of class C^2 in each open quadrant Γ_i, $1 \le i \le 4$, and satisfies (3.5) there. In fact, a simple inspection of the formula for Q_0 shows that, in each quadrant, the first and second derivatives of Q_0 extend continuously to $\bar{\Gamma}_i \setminus \{\mathbf{0}\}$, and hence Q_0 satisfies (3.5) in $\mathscr{R}^2 \setminus \{\mathbf{0}\}$. To show $Q_0 \in \mathscr{D}$, it remains to prove that the first and second derivatives of Q_0 extend continuously *to the origin* in each quadrant. Of course, it suffices to consider the quadrant Γ_1; this is the object of Proposition 4.5, which is proved in Section 9.

PROPOSITION 4.5. *The functions M^g and N^g are of class $C^2[0,\infty)$, and*

$$\lim_{y\downarrow 0}[Q_0]_y(y,0+) = \lim_{y\downarrow 0}(N^g)'(y) = 0,$$
$$\lim_{\xi\downarrow 0}[Q_0]_\xi(0+,\xi) = \lim_{\xi\downarrow 0}(M^g)'(\xi) = 0, \tag{4.27}$$

$$\lim_{y\downarrow 0}[Q_0]_{yy}(y,0+) = \lim_{\xi\downarrow 0}[Q_0]_{yy}(0+,\xi)$$
$$= 2\lambda\int_0^\infty \frac{G^g(u)\cosh(u\sqrt{2\lambda}) - F^g(u)}{\sinh^3(u\sqrt{2\lambda})}\,du - g(0,0), \tag{4.28}$$

$$\lim_{\xi\downarrow 0}[Q_0]_{\xi\xi}(0+,\xi) = \lim_{y\downarrow 0}[Q_0]_{\xi\xi}(y,0+)$$
$$= 2\lambda\int_0^\infty \frac{F^g(u)\cosh(u\sqrt{2\lambda}) - G^g(u)}{\sinh^3(u\sqrt{2\lambda})}\,du - g(0,0). \tag{4.29}$$

Moreover, Q_0 is of class $\mathscr{D}$.

Now that we know $Q_0 \in \mathscr{D}$ and solves (3.5), we have from Proposition 3.2,

$$Q_0(y,\xi) = E\int_0^\infty e^{-\lambda t} g(Y_t^{y,\xi}, Z_t^{y,\xi})\,dt =: V^0(y,\xi)$$

for $(y,\xi) \in \mathscr{R}^2 \setminus \{\mathbf{0}\}$, thus completing the proof of Theorem 3.4 for the case in which g is even in both y and ξ.

The other cases Q_1, Q_2 and Q_3 of Theorem 3.4 follow by arguments similar to those presented above, and we shall sketch briefly that for Q_1. Thus, we shall assume now that *$g \in \mathscr{D}$ is even in y and odd in ξ*: $g \equiv g_1$. This property is then inherited by the function $V^1(y,\xi) \triangleq V^{g_1}(y,\xi)$; in particular, $V^1(y,0) = 0$. Moreover, Proposition 4.1 continues to be valid in $\bar{\Gamma}_1 \setminus \{\mathbf{0}\}$, and leads to

$$V^1(y,\xi) = J^{g_1}(y,\xi) + \frac{\sinh(\xi\sqrt{2\lambda})}{\sinh((y+\xi)\sqrt{2\lambda})}\tilde{M}_1(y+\xi), \tag{4.30}$$

where $\tilde{M}_1(s) \triangleq V^1(0,s)$. To derive an equation for $\tilde{M}_1$, we impose again the Ansätze (4.12) and (4.13). Because V^1 is odd in ξ, V_ξ^1 will automatically be continuous across the y-axis. However, to obtain continuity of V_y^1 across the ξ-axis, we must require again that (4.17)(ii) hold; by way of (4.18), this leads to

$$(4.31)\quad M'(s) = \sqrt{2\lambda}\coth(s\sqrt{2\lambda})M(s) - \frac{F^g(s)}{\sinh(s\sqrt{2\lambda})}, \qquad 0 < s < \infty,$$

by analogy with (4.21). The solution of (4.31), subject to the boundary condition (4.23), is given by

$$(4.32)\qquad \tilde{M}_1(s) = \sinh(s\sqrt{2\lambda})\int_s^\infty \frac{F^g(s)}{\sinh^2(u\sqrt{2\lambda})}\,du, \qquad 0 < s < \infty.$$

The end result from (4.30) and (4.32) is that the assumptions (4.12) and (4.13) imply $V^1(y,\xi) = Q_1(y,\xi)$ in the notation of (3.21). To verify that this is indeed the case, we must show that Q_1 is of class $\mathscr{D}$ and satisfies (3.5). We omit the details, which are similar to the corresponding arguments presented above.

5. Special properties. The purpose of this section is to obtain further properties for the function Q of Theorem 3.4, corresponding to the special choice

$$(5.1)\qquad g(y,\xi) = c(y)\cosh(\theta\xi), \qquad (y,\xi) \in \mathscr{R}^2.$$

Here we take $c\colon \mathscr{R} \to [0,\infty)$ to be an even, convex function of class C^2 with $c'(y) > 0$ for $y > 0$, satisfying the exponential growth condition

$$(5.2)\qquad |c^{(j)}(y)| \le Ke^{\theta_1|y|}, \qquad \forall\, y \in \mathscr{R},\ j = 0,1,2$$

for some positive, real numbers β, K and θ. Throughout we fix $\lambda > \frac{1}{2}(\theta + \theta_1)^2$.

The function of (5.1) is even in both y and ξ, and thus the corresponding $Q \equiv Q^0$ is given by (3.19). The results obtained in this section for Q, particularly Propositions 5.1 and 5.4, will be used heavily in the analysis of the control problem with partial observations, which is the subject of the next two sections. The proofs of Propositions 5.1 and 5.2 are given in Section 9.

PROPOSITION 5.1. *The function*

$$(5.3)\qquad U(y,\xi) \triangleq Q_{y\xi}(y,\xi)$$

satisfies

$$(5.4)\qquad \begin{aligned} &U > 0 \quad \text{in } \Gamma_1, \Gamma_3,\\ &U < 0 \quad \text{in } \Gamma_2, \Gamma_4,\\ &U = 0 \quad \text{on the axes}. \end{aligned}$$

PROPOSITION 5.2. *The second partial derivative $U_{\xi\xi}(y,\xi)$ exists in Γ_1, and the function*

$$(5.5)\qquad W(y,\xi) \triangleq U_{\xi\xi}(y,\xi) - \theta^2 U(y,\xi)$$

is positive in Γ_1.

REMARK 5.3. (i) It is quite easy to see [from (9.9) and (5.2)] that the mixed partial derivative $U = Q_{y\xi}$ satisfies a growth condition of the type

$$U(y,\xi) \le Ke^{\theta_1|y|+\theta|\xi|}, \qquad (y,\xi) \in \mathscr{R}^2. \tag{5.6}$$

(ii) Suppose that $c''(\cdot)$ fails to exist at a finite number of points, excluding $y = 0$. An examination of (3.20) shows that Q will still be of class C^2 on $\mathscr{R}^2$. Furthermore, $U = Q_{y\xi}$ and $U_{\xi\xi}$ will continue to exist, and will satisfy the conclusions of Propositions 5.1 and 5.2, as is easily seen.

PROPOSITION 5.4. *The function Q of* (3.8) *satisfies*

$$\theta^2 Q > Q_{\xi\xi} \quad in\ \mathscr{R}^2. \tag{5.7}$$

PROOF (adapted from [2]). Because of symmetry, it suffices to prove (5.7) in the quadrant $\overline{\Gamma}_1$. From (3.5) written in the form

$$\tfrac{1}{2}\left[Q_{yy} + Q_{\xi\xi}\right] + H = \lambda Q \quad \text{in } \Gamma_1, \tag{5.8}$$

with

$$H(y,\xi) \triangleq c(y)\cosh(\theta\xi) - U(y,\xi), \tag{5.9}$$

and the boundary conditions

$$Q_\xi(y,0+) = 0, \qquad Q_y(0+,\xi) = 0 \tag{5.10}$$

[cf. (4.17)(i) and (ii)], one can obtain the representation

$$\begin{aligned} Q(y,\xi) &= E\int_0^\infty e^{-\lambda t} H(|y + W_1(t)|, |\xi + W_2(t)|)\,dt \\ &= \iiint_{\mathscr{R}_+^3} e^{-\lambda t} p(t;y,u)p(t;\xi,\eta)H(u,\eta)\,du\,d\eta\,dt, \end{aligned} \tag{5.11}$$

where $\mathbf{W} = (W_1, W_2)$ is a standard, two-dimensional Brownian motion process and

$$p(t;x,y) = \frac{1}{\sqrt{2\pi t}}\left[\exp\left\{-\frac{(x-y)^2}{2t}\right\} + \exp\left\{-\frac{(x+y)^2}{2t}\right\}\right], \qquad x > 0, y > 0,$$

is the transition probability density function for Brownian motion in the positive quadrant, with reflection on its sides. [The derivation of (5.11) is accomplished in a straightforward manner, by applying Itô's rule to the semimartingale $e^{-\lambda t}Q(|y + W_1(t)|, |\xi + W_2(t)|)$ and using (5.8) and (5.10), as well as the growth conditions (3.8) and (5.6).]

Integrating by parts twice in (5.11), we obtain, after some calculus,

$$\begin{aligned} \theta^2 Q(y,\xi) - Q_{\xi\xi}(y,\xi) &= \iiint_{\mathscr{R}_+^3} e^{-\lambda t} p(t;y,u)p(t;\xi,\eta) \\ &\qquad \times\left[\theta^2 H(u,\eta) - H_{\eta\eta}(u,\eta)\right] du\,d\eta\,dt \\ &\quad - 2\iint_{\mathscr{R}_+^2} e^{-\lambda t} p(t;y,u)\frac{e^{-\xi^2/2t}}{\sqrt{2\pi t}} H_\eta(u,0+)\,du\,dt \end{aligned} \tag{5.12}$$

(cf. [7], Chapter 1, Theorems 3 and 4). Now from (5.9) and Proposition 5.2, $\theta^2 H - H_{\xi\xi} = U_{\xi\xi} - \theta^2 U = W > 0$ in $(y > 0,\ \xi \geq 0)$, and from (9.13), $-H_{\xi}(y, 0+) = U_{\xi}(y, 0+) > 0$ for $y > 0$. It follows that the expression of (5.12) is positive. □

6. A control problem with partial observations. Let us consider now a probability space $(\Omega, \mathscr{F}, P), \{\mathscr{F}_t\}$, on which a solution of Problem 2.1 has been constructed with $(Y_0, Z_0) = (y, \xi) \in \mathscr{R}^2$. We may always assume that this space is rich enough to support a random variable z *independent of* $\mathscr{F}_\infty$, with given distribution μ. We shall denote by $\{\mathscr{G}_t\}$ and $\{\mathscr{F}_t^{\,Y}\}$ the P-augmentations of the filtrations $\{\sigma(z) \vee \mathscr{F}_t\}$ and $\{\sigma(Y_s),\ 0 \leq s \leq t\}$, respectively.

DEFINITION 6.1. The class $\mathscr{U}$ of *wide-sense admissible control processes* consists of all $\{\mathscr{F}_t\}$-progressively measurable processes $u = \{u_t,\ 0 \leq t < \infty\}$ with values in $[-1, 1]$. The class $\mathscr{U}_s$ of *strict-sense admissible control processes* consists of all processes $u \in \mathscr{U}$ which are adapted to $\{\mathscr{F}_t^{\,Y}\}$.

The terminology employed here comes from [6].

For every $u \in \mathscr{U}$, we introduce the exponential $\{\mathscr{G}_t\}$-martingale

$$\Lambda_t^u \triangleq \exp\left\{z\int_0^t u_s\,dY_s - \tfrac{1}{2}z^2\int_0^t u_s^2\,ds\right\}, \qquad 0 \leq t < \infty, \tag{6.1}$$

and the process

$$W_t^{\,u} \triangleq Y_t - y - z\int_0^t u_s\,ds, \qquad 0 \leq t < \infty. \tag{6.2}$$

For every given $T \in (0, \infty)$, the process $\{W_t^{\,u}, \mathscr{G}_t,\ 0 \leq t \leq T\}$ is a Brownian motion on the interval $[0, T]$ and independent of the random variable z, under the probability measure

$$P_T^u(A) \triangleq E[\Lambda_T^u 1_A], \qquad A \in \mathscr{G}_T, \tag{6.3}$$

by virtue of the Girsanov theorem (cf. [9], Section 3.5).

We can formulate now our control problem with partial observations.

PROBLEM 6.2. *Minimize the expected discounted cost*

$$J(u) \triangleq \lim_{T\to\infty} E_T^u\int_0^T e^{-\alpha t}c(Y_t)\,dt = \int_0^\infty e^{-\alpha t}E_t^u c(Y_t)\,dt \tag{6.4}$$

over $u \in \mathscr{U}$. *Here* $\alpha > 0$ *is a given real constant, and the cost function* $c(\cdot)$ *is as in Section* 3.

Put differently, one seeks to minimize the cost functional of (6.4), subject to the dynamics

$$dY_t = zu_t\,dt + dW_t^{\,u}, \qquad Y_0 = y, \tag{6.2$'$}$$

with W^u a Brownian motion independent of the random variable z. The minimization is to be over controls u which take values in the interval $[-1,1]$ and are adapted to the "observation filtration" $\{\mathscr{F}_t\}$. Because z is independent of $\{\mathscr{F}_t\}$, it is called an "unobservable" variable, and the stochastic control problem is one of *partial* (or incomplete) *observations*.

This independence of z and $\{\mathscr{F}_t\}$ allows us to cast (6.4) as

$$\begin{aligned} J(u) &= \int_0^\infty e^{-\alpha t} E[c(Y_t)\Lambda_t^u]\,dt = E\int_0^\infty e^{-\alpha t} c(Y_t) E[\Lambda_t^u | \mathscr{F}_t\,]\,dt \\ &= E\int_0^\infty e^{-\alpha t} c(Y_t) F\left(\int_0^t u_s^2\,ds, \int_0^t u_s\,dY_s\right) dt, \end{aligned} \tag{6.5}$$

where

$$F(t,x) \triangleq \int_{\mathscr{R}} \exp\{yx - \tfrac{1}{2}y^2 t\}\mu(dy), \qquad (t,x) \in (0,\infty) \times \mathscr{R}. \tag{6.6}$$

On the other hand, the least-squares estimate $\hat{z}_t^u = E_t^u(z|\mathscr{F}_t)$ of the random variable z, given the observations $\mathscr{F}_t$ up to time t, is expressed by the Bayes rule ([9], page 193) as

$$\hat{z}_t^u = \frac{E[z\Lambda_t^u | \mathscr{F}_t\,]}{E[\Lambda_t^u | \mathscr{F}_t\,]} = G\left(\int_0^t u_s^2\,ds, \int_0^t u_s\,dY_s\right), \tag{6.7}$$

where

$$G(t,x) \triangleq \frac{1}{F(t,x)} \int_{\mathscr{R}} y \exp\{yx - \tfrac{1}{2}y^2 t\}\mu(dy), \qquad (t,x) \in (0,\infty) \times \mathscr{R}. \tag{6.8}$$

EXAMPLE 6.3. In the special case of a Bernoulli random variable z, with

$$P[z = \theta] = \rho, \qquad P[z = -\theta] = 1 - \rho \tag{6.9}$$

for some $\theta \in (0,\infty)$ and $\rho \in (0,1)$, (6.5)–(6.8) become

$$F(t,x;\theta) \triangleq \frac{\cosh(b + \theta x)}{\cosh b} e^{-t\theta^2/2}, \qquad G(t,x;\theta) \triangleq \theta \tanh(b + \theta x), \tag{6.10}$$

$$\begin{aligned} J(u;\theta) \triangleq \frac{1}{\cosh(\theta\xi)} E\int_0^\infty \exp\left[-\int_0^t \left(\alpha + \frac{\theta^2}{2} u_s^2\right) ds\right] \\ \times c(Y_t)\cosh(\theta \xi_t^u)\,dt, \end{aligned} \tag{6.11}$$

$$\hat{z}_t^u = \theta \tanh(\theta \xi_t^u), \tag{6.12}$$

where

$$b = \tanh^{-1}(2\rho - 1), \qquad \xi = \frac{b}{\theta}, \tag{6.13}$$

and

$$\xi_t^u = \xi + \int_0^t u_s\,dY_s, \qquad 0 \le t < \infty. \tag{6.14}$$

In the *completely observable case*, where z is almost surely equal to a real constant, it is well known that the optimal control law is of the form

$$u_t^{\text{opt}} = -\operatorname{sgn}(zY_t). \tag{6.15}$$

This result, first proved by Beneš [1], was later established by different methods in [8], [4], [3] and [9], Section 6.5. In the partially observable setting of the present section, it is natural to guess that an optimal law can be obtained if one replaces in (6.15) z by its least-squares estimate $\hat{z}_t \triangleq E_t^{u^*}(z|\mathscr{F}_t)$, that is,

$$u_t^* = -\operatorname{sgn}(\hat{z}_t Y_t). \tag{6.16}$$

This was shown in [2] for a symmetric distribution on μ, and $c(y) = y^2$. It will be established in Section 7 for a general cost function $c(\cdot)$ obeying the assumptions of Section 5.

EXAMPLE 6.3 (Continued). For a Bernoulli random variable z of the type (6.9), we have from (6.12), (6.14) and (6.16), $u_t^* = -\operatorname{sgn}(Y_t \xi_t^*)$ and

$$\xi_t^* = \xi - \int_0^t \operatorname{sgn}(Y_s \xi_s^*)\, dY_s,$$

which is (2.1). Thus, we can make the identification $\xi^* \equiv Z$, and try to show that the process

$$u_t^* = -\operatorname{sgn}(Y_t Z_t), \qquad 0 \le t < \infty, \tag{6.17}$$

is optimal for the Problem 6.2, in the case (6.9) of a Bernoulli random variable. This will be proved in Theorem 7.1.

REMARK 6.4. For any $u \in \mathscr{U}$, the *innovations process*

$$\nu_t^u \triangleq Y_t - y - \int_0^t u_s \hat{z}_s^u\, ds, \qquad 0 \le t < \infty, \tag{6.18}$$

is adapted to $\{\mathscr{F}_t\}$ and, for every $T \in (0, \infty)$, its restriction $\{\nu_t^u, \mathscr{F}_t, 0 \le t \le T\}$ is Brownian motion on $[0, T]$ under the probability measure P_T^u of (6.3). In the case of Example 6.3, it follows from (6.18) and (6.12) that the processes Y and ξ^u [of (6.14)] satisfy the innovations-driven equations

$$dY_t = u_t \theta \tanh(\theta \xi_t^u)\, dt + d\nu_t^u, \qquad Y_0 = y, \tag{6.19}$$

$$d\xi_t^u = u_t^2 \theta \tanh(\theta \xi_t^u)\, dt + u_t\, d\nu_t^u, \qquad \xi_0^u = \xi. \tag{6.20}$$

7. Solution to the control problem. Let us concentrate first on the Bernoulli case of (6.9). Let Q be as in Section 5, and introduce the function

$$\Phi(y, \xi) \triangleq \frac{Q(y, \xi)}{\cosh(\theta \xi)}, \qquad (y, \xi) \in \mathscr{R}^2, \tag{7.1}$$

which is of class C^2 in $\mathscr{R}^2$ and satisfies, with $\alpha \triangleq \lambda - \theta^2/2 > 0$, the analogues

$$\tag{7.2} \begin{aligned} &\tfrac{1}{2}[\Phi_{yy} + \Phi_{\xi\xi}] - \operatorname{sgn}(y\xi)[\Phi_{y\xi} + \theta \tanh(\theta\xi)\Phi_y] \\ &\qquad + \theta\tanh(\theta\xi)\Phi_\xi + c(y) = \alpha\Phi, \end{aligned}$$

$$\tag{7.3} 0 \le \Phi(y,\xi) \le Ke^{\theta_1|y|},$$

$$\tag{7.4} \tfrac{1}{2}\Phi_{\xi\xi} + \theta\tanh(\theta\xi)\Phi_\xi < 0, \qquad \operatorname{sgn}[\Phi_{y\xi} + \theta\tanh(\theta\xi)\Phi_y] = \operatorname{sgn}(y\xi)$$

of (3.5), (5.7) and (5.4) [for (7.3), recall Remark 3.5(ii)]. Due to (7.4), (7.2) can be rewritten as

$$\tag{7.5} \begin{aligned} &\tfrac{1}{2}\Phi_{yy} + \min_{|u|\le 1}\Big[u\{\Phi_{y\xi} + \theta\tanh(\theta\xi)\Phi_y\} \\ &\qquad + u^2\{\tfrac{1}{2}\Phi_{\xi\xi} + \theta\tanh(\theta\xi)\Phi_\xi\}\Big] + c(y) = \alpha\Phi, \end{aligned}$$

the formal *Hamilton–Jacobi–Bellman* (*HJB*) *equation* corresponding to the problem of minimizing the expected discounted cost (6.4), subject to the dynamics (6.19) and (6.20). The minimization in (7.5) is achieved by

$$\begin{aligned} u^* &= -\operatorname{sgn}[\Phi_{y\xi} + \theta\tanh(\theta\xi)\Phi_y] \\ &= -\operatorname{sgn}(y\xi), \end{aligned}$$

again suggesting the process (6.17) is optimal.

THEOREM 7.1 [2]. *For the Bernoulli case* (6.9), *we have*

$$\tag{7.6} J(u;\theta) \ge J(u^*;\theta) = \Phi(y,\xi), \qquad \forall\, u \in \mathscr{U},$$

for any given $y \in \mathscr{R}$, *where* ξ *is given by* (6.13), u^* *is the control process of* (6.17) *and* $c(\cdot)$ *is an even convex function of class* $C^2(\mathscr{R})$ *with* $c'(y) > 0$ *for* $0 < y < \infty$ *satisfying the growth condition* (5.2).

PROOF. From (6.11) and $\lambda = \alpha + \theta^2/2$, we have

$$J(u^*;\theta) = \frac{1}{\cosh(\theta\xi)} E\int_0^\infty e^{-(\alpha+\theta^2/2)t} c(Y_t)\cosh(\theta Z_t)\,dt = \frac{V(y,\xi)}{\cosh(\theta\xi)},$$

and so the equality in (7.6) is obvious from Theorem 3.4(ii). Now for an arbitrary $u \in \mathscr{U}$ we apply Itô's rule to $e^{-\alpha t}\Phi(Y_t, \xi_t^u)$ and obtain, in conjunction with (6.19) and (6.20),

$$\tag{7.7} \begin{aligned} &e^{-\alpha(T\wedge\tau_n)}\Phi\big(Y_{T\wedge\tau_n}, \xi^u_{T\wedge\tau_n}\big) + \int_0^{T\wedge\tau_n} e^{-\alpha t} c(Y_t)\,dt \\ &\qquad = \Phi(y,\xi) + \int_0^{T\wedge\tau_n} \beta_t^u\,dt \\ &\qquad\quad + \int_0^{T\wedge\tau_n} e^{-\alpha t}[\Phi_y(Y_t,\xi_t^u) + u_t\Phi_\xi(Y_t,\xi_t^u)]\,d\nu_t^u, \quad \text{a.s.}, \end{aligned}$$

where $T > 0$ is a constant,

$$\tau_n \triangleq \inf\{t \in [0,\infty),\ |Y_t| \ge n \text{ or } |\xi_t^u| \ge n\}, \qquad n = 1,2,\dots,$$

and the process

$$\begin{aligned}\beta_t^u \triangleq\ & \tfrac{1}{2}\Phi_{yy}(Y_t,\xi_t^u) + c(Y_t) - \alpha\Phi(Y_t,\xi_t^u)\\ &+ u_t\left[\Phi_{y\xi}(Y_t,\xi_t^u) + \theta\tanh(\theta\xi_t^u)\Phi_y(Y_t,\xi_t^u)\right]\\ &+ u_t^2\left[\tfrac{1}{2}\Phi_{\xi\xi}(Y_t,\xi_t^u) + \theta\tanh(\theta\xi_t^u)\Phi_\xi(Y_t,\xi_t^u)\right]\end{aligned}$$

is nonnegative, by virtue of (7.5). If we now take expectations in (7.7) with respect to P_T^u, that of the stochastic integral is 0, and we obtain

$$\Phi(y,\xi) \le E_T^u\int_0^{T\wedge\tau_n} e^{-\alpha t}c(Y_t)\,dt + E_T^u\left[e^{-\alpha(T\wedge\tau_n)}\Phi\left(Y_{T\wedge\tau_n},\xi_{T\wedge\tau_n}^u\right)\right], \quad \forall\, n \in \mathbb{N}. \tag{7.8}$$

Suppose that the cost function $c(\cdot)$ satisfies, instead of (5.2), a polynomial growth condition of the type $0 \le c(y) \le K(1+|y|^\nu)$, $\forall\, y \in \mathscr{R}$ for some $\nu > 0$. Then $\Phi(y,\xi)$ satisfies a similar growth condition [instead of just (7.3)], and we have

$$\begin{aligned}0 &\le E_T^u\left[e^{-\alpha(T\wedge\tau_n)}\Phi\left(Y_{T\wedge\tau_n},\xi_{T\wedge\tau_n}^u\right)\right] \le a_n(T)\\ &\triangleq KE_T^u\left[e^{-\alpha(T\wedge\tau_n)}\left(1+|y|+\theta T+\max_{0\le t\le T}|W_t^u|\right)^\nu\right].\end{aligned}$$

Thus, letting $n \to \infty$ in (7.8), we obtain

$$\begin{aligned}\Phi(y,\xi) &\le E_T^u\int_0^T e^{-\alpha t}c(Y_t)\,dt + a_\infty(T)\\ &= \int_0^T e^{-\alpha t}E_t^u c(Y_t)\,dt + a_\infty(T).\end{aligned} \tag{7.9}$$

Finally, letting $T \to \infty$ in (7.9) and observing that $\lim_{T\to\infty} a_\infty(T) = 0$, we conclude

$$\Phi(y,\xi) = \int_0^\infty e^{-\alpha t}E_t^* c(Y_t)\,dt \le \int_0^\infty e^{-\alpha t}E_t^u c(Y_t)\,dt, \qquad \forall\, u \in \mathscr{U}, \tag{7.10}$$

which proves (7.6) in this case.

For a general $c(\cdot)$ as in Section 5, introduce the sequence $\{c_k(\cdot)\}_{k\in\mathbb{N}}$ of functions

$$c_k(y) \triangleq \begin{cases} c(y), & 0 \le y \le k,\\ c(k) + c'(k)(y-k), & y > k,\\ c_k(-y), & y < 0.\end{cases}$$

These functions are even, convex, strictly increasing on $(0,\infty)$, of class

$C^1(\mathscr{R}) \cap C^2(\mathscr{R} \setminus \{-k, k\})$ and increase monotonically to $c(\cdot)$ as $k \nearrow \infty$. Recalling Remark 5.3(ii) and denoting $Q_k \equiv Q_0^{c_k}$ as in (3.19), we obtain the analogue

$$\Phi_k(y,\xi) \triangleq \frac{Q_k(y,\xi)}{\cosh(\theta\xi)} = \int_0^\infty e^{-\alpha t} E_t^* c_k(Y_t)\, dt \le \int_0^\infty e^{-\alpha t} E_t^u c_k(Y_t)\, dt, \qquad \forall\, k \in \mathbb{N},$$

of (7.10), for every $u \in \mathscr{U}$. Now (7.6) follows by letting $k \nearrow \infty$ and using the monotone convergence theorem. □

REMARK 7.2. The assumptions on $c(\cdot)$ in Theorem 7.1 can be weakened. In fact, in the Bernoulli case (6.9), we have

$$J(u;\theta) \ge J(u^*;\theta), \qquad \forall\, u \in \mathscr{U}, \tag{7.6'}$$

for any even, convex function $c(\cdot)$ satisfying an exponential growth condition at infinity.

Indeed, (7.6′) can be established first for piecewise-linear, even, convex functions $c(\cdot)$ with a finite number of linear segments; this can be done by smoothing the corners of $c(\cdot)$ to obtain a sequence of functions $\{c_n(\cdot)\}_{n \in \mathbb{N}}$ that satisfy the conditions of Theorem 7.1 and decrease pointwise to $c(\cdot)$, and then taking limits and appealing to the dominated convergence theorem. Any even, convex function $c(\cdot)$ that satisfies an exponential growth condition at ∞ is the pointwise limit of an increasing sequence of even, convex, piecewise-linear functions; this may be used to complete the proof of (7.6′) in the general case.

The control u^* of (6.17) does *not* belong to $\mathscr{U}_s$, the class of strictly admissible control processes [recall (2.18)]. It can be shown, as in [2], that

$$\inf_{u \in \mathscr{U}_s} J(u;\theta) = \inf_{u \in \mathscr{U}} J(u;\theta) = J(u^*;\theta) \tag{7.11}$$

and that

(7.12) no control process in $\mathscr{U}_s$ can be optimal.

For a general *symmetric* distribution μ on the random variable z, the function $F(t,x)$ of (6.6) and the expected discounted cost $J(u)$ of (6.5) become, respectively,

$$F(t,x) = 2\int_0^\infty e^{-\theta^2 t/2} \cosh(\theta x) \mu(d\theta), \qquad J(u) = 2\int_0^\infty J(u;\theta)\mu(d\theta). \tag{7.13}$$

It follows then from Theorem 7.1 that

$$J(u) \ge J(u^*) = 2\int_0^\infty J(u^*;\theta)\mu(d\theta) \tag{7.14}$$

holds for any $u \in \mathscr{U}$, thus proving the optimality of u^* in this class.

8. Semigroup and ramifications. In this section we construct the Markov semigroup associated with the diffusion process $(Y^{y,\xi}, Z^{y,\xi})$ of Problem 2.1 and Theorem 2.3, for $(y,\xi) \ne \mathbf{0}$. We then use the fact that the resolvent $V^g(y,\xi)$ in (3.1) of this semigroup is *continuous* at $(y,\xi) = \mathbf{0}$ for continuous functions $g\colon \mathscr{R}^2 \to \mathscr{R}$ [a consequence of (3.27) and of the explicit formulas (3.19)–(3.23) for Q^g], to extend the semigroup to $(y,\xi) = \mathbf{0}$. This methodology allows us to produce a weak solution $(Y^{0,0}, Z^{0,0})$ to Problem 2.1 for $(y,\xi) = \mathbf{0}$.

Let $C(\overline{\mathscr{R}}^2)$ denote the space of continuous functions $\phi\colon \mathscr{R}^2 \to \mathscr{R}$ which admit a limit $\phi_\infty \triangleq \lim_{(y,\xi)\to\infty} \phi(y,\xi)$ at ∞, and define

$$\hat{C}(\mathscr{R}^2) \triangleq \{\phi \in C(\overline{\mathscr{R}}^2),\ \phi_\infty = 0\}. \tag{8.1}$$

Both $C(\overline{\mathscr{R}}^2)$ and $\hat{C}(\mathscr{R}^2)$ are Banach spaces under the sup-norm $\|\phi\|$. *We wish to associate to the operator L of* (3.6) *a strongly continuous, positive, conservative, contraction* (*i.e., Feller*) *semigroup* $(T_t)_{t\ge 0}$, acting on continuous functions $g\colon \mathscr{R}^2 \to \mathscr{R}$. See [5], Chapter 1, for definitions of the semigroup concepts used in this section.

The natural candidate for this semigroup is given as

$$(T_t g)(y,\xi) \triangleq Eg\left(Y_t^{y,\xi}, Z_t^{y,\xi}\right), \tag{8.2}$$

but this is well defined only for $(y,\xi) \neq \mathbf{0}$. *We intend to show that this definition extends to* $\mathbf{0}$ *and that, in fact,* $(T_t)_{t\ge 0}$ *acts on* $C(\overline{\mathscr{R}}^2)$. This latter space is a likely candidate on which to define $(T_t)_{t\ge 0}$, because $V^g \in C(\overline{\mathscr{R}}^2)$ for every $g \in C(\overline{\mathscr{R}}^2)$; this can be checked easily from the explicit representation of Theorem 3.4.

Let us recall the notation $\mathscr{D}$ of (3.7), and introduce the space

$$\mathscr{D}_L \triangleq \left\{\phi \in \mathscr{D} \cap \hat{C}(\mathscr{R}^2),\ L_\phi \in \hat{C}(\mathscr{R}^2)\right\}. \tag{8.3}$$

THEOREM 8.1. *The linear operator* $L\colon \mathscr{D}_L \to \hat{C}(\mathscr{R}^2)$ *of* (3.6) *is closable; its closure* $\overline{L}$ *is conservative, and generates a strongly continuous, positive, contraction semigroup* $(T_t)_{t\ge 0}$ *on* $\hat{C}(\mathscr{R}^2)$. *This extends to a Feller semigroup* $(T_t)_{t\ge 0}$ *on* $C(\overline{R}^2)$ *by the definition* $T_t f = f_\infty + T_t(f - f_\infty)$, *for* $f \in C(\overline{\mathscr{R}}^2)$. *If* $g \in C(\overline{\mathscr{R}}^2)$ *and* $(y,\xi) \neq \mathbf{0}$, *then* (8.2) *holds.*

In proving Theorem 8.1 we shall use heavily the fact that $Q^g(y,\xi)$, our candidate for $(\lambda I - \overline{L})^{-1}g$, $g \in \hat{C}(\mathscr{R}^2)$, is represented as in (3.27) on $\mathscr{R}^2 \setminus \{\mathbf{0}\}$, and that $Q^g(\cdot)$ is *continuous on* $\mathscr{R}^2$ (in particular, at $\mathbf{0}$) for every $g \in \hat{C}(\mathscr{R}^2)$, by virtue of its explicit representation (3.19)–(3.23). We shall deal first with the closability of L.

LEMMA 8.2. *The closure* $\overline{L}$ *of* L (*as in Theorem* 8.1) *exists, and generates a strongly continuous, contraction semigroup* $(T_t)_{t\ge 0}$ *on* $\hat{C}(\mathscr{R}^2)$.

PROOF. By the Hille–Yosida theorem (cf. [5], Chapter 1, Theorem 2.12), it suffices to check that:

(i) Range$(\lambda I - L)$ is dense in $\hat{C}(\mathscr{R}^2)$, for some $\lambda > 0$.
(ii) L is dissipative, that is, $\|(\lambda I - L)\phi\| \ge \lambda\|\phi\|$, $\forall\, \phi \in \mathscr{D}_L$, $\lambda > 0$.
(iii) $\mathscr{D}_L$ is dense in $\hat{C}(\mathscr{R}^2)$.

To prove (i), for any given $g \in \hat{C}(\mathscr{R}^2) \cap C^2(\mathscr{R}^2)$, $\lambda > 0$ denote by $Q^g \in \mathscr{D}$ the solution to (3.5) given by Theorem 3.4. It is easily seen that $Q^g \in \hat{C}(\mathscr{R}^2)$. Thus, $Q^g \in \mathscr{D}_L$ and $(\lambda I - L)Q^g = g$, whence $\hat{C}(\mathscr{R}^2) \cap C^2(\mathscr{R}^2) \subseteq$ Range$(\lambda I - L)$. Therefore, Range$(\lambda I - L)$ is dense in $\hat{C}(\mathscr{R}^2)$.

For (ii), let us take arbitrary $\phi \in \mathscr{D}_L$, $\lambda > 0$ and set $g \triangleq (\lambda I - L)\phi$. It follows from (8.3) that $g \in \hat{C}(\mathscr{R}^2)$, and from Proposition 3.2,

$$\phi(y,\xi) = E\int_0^\infty e^{-\lambda t} g\left(Y_t^{y,\xi}, Z_t^{y,\xi}\right) dt \quad \text{on } \mathscr{R}^2 \setminus \{\mathbf{0}\}.$$

From this, and the continuity of ϕ, we obtain

$$\|\phi\| \le \frac{1}{\lambda}\|g\| = \frac{1}{\lambda}\|(\lambda I - L)\phi\|.$$

Finally, in order to show (iii), it suffices to prove that $\mathscr{D}_L$ is dense in $C_0^2(\mathscr{R}^2)$. With an arbitrary $\phi \in C_0^2(\mathscr{R}^2)$, we create a sequence $\{f_n^1\}_{n=1}^\infty$ of C^2-functions $f_n^1\colon \mathscr{R} \to \mathscr{R}$ such that

$$\| f_n^1 - \phi(\cdot, 0)\| \xrightarrow[n\uparrow\infty]{} 0 \quad \text{and} \quad f_n^1(y) = \begin{cases} \phi(0,0), & |y| \le 1/n, \\ \phi(y,0), & |y| \ge 2/n; \end{cases}$$

similarly, we choose C^2-functions $f_n^2\colon \mathscr{R} \to \mathscr{R}$ with

$$\| f_n^2 - \phi(0, \cdot)\| \xrightarrow[n\uparrow\infty]{} 0 \quad \text{and} \quad f_n^1(\xi) = \begin{cases} \phi(0,0), & |\xi| \le 1/n, \\ \phi(0,\xi), & |\xi| \ge 2/n. \end{cases}$$

With these functions, we define

$$\psi_n(y,\xi) \triangleq \begin{cases} \phi(0,0), & |y| \le 1/n,\ |\xi| \le 1/n, \\ f_n^1(y), & |y| \le 1/n,\ |\xi| > 1/n, \\ f_n^2(\xi), & |y| > 1/n,\ |\xi| \le 1/n, \\ 0, & \text{otherwise}. \end{cases}$$

Also, for any given $n \ge 1$, let $\rho_n\colon \mathscr{R}^2 \to [0,1]$ be a C^∞-function with

$$\rho_n(y,\xi) = \begin{cases} 0, & |y| \ge 2/n \text{ and } |\xi| \ge 2/n, \\ 1, & |y| \le 1/n \text{ or } |\xi| \le 1/n. \end{cases}$$

Then $\phi_n \triangleq (1-\rho_n)\phi + \rho_n\psi_n$ belongs to $C_0^2(\mathscr{R}^2)$ and satisfies $(\phi_n)_{y\xi} = 0$ on the axes. It follows that $\phi_n \in \mathscr{D}_L$ and $\|\phi_n - \phi\| \le \|\rho_n(\psi_n - \phi_n)\| \le \sup\{|\phi(y,\xi) - \psi_n(y,\xi)|;\ |y| \le 1/n \text{ or } |\xi| \le 1/n\} \to_{n\to\infty} 0$, from the uniform continuity of ϕ. □

Proof of Theorem 8.1. First, we have to show that the semigroup $(T_t)_{t\ge 0}$ of Lemma 8.2 is *positive* and *conservative*. The continuity of the map $g \mapsto Q^g$ (with respect to the sup-norm topologies) gives $(\lambda I - \bar{L})^{-1}g = Q^g$, $\forall\, g \in \hat{C}(\mathscr{R}^2)$. Because $Q^g = (\lambda I - \bar{L})^{-1}g \ge 0$ if $g \ge 0$, it follows that $(T_t)_{t\ge 0}$ is positive. To prove that $(T_t)_{t\ge 0}$ is conservative, let $\phi_n(y,\xi) \triangleq r(|y|^2 + (1/n)|\xi|^2)$, where $r\colon \mathscr{R}_+ \to [0,1]$ is a C^2-function with

$$r(s) = \begin{Bmatrix} 1, & 0 \le s \le 1 \\ 0, & s \ge 2 \end{Bmatrix},$$

and check that bp-$\lim_{n\to\infty}(\phi_n, L\phi_n) = (1,0)$.

Second, we note that the extension of $\{T_t\}_{t\ge 0}$ from $\hat{C}(\mathscr{R}^2)$ to $C(\overline{\mathscr{R}}^2)$ is standard (cf. [5], Chapter 4, Lemma 2.3). Finally, for any $g \in C(\overline{\mathscr{R}}^2)$, $\lambda > 0$, we have

$$E\int_0^\infty e^{-\lambda t} T_t g(y,\xi)\,dt = Q^g(y,\xi), \qquad \forall\,(y,\xi) \in \mathscr{R}^2,$$

and from Corollary 3.7,

$$\int_0^\infty e^{-\lambda T} T_t g(y,\xi)\,dt = \int_0^\infty e^{-\lambda t} Eg\big(Y_t^{y,\xi}, Z_t^{y,\xi}\big)\,dt, \qquad \forall\,(y,\xi) \in \mathscr{R}^2 \setminus \{\mathbf{0}\};$$

thus, (8.2) follows for every $g \in C(\overline{\mathscr{R}}^2)$, $(y,\xi) \ne \mathbf{0}$. □

Let $(T_t)_{t\ge 0}$ be the semigroup of Theorem 8.1; there exists a strong Markov process $\{\tilde{Y}_t^{y,\xi}, \tilde{Z}_t^{y,\xi}),\ t \ge 0\}$ corresponding to $(T_t)_{t\ge 0}$, with sample paths in $D([0,\infty); \mathscr{R}^2)$ and with $(\tilde{Y}_0^{y,\xi}, \tilde{Z}_0^{y,\xi}) = (y,\xi)$, $\forall\,(y,\xi) \in \mathscr{R}^2$ (cf. [5], Chapter 4, Theorem 2.7). We may, and shall, assume that $\{(\tilde{Y}_t^{y,\xi}, \tilde{Z}_t^{y,\xi}),\ t \ge 0\}$ is the coordinate process on the canonical space $D([0,\infty); \mathscr{R}^2)$ provided with the measure $P_{y,\xi}$; then, in fact, this process is strong Markov with respect to $\{\mathscr{G}_{t+}\}$, where $\mathscr{G}_t = \sigma((Y_s(\omega), Z_s(\omega)),\ 0 \le s \le t)$ is the canonical filtration.

We should like to show that $P_{y,\xi}$ is supported by $C([0,\infty); \mathscr{R}^2)$, for every $(y,\xi) \in \mathscr{R}^2$. This is obvious for $(y,\xi) \ne \mathbf{0}$, for then $(\tilde{Y}^{y,\xi}, \tilde{Z}^{y,\xi})$ is equal in law to $(Y^{y,\xi}, Z^{y,\xi})$. We need only worry about the case $(y,\xi) \ne \mathbf{0}$. But notice that for $y \ne 0$, $f \in C(\overline{\mathscr{R}})$ and $g_f(y,\xi) \equiv f(y)$, we have

$$E\int_0^\infty e^{-\lambda t} f(y + W_t)\,dt = E\int_0^\infty e^{-\lambda t} f\big(\tilde{Y}_t^{y,\xi}\big)\,dt = Q^{g_f}(y,\xi), \qquad \forall\,\lambda > 0,$$

where W is Brownian motion, and because Q^{g_f} is continuous:

$$E\int_0^\infty e^{-\lambda t} f(W_t)\,dt = E\int_0^\infty e^{-\lambda t} f\big(\tilde{Y}_t^{0,0}\big)\,dt, \qquad \forall\,\lambda > 0.$$

It develops that for each $t > 0$, $\tilde{Y}_t^{0,0}$ is a normal random variable with mean 0 and variance t; similarly for $\tilde{Z}_t^{0,0}$. Therefore, $\lim_{t\downarrow 0}(1/t)P[|\tilde{Y}_t^{0,0}| \ge \varepsilon, |\tilde{Z}_t^{0,0}| \ge \varepsilon] = 0$, $\forall\ \varepsilon > 0$, which implies that $P_{0,0}$ is also supported by $C([0,\infty); \mathscr{R}^2)$ (cf. [5], Chapter 4, Proposition 2.9). In other words, $\{(\tilde{Y}_t^{0,0}, \tilde{Z}_t^{0,0}),\ t \ge 0\}$ admits a version with continuous sample paths.

Proof of Proposition 3.7. By [9], Proposition 5.4.6, or [5], Chapter 5 Proposition 3.1, in order to prove existence it suffices to show that

$(\tilde{Y}_t^{0,0}, \tilde{Z}_t^{0,0})_{t\geq 0}$ is a solution to the following *martingale problem*:

(8.4)
$$M_t^f = f(\tilde{Y}_t^{0,0}, \tilde{Z}_t^{0,0}) - f(0,0) - \int_0^t Lf(\tilde{Y}_s^{0,0}, \tilde{Z}_s^{0,0})\,ds,\ \mathscr{F}_t,\quad 0 \leq t < \infty,$$
is a martingale for every $f \in C_0^\infty$ with $\mathscr{F}_t = \mathscr{G}_{t+}$, where $\{\mathscr{G}_t\}$ is the augmentation under $P_{0,0}$ of the canonical filtration on $C([0,\ \infty); \mathscr{R}^2)$.

Let $f \in C_0^\infty(\mathscr{R}^2)$ and take $0 < s < t < \infty$. Then, using the Markov property of $(\tilde{Y}^{0,0}, \tilde{Z}^{0,0})$, we have

(8.5)
$$\begin{aligned} E[M_t^f | \mathscr{F}_s] &= -f(0,0) - \int_0^s Lf\big(\tilde{Y}_u^{0,0}, \tilde{Z}_u^{0,0}\big)\,du \\ &\quad + E\left[f\big(\tilde{Y}_t^{0,0}, \tilde{Z}_t^{0,0}\big) - \int_s^t Lf\big(\tilde{Y}_u^{0,0}, \tilde{Z}_u^{0,0}\big)\,du \,\middle|\, \tilde{Y}_s^{0,0}, \tilde{Z}_s^{0,0} \right] \\ &= E\left[f\big(\tilde{Y}_{t-s}^{y,\xi}, \tilde{Z}_{t-s}^{y,\xi}\big) - \int_0^{t-s} Lf\big(\tilde{Y}_u^{y,\xi}, \tilde{Z}_u^{y,\xi}\big)\,du \right]\Bigg|_{(y,\xi)=(\tilde{Y}_s^{0,0}, \tilde{Z}_s^{0,0})} \\ &\quad - f(0,0) - \int_0^s Lf\big(\tilde{Y}_u^{0,0}, \tilde{Z}_u^{0,0}\big)\,du. \end{aligned}$$

Now, when $(y, \xi) \neq \mathbf{0}$, we know by Itô's rule that

$$E\left[f\big(\tilde{Y}_{t-s}^{y,\xi}, \tilde{Z}_{t-s}^{y,\xi}\big) - \int_0^{t-s} Lf\big(\tilde{Y}_u^{y,\xi}, \tilde{Z}_u^{y,\xi}\big)\,du \right] = f(y, \xi),$$

because $(\tilde{Y}^{y,\xi}, \tilde{Z}^{y,\xi})$ may be replaced by $(Y^{y,\xi}, Z^{y,\xi})$.

We also have $\mathbb{P}[(\tilde{Y}_s^{0,0}, \tilde{Z}_s^{0,0}) = (0,0)] = 0$ for any $s > 0$, because we already know that $\tilde{Y}_s^{0,0}$ and $\tilde{Z}_s^{0,0}$ are normal with mean 0 and variance s. Thus, from (8.5) we learn that

$$E[M_t^f | \mathscr{F}_s] = f\big(\tilde{Y}_s^{0,0}, \tilde{Z}_s^{0,0}\big) - f(0,0)) - \int_0^s Lf\big(\tilde{Y}_u^{0,0}, \tilde{Z}_u^{0,0}\big)\,du = M_s^f, \quad \text{a.s.},$$

if $0 < s < t < \infty$. By letting $s \downarrow 0$ and exploiting the continuity of f and $(\tilde{Y}^{0,0}, \tilde{Z}^{0,0})$, we obtain $E[M_t^f | \mathscr{F}_0] = 0 = M_0^f$. Thus, $\{M_t^f, \mathscr{F}_t,\ 0 \leq t < \infty\}$ is indeed a martingale, and the existence proof is complete.

To prove uniqueness, let $(\tilde{Y}^{0,0}, \tilde{Z}^{0,0})$ be any solution to the martingale problem for $(y, \xi) = \mathbf{0}$. From (8.4),

$$\left\{ h\big(\tilde{Y}_t^{0,0}\big) - h(0) - \frac{1}{2}\int_0^t h''\big(\tilde{Y}_s^{0,0}\big)\,ds, \mathscr{F}_t; 0 \leq t < \infty \right\}$$

is a martingale for every $h \in C_0^\infty(\mathscr{R})$; therefore, $\tilde{Y}^{0,0}$ and $\tilde{Z}^{0,0}$ are $\{\mathscr{F}_t\}$-Brownian motions, and $(\tilde{Y}^{0,0}, \tilde{Z}^{0,0})$ spends zero time at the origin. Now consider an arbitrary bounded, measurable function $g\colon \mathscr{R}^2 \to \mathscr{R}$, and notice that for any $s > 0$, $\lambda > 0$, we have

$$\begin{aligned}
&E\int_0^\infty e^{-\lambda t} g\left(\tilde{Y}_t^{0,0}, \tilde{Z}_t^{0,0}\right) dt - E\int_0^s e^{-\lambda t} g\left(\tilde{Y}_t^{0,0}, \tilde{Z}_t^{0,0}\right) dt \\
&\quad = E\int_s^\infty e^{-\lambda t} g\left(\tilde{Y}_t^{0,0}, \tilde{Z}_t^{0,0}\right) dt \\
&\quad = E\left[e^{-\lambda s}\int_0^\infty e^{-\lambda u} g\left(\tilde{Y}_{s+u}^{0,0}, \tilde{Z}_{s+u}^{0,0}\right) du\right] \\
&\quad = e^{-\lambda s} E\left[E\int_0^\infty e^{-\lambda u} g\left(\tilde{Y}_u^{y,\xi}, \tilde{Z}_u^{y,\xi}\right) du\bigg|_{y=\tilde{Y}_s^{0,0},\ \xi=\tilde{Z}_s^{0,0}}\right] \\
&\quad = e^{-\lambda s} E\left[E\int_0^\infty e^{-\lambda u} g\left(Y_u^{y,\xi}, Z_u^{y,\xi}\right) du\bigg|_{y=\tilde{Y}_s^{0,0},\ \xi=\tilde{Z}_s^{0,0}}\right] \\
&\quad = e^{-\lambda s} E Q^g\left(\tilde{Y}_s^{0,0}, \tilde{Z}_s^{0,0}\right);
\end{aligned}$$

we have used Lemma 5.4.19 in [9], the uniqueness-in-law for the martingale problem starting away from the origin, and (3.27). Now let $s \downarrow 0$ in

$$E\int_0^\infty e^{-\lambda t} g\left(\tilde{Y}_t^{0,0}, \tilde{Z}_t^{0,0}\right) dt = E\int_0^s e^{-\lambda t} g\left(\tilde{Y}_t^{0,0}, \tilde{Z}_t^{0,0}\right) dt + e^{-\lambda s} E Q^g\left(\tilde{Y}_s^{0,0}, \tilde{Z}_s^{0,0}\right);$$

the continuity of $(\tilde{Y}^{0,0}, \tilde{Z}^{0,0})$ and the continuity at the origin of the function Q^g for any bounded, measurable $g\colon \mathscr{R}^2 \to \mathscr{R}$ (cf. Corollary 3.6) lead to

$$E\int_0^\infty e^{-\lambda t} g\left(\tilde{Y}_t^{0,0}, \tilde{Z}_t^{0,0}\right) dt = Q^g(0,0), \qquad \forall\, \lambda > 0.$$

In other words, the distribution of $(\tilde{Y}_t^{0,0}, \tilde{Z}_t^{0,0})$ is uniquely determined, for every $t \in [0,\infty)$; from Proposition 5.4.27 in [9], it follows that uniqueness-in-law holds for the martingale problem, and thus also for Problem 2.1, corresponding to $(y, \xi) = \mathbf{0}$. □

APPENDIX

9. Proofs of selected results. In this section we give the proofs of Propositions 4.5, 5.1 and 5.2.

PROOF OF PROPOSITION 4.5. For simplicity of notation, we suppress in this proof the subscript from Q_0. Recall from the proof of Corollary 3.6 [in particular, (3.31)] that $M^g, N^g \in C[0,\infty)$. Next, from (4.22) we obtain, for $s > 0$,

$$\begin{aligned}
Q_y(s, 0+) = (N^g)'(s) = \sqrt{2\lambda}\left[\frac{\cosh(s\sqrt{2\lambda})N^g(s) - M^g(s)}{\sinh(s\sqrt{2\lambda})}\right] \\
- \frac{G^g(s)}{\sinh(s\sqrt{2\lambda})}.
\end{aligned} \tag{9.1}$$

Letting $s \downarrow 0$ and using (3.29) and (3.30), we obtain $\lim_{s \downarrow 0} Q_y(s, 0+) = 0$. We prove the other statement of (4.27), $\lim_{s \downarrow 0} Q_\xi(0+, s) = 0$, in a similar manner.

To establish (4.28) and (4.29), we first use (4.25) to observe that

$$\begin{aligned}
\lim_{s \downarrow 0} & \frac{M^g(s) - N^g(s)}{s \sinh(s\sqrt{2\lambda})} \\
&= \lim_{s \downarrow 0} \frac{1}{s} \tanh\left(\frac{s}{2}\sqrt{2\lambda}\right) \int_s^\infty \frac{F^g(u) - G^g(u)}{2\sinh(u\sqrt{2\lambda})\sinh^2\left(\frac{u}{2}\sqrt{2\lambda}\right)} du \\
&= \sqrt{\frac{\lambda}{2}} \int_0^\infty \frac{F^g(u) - G^g(u)}{2\sinh(u\sqrt{2\lambda})\sinh^2\left(\frac{u}{2}\sqrt{2\lambda}\right)} du \qquad (9.2) \\
&= \sqrt{2\lambda} \int_0^\infty \frac{[F^g(u) - G^g(u)]\cosh^2\left(\frac{u}{2}\sqrt{2\lambda}\right)}{\sinh^3(u\sqrt{2\lambda})} du.
\end{aligned}$$

It follows from (4.21) and some calculation that

$$\begin{aligned}
[M^g]''(0) &= \lim_{s \downarrow 0} \frac{[M^g]'(s)}{s} \\
&= \lim_{s \downarrow 0} \left[\sqrt{2\lambda}\, \frac{(\cosh(s\sqrt{2\lambda}) - 1)M^g(s) + M^g(s) - N^g(s)}{s\sinh(s\sqrt{2\lambda})} \right. \qquad (9.3) \\
&\qquad \left. - \frac{F^g(s)}{s\sinh(s\sqrt{2\lambda})} \right] \\
&= 2\lambda \int_0^\infty \frac{F^g(u)\cosh(u\sqrt{2\lambda}) - G^g(u)}{\sinh^3(u\sqrt{2\lambda})} du - g(0,0).
\end{aligned}$$

Likewise,

$$\begin{aligned}
[N^g]''(0) &= \lim_{s \downarrow 0} \frac{[N^g]'(s)}{s} \qquad (9.4) \\
&= 2\lambda \int_0^\infty \frac{G^g(u)\cosh(u\sqrt{2\lambda}) - F^g(u)}{\sinh^3(u\sqrt{2\lambda})} du - g(0,0).
\end{aligned}$$

Then, using (4.21), (4.22), (9.3) and (9.4), it is not difficult to establish that $\lim_{y \downarrow 0} Q_{yy}(y, 0+) = \lim_{y \downarrow 0}[N^g]'' = [N^g]''(0)$ and $\lim_{\xi \downarrow 0} Q_{\xi\xi}(0+, \xi) = \lim_{\xi \downarrow 0}[M^g]''(\xi) = [M^g]''(0)$, so that $M^g, N^g \in C^2[0, \infty)$. On the other hand, the identity $Q_{y\xi}(0+, \xi) \equiv 0$ and (4.14) yield

$$\begin{aligned}
Q_{\xi\xi}(y, 0+) &= 2\lambda N^g(y) - (N^g)''(y) - 2g(y,0) \\
&\xrightarrow[y \downarrow 0]{} 2\lambda N^g(0) - [N^g]''(0) - 2g(0,0) = [M^g]''(0),
\end{aligned}$$

which completes the proof of (4.29). The proof of (4.28) is completed by a similar argument.

Finally, we wish to show that $Q \in \mathscr{D}$. It suffices to show that Q and its first and second derivatives extend continuously from Γ_1 to its boundary $\partial\Gamma_1 = \{(0,\xi),\ \xi \geq 0\} \cup \{(y,0),\ y \geq 0\}$. In fact, it is easily seen that Q and its derivatives up to order 2 extend continuously to $\partial\Gamma_1 \setminus \{\mathbf{0}\}$. Checking continuity at the origin is the only part that requires some work. We shall use the following lemma, which will be useful later as well.

LEMMA 9.1. (a) *Assume that* $h \in C(\overline{\Gamma}_1)$ *and consider two functions* $M, N \in C[0,\infty)$, *with* $M(0) = N(0)$ (*not necessarily equal to* M^h *and* N^h, *respectively*). *Define*

$$U(y,\xi) \triangleq J^h(y,\xi) + \frac{\sinh(\xi\sqrt{2\lambda})}{\sinh\left[(y+\xi)\sqrt{2\lambda}\right]} M(y+\xi) + \frac{\sinh(y\sqrt{2\lambda})}{\sinh\left[(y+\xi)\sqrt{2\lambda}\right]} N(y+\xi) \tag{9.5}$$

for $(y,\xi) \in \overline{\Gamma}_1 \setminus \{\mathbf{0}\}$. *Then* $\lim_{(y,\xi)\to\mathbf{0}} U(y,\xi) = M(0)$. *Hence, U extends continuously to* $\overline{\Gamma}_1$.

(b) *Assume in* (4.34) *that* $h \in C^1(\overline{\Gamma}_1 \setminus \{\mathbf{0}\})$ *and that* $M, N \in C^1(0,\infty)$. *Then* $U \in C^1(\overline{\Gamma}_1 \setminus \{\mathbf{0}\})$ *and, for* $(y,\xi) \in (\overline{\Gamma}_1 \setminus \{\mathbf{0}\})$,

$$U_y(y,\xi) = J^{h_y}(y,\xi) + \frac{\sinh(\xi\sqrt{2\lambda})}{\sinh((y+\xi)\sqrt{2\lambda})} U_y(0+, y+\xi) + \frac{\sinh(y\sqrt{2\lambda})}{\sinh((y+\xi)\sqrt{2\lambda})} U_y(y+\xi, 0+). \tag{9.6}$$

A similar formula holds for $U_\xi(y,\xi)$.

PROOF. It is easy to see that $\lim_{(y,\xi)\to\mathbf{0}} J^h(y,\xi) = 0$. Thus, part (a) follows easily, because $M(0) = N(0)$. On the other hand, (9.6) is intuitively clear from (4.3) because, in the case of sufficient higher-order differentiability,

$$\tfrac{1}{2}\left[(U_y)_{yy} + (U_y)_{\xi\xi}\right] - (U_y)_{y\xi} + h_y = \lambda U_y \quad \text{in } \Gamma_1.$$

For the rigorous proof, one computes $U_y(y,\xi)$ explicitly using the C^1-assumptions on h, M and N, and simply checks (9.6). □

To complete the proof of Proposition 4.5, first observe that, because $M^g(0) = N^g(0)$ [see (3.30)], Lemma 9.1(a) implies $Q \in C(\overline{\Gamma}_1)$. By (9.6), (4.17)(ii) and the identity $Q_y(y, 0+) = [N^g]'(y)$, we have

$$Q_y(y,\xi) = J^{g_y}(y,\xi) + \frac{\sinh(y\sqrt{2\lambda})}{\sinh((y+\xi)\sqrt{2\lambda})}[N^g]'(y+\xi), \qquad (y,\xi) \in \overline{\Gamma}_1 \setminus \{\mathbf{0}\}. \tag{9.7}$$

Hence, because of (4.27), Lemma 9.1(a) applies again to show $Q_y \in C(\overline{\Gamma}_1)$. A similar argument works for Q_ξ, and thus $Q \in C^1(\overline{\Gamma}_1)$. Next, we may again apply (9.6) to (9.7) to get

$$Q_{yy}(y,\xi) = J^{g_{yy}}(y,\xi) + \frac{\sinh(\xi\sqrt{2\lambda})}{\sinh\left[(y+\xi)\sqrt{2\lambda}\right]}Q_{yy}(0, y+\xi) + \frac{\sinh(y\sqrt{2\lambda})}{\sinh\left[(y+\xi)\sqrt{2\lambda}\right]}Q_{yy}(y+\xi, 0)$$

for $(y, \xi) \in \overline{\Gamma} \setminus \{\mathbf{0}\}$. Now (4.28) implies $Q_{yy} \in C(\overline{\Gamma}_1)$. A similar argument shows $Q_{\xi\xi} \in C(\overline{\Gamma}_1)$, and then $Q_{y\xi} \in C(\overline{\Gamma}_1)$ because of (4.14), which implies

$$Q_{y\xi}(y,\xi) = \tfrac{1}{2}\left[Q_{yy}(y,\xi) + Q_{\xi\xi}(y,\xi)\right] + g(y,\xi) - Q(y,\xi), \quad \text{in } \overline{\Gamma}_1.$$

This completes the proof of Proposition 4.5. □

PROOF OF PROPOSITION 5.1. From the even symmetry of Q in both y and ξ, it clearly suffices to show that *U is positive in* Γ_1. However, from Lemma 9.1, (9.6) and (4.17)(ii), we find

$$Q_y(y,\xi) = J^{g_y}(y,\xi) + \frac{\sinh(y\sqrt{2\lambda})}{\sinh((y+\xi)\sqrt{2\lambda})}Q_y(y+\xi, 0+), \tag{9.8}$$

where $g(y, \xi) = c(y)\cosh(\theta\xi)$. However, $Q \in \mathscr{D}$, and hence $Q_y(s, 0+)$ is of class C^1 for $s > 0$. This allows us to apply (9.6) again, and compute $Q_{y\xi}(y, \xi)$ from (9.8). Since $Q_{y\xi}(y, 0+) = 0$, $0 \le y < \infty$, and $Q_{y\xi}(0+, \xi) = 0$, $0 \le \xi < \infty$, we obtain

$$Q_{y\xi}(y,\xi) = J^{g_{y\xi}}(y,\xi). \tag{9.9}$$

But $g_{y\xi}(y, \xi) = \theta c'(y)\sinh(\theta\xi)$ is positive in $(y > 0,\ \xi > 0)$, and thus the same is true for $U(y, \xi) = Q_{y\xi}(y, \xi)$. □

PROOF OF PROPOSITION 5.2. From Lemma 9.1 and the equation (9.9) representing U, we have

$$U_\xi(y,\xi) = J^{g_{y\xi\xi}}(y,\xi) + \frac{\sinh(\xi\sqrt{2\lambda})}{\sinh((y+\xi)\sqrt{2\lambda})}U_\xi(0+, y+\xi) + \frac{\sinh(y\sqrt{2\lambda})}{\sinh((y+\xi)\sqrt{2\lambda})}U_\xi(y+\xi, 0+), \tag{9.10}$$

whereas from (9.9), the definition (3.18) of J^g and a change of variables:

$$
\begin{aligned}
U(y,\xi) = \theta\sqrt{\frac{2}{\lambda}}\Bigg[&\frac{\sinh(y\sqrt{2\lambda})}{\sinh((y+\xi)\sqrt{2\lambda})}\int_y^{y+\xi}\sinh((y+\xi-u)\sqrt{2\lambda})c'(u)\\
&\times\sinh(\theta(y+\xi-u))\,du \qquad (9.11)\\
&+\frac{\sinh(\xi\sqrt{2\lambda})}{\sinh((y+\xi)\sqrt{2\lambda})}\int_0^y\sinh(u\sqrt{2\lambda})c'(u)\sinh(\theta(y+\xi-u))\,du\Bigg].
\end{aligned}
$$

Differentiating in (9.11), we obtain

$$
\begin{aligned}
&\frac{1}{\theta}\sqrt{\frac{\lambda}{2}}\left[\sinh((y+\xi)\sqrt{2\lambda})U_\xi(y,\xi)+\sqrt{2\lambda}\cosh((y+\xi)\sqrt{2\lambda})U(y,\xi)\right]\\
&\quad=\sinh(y\sqrt{2\lambda})\int_y^{y+\xi}c'(u)\left[\sqrt{2\lambda}\cosh((y+\xi-u)\sqrt{2\lambda})\sinh(\theta(y+\xi-u))\right.\\
&\qquad\qquad\left.+\theta\sinh((y+\xi-u))\sqrt{2\lambda})\cosh(\theta(y+\xi-u))\right]du \qquad (9.12)\\
&\qquad+\theta\sinh(\xi\sqrt{2\lambda})\int_0^y\sinh(u\sqrt{2\lambda})c'(u)\cosh(\theta(y+\xi-u))\,du\\
&\qquad+\sqrt{2\lambda}\cosh(\xi\sqrt{2\lambda})\int_0^y\sinh(u\sqrt{2\lambda})c'(u)\sinh(\theta(y+\xi-u))\,du,
\end{aligned}
$$

whence

$$
\begin{aligned}
U_\xi(y,0) &= \frac{2\theta}{\sinh(y\sqrt{2\lambda})}\int_0^y\sinh(u\sqrt{2\lambda})c'(u)\sinh(\theta(y-u))\,du, \qquad (9.13)\\
U_\xi(0,\xi) &= 0.
\end{aligned}
$$

Since $U_\xi(y,0)$ is differentiable in y, Lemma 9.1 may be applied again in (9.10), to obtain

$$
\begin{aligned}
U_{\xi\xi}(y,\xi) = J^{g_{y\xi\xi\xi}}(y,\xi) &+ \frac{\sinh(\xi\sqrt{2\lambda})}{\sinh((y+\xi)\sqrt{2\lambda})}U_{\xi\xi}(0+,y+\xi) \qquad (9.14)\\
&+\frac{\sinh(y\sqrt{2\lambda})}{\sinh((y+\xi)\sqrt{2\lambda})}U_{\xi\xi}(y+\xi,0+).
\end{aligned}
$$

If we combine this with the expression (5.6) for U, and note that $\theta^2 g_{y\xi} = g_{y\xi\xi\xi}$, we obtain

$$
W(y,\xi) = \frac{\sinh(\xi\sqrt{2\lambda})W(0,y+\xi)+\sinh(y\sqrt{2\lambda})W(y+\xi,0)}{\sinh((y+\xi)\sqrt{2\lambda})}. \qquad (9.15)
$$

Thus, *in order to determine W in the positive quadrant, we have only to compute its values on the axes*. Differentiating in (9.12) with respect to ξ and evaluating along the axes, we get

$$U_{\xi\xi}(0+,\xi) = 0 \tag{9.16}$$

and

$$\begin{aligned} &\frac{1}{\theta}\sqrt{\frac{\lambda}{2}}\,\sinh(y\sqrt{2\lambda})U_{\xi\xi}(y,0+) \\ &\quad = \frac{4\lambda}{\sinh(y\sqrt{2\lambda})}\int_0^y c'(y-u)\sinh(u\sqrt{2\lambda})\sinh(\theta u)\,du \\ &\qquad + 2\sqrt{2\lambda}\int_0^y c''(y-u)\sinh((y-u)\sqrt{2\lambda})\sinh(\theta u)\,du. \end{aligned} \tag{9.17}$$

This proves that $U_{\xi\xi}(y,0+) > 0$ for $y > 0$. Hence, $W(y,0) = U_{\xi\xi}(y,0+) - \theta^2 U(y,0+) = U_{\xi\xi}(y,0+) > 0$ for $y > 0$, and $W(0,\xi) = U_{\xi\xi}(0+,\xi) - \theta^2 U(0+,\xi) \equiv 0$ for $\xi > 0$. It follows from (9.15) that $W(y,\xi) > 0$ in Γ_1, as claimed. □

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