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FINITENESS OF WAITING-TIME MOMENTS IN GENERAL STATIONARY SINGLE-SERVER QUEUES

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Conditions for the finiteness of waiting-time moments in queues with a renewal arrival process were established by Kiefer and Wolfowitz. This paper establishes analogous conditions, some necessary, and some sufficient, in single-server queues with a general stationary ergodic arrival process. The feature of the arrival process in contributing to delay is any tendency to form clumps (or, clusters) of arrivals. In the more familiar setting of a renewal arrival process, the regenerative nature of the process severely limits any such tendency.

More generally, strong mixing conditions on the sequence of interarrival times are used to give a sufficient condition for the finiteness of waiting-time moments. The details are worked out for the two important examples where the arrivals are generated by a Cox process and where the sequence of interarrival times contains an embedded stationary regenerative phenomenon. The latter example sheds light on the recent work of Wolff and the range of examples and counterexamples used to elaborate the theoretical results presented.

1. Introduction. The existence of moments in queues is a classical problem of queueing theory. The celebrated theorem of Kiefer and Wolfowitz (1956) deals with the waiting time in GI/GI/k queues, and there are various more or less intricate proofs in the literature [see, e.g., Wolff (1984)]. For the single-server case, most proofs essentially utilize properties of sums of independent identically distributed (i.i.d.) random variables (r.v.'s). For this reason they cannot serve as a basis for generalizations to G/GI/1 queues. To the authors' knowledge the only known extensions are for queues with some special structure such as a queue with a periodic Poisson input [see Afanas'eva (1985) and Rolski (1990)], or a queue which is one of a series of queues in tandem [see Sacks (1960) and Wolfson (1984)]. This paper has been influenced in particular by seeing a preprint of Wolff (1991) whose work bears particularly on tandem queues.

Conditions for the existence of moments are of interest in their own right. We found them essential for developing light traffic theorems for queues: They enable us to state assumptions for such theorems in terms of the basic data, that is, in terms of conditions on the input process and on the finiteness of

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suitable moments of the service time. There are related problems concerning bounds for queueing characteristics. Such bounds frequently entail moment conditions in queues [see, e.g., Kingman (1970) and Lemoine (1976)]. We have also found that the study of finiteness of moments helps us to understand better the relevance of clumps in the point process of arrivals. It is intuitively obvious that "large" clusters of short interarrival times must somehow influence the waiting time in a queue. There are examples of this effect in other papers as, for example, Miyazawa's (1979) example of infinite expected waiting time in some G/D/1 queues, or Wolff's (1991) R/GI/1 queues.

We consider a single-server queue in which the arrivals occur at the epochs of a stationary point process that is not necessarily a renewal process. Except in Section 7 we work below with the Palm version of the input process. Specifically, the interarrival times $\{T_n: n = 0, \pm 1, \dots\}$ are assumed to constitute a strictly stationary ergodic sequence of nonnegative random variables (r.v.'s) independent of the service times $\{S_n: n = 0, \pm 1, \dots\}$ which are assumed to form a sequence of i.i.d. r.v.'s. The waiting-time process we consider has the representation

$$W_n = \sup_{j \geq 0} \left\{ \sum_{i=-j}^{-1} (S_{n+i} - T_{n+i}) \right\}, \qquad n = 0, \pm 1, \dots, \tag{1.1}$$

and this process $\{W_n\}$ is then stationary and nontrivial provided, as we assume, that

$$0 < b = ES < a = ET < \infty. \tag{1.2}$$

Here, S denotes a generic service-time r.v., and T a r.v. with the marginal distribution of any T_n; similarly, we write W for a r.v. distributed like any member of the stationary sequence W_n. The process can be described briefly as a stable G/GI/1 queueing system. As defined in (1.1) it inherits stationarity properties from the underlying processes of interarrival and service times.

In the case that the sequence $\{T_n\}$ is also i.i.d., so that the system is a GI/GI/1 queueing system, Kiefer and Wolfowitz (1956) showed that for $\gamma > 0$,

$$\text{when (1.2) holds, } EW^{\gamma} < \infty \quad \text{if and only if } ES^{\gamma+1} < \infty. \tag{1.3}$$

Our aim is to develop analogues of this results for the more general queueing system G/GI/1. It has long been known [Kiefer and Wolfowitz (1956); cf. Proposition 1 below] that when $EW^{\gamma} < \infty$, the sequence $\{S_n\}$ must necessarily satisfy (1.3), and a sense in which the condition is sufficient is given in Proposition 4. Unfortunately, earlier work on the finiteness problem gives us no guidance as to the conditions that $\{T_n\}$ must satisfy. Examples and counterexamples in Wolff (1991), along with the clustering phenomenon required for positive waiting times in light traffic [Daley and Rolski (1991, 1992)], are consistent with the conditions on $\{T_n\}$ that we deduce in Propositions 2 and 5.

In Section 5 we show that when the sequence $\{T_i\}$ is strongly mixing, finiteness of EW^{γ} can be related to the rate of convergence to 0 of the mixing coefficients. This result is illustrated via processes with embedded regenerative phenomena, a setting that is close to what Wolff used in discussing queues with "regenerative" arrival process, denoted R/·/·. These results, and known bounds on EW in GI/GI/1 systems, all serve to highlight what restrictions are implicit in assuming a renewal arrival process.

Finally, we note that finiteness of moments of the stationary work-load process are closely related to the moments of the stationary waiting-time r.v. We use this connection in Section 7 to deduce properties of the waiting-time moments in a queue with Cox (i.e., doubly stochastic Poisson) arrival process, such as the example of a Markov modulated arrival process.

2. Necessary conditions for finiteness. The function sup{ ⋯ } in (1.1) is convex in its arguments and $(x_+)^\gamma$ is a convex function for $\gamma \geq 1$. Since the sequences $\{S_n\}$ and $\{T_n\}$ are independent, we can deduce (2.1) below by applying Jensen's inequality to the r.v.'s $\{T_n\}$ and, for the equality, by appealing to the i.i.d. nature of the sequence $\{S_n\}$:

$$(2.1)\qquad EW^\gamma \geq E\left[\left(\sup_{j\geq 0}\left\{\sum_{i=-j}^{-1}(S_i - ET)\right\}\right)^\gamma\right]$$

$$= \gamma\text{th moment of waiting time in D/GI/1.}$$

Applying (1.3) then gives the following result; Miyazawa (1979) observed that it is implicit in Kiefer and Wolfowitz (1956).

PROPOSITION 1. *If for some $\gamma \geq 1$, $EW^\gamma < \infty$ for the stationary waiting-time r.v. W in a stable* G/GI/1 *queueing system satisfying* (1.2), *the service-time r.v. S satisfies $E(S^{\gamma+1}) < \infty$.*

We can just as easily apply Jensen's inequality to the sequence $\{S_n\}$ in (1.1), in which case we have, for $\gamma \geq 1$,

$$(2.2)\qquad EW^\gamma \geq E\left[\left(\sup_{j\geq 0}\left\{\sum_{i=-j}^{-1}(ES - T_i)\right\}\right)^\gamma\right]$$

$$= \gamma\text{th moment of waiting time in G/D/1/.}$$

For any positive η define

$$(2.3)\qquad J_\eta = \sup_{j\geq 0}\{j: T_{-1} + \cdots + T_{-j} \leq j\eta\}.$$

Because $\{T_n\}$ is a stationary ergodic sequence, J_η is a finite-valued r.v. for $\eta < a$. Then for G/D/1 we have from (1.1) that

$$(2.4)\qquad W_0 \geq_{\text{a.s.}} J_\eta(ES - \eta) = J_\eta(b - \eta),$$

and therefore

$$(2.5)\qquad EW^\gamma < \infty \quad \text{implies} \quad E(J_\eta^\gamma) < \infty \qquad (\text{all } \eta < b).$$

This yields the following counterpart to Proposition 1.

PROPOSITION 2. *If for some* $\gamma \geq 1$, $EW^{\gamma} < \infty$ *for the stationary waiting-time r.v.* W *in a stable* G/GI/1 *queueing system satisfying* (1.2), *then the stationary ergodic interarrival-time sequence* $\{T_n\}$ *satisfies* (2.5) *and the stronger condition*

$$\sup_{\eta < b} (b - \eta)^{\gamma} E(J_{\eta}^{\gamma}) < \infty. \tag{2.6}$$

In fact, this convexity argument demonstrates the following extremal property of queueing systems with either regular arrivals or constant service times, generalizing a result due to Rogozin for GI/GI/1 systems [see, e.g., Stoyan (1983), Section 5.2].

PROPOSITION 3. *Within the class of stationary queueing systems* G/GI/1 *with independent service-time r.v.'s independent of the stationary ergodic interarrival-time sequence, if there is given either the service-time d.f. and the arrival rate* λ, *or the distribution of the interarrival-time sequence and the mean service time, then all moments of order* $\gamma \geq 1$ *of the stationary waiting-time r.v.* W *are minimized by having constant interarrival times* $(= \lambda^{-1})$ *or constant service times, respectively.*

That Proposition 2 is not vacuous is shown by the following example which Wolff (1991) refers to as well known. He gives another example, adapted from Miyazawa (1979), of a G/GI/1 queue with $ES^2 < \infty$ but $EW = \infty$; this second example also contravenes the necessary condition of Proposition 2 for $EW < \infty$.

EXAMPLE 1 (Folklore counterexample). Consider independent sequences of i.i.d. positive r.v.'s $\{T_n^*\}$ and nonnegative r.v.'s $\{S_n^*\}$, where the generic r.v. S^* is a random sum of n^* i.i.d. nonnegative r.v.'s $\{S_n\}$ and n^* has a distribution $\{\pi_k^*\} \equiv \{\Pr\{n^* = k\}\}$ concentrated on $\{1, 2, \dots\}$ with finite first moment

$$b^* \equiv \sum_{k=1}^{\infty} k\pi_k^* < \infty.$$

Such sequences $\{T_n^*\}$ and $\{S_n^*\}$ arise from a single-server queue with batch arrivals in which the former sequence denotes interarrival times between batches of arrivals, the numbers in distinct batches are positive i.i.d. r.v.'s with distribution $\{\pi_k^*\}$, and service times of arrivals are i.i.d. r.v.'s $\{S_n\}$. The waiting times confronting the first-served arrivals in successive batches are then the same in distribution as the sequence of waiting times of customers in a GI/GI/1 queue with generic interarrival time T^* and service time S^*. A stationary waiting-time r.v. W^*, say, exists for such a system if (1.2) is satisfied, that is, when $ES^* < ET^* < \infty$, equivalently, $ES < ET^*/b^* < \infty$. Define further r.v.'s W_k^* by

$$W_1^* =_d W^* \quad \text{and} \quad W_{k+1}^* =_d W_k^* + S_k = W^* + S_1 + \cdots + S_k, \tag{2.7}$$

where W^* and $S_1, S_2, \dots$ are mutually independent, and each $S_i =_d S$. These r.v.'s have the interpretation that arrival in a batch of size at least k and kth to be served from the batch, has as its waiting time a r.v. whose distribution is that of W_k^*. Clearly, $W_k^* \geq_d W^*$ (all k), so in order that stationary waiting-time r.v.'s of this queueing system with batch arrivals should have finite γth moment, it is necessary by (1.3) that $E[(S^*)^{\gamma+1}] < \infty$. In the simplest case $\gamma = 1$, since

$$E\left[(S^*)^2\right] = E(n^*)ES^2 + E[n^*(n^* - 1)][ES]^2,$$

finiteness of EW^* requires both $ES^2 < \infty$ and that the batch size distribution associated with the process of arrivals should have a finite second moment. This latter condition is just the necessary condition of Proposition 2. Conversely, $E[(n^*)^2] = \infty$ implies that $EW^* = \infty$, so the stationary mean waiting time of a customer is then infinite, even when $ES^2 < \infty$.

3. Sufficient conditions. The sense in which we can establish results converse to those of Propositions 1 and 2 is governed in part by the tools available for studying moments of partial sums of stationary (not necessarily i.i.d.) r.v.'s.

PROPOSITION 4 (Sufficient conditions via decomposition). *If the stationary waiting-time r.v. W_D of a* G/D/1 *queueing system with mean service time $\eta < a$ satisfies $EW_D^\gamma < \infty$, then the stationary waiting-time r.v. W of a* G/GI/1 *system with the same arrival process and for which $b = ES < \eta$ and $ES^{\gamma+1} < \infty$, satisfies $EW^\gamma < \infty$.*

PROOF. For any pair of infinite sequences of reals $\{x_j\}, \{y_j\}$, we have $\sup_j\{x_j + y_j\} \leq \sup_j\{x_j\} + \sup_j\{y_j\}$. Apply this inequality to the partial sums in a representation like (1.1) for $W =_d W_0$ written as

$$\begin{aligned} W &=_d \sup_{j \geq 0}\left\{\sum_{i=-j}^{-1}(S_i - \eta) + \sum_{i=-j}^{-1}(\eta - T_i)\right\} \\ &\leq \sup_{j\geq 0}\left\{\sum_{i=-j}^{-1}(S_i - \eta)\right\} + \sup_{j \geq 0}\left\{\sum_{i=-j}^{-1}(\eta - T_i)\right\}. \end{aligned} \tag{3.1}$$

For any $\gamma \geq 1$, the moment EW^γ is finite if the γth moment of each of the terms of the right-hand side is finite. But these terms are just representations of stationary waiting-time r.v.'s W', W_D say, in D/GI/1 and G/D/1 systems respectively and both these systems are stable because $b < \eta < a$. Since $ES^{\gamma+1} < \infty$, $E(W')^\gamma < \infty$ by the Kiefer–Wolfowitz result, and $EW_D^\gamma < \infty$ by assumption, proving the proposition. □

We shall see in Example 2 that in general the proposition cannot be extended to cover the case $ES = \eta$.

The decomposition underlying the upper bound at (3.1) is not unlike the decoupling technique used in Wolff (1991), Section 4. Indeed, decompositions are the major tools in studying finiteness questions, at least for tandem queues [Sacks (1960) and Wolfson (1984)].

A heuristic argument indicated in Section 4 suggests that the condition (2.6) on the interarrival-time sequence may be sufficient as well as necessary for the finiteness of EW^{γ}. However, as shown below in Example 3, this is not so: We have to be content instead with weaker sufficient conditions, and even the necessary and sufficient condition in (4.8) for the finiteness of EW_D^{γ} in G/D/1 is little more than a tautology, and is not so readily applied as either (2.6) or (3.2).

PROPOSITION 5. *If the sequence of interarrival times of a stable* G/D/1 *queueing system with mean service time b satisfies*

$$\sup_{\eta<b} E(J_\eta^{\gamma}) < \infty, \tag{3.2}$$

then the stationary waiting-time r.v. W_D *satisfies* $EW_D^{\gamma} < \infty$.

PROOF. Observe from (1.1) that W_D has the representation

$$\begin{aligned} W_D &= \sup_{j\geq 0}\{jb - T_{-j} - \cdots - T_{-1}\} \\ &= \nu b - T_{-\nu} - \cdots - T_{-1} \\ &\leq \nu b, \end{aligned} \tag{3.3}$$

for some nonnegative integer-valued r.v. ν, where if equality occurs then $T_{-\nu} + \cdots + T_{-1} = 0$. From (3.3) we have

$$E(W_D/b)^{\gamma} \leq E\nu^{\gamma} \tag{3.4}$$

and also

$$\sup_{\eta<b} J_\eta = \sup_{\eta<b}\sup_{j\geq 0}\{j: T_{-j} + \cdots + T_{-1} \leq j\eta\} \geq \nu,$$

because either equality holds at (3.3) and we have already noted that then $J_0 = \nu$, or else when strict inequality holds we have $J_\eta = \nu$ for some (random) $\eta < b$. Thus $\nu \leq \sup_{\eta<b} J_\eta$. Using this in (3.4), then monotonicity, and finally the monotone convergence theorem, gives

$$E(W_D/b)^{\gamma} \leq E\Big[\sup_{\eta<b}(J_\eta^{\gamma})\Big] = E\Big[\lim_{\eta\uparrow b}(J_\eta^{\gamma})\Big] = \lim_{\eta\uparrow b} E(J_\eta^{\gamma}),$$

proving our assertion. □

4. Two counterexamples and a summary theorem. Both the counterexamples that follow are variations of the same idea from Miyazawa (1979) as used in Wolff (1991). First define the sizes K_i of clumps of arrivals in which the $K_i - 1$ interarrival times *within* each clump are the same (but may depend on the size of the clump), while the interarrival time *between* the last and the first members of adjacent clumps are such as to ensure that the mean interarrival time equals the prescribed value ET, say. By making the r.v.'s K_i

i.i.d, but with a sufficiently heavy tail, by making the "interclump" distance depend on the preceding clump only (and thus, independent of the ensuing clump), and by having a service-time r.v. that is less than the mean interarrival time, we obtain a process in which the system regenerates at certain arrival epochs. This property makes certain algebraic computations possible in closed form, and it is then an easy step to choose the distribution for K_i in such a way that it has the finite or infinite properties dictated by the requirements of the counterexample.

Let $\{K_i: i = 0, \pm 1, \dots\}$ be a doubly infinite sequence of independent positive integer-valued r.v.'s which, apart from K_0, are identically distributed as

(4.1a) $$\Pr\{K_i = k\} = \pi_k, \qquad k = 1, 2, \dots \text{ (all } i \neq 0),$$

for some distributions $\{\pi_k\}$ satisfying

(4.1b) $$EK = \sum_{k=1}^{\infty} k\pi_k < \infty;$$

K_0 has the distribution

(4.1c) $$\Pr\{K_0 = k\} = \frac{k\pi_k}{EK}, \qquad k = 1, 2, \dots,$$

with EK as at (4.1b). Given K_0, let K_0' be a r.v. uniformly distributed on $\{1, \dots, K_0\}$, independent of the other K_i, and define random indices $\{N_i: i = 0, \pm 1, \dots\}$ by

(4.2a) $$N_1 = K_0', \qquad N_{i+1} = N_i + K_i, \qquad i = 1, 2, \dots,$$

(4.2b) $$N_0 = -(K_0 - K_0'), \qquad N_{-i} = N_{-i+1} - K_{-i}, \qquad i = 1, 2, \dots .$$

This construction gives us a stationary renewal point process on the integers with lifetime distribution $\{\pi_k\}$ (i.e., a stationary process of "recurrent events" as in Volume 1 of Feller's treatise), namely, there are unit atoms at the random "times" N_i, designating regeneration epochs, and $\cdots < N_0 \leq 0 < N_1 < \cdots$, and for all integers $n = 0, \pm 1, \dots,$ a unique sequence of random indices $i \equiv i(n) \equiv i(n, \omega)$ is determined such that $N_i < n \leq N_{i+1}$. Then for $k = 1, 2, \dots,$

(4.3a) $$\begin{aligned} \Pr\{K_0' = k\} &= \sum_{l=k}^{\infty} \Pr\{K_0' = k | K_0 = l\} \Pr\{K_0 = l\} \\ &= \sum_{l=k}^{\infty} \frac{1}{l} \frac{l\pi_l}{EK} = \frac{\sum_{l=k}^{\infty} \pi_l}{EK} \end{aligned}$$

and

(4.3b) $$\begin{aligned} \Pr\{K_0 - K_0' = k - 1\} &= \sum_{l=k}^{\infty} \Pr\{K_0' = K_0 + 1 - k | K_0 = l\} \Pr\{K_0 = l\} \\ &= \frac{\sum_{l=k}^{\infty} \pi_l}{EK}. \end{aligned}$$

EXAMPLE 2. This example supports our assertion following Proposition 4. With $i(n)$ as just given, determine a stationary ergodic interarrival-time sequence $\{T_n\}$ with $ET_n = a$ by

$$T_n = \begin{cases} b, & \text{if } N_{i(n)} < n < N_{i(n)+1}, \\ K_{i(n)}a - (K_{i(n)} - 1)b, & \text{if } n = N_{i(n)+1}, \end{cases} \tag{4.4}$$

for some positive constant $b < a$. Observe that a G/D/1 queueing system with such $\{T_n\}$ as its interarrival time sequence and service times equal to b is trivially stable, provided the distribution $\{\pi_k\}$ satisfies (4.1), and that its stationary waiting-time r.v. equals 0 a.s.

Let nonnegative i.i.d. r.v.'s $\{S_n\}$ have mean b and finite second moment. Consider a G/GI/1 queueing system with such $\{S_n\}$ as the service-time sequence and interarrival-time sequence $\{T_n\}$ as in (4.4). Suppose that $b = ES < ET = a$. For the (nonstationary) sequence of waiting times W_n^* defined by $W_1^* = 0$ and otherwise satisfying the recurrence relation $W_{n+1}^* = (W_n^* + S_n - T_n)_+$, we necessarily have $W_n^* \le_d W$, where W denotes the stationary waiting-time r.v. of such a system. For $1 \le r < N_1$, we also have

$$\begin{aligned} W_r^* &= \max\{0, S_{r-1} - b, S_{r-1} + S_{r-2} - 2b, \dots, S_{r-1} + \cdots + S_1 - rb\} \\ &=_d \max_{0 \le j \le r-1} \{S_1 + \cdots + S_j - jb\}. \end{aligned}$$

Now

$$EW \ge E\big(I_{\{K_0' > 1\}} W_{K_0'}^*\big) = \frac{1}{EK} \sum_{k=2}^{\infty} EW_k^* \sum_{j=k}^{\infty} \pi_j = \frac{1}{EK} \sum_{k=2}^{\infty} w_k^* \sqrt{k} \sum_{j=k}^{\infty} \pi_j, \tag{4.5}$$

where

$$w_n^* = \frac{1}{\sqrt{n}} E\Big(\max_{0 \le r \le n-1} \{S_1 + \cdots + S_r - rb\}\Big).$$

Choose $\pi_k = C/k^{2+\alpha}$ for some α in $0 < \alpha < 0.5$ and suitable normalizing constant C. Then because

$$\lim_{n \to \infty} w_n^* = \sqrt{\frac{2\,\mathrm{Var}(S)}{\pi}} > 0$$

[this follows from (8.5.13) and Exercise 6.4.2 of Chung (1974)], the right-hand side of (4.5) is infinite while both (4.1) and $ES^2 < \infty$ hold.

Refer again to (3.3) where the random index $\nu \equiv \nu(\omega)$ is defined. Define $\eta(\omega)$ as follows. Either $\nu(\omega) = 0$, in which case set $\eta(\omega) = 0$, or else $\nu(\omega) > 0$ and

$$\eta(\omega) \equiv \frac{T_{-\nu(\omega)}(\omega) + \cdots + T_{-1}(\omega)}{\nu(\omega)} \tag{4.6}$$

is well defined, with $\eta(\omega) <_{\text{a.s.}} b$. We assert that

$$J_{\eta(\omega)}(\omega) = \nu(\omega). \tag{4.7}$$

To see this, it is clear from (3.3) that $J_{\eta(\omega)}(\omega) \geq \nu(\omega)$, so suppose that $\nu(\omega) < j \leq J_{\eta(\omega)}(\omega)$. Then

$$(4.8) \quad bj - T_{-1}(\omega) - \cdots - T_{-j}(\omega) \leq b\nu(\omega) - T_{-1}(\omega) - \cdots - T_{-\nu(\omega)}(\omega).$$

Since $(\eta(\omega) - b)j < (\eta(\omega) - b)\nu(\omega)$, we have by (4.8) that

$$\begin{aligned} j\eta(\omega) &- T_{-1}(\omega) - \cdots - T_{-j}(\omega) \\ &= j(\eta(\omega) - b) + jb - T_{-1}(\omega) - \cdots - T_{-j}(\omega) \\ &< \nu(\omega)(\eta(\omega) - b) + b\nu(\omega) - T_{-1} - \cdots - T_{-\nu(\omega)}(\omega) \\ &= \nu(\omega)\eta(\omega) - T_{-1}(\omega) - \cdots - T_{-\nu(\omega)}(\omega) = 0, \end{aligned}$$

which shows that we cannot have $J_{\eta(\omega)}(\omega) > \nu(\omega)$. Then for all ω we have the representation

$$W_D(\omega) = \nu(\omega)(b - \eta(\omega)) =_{\text{a.s.}} J_{\eta(\omega)}(b - \eta(\omega))$$

and

$$EW_D^\gamma = E\big([b - \eta(\omega)]^\gamma J_{\eta(\omega)}^\gamma\big).$$

This suggests that (2.6) may be sufficient as well as necessary—after all, since the r.v. J_η is well defined and finite and nondecreasing in η, it is a small step to replace $\eta(\omega)$ here by η. The following example underlines the failure of this heuristic approach. What we do is to show that both (2.6) can hold and $EW = \infty$.

EXAMPLE 3. In place of T_n as in (4.4), define now

$$(4.9) \quad T_n = \begin{cases} b - \varepsilon_{K_{i(n)}}, & \text{if } N_i < n < N_{i+1}, \\ K_{i(n)}ET - (K_{i(n)} - 1)\big(b - \varepsilon_{K_{i(n)}}\big), & \text{if } n = N_{i+1}, \end{cases}$$

where $\{\varepsilon_k\}$ is a monotonic sequence decreasing to 0 with $\varepsilon_2 < b$ and, together with $\{\pi_k\}$ which necessarily satisfies (4.1), satisfies also

$$(4.10) \qquad \sum_{k=1}^{\infty} k^2\pi_k = \infty, \qquad \sup_k \left\{\varepsilon_k \sum_{j=1}^{k} (j-1) \sum_{i=j}^{\infty} \pi_i\right\} < \infty.$$

Then for $\eta < b$,

$$(4.11) \qquad J_\eta = \begin{cases} 0, & \text{if } 0 \leq \eta < b - \varepsilon_{K_0}, \\ K_0 - K_0', & \text{if } b - \varepsilon_{K_0} \leq \eta < b; \end{cases}$$

note that it is possible to have $J_\eta = 0$ in the latter case as well because $K_0 - K_0'$ may equal 0.

Define $k(\eta) = \sup\{k\colon b - \varepsilon_k \le \eta\}$. Referring to Proposition 5, first calculate

$$
\begin{aligned}
EJ_\eta &= E\big[E(K_0 - K_0' | K_0) I\{K_0 \le k(\eta)\}\big] \\
&= E\left[\frac{1}{2}(K_0 - 1) I\{K_0 \le k(\eta)\}\right] \\
&= \frac{1}{2EK} \sum_{k=1}^{k(\eta)} (k-1) \sum_{j=k}^{\infty} \pi_j .
\end{aligned} \tag{4.12}
$$

Then condition (2.6) holds if and only if

$$
\sup_k \left\{ \varepsilon_k \sum_{j=1}^{k} (j-1) \sum_{i=j}^{\infty} \pi_i \right\} < \infty. \tag{4.13}
$$

On the other hand, $b - \eta(\omega) = \varepsilon_{K_0}$, so in a G/D/1 queue with service times $S_n \equiv b$,

$$
\begin{aligned}
EW &= \sum_{k=1}^{\infty} \varepsilon_k E(K_0 - K_0' | K_0 = k) \frac{k\pi_k}{EK} \\
&= \frac{1}{2EK} \sum_{k=1}^{\infty} \varepsilon_k (k-1) \sum_{j=k}^{\infty} \pi_j .
\end{aligned} \tag{4.14}
$$

For an appropriate constant C and $b = 1$, $\{C/k^3\}$ is a distribution satisfying (4.10) when $\varepsilon_k = (1 + \log k)^{-1}$. For this distribution, (4.13) is satisfied, and hence (2.6), but from (4.14) we have $EW = \infty$.

Surveying these results and counterexamples, we summarize as in the following theorem.

THEOREM 6. *Let W be distributed as the stationary waiting-time r.v. in a stable stationary* G/GI/1 *queueing system with stationary and ergodic inter-arrival-time sequence and i.i.d. service-time r.v.'s. If*

$$
EW^\gamma < \infty, \tag{4.15}
$$

then

$$
E(S^{\gamma+1}) < \infty \tag{4.16}
$$

and, with $b = ES$,

$$
E\big([b - \eta(\omega)]^\gamma J_{\eta(\omega)}^\gamma\big) < \infty, \tag{4.17}
$$

where the r.v. $\eta(\omega)$ *is determined by* (4.6). *Conversely, if* (4.16) *holds and* $EJ_\eta^\gamma < \infty$ *for some* $\eta > b$, *then* (4.15) *holds.*

5. Mixing conditions. In this section we impose some mixing conditions on the interarrival times. Recall that a stationary sequence $\{T_n: -\infty < n < \infty\}$ is strongly mixing with mixing coefficients ρ_m if

$$\sup|\Pr(A \cap B) - \Pr(A)\Pr(B)| = \rho_m, \tag{5.1}$$

where the supremum is taken over all $A \in \sigma\{T_r: r \le n\}$ and $B \in \sigma\{T_r: r \ge m + n\}$ and $\rho_m \to 0$.

Lemma 7. *Let the strongly mixing stationary nonnegative sequence* $\{T_n\}$ *have mixing coefficients* $\{\rho_n\}$ *satisfying*

$$\sum_{n=1}^{\infty} n^{\gamma-1}\rho_n < \infty, \tag{5.2}$$

for some $\gamma > 0$. *Then* $EJ_\eta^\gamma < \infty$ *for all* $\eta < a = ET$.

Proof. Suppose first that $T_n \le M$ for some finite constant M. We have $EJ_\eta^\gamma < \infty$ if and only if

$$\sum_{k=1}^{\infty} k^{\gamma-1}\Pr\{J_\eta \ge k\} < \infty.$$

Now

$$\begin{aligned}
\{J_\eta \ge k\} &= \bigcup_{j \ge k}\left\{\sum_{i=1}^{j} T_{-i} \le j\eta\right\} = \bigcup_{j\ge k}\left\{\sum_{i=1}^{j}(T_{-i} - a) \le j(\eta - a)\right\}\\
&= \bigcup_{j\ge k}\left\{\frac{1}{j}\sum_{i=1}^{j}(a - T_{-i}) \ge a - \eta\right\} \qquad (5.3)\\
&= \left\{\sup_{j \ge k}\left\{\frac{1}{j}\sum_{i=1}^{j}(a - T_{-i}) \ge a - \eta\right\}\right\}.
\end{aligned}$$

Set $M_a = \max(a, |a - M|)$ and $\xi_i = (a - T_{-i})/M_a$, so that $|\xi_i| \le 1$. From (5.3) it follows that $EJ_\eta^\gamma < \infty$ if and only if

$$\sum_{k=1}^{\infty} k^{\gamma-1}\Pr\left\{\sup_{j\ge k}\left\{\frac{1}{j}\sum_{i=1}^{j}\xi_i\right\} \ge \frac{a-\eta}{M_a}\right\} < \infty.$$

By Theorem 1.2 of Berbée (1987) this last sum is finite when (5.2) holds, which proves the lemma in the case that $\{T_i\}$ are bounded.

For the unbounded case start by observing that, given $\eta < a$, we can find M such that the sequence $\{T_i^M\} \equiv \{\min(T_i, M)\}$ has $E(T_i^M) > \eta$. For such a sequence the analogous r.v.'s $J_\eta^M \ge J_\eta$, so by the first part of the proof it is enough to show that the stationary nonnegative sequence $\{T_i^M\}$ is strongly mixing with mixing coefficients $\rho_{m,M} \le \rho_m$ as in (5.1). But the backwards and forwards σ-fields in (5.1) are sub-σ-fields of the corresponding σ-fields generated by the sequence $\{T_i\}$, so (5.1) holds with $\rho_{m,M} \le \rho_m$ as required. (We thank Professor M. R. Leadbetter for this argument.) □

REMARK 1. There is a parallel theory for a GI/G/1 queue with a sequence $\{T_n\}$ of i.i.d. interarrival times and a sequence $\{S_n\}$ of service times that constitute a strictly stationary ergodic sequence, with the two sequences mutually independent. For such a system a decomposition analogous to Proposition 4 holds. It is also possible to prove an analogue of Lemma 7, provided that we can assume that the service times are bounded. The problem of unbounded service times is more difficult.

COROLLARY 8. *If the interarrival sequence $\{T_n\}$ of a stable* G/GI/1 *queue is strongly mixing and satisfies* (5.2) *for some given $\gamma > 1$, and $ES^{\gamma+1} < \infty$, then the stationary waiting-time r.v. W satisfies $EW^\gamma < \infty$.*

PROOF. Choose η in $b = ES < \eta < a = ET$. Then the result follows by the theorem and Lemma 7. □

EXAMPLE 4 (Renewal arrival process). In a stationary stable GI/D/1 queueing system, so that $\{T_n\}$ is an i.i.d. sequence of r.v.'s with $ET > b =_{\text{a.s.}} S$,

$$J_b =_d \sup\{j\colon T_1 + \cdots + T_j \le jb\}. \tag{5.4}$$

In order to apply a general result on partial sums of i.i.d. r.v.'s [e.g., Chow and Teicher (1978), Section 10.4, Theorem 3], consider the i.i.d. r.v.'s $\{X_n\} \equiv \{\mathrm{E}T - T_n\}$, for which a generic such r.v. X has zero mean and $E(X_+^{\gamma+1}) \le (ET)^{\gamma+1} < \infty$ (all $\gamma > 0$). Then

$$J_b =_d \sup\{n \ge 0\colon X_1 + \cdots + X_n \ge (ET - b)n\}, \tag{5.5}$$

and by equation (23) in the cited theorem, $E(J_b^\gamma) < \infty$. The condition (3.2) is satisfied, and Proposition 5 applies.

A much shorter proof is available via Proposition 8 because a renewal process is strongly mixing with mixing coefficients all 0.

EXAMPLE 5 (Markovian interarrival times). Suppose the interarrival times form a Harris-recurrent Markov chain on $\mathbb{R}_+$ with transition kernel $p(\cdot,\cdot)$ and stationary distribution $\pi(\cdot)$. As usual let p^m denote the m-fold convolution of p. Set

$$K_m(y) = \| p^m(y,\cdot) - \pi(\cdot)\|.$$

From the proof of Theorem 1 of Athreya and Pantula (1986), we have

$$\rho_m \le \beta_m \equiv 2 \sup_{y \ge 0} K_{m-1}(y).$$

Then by Proposition 8, we have $EW^\gamma < \infty$ if

$$\sum_{n=1}^{\infty} n^{\gamma-1}\beta_n < \infty. \tag{5.6}$$

Condition (5.6) is automatically satisfied for chains satisfying the Doeblin condition, in which case the β_n's decrease exponentially.

Strong mixing of Harris-recurrent Markov chains dates back at least to Davydov (1969, 1973). Aspects of Davydov's results are developed further by Davies and Grübel (1981) using Banach algebra methods (as did Davydov). Harris-recurrent Markov chains are known to be regenerative in a generalized sense [see, e.g., Asmussen (1987)]. This notion contains regeneration in the classical sense, meaning that there exists a discrete-time renewal sequence $\{N_n\}$ such that consecutive cycles

$$\mathscr{C}_n = \{N_{n+1} - N_n, \{X_i\colon N_n \le i < N_{n+1}\}\}, \qquad n = 0, \pm 1, \dots,$$

are independent and identically distributed for $n \neq 0$. The results of Miyazawa (1979) or Wolff (1991) also motivate the study of queues with interarrival times having a regenerative structure.

We now introduce another concept of regeneration, for which we are able to characterize mixing coefficients in terms of moments of lifetime distributions. This notion of regeneration again contains regenerative sequences in the classical sense. Our setting uses the notion of a regenerative phenomenon as in, for example, Kingman (1972).

Let $(\Omega, \mathscr{F}, \mathscr{P})$ be a probability space supporting a discrete-time process $\{X_n\}$. We say that the process $\{X_n\}$ *contains an embedded regenerative phenomenon* $\Phi \equiv \{E_n\}$ if for any $A \in \mathscr{B}_m \equiv \sigma\{X_r\colon r \le m\}$ and $B \in \mathscr{F}_{m+n} \equiv \sigma\{X_r\colon r \ge m + n\}$, and any E_s with $m < s < n + m$,

$$(5.7) \qquad \mathscr{P}(A \cap E_s \cap B)\mathscr{P}(E_s) = \mathscr{P}(A \cap E_s)\mathscr{P}(B \cap E_s).$$

Equivalently, provided $\mathscr{P}(E_s) > 0$ (all s),

$$(5.7') \qquad \mathscr{P}(A \cap B|E_s) = \mathscr{P}(A|E_s)\mathscr{P}(B|E_s).$$

Let Φ have lifetime distribution $\{f_n\}$ so that

$$(5.8) \qquad f_n = \mathscr{P}\{E_n E_{n-1}^c \dots E_1^c | E_0\}, \qquad n = 1, 2, \dots,$$

set $g_n = f_n + f_{n+1} + \cdots$, and define its associated renewal sequence $\{u_n\}$ by

$$(5.9) \qquad u_n = \mathscr{P}(E_n|E_0) = \begin{cases} 1, & n = 0, \\ \sum_{m=1}^{n} f_m u_{n-m}, & n = 1, 2, \dots. \end{cases}$$

Put

$$Z_n(\omega) = \begin{cases} 1, & \text{if } \omega \in E_n, \\ 0, & \text{otherwise.} \end{cases}$$

Assume that:

(i) $\{(X_n, Z_n)\}$ forms a jointly stationary sequence,

(ii) the regenerative phenomenon is positive recurrent (i.e., $\sum n f_n < \infty$),

(iii) the regenerative phenomenon is aperiodic (i.e., the highest common factor of $\{n \ge 1\colon f_n > 0\}$ equals 1).

When we consider processes $\{X_n\}$ containing an embedded regenerative phenomenon below, we shall assume that conditions (i)–(iii) are satisfied.

When conditions (ii) and (iii) hold, the Erdös–Feller–Pollard theorem [see, e.g., Feller (1968), Volume 1, or Kingman (1972), Chapter 1, for discussion] implies that

$$u_n \to \pi \equiv \mathscr{P}(E_i) = \left(\sum_{m=1}^{\infty} m f_m \right)^{-1} > 0, \qquad n \to \infty; \tag{5.10}$$

equivalently,

$$\alpha_n \equiv \sup_{m \ge n} |u_m - \pi| \to 0, \qquad n \to \infty. \tag{5.11}$$

The following lemma coupled with Proposition 8 is essentially a form of a result of Wolff (1991) on R/GI/1 queues.

LEMMA 9. *If an interarrival-time sequence $\{T_i\}$ contains an aperiodic regenerative phenomenon for which the lifetime distribution $\{f_m\}$ has*

$$\sum_{m=1}^{\infty} m^{\gamma+1} f_m < \infty, \tag{5.12}$$

for some $\gamma > 0$, then $\{T_i\}$ is strongly mixing with coefficients $\{\rho_n\}$ satisfying (5.2) for such γ.

PROOF. We use Lemma A in the appendix and a result of Davydov (1973), Theorem 5(iii), stating that for given $\delta > 0$, the renewal sequence $\{u_n\}$ generated by the lifetime distribution $\{f_m\}$ of an aperiodic discrete-time renewal process satisfies

$$\sum_{m=1}^{\infty} |u_m - u_{m-1}| m^{\delta} < \infty \quad \text{if and only if} \quad \sum_{m=1}^{\infty} m^{\delta+1} f_m < \infty \tag{5.13}$$

(we have expressed Davydov's result in terms of $\{f_m\}$ rather than $\{g_m\}$). Directly from (A.1) of Lemma A, (5.2) is satisfied if both

$$\text{(a)} \quad \sum_{m=1}^{\infty} m^{\gamma-1} G_m < \infty \quad \text{and} \quad \text{(b)} \quad \sum_{m=1}^{\infty} m^{\gamma-1} \alpha_m < \infty, \tag{5.14}$$

where $G_m = \pi(g_{m+1} + g_{m+2} + \cdots)$. Since $\alpha_m \le \sum_{t=m}^{\infty} |\Delta u_t|$ where $\Delta u_t = u_t - u_{t+1}$,

$$\sum_{m=1}^{\infty} m^{\gamma-1} \alpha_m \le \sum_{m=1}^{\infty} m^{\gamma-1} \sum_{t=m}^{\infty} |\Delta u_t| \le C_\gamma \sum_{m=1}^{\infty} m^{\gamma} |\Delta u_m|,$$

for some positive constant C_γ, provided here that $\gamma > 0$, and this is finite by

(5.13) when (5.12) is satisfied. Since G_n in part (a) of (5.14) is a sum of tail sums of $\{f_n\}$, finiteness of the sum in (a) is certainly implied by (5.12). □

REMARK 2. Because a regenerative phenomenon of period greater than or equal to 2 cannot be ergodic [i.e., (5.11) cannot hold], it cannot be strongly mixing, so the assumption of aperiodicity is made without loss of generality. Nevertheless, as is implicit in example (b) of Wolff (1991), the finiteness of waiting-time moments for R/GI/1 with periodic arrival processes can be considered outside the context of mixing.

EXAMPLE 6 (M/GI/1 → GI/1 tandem queue). Customers arrive at the first system which is an M/GI/1 queue, and upon completing their service they proceed to the second system, which also consists of one server. The characteristic of interest is the stationary waiting time at the second system. We assume that Poisson arrivals at the first queue are of intensity λ, that the service times $\{S_n^{(1)}\}, \{S_n^{(2)}\}$ at the first and second system respectively consist of i.i.d. r.v.'s, and that the input and service sequences at the stations are all mutually independent. The interarrival times $\{T_n\}$ at the second system are interdeparture times from the first one. Suppose that customers initiating busy cycles have indices $\dots, N_{-1}, N_0, N_1, \dots$. Using the definition of $i(n)$ from Section 4, we have

$$T_n = \begin{cases} S_n^{(1)}, & \text{if } N_{i(n)} < n < N_{i(n)+1}, \\ M_{i(n)} + S_n^{(1)}, & \text{if } n = N_{i(n)}, \end{cases} \tag{4.4}$$

where $\{M_n\}$ is an i.i.d. sequence of exponentially distributed random variables with parameter λ. The length K of a generic cycle is equal in distribution to the generic number of customers in a busy cycle. Thus $EK^{\gamma+1} < \infty$ if $E(S^{(1)})^{\gamma+1} < \infty$. Appealing to Proposition 9, we deduce that for the stationary waiting-time r.v. $W^{(2)}$ at the second station, $E(W^{(2)})^{\gamma} < \infty$ provided $E(S^{(2)})^{\gamma+1} < \infty$. This is implicit in Wolfson (1984) and is also shown in Wolff (1991). Note that generalizations to systems like GI/GI/1 → GI/1 are not straightforward because, following work of Nummelin (1981) or Sigman (1988), the "regenerative phenomenon" that is involved is not immediately the process under study.

6. The limited effect of clumping in renewal arrival processes. We stressed in Daley and Rolski (1991) an interpretation of delays in single-server queues with renewal arrival process arising from the combination of the two influences of the tail behaviour of the service-time distribution and any clustering tendency in the interarrival-time distribution. What our Propositions 2 and 5 and Examples 1–3 show is that, more generally, waiting times are influenced by *clumping* behaviour of the interarrival times, though this effect can never be strong enough to give infinite mean waiting times with the independence structure of a renewal arrival process [Kiefer and Wolfowitz (1956); Example 4 above]. We consider Kingman's (1962) upper bound on mean waiting times which in its simple form

$$(6.1)\qquad \begin{aligned} EW &\le \frac{\operatorname{Var}(T) + \operatorname{Var}(S)}{ET - ES} \\ &= \frac{\operatorname{Var}(T - S)}{2E(T - S)} = \frac{\operatorname{Var}(T - S)}{2a(1 - \rho)}, \quad \text{where } \rho \equiv \frac{ES}{ET} = \frac{b}{a}, \end{aligned}$$

is a function of $T - S$ alone, and hence "almost" symmetrical in S and T, unlike the Kiefer and Wolfowitz result at (1.3); this symmetry disappears in refined forms such as

$$(6.2)\qquad EW \le \frac{\rho(2 - \rho)\operatorname{Var}(T) + \operatorname{Var}(S)}{2ET(1 - \rho)}$$

[Daley (1977)]. Here we can interpret the term involving $\operatorname{Var}(S)$ as arising from the (occasional) occurrence of large service times and the consequential delays to subsequent arrivals; indeed, there exists a sequence of stable stationary D/GI/1 systems, with the same interarrival time and first two moments of S, in which EW can be arbitrarily close to the bound

$$(6.3)\qquad \frac{\operatorname{Var}(S)}{2ET(1 - \rho)}.$$

To interpret the other term in (6.2), observe that in a GI/D/1 system it takes the form

$$(6.4)\qquad EW \le \frac{\rho(2 - \rho)\operatorname{Var}(T)}{2ET(1 - \rho)}.$$

Consider a family of GI/D/1 queues, all with the same $b = ES$, $a = ET$ and ET^2, and indexed by x in $0 \le x < a$, for which the i.i.d. r.v.'s T_n satisfy

$$(6.5)\qquad T_n =_d T \equiv \begin{cases} x, & \text{with probability } \varpi_x \equiv \dfrac{\operatorname{Var}(T)}{\operatorname{Var}(T) + (a - x)^2}, \\ a + \dfrac{\operatorname{Var}(T)}{a - x}, & \text{with probability } 1 - \varpi_x. \end{cases}$$

For all such x, arrivals occur in clumps of i.i.d. random sizes that are geometrically distributed. As x increases from 0 to a, the mean number in each clump increases but both the distance between clumps and the distance between adjacent elements within clumps increase. Visually, the pattern of arrivals tends to look more highly clustered the smaller the value of x. For all $x \ge b$, we have $W =_{\text{a.s.}} 0$, while for smaller x intuition suggests [and this is supported by direct computations noted in Daley (1990)] that EW should increase as $x \downarrow 0$. Note that for such a system, $J_\eta =_{\text{a.s.}} 0$ for $\eta < x$; further inspection shows that

$$(6.6)\qquad \Pr\{J_x = j\} = (1 - \varpi_x)\varpi_x^j, \qquad j = 0, 1, \dots,$$

and $J_\eta \geq_d J_x$, $x \leq \eta < a$. Then from (2.4), for $0 \leq x < b$,

$$EW \geq \frac{(b-x)\varpi_x}{1-\varpi_x} = \frac{(b-x)\operatorname{Var}(T)}{(a-x)^2}. \tag{6.7}$$

We infer from this relation that the presence of the term $\operatorname{Var}(T)$ in the bounds (6.1) and (6.2) is attributable to its being a surrogate for a measure of whatever clumpiness is possible in a queue with a renewal arrival process: It gives a uniform bound on the effect of such clumping on EW when $\operatorname{Var}(T) < \infty$. Some elementary algebra shows that the maximum in $0 \leq x < b$ of the right-hand side of (6.7) is at least $\frac{2}{3}$ times the upper bound at (6.4) for all ρ in $0 < \rho < 1$.

7. Queues with Cox input. Suppose that N is the stationary point process which corresponds to the Palm version (or, synchronous process) of $N^0 = \sum_i \delta_{\tau_i}$ where

$$\dots, \tau_{-1} = -T_0, \tau_0 = 0, \tau_1 = T_1, \tau_2 = T_1 + T_2, \dots$$

[see, e.g., Rolski (1981) or Daley and Vere-Jones (1988) for these notions on stationary point processes and facts used below]. Parallel to (1.1) the work-load process $\{V(t)\colon -\infty < t < \infty\}$ has the representation

$$V(t) = \sup_{s \leq t} \left\{ \sum_{i=N(0,s]+1}^{N(0,t]} S_i - (t-s) \right\}. \tag{7.1}$$

From Miyazawa (1979) we know that in a stable G/GI/1 queueing system, the stationarity work-load r.v. V has $EV^\gamma < \infty$ if and only if $EW^\gamma < \infty$.

Let $\{\lambda(t)\colon t \in \mathbb{R}\}$ be an ergodic stationary nonnegative process. Suppose further that it is the arrival rate process of a queueing system into which customers arrive at the epochs of a Cox (i.e., doubly stochastic Poisson) process directed by this process as its arrival rate. From (7.1) it follows easily that

$$V \equiv V(0) =_d \sup_{t \geq 0} \left\{ \sum_{i=1}^{\Pi \circ \Lambda^-(t)} S_i - t \right\}, \tag{7.1$'$}$$

where $\Pi(t)$ denotes the number of points in $(0, t]$ of a stationary homogeneous Poisson process with unit rate and

$$\Lambda^-(t) = \int_0^t \lambda^-(s)\, ds \equiv \int_0^t \lambda(-s)\, ds.$$

Following Section 3 of Rolski (1990), set

$$D \equiv \sup_{t \geq 0} \left\{ \Lambda^-(t) - (1+\varepsilon)\bar{\lambda}(t) \right\}, \tag{7.2}$$

where $\bar{\lambda} = E\Lambda^-(1)$ and $\varepsilon > 0$ is such that $(1+\varepsilon)\bar{\lambda} ES < 1$. We tacitly assume that $\lambda(t)$ is locally integrable and such that D is a r.v. Note that D is finite because λ^- is stationary and ergodic. Then for any $h > 0$,

$$
\begin{aligned}
D &= \sup_{n \ge 0} \left\{ \sup_{nh \le t < (n+1)h} \{\Lambda^-(t) - (1+\varepsilon)\bar{\lambda} t\} \right\} \\
&\le \sup_{n \ge 0} \{\Lambda^-((n+1)h) - (1+\varepsilon)\bar{\lambda} nh\} \\
&\le \Lambda^-(h) + \sup_{n \ge 1} \left\{ \int_h^{nh} \lambda^-(s)\, ds - (1+\varepsilon)\bar{\lambda} nh \right\} \\
&= \Lambda^-(h) + \sup_{n \ge 1} \{\xi_1 + \cdots + \xi_n - (1+\varepsilon)\bar{\lambda} nh\}.
\end{aligned}
\tag{7.3}
$$

Rolski (1990) proved the special case $\gamma = 1$ of our next result.

PROPOSITION 10. *If $ED^\gamma < \infty$ and $ES^{\gamma+1} < \infty$, then $EV^\gamma < \infty$.*

PROOF. Use the inequality

$$
V \le_d \sum_{i=1}^{\Pi'(D)} S_i' + \sup_{t \ge 0} \left\{ \sum_{i=1}^{\Pi((1+\varepsilon)t)} S_i - t \right\}, \tag{7.4}
$$

where all S_i''s, all S_i's, $\Pi'(D)$ and $\{\Pi((1+\varepsilon)t)\colon t \ge 0\}$ are independent, with each element of the first two sequences distributed like S and $\Pi'(D) =_d \Pi(D)$ [see (3.4) in Rolski (1990)]. From the standard theory of queues, the second term on the right-hand side of (7.4) is the waiting time in M/GI/1 queues, so its γth moment is finite provided $ES^{\gamma+1}$ is finite. The γth moment of the first term is finite because

$$
E\left(\sum_{i=1}^{\Pi'(D)} S_i' \right)^\gamma \le ES^\gamma E\big([\Pi'(D)]^\gamma\big) < \infty.
$$

Define the strong mixing coefficient function of the stationary process $\lambda^-(t) = \lambda(-t)$, $t \ge 0$, by

$$
\rho(t) = \sup |\mathscr{P}(A \cap B) - \mathscr{P}(A)\mathscr{P}(B)|,
$$

where the supremum is taken over all $A \in \sigma\{\lambda^-(v),\ 0 \le v \le s\}$ and $B \in \sigma\{\lambda^-(v),\ v \ge s + t\}$. □

PROPOSITION 11. *For a stationary queue with Cox input with a bounded arrival rate process, let the mixing coefficient function satisfy*

$$
\sum_{n=1}^{\infty} n^{\gamma-1} \rho(nh) < \infty,
$$

for some $h > 0$. If $ES^{\gamma+1} < \infty$, then EV^γ and EW^γ are both finite.

PROOF. The sequence $\{\xi_i\}$ is strong mixing with mixing coefficients $\rho_n = \rho(nh)$ satisfying (5.2). Hence by Remark 2,

$$E\Big(\sup_{n\geq 1}\{\xi_1 + \cdots + \xi_n - (1+\varepsilon)\bar{\lambda}nh\}\Big)^{\gamma} < \infty.$$

The proof is complete now by Proposition 10. □

Proposition 11 contains as a special case the G/GI/1 queue with periodic arrivals [cf. also Rolski (1990) and Afanas'eva (1984)]. Another important example is the following.

EXAMPLE 7 (Markov modulated arrivals). Arrivals occur at the epochs of a doubly stochastic Poisson process with a random arrival rate function of the form $f(X(t))$, where $\{X(t)\}$ is an irreducible finite-state Markov process and $f(\cdot)$ is a nonnegative function defined on its state space. For such an arrival rate function, the mixing coefficient function decreases exponentially, so for the finiteness of the γth moment of the waiting time it is sufficient that $ES^{\gamma+1} < \infty$.

APPENDIX

Strong mixing of a process containing an embedded regenerative phenomenon. The counterexamples in Section 4 above and some examples in Wolff (1991) fall into the framework of processes containing an embedded regenerative phenomenon as defined in Section 5. The following lemma relates strong mixing coefficients ρ_n with G_n's and α_n's defined also in Section 5.

LEMMA A. *A stochastic process $\{X_n\}$ containing an embedded regenerative phenomenon is strongly mixing with mixing coefficients ρ_n satisfying*

$$\rho_n \leq 2(G_r + G_s) + \alpha_t/\pi, \tag{A.1}$$

where r, s and t are any positive integers with $r+s+t=n$.

REMARK 3. For (A.1) we use only the joint stationary property as in assumption (i) of Section 5.

PROOF OF LEMMA A. Because of stationarity, take $m = 0$ without loss of generality. We have to show that for any $A \in \mathscr{B}_0$ and $B \in \mathscr{F}_n$, the right-hand side of (A.1) is a bound for $|\mathscr{P}(A\cap B) - \mathscr{P}(A)\mathscr{P}(B)|$. Define

$$D_r^s = \begin{cases} \bigcap_{n=r}^{s} E_n^c, & \text{if } r \leq s,\\ \Omega, & \text{otherwise,}\end{cases} \tag{A.2}$$

so that each of the equations

$$\Omega = E_1 \cup D_1^1 \cap E_2 \cup \cdots \cup D_1^{r-1} \cap E_r \cup D_1^r \\ = D_1^s \cup E_1 \cap D_2^s \cup \cdots \cup E_{s-1} \cap D_s^s \cup E_s \tag{A.3}$$

represents Ω as the union of mutually exclusive events. Then for any $0 < r < n - s$ we have

$$\begin{aligned} \mathscr{P}(A \cap B) = {} & \sum_{i=1}^{r} \sum_{j=1}^{s} \mathscr{P}\big(A \cap D_1^{i-1} \cap E_i \cap E_{n-j} \cap D_{n-j+1}^{n-1} \cap B\big) \\ & + \sum_{i=1}^{r} \mathscr{P}\big(A \cap D_1^{i-1} \cap E_i \cap D_{n-s}^{n-1} \cap B\big) \\ & + \sum_{j=1}^{s} \mathscr{P}\big(A \cap D_1^{r} \cap E_{n-j} D_{n-j+1}^{n-1} \cap B\big) \\ & + \mathscr{P}\big(A \cap D_1^r \cap D_{n-s}^{n-1} \cap B\big) \end{aligned} \tag{A.4}$$

$$= S_1 + S_2 + S_3 + S_4 \quad \text{say.} \tag{A.4$'$}$$

Observe that by the stationarity property of the regenerative phenomenon, (5.10), and the identity

$$\sum_{i=1}^{\infty} \sum_{j=i}^{\infty} f_j = \sum_{i=1}^{\infty} g_i = \pi^{-1},$$

we have

$$\begin{aligned} \mathscr{P}(D_1^r) &= 1 - \sum_{i=1}^{r} \sum_{j=0}^{-\infty} \mathscr{P}\big(E_j \cap E_{j+1}^c \cap \cdots \cap E_0^c \cap D_1^{i-1} \cap E_i\big) \\ &= 1 - \pi(g_1 + \cdots + g_r) = G_r. \end{aligned}$$

Thus $\mathscr{P}(A \cap D_1^r \cap B) \le \mathscr{P}(D_1^r) = G_r$. Then

$$\begin{aligned} 0 &\le S_2 + S_3 + S_4 \le S_2 + S_3 + 2S_4 \\ &= \mathscr{P}(A \cap D_1^r \cap B) + P\big(A \cap D_{n-s}^{n-1} \cap B\big) \le G_r + G_s. \end{aligned} \tag{A.5}$$

Use (5.7) twice to rewrite the typical term in the double sum S_1 at (A.4) as

$$\mathscr{P}\big(A \cap D_1^{i-1} | E_i\big) \mathscr{P}(E_i) \mathscr{P}\big(E_{n-j} | E_i\big) \mathscr{P}\big(D_{n-j+1}^{n-1} \cap B | E_{n-j}\big). \tag{A.6}$$

Recall α_n in (5.11). Then the expression in (A.6) bounded by

$$(1 \pm \alpha_{n-r-s}/\pi) \mathscr{P}\big(A \cap D_1^{i-1} | E_i\big) \pi \mathscr{P}\big(D_{n-j+1}^{n-1} \cap B | E_{n-j}\big) \pi. \tag{A.7}$$

It follows therefore that the sum S_1 lies in the range

$$(1 \pm \alpha_{n-r-s}/\pi) \mathscr{P}(A \cap (\Omega - D_1^r)) \mathscr{P}\big((\Omega - D_{n-s}^{n-1}) \cap B\big), \tag{A.8}$$

where the product of probabilities lies between

$$(\mathscr{P}(A) - G_r)_+ (\mathscr{P}(B) - G_s)_+ \quad \text{and} \quad \mathscr{P}(A)\mathscr{P}(B). \tag{A.9}$$

Thus,

$$|\mathscr{P}(A \cap B) - \mathscr{P}(A)\mathscr{P}(B)| \leq 2(G_r + G_s) + \alpha_{n-r-s}/\pi, \tag{A.10}$$

proving the lemma. □

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