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LARGE DEVIATION RESULTS FOR A CLASS OF MARKOV CHAINS WITH APPLICATION TO AN INFINITE ALLELES MODEL OF POPULATION GENETICS

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Let $\{X_n\} = \{X_n^{(N)}\}$ be a Markov chain in the probability measures $\mathbf{P}[0,1]$, equipped with a certain metric for the topology of weak convergence, and denote $\mathbf{E}_x(X_1) = \mathbf{f}_N(x)$. Define a projection $\xi = \pi_d x$ on $\mathbf{P}[0,1]$ by defining ξ to be absolutely continuous with respect to Lebesgue measure with constant density $(d+1)x(A)$ on each interval A of an equipartition of $[0,1]$ into $d+1$ intervals, and let $\pi_d \mathbf{P}[0,1] \subset \mathbb{R}^d$ be the natural embedding. Assume $\pi_d \mathbf{f}_N(\xi) \simeq \xi + \beta_N h_d(\xi)$, $h_d(\xi_{0,d}) = \mathbf{0}$ and $\mathrm{Cov}_\xi(\pi_d X_1) \simeq \sigma_d^2(\xi)/N$, in certain senses as $N \to \infty$, where $\xi_{0,d}$ is an asymptotically stable fixed point of $\pi_d \mathbf{f}_N(\xi) = \xi$ and $x_0 = \lim_{d\to\infty} \xi_{0,d}$ exists in $\mathbf{P}[0,1]$. Assuming various regularity conditions and $\beta = \beta_N \to 0$, $N\beta/\log N \to \infty$, it is shown that the expected time it takes the Markov chain to exit a fixed open ball D about x_0 once $X_0 \in D$ is logarithmically equivalent to $\exp[N\beta_N V]$, where $V > 0$ is a limit of solutions $V_d = V_d(h_d, \sigma_d)$ of variational problems of Wentzell–Freidlin type in $\mathbb{R}^d$ as $d \to \infty$. These results apply to an infinite alleles model in population genetics, where $\{X_n\}$ represents the evolution of distributions of types among a population of N randomly mating genes, and where forces of mutation and selection are stronger than effects due to finite population size.

1. Main results. This paper establishes a large deviation result for a population genetics model that can be viewed either as Kingman's house of cards model with genetic drift [Kingman (1978, 1980)], or as a Markov chain version of the measure-diffusion infinite alleles model with stabilizing selection [Ethier and Kurtz (1986)]. The result gives the asymptotic expected time for the population distribution to wander a fixed distance from its equilibrium in a particular metric for the topology of weak convergence of distributions.

To introduce our ideas, consider a class of $\mathbb{R}^d$-valued Markov chains $\{\boldsymbol{\xi}_n\}$ which are strongly attracted to a stable fixed point as follows. Assume that

$$\mathbf{E}_\xi(\boldsymbol{\xi}_1) = f_{N,d}(\xi) \simeq \xi + \beta h_d(\xi), \qquad h_d(\xi_{0,d}) = \mathbf{0}, \quad \text{and}$$

$$\text{(1.1)} \qquad \mathrm{Cov}_\xi(\boldsymbol{\xi}_1) \simeq \frac{\sigma_d^2(\xi)}{N}$$

in senses made precise in Section 2.2, where $N \to \infty$ and $\beta = \beta_N \to 0$, with

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$N\beta \to \infty$. Assume also

$$(1.2)\qquad |f_{N,d}(\xi) - \xi_{0,d}^{(N)}| \le (1 - \beta\kappa)|\xi - \xi_{0,d}^{(N)}|, \qquad \xi \in U,$$

with some $\kappa > 0$ and some bounded connected open set U containing $\xi_{0,d} = \lim_{N\to\infty} \xi_{0,d}^{(N)}$. Under additional regularity assumptions on the Markov chain $\{\boldsymbol{\xi}_n\}$, Morrow and Sawyer (1989) prove that there are constants $V_d > 0$ and $\omega_N \to 0$ such that, for each $\xi \in U$,

$$(1.3)\qquad \lim_{N\to\infty} \mathbf{P}_\xi\big[\exp(N\beta(V_d - \omega_N)) \le T_e \le \exp(N\beta(V_d + \omega_N))\big] = 1,$$

where $T_e = \inf\{n: \boldsymbol{\xi}_n \notin U\}$. This result extends the large deviation theory for solutions of the stochastic differential equation

$$(1.4)\qquad d\boldsymbol{\xi}_t = h_d(\boldsymbol{\xi}_t)\,dt + \sqrt{\varepsilon}\,\sigma_d(\boldsymbol{\xi}_t)\,dW_t$$

with $\varepsilon = 1/N\beta$ and scaling $t = n\beta$ [Freidlin and Wentzell (1984)].

A special motivating case for the class (1.1) is the Wright–Fisher model of population genetics, defined by

$$(1.5)\qquad \boldsymbol{\xi}_{n+1} \approx B(N, f_N(\xi))/N, \quad \text{given } \boldsymbol{\xi}_n = \xi,$$

where $B(N, f)$ denotes a multinomial random variable with frequencies $f \in \{\xi \in \mathbb{R}^d: \xi_i \ge 0, \Sigma\xi_i \le 1\}$. Then $\boldsymbol{\xi}_n$ gives the joint frequencies of $d + 1$ alleles in generation n, and $N\beta \to \infty$ in (1.1) provides a model for situations in which the average stabilizing effect of forces such as mutation and selection is stronger than random effects due to a finite population size N.

The present paper is in turn an extension of the work of Morrow and Sawyer (1989), as follows. Let $\{X_n\} = \{X_n^{(N)}\}$ be a family of Markov chains taking values in the probability measures $\mathbf{P}[0,1]$ on the unit interval. Let $\xi = \pi_d x$ be the conditional expectation of $x \in \mathbf{P}[0,1]$ given an equipartition of the Lebesgue unit interval into $d + 1$ sets [see (1.15)]. We define a metric ρ on $\mathbf{P}[0,1]$ [see (1.14)] that is consistent with the topology of weak convergence and such that

$$(1.6)\qquad \sup_{x\in\mathbf{P}[0,1]} \rho(x, \pi_d x) \to 0 \quad \text{as } d \to \infty.$$

Our extension rests on the basic assumption that the typically non-Markov chains $\{\boldsymbol{\xi}_n\} = \{\pi_d X_n\}$ satisfy (1.1) and (1.2) for the natural embeddings $\pi_d\mathbf{P}[0,1] \subset \mathbb{R}^d$. This approach is consistent with the theory for (1.1) in the following sense. There exist Wright–Fisher models (1.5) that satisfy (1.3) with exponents V_d such that $V_\infty = \lim_{d\to\infty} V_d$ exists and such that $V_\infty > 0$ plays a similar role as V_d in (1.3) but for an infinite alleles model $\{X_n\}$.

1.1. *Infinite alleles model.* A good deal of the perspiration and novelty in this paper lies in its application to a certain infinite alleles model of population genetics. The simplest case of this model is the infinitely many neutral alleles model of Kimura and Crow (1964). Starting with the Wright–Fisher model (1.5) with symmetric multiple way mutation between $d + 1$ allelic types, one may attempt to understand the neutral infinite alleles model by setting $d = \infty$. However, in passing to the case $d = \infty$ it is assumed that any mutant type will be totally new to the population, contrary to the case of symmetric mutation.

To handle this notion, one formulates a general infinite alleles Wright–Fisher model as follows. Let the unit interval $[0, 1]$ be the labels of all possible allelic types. Let $\{t_i\}_{i=1}^N$, $t_i \in [0, 1]$, denote the labels, counting multiplicities, currently present in a population of N randomly mating genes. Let $x = \sum_{i=1}^N \delta(t_i)/N$ denote the empirical distribution of these labels on types. Construct a Markov chain $\{X_n\} = \{X_n^{(N)}\}$ in $\mathbf{P}[0, 1]$ by the following transition mechanism: Given $X_n = x$ in generation n, let X_{n+1} be a random frequency distribution of types in generation $n + 1$ defined by

$$X_{n+1} \approx \sum_{i=1}^{N} \delta(\tau_i)/N, \tag{1.7}$$

where $\{\tau_i\}_{i=1}^N$ are i.i.d. in $[0, 1]$ with common distribution $\mathbf{f}_N(x) \in \mathbf{P}[0, 1]$. The original model of Kimura and Crow (1964) is represented by the formula

$$\mathbf{f}_N(x) = (1 - u)x + u\lambda, \tag{1.8}$$

for a mutation parameter $0 < u < 1$ and Lebesgue measure λ on $[0, 1]$. Note that the process of projections $\{\pi_d X_n\}$ gives back the neutral Wright–Fisher model with symmetric mutation. Unfortunately one does not get back a Markov chain in general under projection, especially in case selection is incorporated in the model; see Section 1.2.

The model (1.7)–(1.8) determines the stochastic dynamics of a population of N genes undergoing random mating in each generation with mutation to totally novel types at rate u per gene per generation. Even though there can be no stationary distribution of an individual type (as each eventually dies out), in the diffusion limit $2Nu \to \theta$ there is a stationary distribution of probability measures on labels which is summarized by the Ewens sampling formula [Ewens (1979) and Ethier and Kurtz (1986), Theorems 10.4.6, 10.4.7]. In fact our work can be regarded as an extension of the well-developed theory of diffusion approximation that arises for the case $\beta = 1/N$ in (1.1) and that is established in general for measure-valued Markov chains [Ethier and Kurtz (1986)]. However, our work embarks with the assumption that the diffusion limit degenerates to the limiting equilibrium of $\mathbf{f}_N(x) = x$.

The infinite alleles model is appealing by the fact that it allows mutations to arise in a multitude of ways, none of which might be successfully modeled alone [Sawyer (1989)]. Thus the gene can be modeled in a unified way by mapping its various traits into the labelling space $[0, 1]$ (or other appropriate compact space). This view tends to support our assumption (1.11) that the total mutation rate Nu tends to infinity. Yet it seems for the present theory that it is not feasible to do this mapping directly in terms of nucleotide construction.

1.2. *Selection.* We introduce selection in the model (1.7) as follows. Define $\mathbf{f}_N(x) \in \mathbf{P}[0,1]$ for all $x \in \mathbf{P}[0,1]$ by its action on measurable sets $A \subset [0,1]$:

$$(1.9) \qquad \mathbf{f}_N(x)(A) = u\lambda(A) + (1-u)\frac{\left[\int_A x(dt)\int_0^1 W(\tau,t)x(d\tau)\right]}{\left[\int_0^1\int_0^1 x(dt)W(\tau,t)x(d\tau)\right]}.$$

Here $W(\tau,t) = 1 + s(\tau,t)$ is a symmetric function that determines the relative fitnesses of the zygotic pairs (τ,t). We assume W is a product function:

$$(1.10) \qquad W(\tau,t) = w(\tau)w(t) \quad \text{for } w(t) = 1 + s(t).$$

Thus the selection acts independently on genes and (1.9) becomes

$$\mathbf{f}_N(x)(A) = u\lambda(A) + (1-u)\frac{\int_A(1+s(t))x(dt)}{\int_0^1(1+s(t))x(dt)}.$$

Let the parameters u and $s(t)$ have the scaling

$$(1.11) \qquad u = \mu\beta \quad \text{and} \quad s(t) = \beta\sigma(t).$$

Assume σ is Lipschitz continuous with

$$(1.12) \qquad \text{(i) } \|\sigma\|_\infty < \mu/3 \quad \text{and} \quad \text{(ii) } \|\sigma\|_{\mathrm{Lip}} < 2\mu/3.$$

Condition (1.12)(i) is used to establish stability of the transition $\xi \to \pi_d \mathbf{f}_N(\xi)$ in Section 2.2.

To find an infinite-dimensional analogue x_0 of $\xi_{0,d}$ in (1.1), we proceed as follows. It is easy to see that the mapping $x \to \mathbf{f}_N(x)$ of (1.9)–(1.11) has an equilibrium $x_0^{(N)}$ with density $x_0^{(N)}(t) = u/[1 - c_0 w(t)]$ relative to Lebesgue measure, where c_0 is chosen to make this expression a probability density. Here, with a slight abuse of notation, we have chosen to identify the probability measure with its density. Calculate, by writing $c_0 = 1 - \beta\alpha_0$, and by (1.11) and (1.12)(i), that $x_0^{(N)}(t)$ tends to $x_0 = x_0(t)$ uniformly on $[0,1]$ for

$$(1.13) \qquad x_0(t) = \mu/[\alpha_0 - \sigma(t)], \quad \text{with } \int_0^1 (\mu/[\alpha_0 - \sigma(b)])\, db = 1.$$

Note that the infinite alleles model (1.7) and (1.9)–(1.12) is a case of Kingman's house of cards [Kingman (1978) or (1980), pages 15–18], wherein an equilibrium fitness distribution for $\mathbf{f}_N(x) = x$ is absolutely continuous with respect to Lebesgue measure.

1.3. *Projection method.* We define a suitable metric on $\mathbf{P}[0,1]$. One approach to this problem is to embed the probability measures in a normed linear space, namely, $\mathbf{Lip}[0,1]^*$, such that the inherited topology on $\mathbf{P}[0,1]$ is just the usual topology of weak convergence of probability measures. Indeed, let Λ be the subset of Lipschitz functions φ on $[0,1]$ with both $\|\varphi\|_\infty \le 1$ and $\|\varphi\|_{\mathrm{Lip}} \le 1$. Observe by the Arzela–Ascoli theorem that Λ is compact in $\mathbf{C}[0,1]$ with the supremum norm. For each $x \in \mathbf{P}[0,1]$ we define the linear functional

$\varphi \to \int_0^1 \varphi(t)x(dt)$, $\varphi \in \Lambda$. By the fact that **Lip**[0, 1] is dense in **C**[0, 1], there is a one–one correspondence between these functionals and **P**[0, 1]. We thus define the metric

$$(1.14) \qquad \rho(x,y) = \sup_{\varphi \in \Lambda} \left| \int_0^1 \varphi(t)[x(dt) - y(dt)] \right|$$

on **P**[0, 1]. Then it follows by the fact that **Lip**[0, 1] is dense in **C**[0, 1] and by the compactness of Λ that the $\rho(\cdot, \cdot)$-topology on **P**[0, 1] is just the topology of weak convergence [cf. Hutchinson (1981)]. A referee points out another way to get at this metric. Indeed, by writing $\varphi(t) = \varphi(0) + \int_0^t \varphi'(s)\,ds$, substituting the definition (1.14) of $\rho(x,y)$ and changing the order of integration in the resulting double integral, it holds that $\rho(x,y) = \int_0^1 |F_x(t) - F_y(t)|\,dt$, for all $x, y \in$ **P**[0, 1], where $F_x(t) = x([0,t])$.

The projection π_d: **P**[0, 1] $\to$ **P**[0, 1] is defined to be the conditional expectation relative to an equipartition $\{A_k\}_{k=1}^{d+1}$ of the Lebesgue unit interval, that is,

$$(1.15) \quad \int_0^1 \varphi(t)[\pi_d x](dt) = (d+1) \sum_{k=1}^{d+1} x(A_k) \int_{A_k} \varphi(t)\lambda(dt), \quad \text{for all } \varphi \in \Lambda,$$

where

$$A_k = [(k-1)/(d+1), k/(d+1)), \; k = 1, \ldots, d, \quad \text{and}$$
$$A_{d+1} = [d/(d+1), 1],$$

and where λ denotes Lebesgue measure. Obviously π_d extends to the finite signed measures on [0, 1]. Embed π_d**P**[0, 1] $\subset \mathbb{R}^d$ by identifying the extremal points of π_d**P**[0, 1] with the extremal points of a d-dimensional simplex in $\mathbb{R}^d$. Specifically, let b_k be the probability measure with density $(d+1)1_{A_k}(t)$ relative to Lebesgue measure, $k = 1, \ldots, d+1$. The embedding is given by linearity and the correspondence

$$(1.16) \qquad \begin{aligned} b_k &\to (0, \ldots, 1, 0, \ldots, 0) \in \mathbb{R}^d, \qquad k = 1, \ldots, d, \\ b_{d+1} &\to \mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^d. \end{aligned}$$

Hence an element $\xi \in \pi_d$**P**[0, 1] $\subset \mathbb{R}^d$ is a subprobability vector $\xi = (\xi_1, \ldots, \xi_d)$. Notice that the finite-dimensional process $\{\pi_d X_n\}_{n \ge 1}$ is not Markovian for the infinite alleles model (with selection) defined by (1.7) and (1.9)–(1.12).

The uniform approximation (1.6) is verified as follows. Let $x \in$ **P**[0, 1]. Then by (1.14) and (1.15),

$$\rho(x, \pi_d x) = \sup_{\varphi \in \Lambda} \left| \sum_{k=1}^{d+1} \int 1_{A_k}(t) \left[\varphi(t) - (d+1) \int_{A_k} \varphi(\tau)\lambda(d\tau) \right] x(dt) \right|.$$

Here, for each k, the term inside the square brackets is bounded uniformly in absolute value by

$$\sup_{\varphi\in\Lambda}\left|(d+1)\int_{A_k}(\varphi(t)-\varphi(\tau))\lambda(d\tau)\right| \le \sup_{\varphi\in\Lambda}\|\varphi\|_{\mathrm{Lip}}(d+1)\int_0^{1/(d+1)}t\lambda(dt)$$
$$= 1/[2(d+1)].$$

Therefore condition (1.6) is verified with the following rate of convergence:

(1.17) $$\sup_{x\in\mathbf{P}[0,1]}\rho(x,\pi_d x)\le\delta,\quad \text{for } d=d(\delta):=[1/(2\delta)].$$

Notice that the Euclidean metric is continuous with respect to ρ on $\pi_d\mathbf{P}[0,1]$. This is seen by choosing φ to be linear on A_k and zero otherwise. The corresponding continuity of ρ relative to the Euclidean metric is clear by (2.49). However, note that if $x_\varepsilon=\delta_{t(\varepsilon)}$ and $y_\varepsilon=\delta_{t(\varepsilon)+\varepsilon}$ such that $t(\varepsilon)$ and $t(\varepsilon)+\varepsilon$ belong to A_k and A_{k+1}, respectively, for some $k\le d$, then $\rho(x_\varepsilon,y_\varepsilon)\to 0$ as $\varepsilon\to 0$, while $\rho(\pi_d x_\varepsilon,\pi_d y_\varepsilon)=1/(d+1)$ for all $\varepsilon>0$. Hence π_d is not continuous on $(\mathbf{P}[0,1],\rho(\cdot,\cdot))$.

We introduce next the variational problem that describes the rough characteristics of the time it takes $\{X_n\}$ to leave an open ball D about the limiting equilibrium x_0. In the finite-dimensional case (1.1), this problem is defined in terms of the large deviation exponents

(1.18) $$V_d(U)=\inf\tfrac{1}{2}\int_0^T\left(\dot u-h_d(u),\sigma_d^{-2}(u)(\dot u-h_d(u))\right)dt,$$

where U is a domain containing $\xi_{0,d}$ and where the infimum is extended over all $T>0$ and all trajectories u such that $u(0)=\xi_{0,d}$, $u(t)\in U$, $0\le t<T$ and $u(T)\in\partial U$. Define the open ball

(1.19) $$D=D(r)=\{x\in\mathbf{P}[0,1]:\rho(x,x_0)<r\},\quad\text{for all } r>0.$$

If $d+1=2^l$ in (1.15), then it is shown in Sections 2.4 and 5 that

$$\lim_{d\to\infty}V_d(\operatorname{int}(\pi_d D)) \text{ exists and is equal to}$$

(1.20) $$V_\infty(D)=\inf-2\left\{\int\sigma(t)[y(dt)-x_0(t)\,dt]+\mu\int\log[(dy_{\mathrm{ac}}/dt)/x_0(t)]\,dt\right\},$$

where μ, $\sigma(t)$ and $x_0(t)$ are defined by (1.9)–(1.13), and where the infimum is extended over all probability measures y with $\rho(y,x_0)=r$, and y_{ac} denotes the absolutely continuous part in the Lebesgue decomposition of y. Note that $V_\infty(D)<\infty$ if and only if $r<\sup\{\rho(y,x_0): y\in\mathbf{P}[0,1]\}$. Intuitively, $\exp(-N\beta V_\infty(D))$ is the instantaneous rate with which the process $\{X_n\}$ tunnels out to the boundary of D from its equilibrium x_0. In Section 2.4 it is shown that $V_\infty(D)>0$. Moreover, it is shown in Section 5 that $V_\infty(D)$ is continuous as a function of the radius r of D. This property is crucial for the following result.

THEOREM 1.1. *Let $\{X_n^{(N)}\}$ be the infinite alleles model with selection defined by* (1.7) *and* (1.9)–(1.12), *and let $N \to \infty$, $\beta = \beta_N \to 0$ with $N\beta/\log N \to \infty$. Define the exit time $T_e = \inf\{n: X_n \notin D\}$ for some radius $r > 0$ of D in* (1.19). *Then* (1.20) *holds and*

$$\text{(i)} \qquad \lim_{N\to\infty} (\log \mathbf{E}_x T_e)/N\beta = V_\infty(D), \quad \text{for all } x \in D.$$

Further, if $V_\infty(D) < \infty$, then there exist constants $\omega_N \to 0$ such that, for all $x \in D$,

$$\text{(ii)} \quad \lim_{N\to\infty} \mathbf{P}_x\big[\exp\big(N\beta(V_\infty(D) - \omega_N)\big) \le T_e \le \exp\big(N\beta(V_\infty(D) + \omega_N)\big)\big] = 1.$$

REMARK 1.1. We are also interested in the probability that the Markov chain $\{X_n\}$ is found a fixed distance from the limiting equilibrium x_0 after many generations. In the neutral case $[\sigma(t) \equiv 0]$ we know that $\{\pi_d X_n\}$ is ergodic for each fixed d, and thus there is a unique limiting distribution $\mu_N \in \mathbf{P}(\mathbf{P}[0,1])$ for $\{X_n^{(N)}\}$. Hence, by the standard argument of Freidlin and Wentzell [(1984), Chapter 4], if $F = D^c$ is the closed annulus of inner radius $r > 0$ about x_0 in $\mathbf{P}[0,1]$, then, under the neutral case of Theorem 1.1,

$$(1.21) \qquad \lim_{N\to\infty} \log[\mu_N(F)]/N\beta = -V_\infty(D).$$

Since revising this paper the author found an ergodicity result for the infinite alleles model with selection based on the method of Barnsley and Elton (1988). It turns out that the Markov chain (1.7) is ergodic whenever $\mathbf{f}_N$ is a strict contraction on $(\mathbf{P}[0,1], \rho(\cdot,\cdot))$. This holds under (1.9)–(1.11) if $3\|\sigma\|_\infty + 2\|\sigma\|_{\mathrm{Lip}} < \mu$. Thus (1.21) continues to hold when this last condition is added to the hypothesis of Theorem 1.1.

Theorem 1.1 is extended in Section 2 to a class of Markov chains taking values in a convex compact metric space $(Q, \rho(\cdot,\cdot))$ embedded in a linear space such that for a sequence of linear projections π_d, $d \to \infty$, (1.6) holds with Q in place of $\mathbf{P}[0,1]$ and (1.1) holds in an appropriate sense for the finite-dimensional projected processes $\{\xi_n\} = \{\pi_d X_n\}$. This framework is particularly useful in organizing our application to the infinite alleles model. However, the question of whether the general analogue of $V_\infty(D)$ in (1.20) is continuous in the radius r of an open ball in Q is left open. Nevertheless, in finite dimensions the regularity of the large deviation exponent $V_d(U)$ as a function of the domain U is established in Morrow and Sawyer (1989) for a class of domains satisfying the well-sharpened-pencil condition. Hence the present work improves the condition $N\beta^2/\log N \to \infty$ assumed by these authors. Finally, our approach is expected to yield large deviation exponents for certain measure-valued diffusions as well as related measure-valued birth and death processes.

2. General framework. This section is devoted to formulating a general set of hypotheses for the projection method outlined in Section 1. For the sake of continuity these hypotheses are established for the infinite alleles model in turn within this framework. Here and in the following the infinite alleles model means the model (1.7) and (1.9)–(1.12).

To motivate the development, the main steps in the Proof of Theorem 1.1 are outlined. First, note that a key difficulty in extending the continuous time theory for (1.4) even in the finite-dimensional case arises from a lack of scaling in the noise of the Markov chain (1.1). This problem is solved in Morrow and Sawyer (1989) by means of certain tightness arguments for the exponents V_d in (1.18). Since these arguments are independent of the Markov property, they carry over to the process $\{\xi_n\} = \{\pi_d X_n\}$ under precise versions of conditions (1.1) and (1.2). However, a new difficulty for our extension is to obtain large deviation estimates for the finite-dimensional process. This is done by extending the approaches of Ventsel' (1976) and Darden (1983), especially for the lower large deviation estimate of Section 3.3. This latter estimate is utilized in an approximation scheme in Section 4 to obtain an upper bound on the exit time. Our key idea for these last two steps is to link the dimension d with certain convergence properties of the Legendre transform of the finite-dimensional process as well as with regularity properties of the variational problems V_d. These properties are made precise by conditions (2.28) and (2.29) of Section 2.3 and by conditions (2.40)–(2.42) of Section 2.4. In fact the exit times are controlled from above and below by limits of d-dimensional large deviation exponents corresponding to domains that are respectively larger and smaller than the given domain. This program is successfully carried to a conclusion for the infinite alleles model in Section 5, where it is shown that these limiting exponents vary continuously with the radius of the domain. Thus the formula (1.20) is established.

2.1. *Setting up the model.* For each $N \geq 1$, let $\{X_n\} = \{X_n^{(N)}\}_{n \geq 1}$ be a Markov chain taking values in a convex compact metric space $(Q, \rho(\cdot, \cdot))$ embedded in a vector space L. Assume that

$$\begin{aligned} &\mathbf{E}_x(X_1) = \mathbf{f}_N(x) = x + \beta \mathbf{h}_N(x), \qquad x \in Q, \\ &\mathbf{h}_N(x_0^{(N)}) = \mathbf{0}, \end{aligned} \tag{2.1}$$

where $\beta = \beta_N \to 0$ and $N\beta \to \infty$ as $N \to \infty$. Assume that $x_0 = \lim_{N \to \infty} x_0^{(N)}$ exists in Q and let $D \subset Q$ be an open connected subset containing x_0.

For each d in a sequence of integers tending to ∞, let π_d: $Q \to Q$ be maps with d-dimensional range for all d and such that π_d extends to a linear mapping of the subspace of L generated by Q. We also assume that there are distinguished metric space embeddings $\pi_d Q \subset \mathbb{R}^d$ and that $\pi_d Q$ is compact for all d. Suppose further that for every $\delta > 0$ there exists an integer $d(\delta)$ with $d(\delta) \to \infty$ as $\delta \to 0$ such that

$$\sup_{x \in Q} \rho(x, \pi_{d(\delta)} x) \leq \delta. \tag{2.2}$$

By the computations leading up to (1.13) and by (1.17), it follows that (2.1) and (2.2) hold for the infinite alleles model.

If the finite-dimensional approximant is degenerate, then technical difficulties arise, which unfortunately cannot be avoided near the boundary of $\mathrm{int}(\pi_d D)$. Under the embedding $\pi_d Q \subset \mathbb{R}^d$, define

$$\Sigma_{N,d}^2(x) = \mathrm{Cov}_x(\pi_d X_1),$$

for all $x \in Q$. If $\varepsilon > 0$, define $D^\varepsilon = \{x \in Q\colon \rho(x, D) < \varepsilon\}$ and $D^{-\varepsilon} = \{x \in D\colon \rho(x, D^c) > \varepsilon\}$, with $D^0 = D$. Also, for any domain $G \subset Q$, let $(G_{-\gamma,d})_{\gamma>0}$ be the collection of subdomains of $\mathrm{int}(\pi_d Q)$ defined by $G_{-\gamma,d} = \{x \in \mathrm{int}(\pi_d G)\colon |x - \partial(\mathrm{int}(\pi_d Q))| > \gamma\}$. Here and in the sequel if $A \subset \pi_d Q$, then $\mathrm{int}(A)$ denotes the interior of $A \subset \mathbb{R}^d$ and $|\cdot|$ denotes the Euclidean norm. In the proof of our main result we take $G = D^\varepsilon$ and we send the parameters introduced above to their respective limits in the following order: $N \to \infty$, $\gamma \to 0$, $d \to \infty$ ($\delta \to 0$) and $\varepsilon \to 0$.

Assume that for each $\gamma > 0$ and $d \geq 1$ there exists $\lambda_{\gamma,d} > 0$ and a big-O function O_d such that, for all $x \in Q$ such that $\xi = \pi_d x \in Q_{-\gamma,d}$,

$$\lambda(x) := \left[\text{minimum eigenvalue of } \Sigma_{N,d}^2(x)\right] \geq \lambda_{\gamma,d}/N. \tag{2.3}$$

Assume further that for each $d \geq 1$ there is a constant $c_d > 0$ and a big-O function O_d such that, uniformly for $\xi, \eta \in \pi_d Q$,

$$\left|N\Sigma_{N,d}^2(\xi) - N\Sigma_{N,d}^2(\eta)\right|_{\mathbb{R}^{d\times d}} \leq c_d|\xi - \eta| + O_d(\beta), \quad \text{as } N \to \infty. \tag{2.4}$$

We also assume that, uniformly for $\xi \in \pi_d Q$,

$$\lim_{N\to\infty} N\Sigma_{N,d}^2(\xi) = \sigma_d^2(\xi) \tag{2.5(i)}$$

and

$$\lim_{N\to\infty} \pi_d \mathbf{h}_N(\xi) = h_d(\xi) \tag{2.5(ii)}$$

and that h_d and σ_d^2 are continuously differentiable on $\pi_d Q \subset \mathbb{R}^d$.

These conditions are verified for the infinite alleles model as follows. Under $\mathbf{P}_x$ and the embedding (1.15) and (1.16), $N\pi_d X_1$ is a multinomial variable with size N and parameters $\mathbf{f}_N(x)(A_k)$ for $\mathbf{f}_N(x)$ defined by (1.9) and (1.10). Furthermore, by (1.10)–(1.12),

$$\mathbf{f}_N(x)(A_k) = (1-u)\left[\frac{\int_{A_k} w(t)x(dt)}{\int_0^1 w(t)x(dt)}\right] + \frac{u}{d+1} = x(A_k) + O(\beta), \tag{2.6}$$

with $O(\beta)$ uniform in $x \in Q$. It follows from (2.6) that there is a number $\varepsilon(\gamma, d) > 0$ such that for all $x \in \pi_d Q$ such that $\xi = \pi_d x \in Q_{-\gamma,d}$,

$$1 - \varepsilon(\gamma, d) \geq \mathbf{f}(x)(A_k) \geq \varepsilon(\gamma, d), \quad \text{for all } k = 1, \dots, d+1. \tag{2.7}$$

Therefore, since the covariance of $\pi_d X_1$ is given by

$$\left(\Sigma_{N,d}^2(x)\right)_{ij}^{d\times d} = \frac{1}{N}\mathbf{f}(x)(A_i)\left(\delta_{ij} - f(x)(A_j)\right), \qquad i, j = 1, \dots, d,$$

we get by (2.6) and (2.7) that (2.3), (2.4) and (2.5)(i) hold with

$$\sigma_d^2(\xi)_{ij} = \xi_i(\delta_{ij} - \xi_j), \qquad i, j = 1, \dots, d. \tag{2.8}$$

By (1.9)–(1.12) it is not difficult to compute that (2.5)(ii) holds for

$$(2.9)\quad (h_d)_i(\xi) = (\bar{\sigma}_i - \sigma^*(\xi))\xi_i + \mu\big((d+1)^{-1} - \xi_i\big), \qquad i = 1, \dots, d+1,$$

where $\bar{\sigma}_i = (d+1)\int_{A_i}\sigma(t)\,dt$ and $\sigma^*(\xi) = \sum_{i=1}^{d+1}\bar{\sigma}_i\xi_i$. Here and in the sequel, by a slight abuse of notation, $\xi = \pi_d x$ is viewed as a probability vector under (1.16) by introducing the extra component $\xi_{d+1} = 1 - \sum_{i=1}^{d}\xi_i$. It is easy to check that the $d+1$ components of h_d so defined under this convention sum to zero.

Define the cumulant generating function of the finite-dimensional approximant for all $z \in \mathbb{R}^d$ and all $x \in Q$ by

$$(2.10)\qquad G_d(N, x, z) = \log \mathbf{E}_x \exp(z \cdot \pi_d(X_1 - x)).$$

The second derivative of G_d in z is a weighted covariance, which is denoted by

$$(2.11)\qquad \tilde{\Sigma}^2_{N,d,z}(x) = \nabla_z^2 G_d(N, x, z).$$

Assume that for any constant $\zeta_d > 0$ there is a big-O function $O_d = O_{\zeta_d}$ such that, for all $x \in Q$ and all $|z| \le \zeta_d N$,

$$(2.12)(\mathrm{i})\qquad \tilde{\Sigma}^2_{N,d,z}(x) = \Sigma^2_{N,d}(x) + O_d\left(\frac{|z| + N\beta}{N^2}\right) \quad \text{as } N \to \infty,$$

and there is a constant $C_0 = C_0(d)$ such that, for all $x \in Q$ and any z,

$$(2.12)(\mathrm{ii})\qquad \left[\text{largest eigenvalue of } \tilde{\Sigma}^2_{N,d,z}(x)\right] \le \frac{C_0}{N}.$$

Define the Legendre transform of $G_d(N, x, \cdot\,)$ for all $x \in Q$ and $u \in \mathbb{R}^d$ such that $\pi_d x + u \in \pi_d Q$ by

$$(2.13)\qquad H_d(N, x, u) = \sup_{z \in \mathbb{R}^d} (z \cdot u - G_d(N, x, z)).$$

Assume that this supremum is uniquely achieved with $z = z_d(N, x, u)$ and that for $\gamma > 0$ and $d \ge 1$ there is a constant $\zeta_{\gamma,d} < \infty$ so that, for all $x \in Q$, all $\xi = \pi_d x$ and all $u \in \mathbb{R}^d$ such that both $\xi, \xi + u \in Q_{-\gamma,d}$,

$$(2.14)\qquad |z_d(N, x, u)| \le \zeta_{\gamma,d} N.$$

Hypotheses (2.3), (2.12)(i), (2.12)(ii) and (2.14) are more or less typical assumptions for proving large deviation bounds for $\mathbb{R}^d$-valued Markov chains of type (1.1) [Morrow and Sawyer (1989)]. Regularity of the Legendre transform is also typically assumed, but this is deferred until Section 2.3.

The assumptions (2.12)(i), (2.12)(ii) and (2.14) simplify for Markov chains of the form

$$(2.15)\qquad X_1 \approx \frac{1}{N}\sum_{i=1}^{N} Y_{i,x}^{(N)}, \quad \text{given } X_0 = x,$$

where $\{Y_{i,x}^{(N)};\ 1 \le i \le N\}$ are i.i.d. variables taking values in Q. Indeed, since $\tilde{\Sigma}^2_{N,d,z}(x)$ is the covariance of $\pi_d X_1$ under $\tilde{\mathbf{P}}_x$ for $d\tilde{\mathbf{P}}_x/d\mathbf{P}_x = \exp(z \cdot \pi_d X_1)/\mathbf{E}_x[\exp(z \cdot \pi_d X_1)]$, and ξ is bounded for $\xi \in \pi_d Q$, it is easy to see that (2.15) implies (2.12)(i). Also (2.14) is satisfied if a change in the mean of $\pi_d Y_x$ to $\xi + u$ is possible under a change of measure with weight $\exp(z \cdot \pi_d Y_x)$ for some $|z| \le \zeta_{\gamma,d}$ and some finite constant $\zeta_{\gamma,d}$.

By these last remarks, it follows that (2.12)(i), (2.12)(ii), and (2.14) hold for the infinite alleles model since by (2.7) we can find the constant $\zeta_{\gamma,d}$ and because (2.12)(ii) is satisfied with $C_0 = \frac{1}{4}$. In fact the maximum eigenvalue of a covariance matrix with form $\eta_i(\delta_{ij} - \eta_j)$, $i, j = 1, \dots, d$, for subprobability vectors $\eta = (\eta_1, \dots, \eta_d)$, is bounded by $\frac{1}{4}$ for all d by the identity $\sum\sum_{i,j=1}^d v_i \eta_i(\delta_{ij} - \eta_j)v_j = \sum_{i=1}^d \eta_i(1 - \eta_i)v_i^2 - (\sum_{i=1}^d v_i \eta_i)^2$.

2.2. *Stability*. The analogue of the conditions (1.1) and (1.2) for the process $\{\pi_d X_n\}$ is made precise as follows. Assume that for each d there is an asymptotically fixed stable point $\xi_{0,d}^{(N)}$ for the dynamic $\xi \to \pi_d \mathbf{f}_N(\xi)$ on $\pi_d Q$ as follows. Assume that $\pi_d \mathbf{h}_N(\xi_{0,d}^{(N)}) = \mathbf{0}$ and that there exist constants $\kappa_0 > 0$, K_d and C_1, big-O functions O_d, and $N_0 < \infty$ such that the following hold:

(2.16)
(i) for all $\xi \in \pi_d Q$, $d \ge 1$ and $N \ge N_0$,
$|\pi_d \mathbf{f}_N(\xi) - \xi_{0,d}^{(N)}| \le (1 - \kappa_0 \beta)|\xi - \xi_{0,d}^{(N)}|$;

(ii) uniformly for $\xi, \eta \in \pi_d Q$, $|\pi_d(\mathbf{h}_N(\xi) - \mathbf{h}_N(\eta))| \le K_d|\xi - \eta| + O_d(\beta)$ as $N \to \infty$;

(iii) for all $\gamma > 0$ and $d \ge 1$,
$$\sup_{N \ge 1} \sup_{x \in Q,\, \pi_d x \in Q_{-\gamma,d}} |\pi_d \mathbf{h}_N(x)| < \infty,$$

where $\mathbf{f}_N$ and $\mathbf{h}_N$ are defined by (2.1). Suppose further that there is a point $\xi_{0,d} \in \operatorname{int}(\pi_d Q)$ such that

$$\text{(i)}\ \xi_{0,d} = \xi_{0,d}^{(N)} + o(1) \quad \text{as } N \to \infty \quad \text{and} \quad \text{(ii)}\ h_d(\xi_{0,d}) = \mathbf{0}, \tag{2.17}$$

where h_d is defined by (2.5)(ii). Assume moreover that the asymptotic equilibrium $x_0 \in D$ from (2.1) is the limit of the asymptotic stable fixed points $\xi_{0,d}$ in (2.17):

$$\rho(\xi_{0,d}, x_0) \to 0 \quad \text{as } d \to \infty. \tag{2.18}$$

These stability criteria are now established for the infinite alleles model. In general if $w(t)$ is a function on the unit interval, define

$$\pi_d w(t) = \sum_{k=1}^{d+1} \overline{w}_k 1_{A_k}(t), \quad \text{for } \overline{w}_k = (d+1)\int_{A_k} w(t)\lambda(dt).$$

Define $\xi_0 = \xi_{0,d}^{(N)}$ by writing its density $\xi_0(t)$ with respect to Lebesgue measure as

$$\xi_0(t) = \sum_{k=1}^{d+1} \bar{\xi}_k 1_{A_k}(t), \tag{2.19}$$

subject to the condition

$$(\xi_0(t) - u)w^* = (1 - u)\xi_0(t)\pi_d w(t),$$

where we denote

$$w^* = \int_0^1 w(t)\xi_0(t)\lambda(dt),$$

and where u and $w(t)$ are given by (1.9)–(1.11). By (2.19), one computes that $\bar{\xi}_k = u/[1 - c_0\overline{w}_k]$ for a number $c_0 = (1 - u)/w^*$ chosen to make $\xi_0(t)$ a density. Since $\xi_0(t)$ is a density this in turn implies by (1.11), (1.12)(i) that

$$\begin{aligned} 1 - 4\frac{\mu}{3}\beta(1 + o(1)) \le c_0 \le 1 - 2\frac{\mu}{3}\beta(1 + o(1)) \\ \frac{3}{5}(1 + o(1)) \le \bar{\xi}_k \le 3(1 + o(1)). \end{aligned} \tag{2.20}$$

In what follows, let $\xi(t) \in \pi_d Q$ be a fixed density different from $\xi_0(t)$ and denote $w^{**} = \int_0^1 w(t)\xi(t)\,dt$. To establish (2.16)(i), use $w^*(\xi_0 - u\lambda) = (1 - u)\pi_d(w\xi_0)$ to calculate that

$$\begin{aligned} &\pi_d \mathbf{f}_N(\xi) - \xi_0 \\ &= (1 - u)\frac{\pi_d(w\xi)}{w^{**}} + u\lambda - \xi_0 \\ &= \frac{(1 - u)\pi_d(w\xi) - w^*(\xi_0 - u\lambda)}{w^*} + \left[\frac{1}{w^{**}} - \frac{1}{w^*}\right](1 - u)\pi_d(w\xi) \\ &= (1 - u)\frac{\pi_d(w\xi - w\xi_0)}{w^*} + (1 - u)(w^* - w^{**})\frac{\pi_d[(w\xi)/w^{**}]}{w^*} \\ &= c_0\pi_d[w(\xi - \xi_0)] + c_0(w^* - w^{**})\pi_d(\text{probability}). \end{aligned} \tag{2.21}$$

Thus, by (2.21) and (1.10),

$$|\pi_d \mathbf{f}_N(\xi) - \xi_0|_{\mathbb{R}^d} \le c_0|\pi_d[w(\xi - \xi_0)]|_{\mathbb{R}^d} + c_0\left|\int s(t)(\xi - \xi_0)(t)\lambda(dt)\right|. \tag{2.22}$$

Hence from (1.11), (1.12)(i), (2.20) and (2.22) we have

$$|\pi_d \mathbf{f}_N(\xi) - \xi_0|_{\mathbb{R}^d} \le c_0(1 + 2\|s\|_\infty)|\xi - \xi_0|_{\mathbb{R}^d} \le (1 - \kappa_0\beta)|\xi - \xi_0|_{\mathbb{R}^d}, \tag{2.23}$$

for all $N \ge N_0$. Therefore, (2.16)(i) holds since $\xi_0 = \xi_{0,d}^{(N)}$ in (2.23). Note also that, by (2.19)–(2.21), inequalities (2.22) and (2.23) hold with $\rho(\cdot,\cdot)$ in place of the Euclidean norm as follows:

$$\rho\big(\pi_d \mathbf{f}_N(\xi), \xi_{0,d}^{(N)}\big) \le (1-\kappa_0\beta)\rho\big(\xi, \xi_{0,d}^{(N)}\big). \tag{2.24}$$

Condition (2.16)(ii) follows by elementary computations from (1.9)–(1.12) and the Cauchy–Schwarz inequality with

$$K_d = \text{const.}(\sqrt{d}\,). \tag{2.25}$$

Condition (2.16)(iii) is likewise easily verified. Condition (2.17) is satisfied for $\xi_0 = \xi_{0,d}^{(N)}$ defined by (2.19). By the continuity of $\sigma(t)$, (2.18) holds with $\xi_{0,d}(t) = \mu/[\alpha_{0,d} - \pi_d\sigma(t)]$, for a constant $\alpha_{0,d}$ chosen to make this expression a density relative to Lebesgue measure, and with x_0 given by (1.13).

The stability estimate (2.23) and (2.24) is now used to derive some further regularity properties of the continuous time dynamical system (1.4) with $\varepsilon = 0$. In general, let $u(t)$, $t \ge 0$, be a trajectory in $\mathbb{R}^d$ satisfying $\dot{u}(t) = h_d(u(t))$, where h_d is defined in (2.5)(ii). Suppose there exist $t_0(d) > 0$, $\gamma(t,d) > 0$ for $t > 0$, $\gamma(d) > 0$, and $r(x) > 0$ for $x \in D$, such that the following hold:

(2.26)

(i) if $d \ge 1$, $0 < t_0 < t_0(d)$, and $\xi \in \pi_d Q$, then $u(0) = \xi$ implies $u(t) \in Q_{-\gamma,d}$ for $\gamma = \gamma(t_0, d)$ and all $t \ge t_0$;

(ii) if $d \ge 1$ and $0 < \gamma < \gamma(d)$, then $u(0) \in Q_{-\gamma,d}$ implies $u(t) \in Q_{-\gamma,d}$ for all $t \ge 0$;

(iii) if $x \in D$ and d is sufficiently large depending on x, then $u(0) = \pi_d x$ implies $u(t) \in D^{-r}$ for $r = r(x)$ and all $0 \le t \le 1$.

Here, in (2.26)(ii) and (iii), the embedding $\pi_d Q \subset \mathbb{R}^d$ is used to view $u(t)$ as an element of Q.

These conditions are verified for the infinite alleles model with D in (1.19) by applying (2.24) and the following Gronwall-type inequality. Let $(a_n)_{n\ge 0}$ be nonnegative real numbers.

$$\text{If } a_n \le A + B\sum_{m=0}^{n-1} a_m \quad \text{for all } n \ge 0, \quad \text{then } a_n \le A(1+B)^n \text{ for all } n \ge 0.$$

As in (2.9) it is convenient to work in the set of probability vectors

$$\Delta^d = \Big\{(\xi_1, \dots, \xi_{d+1}) \in \mathbb{R}^{d+1} \colon \xi_i \ge 0,\ \sum \xi_i = 1\Big\}.$$

Define

$$\begin{aligned}\partial(\Delta^d) &= \big\{\xi \in \Delta^d \colon \xi_i = 0 \text{ for at least one } i\big\}\\ &= \big\{\xi \in \Delta^d \colon (\xi_1, \dots, \xi_d) \in \partial(\pi_d \mathbf{P}[0,1]) \subset \mathbb{R}^d\big\}.\end{aligned}$$

Let $\xi \in \partial(\Delta^d)$ and define a trajectory $u(t)$, $t \geq 0$, in Δ^d by $\dot{u} = h_d(u)$, $u(0) = \xi$, for h_d defined by (2.9). When convenient we regard $u(t) \in \pi_d \mathbf{P}[0, 1]$ by dropping the coordinate $u_{d+1}(t)$. By (2.9) and (1.12)(i) we compute that for any point ξ not at a corner of $\partial(\Delta^d)$, a unit inner normal $\mathbf{n} \in \mathbb{R}^{d+1}$ at ξ satisfies $\mathbf{n} \cdot h_d(\xi) > \mu/[3(d+1)]$. In fact, on a face of $\partial(\Delta^d)$ with $\xi_{k_1}, \dots, \xi_{k_j} = 0$, for some $j \geq 1$, an inner unit normal is given by $\mathbf{n}_k = c/(d+1)$ for $k = k_i$, $i = 1, \dots, j$, and $\mathbf{n}_k = -cj/[(d+1-j)(d+1)]$ for all other k, with $c = [(d+1-j)(d+1)/j]^{1/2}$. It follows that

$$\mathbf{n} \cdot h_d(\xi) = \sum (\bar{\sigma}_k - \sigma^*(\xi) - \mu)\xi_k \mathbf{n}_k > \mu/(3c) > \mu/[3(d+1)].$$

Hence $u(t) \in \operatorname{int}(\pi_d \mathbf{P}[0, 1])$ for all $t > 0$. Define

$$\gamma(t_0, d) = \mu t_0/[3(d+1)]. \tag{2.27}$$

The above estimate shows that $u(t_0) \in Q_{-\gamma, d}$ for $\gamma = \gamma(t_0, d)$ and all sufficiently small t_0. It also shows by continuity that $u(0) \in Q_{-\gamma, d}$ implies $u(t) \in Q_{-\gamma, d}$ for all $t \geq 0$ and all $0 < \gamma < \gamma(d)$ for some sufficiently small $\gamma(d) = \gamma(d, \mu)$. Hence (2.26)(i) and (ii) hold.

Next, let $x \in D$ and $u(0) = \xi = \pi_d x \in \Delta^d$, so that by (1.17), $u(0) \in D$ for large d depending on x. Define $f_d(u) = u + \beta h_d(u)$ and $p(n\beta) = f_d^{\circ n}(\xi)$ with $p(0) = \xi$. Since h_d is continuously differentiable, $u(t + \beta) = f_d(u(t)) + o(\beta)$, with the constant implied by little-o depending only on d. Hence

$$\begin{aligned} & u(n\beta + \beta) - p(n\beta + \beta) \\ & \quad = f_d(u(n\beta)) + o(\beta) - f_d(p(n\beta)) \\ & \quad = u(n\beta) - p(n\beta) + \beta[h_d(u(n\beta)) - h_d(p(n\beta))] + o(\beta). \end{aligned}$$

Therefore since f_d preserves Δ^d, we have $p(n\beta) \in \Delta^d$, so that by (2.5)(ii) and (2.16)(ii),

$$|u(n\beta + \beta) - p(n\beta + \beta)| \leq (1 + K_d \beta)|u(n\beta) - p(n\beta)| + o(\beta).$$

Also, by the same calculation and by definition of $\rho(\cdot, \cdot)$ in (1.14),

$$\rho(u(n\beta + \beta), p(n\beta + \beta)) \leq (1 + L_d \beta)\rho(u(n\beta), p(n\beta)) + o(\beta)$$

for some constant $L_d < \infty$, taking into account the equivalence of norms in finite-dimensional spaces. Hence, it follows by the Gronwall inequality that

$$\rho(u(n\beta), p(n\beta)) \leq o(n\beta)(1 + L_d \beta)^n.$$

Next, define $p_N(n\beta) = (\pi_d \mathbf{f}_N)^{\circ n}(\xi)$ with $p_N(0) = \xi$. We know $p_N(n\beta) \in \Delta^d$ since $\pi_d \mathbf{f}_N(\xi) = \mathbf{E}_\xi \pi_d(X_1) \in \Delta^d$. Since also $\{p(n\beta): n \geq 0\} \subset \Delta^d$, it follows by (2.16)(ii) and the Gronwall inequality as above that $\rho(p_N(n\beta), p(n\beta)) \leq$

$o(n\beta)(1 + L_d\beta)^n$. But by (2.24), $\rho(p_N(n\beta), \xi_{0,d}^{(N)}) \le (1 - \kappa_0\beta)^n \rho(\xi, \xi_{0,d}^{(N)})$. Therefore, putting these estimates together, it holds that

$$\rho\big(u(t), \xi_{0,d}^{(N)}\big) \le (1 + o(1))\exp(-\kappa_0 t)\rho\big(\xi, \xi_{0,d}^{(N)}\big) + o(1), \quad \text{for all } 0 \le t \le 1.$$

Hence, letting first $N \to \infty$ and then $d \to \infty$ it follows by (1.17), (2.17), (2.18) and the choice of D that (2.26)(iii) holds.

2.3. *Legendre transform.* In this section certain dimension-dependent estimates are verified for the Legendre transform (2.13) in the infinite alleles model. These estimates are assumed to take the following general form. There is a sequence (a_d) depending only on d such that $a_d \to 0$ as $d \to \infty$, and for each $\gamma > 0$, $d \ge 1$ there are constants $\boldsymbol{\Delta}_{\gamma,d} > 0$ and big-O functions $O_{\gamma,d}$ with implied constants depending on γ and d such that the following hold:

(2.28)

(i) for all $x \in Q$, $\xi = \pi_d x$, and $|u| \le \boldsymbol{\Delta}_{\gamma,d}$ with both $\xi, \xi + u \in Q_{-\gamma,d}$,

$$\frac{1}{N}|H_d(N,x,u) - H_d(N,\xi,u)| \le O_{\gamma,d}\big((|u| + \beta)^3\big) + a_d\beta^2;$$

(ii) for all $\xi, \eta \in Q_{-\gamma,d}$ and $|u| \le \boldsymbol{\Delta}_{\gamma,d}$ with both $\xi + u, \eta + u \in Q_{-\gamma,d}$,

$$\frac{1}{N}|H_d(N,\eta,u) - H_d(N,\xi,u)| \le O_{\gamma,d}\left(|\eta - \xi|\left(\beta^2 + \frac{H_d(N,\xi,u)}{N}\right) + (|u| + \beta)^3\right).$$

A mild condition on the rate of decay of the sequence (a_d) is imposed in (2.40).

Assume in addition that there is a constant C_1 and big-O functions O_d such that, for $\xi = \pi_d x$ and all $x \in Q$,

$$|\pi_d \mathbf{h}_N(x) - \pi_d \mathbf{h}_N(\xi)| \le C_1\delta + O_d(\beta), \quad \text{with } d = d(\delta). \tag{2.29}$$

Conditions (2.28)(i) and (ii), respectively, are the key new conditions in our extension of the lower and upper large deviation bounds for the process $\{\pi_d X_n\}$. While these bounds are proved assuming only $N \to \infty$, $\beta \to 0$ and $N\beta \to \infty$, the corresponding expectation bounds in Theorem 1.1 are proved under the slightly stronger condition $N\beta/\log N \to \infty$. Condition (2.29) plays a role in our approximation scheme for estimating exit times of the original Markov chain via these large deviation bounds.

These conditions are now verified by direct computation for the infinite alleles model. Let $\gamma > 0$, $\delta > 0$ and $x \in \mathbf{P}[0,1]$. Denote $\xi = \pi_d x$ and $d = d(\delta)$ and assume $\xi \in (\mathbf{P}[0,1])_{-\gamma,d}$. First some estimates for the function $\pi\mathbf{h}(x) = \pi_d \mathbf{h}_N(x)$ are made. For notational convenience the probability vector $\xi = (\xi_1, \dots, \xi_{d+1})$ is viewed also as a measure $\xi(dt)$ with $\xi(A_k) = \xi_k$. By (2.1) and

(1.9)–(1.11),

$$\begin{aligned}
&\frac{\beta[\pi\mathbf{h}(x)-\pi\mathbf{h}(\xi)]_k}{(1-\mu\beta)}\\
&\quad=\frac{[\pi_d\mathbf{f}_N(x)-\pi_d\mathbf{f}_N(\xi)-(\pi x-\xi)]}{(1-\mu\beta)}\\
&\quad=\frac{\int_{A_k}(1+s(t))x(dt)}{\int_0^1(1+s(t))x(dt)}-\frac{\int_{A_k}(1+s(t))\xi(dt)}{\int_0^1(1+s(t))\xi(dt)}\\
&\quad=\left\{\int_0^1(1+s(t))x(dt)\int_0^1(1+s(t))\xi(dt)\right\}^{-1}\\
&\qquad\times\left\{\left[1+\int_0^1 s(t)\xi(dt)\right]\left[x(A_k)+\int_{A_k}s(t)x(dt)\right]\right.\\
&\qquad\left.-\left[1+\int_0^1 s(t)x(dt)\right]\left[\xi(A_k)+\int_{A_k}s(t)\xi(dt)\right]\right\}\\
&\quad=(1+O(\beta))^{-1}\xi(A_k)\left[\int_0^1 s(t)(\xi(dt)-x(dt))+O(\beta^2)\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&[\pi\mathbf{h}(x)-\pi\mathbf{h}(\xi)]_k\\
&\quad=\xi(A_k)\left[\int_0^1\sigma(t)(\xi(dt)-x(dt))\right]+O(\beta) \qquad (2.30)\\
&\quad=\theta C_\sigma\xi_k\delta+O(\beta),\quad \text{where } C_\sigma=\max\{\|\sigma\|_\infty,\|\sigma\|_{\mathrm{Lip}}\} \text{ and } |\theta|\le 1,
\end{aligned}$$

and in all cases the constants implied in the big-O function depend only on μ and σ. By (2.30), condition (2.29) holds with $C_1 = C_\sigma$.

Turn now to condition (2.28). Because $N\pi_d X_1$ is multinomial, it follows that

$$H_d(N,x,u)=N\sum_{k=1}^{d+1}(\xi_k+u_k)\log\left\{\frac{\xi_k+u_k}{[\pi\mathbf{f}(x)]_k}\right\}.$$

Therefore, after subtracting and expanding the logarithms,

$$\begin{aligned}
&\frac{1}{N}(H_d(N,\xi,u)-H_d(N,x,u))\\
&\quad=\sum_{k=1}^{d+1}(\xi_k+u_k)\log\left\{\frac{\xi_k+\beta[\pi\mathbf{h}(x)]_k}{\xi_k+\beta[\pi\mathbf{h}(\xi)]_k}\right\}\\
&\quad=\sum_{k=1}^{d+1}(\xi_k+u_k)\log\left\{1+\frac{\beta[\pi\mathbf{h}(x)-\pi\mathbf{h}(\xi)]_k}{\xi_k+\beta[\pi\mathbf{h}(\xi)]_k}\right\} \qquad (2.31)\\
&\quad=\sum_{k=1}^{d+1}(\mathrm{I}_k+\mathrm{II}_k+\mathrm{III}_k)+O_{\gamma,d}(\beta^3),
\end{aligned}$$

with

$$\mathrm{I}_k = \beta[\pi\mathbf{h}(x) - \pi\mathbf{h}(\xi)]_k,$$

$$\mathrm{II}_k = \beta(u_k - \beta[\pi\mathbf{h}(\xi)]_k)\frac{[\pi\mathbf{h}(x) - \pi\mathbf{h}(\xi)]_k}{\xi_k + \beta[\pi\mathbf{h}(\xi)]_k}$$

and

$$\mathrm{III}_k = -\frac{1}{2}(\xi_k + u_k)\beta^2\left(\frac{[\pi\mathbf{h}(x) - \pi\mathbf{h}(\xi)]_k}{\xi_k + \beta[\pi\mathbf{h}(\xi)]_k}\right)^2.$$

The sums in (2.31) are studied in turn. First, use $\Sigma_{k=1}^{d+1}[\pi\mathbf{h}]_k = 0$ to get, by (2.28),

$$\sum_{k=1}^{d+1} I_k = 0. \tag{2.32}$$

Next, by (2.30) and (2.31), for $N \geq N_0$,

$$\mathrm{II}_k = \beta(u_k - \beta[\pi\mathbf{h}(\xi)]_k)\frac{[\theta C_\sigma \xi_k \delta + O_{\sigma,\mu}(\beta)]}{\xi_k + \beta[\pi\mathbf{h}(\xi)]_k}$$
$$= \beta(u_k - \beta[\pi\mathbf{h}(\xi)]_k)(\theta C_\sigma \delta + O_{\gamma,d}(\beta)).$$

So, using again $\Sigma[\pi\mathbf{h}]_k = 0$ as well as $\Sigma u_k = 0$, we obtain that

$$\left|\sum_{k=1}^{d+1} \mathrm{II}_k\right| \leq O_{\gamma,d}\big((|u| + \beta)^3\big). \tag{2.33}$$

Finally, by (2.30) and (2.31),

$$\mathrm{III}_k = (\xi_k + u_k)\big(-\theta^2 C_\sigma^2 \beta^2 \delta^2/2 + O_{\gamma,d}(\beta^3)\big).$$

Hence, because $\xi + u$ is a probability vector,

$$\left|\sum_{k=1}^{d+1} \mathrm{III}_k\right| \leq C_\sigma^2 \beta^2 \delta^2/2 + O_{\gamma,d}(\beta^3). \tag{2.34}$$

Putting together (1.17) and (2.31)–(2.34), it follows that (2.28)(i) holds with

$$a_d = \text{const.}(1/d^2). \tag{2.35}$$

To establish (2.28)(ii), note by calculations similar to those above that

$$\frac{1}{N} H_d(N, \xi, u) = \frac{1}{2}\sum \frac{(u_k - \beta[\pi\mathbf{h}(\xi)]_k)^2}{\xi_k + \beta[\pi\mathbf{h}(\xi)]_k} + O_{\gamma,d}\big((|u| + \beta)^3\big). \tag{2.36}$$

Condition (2.28)(ii) follows after some algebra from (2.36), (2.16)(ii) and Lemma

3.1 (in Section 3.1) as follows:

$$\frac{\left(u_k - \beta[\pi\mathbf{h}(\xi)]_k\right)^2}{\xi_k + \beta[\pi\mathbf{h}(\xi)]_k} - \frac{\left(u_k - \beta[\pi\mathbf{h}(\eta)]_k\right)^2}{\eta_k + \beta[\pi\mathbf{h}(\eta)]_k}$$

$$= O_{\gamma,d}\Big(|\eta_k - \xi_k|\left(u_k - \beta[\pi\mathbf{h}(\xi)]_k\right)^2$$

$$+\xi_k\{\beta[\pi\mathbf{h}(\eta) - \pi\mathbf{h}(\xi)]_k\left(2(u_k - \beta[\pi\mathbf{h}(\xi)]_k) + \beta[\pi\mathbf{h}(\xi) - \pi\mathbf{h}(\eta)]_k\right)\}\Big)$$

$$= O_{\gamma,d}\left(|\eta - \xi|\left(\frac{H_d(N,\xi,u)}{N} + \beta\left(\frac{H_d(N,\xi,u)}{N}\right)^{1/2} + \beta^2\right) + (|u| + \beta)^3\right).$$

2.4. *Variational problem.* In this section certain conditions on the regularity of the finite-dimensional variational problems are introduced. It is shown how these problems give rise to a limiting large deviation exponent for the infinite alleles model in Section 5.

Let $V_d(U)$ be the large deviation exponents defined by (1.18). Assume there exists an $\varepsilon_0 > 0$ such that for all $G = D^\varepsilon$ with $|\varepsilon| \le \varepsilon_0$ and all $\gamma > 0$ and $d \ge 1$, an extremal solution u_γ^* for the problem $V_d(G_{-\gamma,d})$ satisfies the Euler–Lagrange equations $\dot{u}_\gamma^* = -h_d(u_\gamma^*)$. Denote by $u_\gamma^*(\infty)$ the point where u_γ^* exits $G_{-\gamma,d}$. Assume that there exist $\gamma > 0$ and $r(d) > 0$ with $r(d) \to 0$ as $d \to \infty$ such that, for all $0 < \varepsilon < \varepsilon_0$ and any $d = d(\delta)$ in (2.2),

(2.37)(i) $$u_\gamma^*(\infty) \in \left(G^{-r(d)}\right)^c$$

and

(2.37)(ii) $$\lim_{\gamma\downarrow 0} V_d(G_{-\gamma,d}) = V_d\left(\operatorname{int}(\pi_d G)\right).$$

The meaning of (2.37)(i) is that u_γ^* should not exit $G_{-\gamma,d}$ too far from the boundary of $G \subset Q$. For convenience in the following we denote

$$V_d(G) := V_d\left(\operatorname{int}(\pi_d G)\right).$$

Assume further that there exists $b_d \downarrow 0$ as $d \to \infty$ such that the integral $I_d(G)$, defined as the right-hand side of (1.18) integrated over the set of times that the solution u^* to the problem $V_d(G)$ is no longer in the ball $\{\xi\colon |\xi - \xi_{0,d}| \le b_d\}$, satisfies

(2.38) $$I_d(G) - V_d(G) \to 0 \quad \text{as } d \to \infty.$$

Define

(2.39) $$\lambda_d = \lim_{N\to\infty} \inf\{N\lambda(\xi)\colon |\xi - \xi_{0,d}| \le b_d\}$$

for $\lambda(\xi)$ defined by (2.3).

Assume that the parameters $d(\delta)$, a_d, K_d, C_1, $\gamma(t_0, d)$, b_d and λ_d of (2.2), (2.28)(i), (2.16)(ii), (2.29), (2.26)(i), (2.38) and (2.39), respectively, satisfy the following:

$$
(2.40)\qquad
\begin{aligned}
&\text{(i)}\ \ a_d(1+|\log(b_d)|)\to 0 \quad \text{as } d\to\infty;\\
&\text{(ii)}\ \ \delta K_{d(\delta)}\to 0 \quad \text{as } \delta\to 0;\\
&\text{(iii)}\ \ b_d^2(1+K_d^2)/\lambda_d\to 0 \quad \text{as } d\to\infty;\\
&\text{(iv)}\ \ \text{for each } \delta>0,\quad \liminf_{t\to 0}\gamma(t,d(\delta))/[C_1t\delta]>1;\\
&\text{(v)}\ \ \text{there exists } q(\delta)=O(1/\delta) \text{ such that } \rho(\xi,\eta)\le\\
&\qquad q(\delta)|\xi-\eta|_{\mathbb{R}^{d(\delta)}} \text{ for all } \xi,\eta\in\pi_{d(\delta)}Q.
\end{aligned}
$$

Condition (2.40)(i) is employed in the proof of the large deviation lower bound in Section 3.3, while conditions (2.40)(ii)–(v) are basic elements of the approximation scheme in Section 4.

Define limiting large deviation exponents

$$
(2.41)\qquad
\begin{aligned}
V_\infty^+(D) &= \lim_{\varepsilon\downarrow 0}\limsup_{d\to\infty} V_d(D^\varepsilon),\\
V_\infty^-(D) &= \lim_{\varepsilon\downarrow 0}\liminf_{d\to\infty} V_d(D^{-\varepsilon}).
\end{aligned}
$$

Assume finally that

$$
(2.42)\qquad V_\infty^-(D)>0.
$$

To verify these conditions for the infinite alleles model, a representation for V_d in (1.18) is computed. We begin with a lemma to compute the integrand in (1.18) for a general Wright–Fisher model in d dimensions.

LEMMA 2.1. *Let $\xi=(\xi_1,\dots,\xi_d)$ for a probability vector $(\xi_1,\dots,\xi_{d+1})\in\Delta^d$ and let $b=(b_1,\dots,b_d)$ for probability vectors $(\xi_1+b_1(\xi),\dots,\xi_{d+1}+b_{d+1}(\xi))$, so that $b_{d+1}=-\sum_{i=1}^d b_i$. Let also $\sigma_d^2(\xi)^{d\times d}$ be given by (2.8). Define $\eta_{d+1}=-\sum_{i=1}^d\eta_i$ for all $\eta=(\eta_1,\dots,\eta_d)\in\mathbb{R}^d$. Then we have the algebraic identity*

$$
\big(\eta-b(\xi),\sigma_d^{-2}(\xi)(\eta-b(\xi))\big)=\sum_{i=1}^{d+1}(\eta_i-b_i(\xi))^2/\xi_i.
$$

PROOF. First see that $\sigma_d^{-2}(\xi)_{ij}=\xi_{d+1}^{-1}+\xi_j^{-1}\delta_{ij}$. Check this by the calculation

$$
\begin{aligned}
&\sum_{k=1}^d(\xi_{d+1}^{-1}+\xi_k^{-1}\delta_{ik})\xi_k(\delta_{kj}-\xi_j)\\
&\quad=\sum_{k=1}^d\delta_{ik}(\delta_{kj}-\xi_j)+\xi_{d+1}^{-1}\sum_{k=1}^d\xi_k(\delta_{kj}-\xi_j)\\
&\quad=(\delta_{ij}-\xi_j)+\xi_{d+1}^{-1}\Big[\xi_j(1-\xi_j)-\xi_j\sum_{k\ne j,\,d+1}\xi_k\Big]\\
&\quad=\delta_{ij}-\xi_j+\xi_j\\
&\quad=\delta_{ij}.
\end{aligned}
$$

Therefore,

$$\begin{aligned}(\eta - b(u), \sigma_d^{-2}(u)(\eta - b(u))) &= \sum_{i,j=1}^{d} (\eta_i - b_i)\big(\xi_{d+1}^{-1} + \xi_j^{-1}\delta_{ij}\big)(\eta_j - b_j)\\ &= \sum_{i=1}^{d} (\eta_i - b_i)^2/\xi_i + \xi_{d+1}^{-1}\left(\sum_{i=1}^{d}(\eta_i - b_i)\right)^2\\ &= \sum_{i=1}^{d+1} (\eta_i - b_i)^2/\xi_i. \qquad \square\end{aligned}$$

It follows from Lemma 2.1 and (2.8) that (1.18) can be rewritten

$$V_d(U) = \inf \frac{1}{2}\int \sum_{k=1}^{d+1} \big(\dot{u}_k - [h_d(u)]_k\big)^2/u_k\,dt, \quad \text{where } u \in \Delta^d.$$

We want now to show that an extremal $u \in \Delta^d$ for the problem $V_d(U)$ is defined by

$$\dot{u} = -h_d(u), \qquad t > -\infty. \tag{2.43}$$

This can be done directly as follows. Set $\dot{u}(t) = -b(u(t)) + l(t)$, with b as in Lemma 2.1. Then by the identity $(-2b + l)^2 = -2b(-2b + 2l) + l^2$, it obtains that $(\dot{u}_i - b_i)^2 = (-2b_i + l_i)^2 = -2b_i(2\dot{u}_i) + l_i^2$. Therefore

$$\int_{-\infty}^{T} \frac{1}{2}\sum_{i=1}^{d+1} u_i^{-1}(\dot{u}_i - b_i)^2\,dt \geq \int_{-\infty}^{T} \frac{1}{2}\sum_{i=1}^{d+1} u_i^{-1}(-4b_i(u)\dot{u}_i)\,dt. \tag{2.44}$$

Now substitute $b = h_d$ from (2.9) in (2.44). Then the last integral becomes

$$\begin{aligned}&\frac{1}{2}\int \sum_{i=1}^{d+1} u_i^{-1}(-4(\bar{\sigma}_i - \sigma^*(u))u_i)\dot{u}_i\,dt\\ &\quad + \frac{1}{2}\mu\int \sum_{i=1}^{d+1} -4u_i^{-1}(1/(d+1) - u_i)\dot{u}_i\,dt \\ &= -2\int \sum_{i=1}^{d+1} \bar{\sigma}_i\dot{u}_i\,dt - 2\mu\int \sum_{i=1}^{d+1} u_i^{-1}\dot{u}_i\,dt/(d+1),\end{aligned} \tag{2.45}$$

where we have used $\Sigma_{i=1}^{d+1}\dot{u}_i = 0$. Fortunately both of these last integrals can be evaluated explicitly, as expressions depending only on the starting point and endpoint of the trajectory u. Therefore by Lemma 2.1, (2.44) and (2.45) and by reversing the direction of a trajectory that goes into the stable point $\xi_{0,d}$ from a point ξ on the boundary of $\operatorname{int}(\pi_d D)$, it follows that (2.43) holds and that

$$V_d(D) = \inf -2\sum_{i=1}^{d+1} \big[\bar{\sigma}_i(\xi_i - (\xi_{0,d})_i) + \mu\log(\xi_i/(\xi_{0,d})_i)/(d+1)\big], \tag{2.46}$$

where the infimum is extended over all $\xi \in \partial(\operatorname{int}(\pi_d D))$.

The result (2.46) can also be gotten by applying Theorem 4.3 of Freidlin and Wentzell [(1984), Chapter 5]. To see this, define $m(\xi, z) = (h_d(\xi), z) + \frac{1}{2}(z, \sigma_d^2(\xi)z)$. For each ξ, $\exp(m(\xi, \cdot\,))$ is the moment generating function of the Gaussian measure on $\mathbb{R}^d$ with mean $h_d(\xi)$ and variance $\sigma_d^2(\xi)$. One wants to find a function g: $\mathrm{int}(\pi_d D) \to \mathbb{R}$ with $g(\xi_{0,d}) = 0$, $\nabla g \neq \mathbf{0}$ for $\xi \neq \xi_{0,d}$ and $m(\xi, \nabla g(\xi)) = 0$. Then $V_d(\mathrm{int}(\pi_d D)) = \inf_\xi g(\xi)$, and an extremal $u(t)$ for this problem is defined by $\dot{u}_t = \nabla_z m(u_t, \nabla g(u_t))$. One can easily solve the differential equation for g by assuming the special form $g = \sum g_i(\xi_i)$.

Conditions (2.38) and (2.40)(i) are verified by choosing b_d. In fact, since $|\xi - \xi_{0,d}| \leq b_d$ implies the same inequality coordinatewise, and since $|\log(a) - \log(b)| \leq |a - b|/\min\{a, b\}$, it holds by (2.19), (2.20), (2.46) and the Cauchy–Schwarz inequality that

$$(2.47) \qquad |I_d(G) - V_d(G)| \leq \text{const.}\Big(\sum |\xi_i - (\xi_{0,d})_i|\Big) \leq \text{const.}\, b_d\sqrt{d}\,.$$

Therefore, by (2.35) and (2.47), conditions (2.38) and (2.40)(i) hold with

$$(2.48) \qquad b_d = 1/d^2.$$

Also, with this choice of b_d it is clear that $\lambda_d \geq \text{const.}/d$ in (2.39), so that by (2.25), condition (2.40)(iii) is verified as well. Condition (2.40)(iv) holds by (1.12), (1.17), (2.27), (2.29) and (2.30).

To verify (2.42), note first that by definition of the $\rho(\cdot, \cdot\,)$ in (1.14),

$$(2.49) \qquad \rho(\eta, \xi) \leq \sum_{i=1}^{d+1} |\eta_i - \xi_i|, \quad \text{for all } \xi, \eta \in \Delta^d.$$

Let $u(t) \in \pi_d \mathbf{P}[0,1]$, $-\infty < t < \infty$, be extremal for the problem $V_d(G)$ with $G = D^{-\varepsilon}$ for some $0 < \varepsilon < \varepsilon_0$, and as before set $u_{d+1} = 1 - \sum_{i=1}^d u_i$. Here we simply fix $u(t) = u(\infty)$, where $u(t)$ first exits $\partial(\mathrm{int}(\pi_d G))$. Because by (2.46) $u(t)$ does not leave $\mathrm{int}(\pi_d G)$ by the boundary of $\pi_d \mathbf{P}[0,1]$, we deduce by (1.19), (2.18) and Remark 2.1(b) that if d is large enough and ε_0 small enough, then $u(t)$ must pass through a point in the boundary of a ball about $\xi_{0,d}$ in $\mathbf{P}[0,1]$ of radius $r_0 > 0$ independent of d: $\rho(u(\infty), \xi_{0,d}) \geq r_0$. Next, by the Cauchy–Schwarz inequality,

$$(2.50) \qquad \sum_{i=1}^{d+1} \frac{\dot{u}_i^2}{u_i} = \sum_{i=1}^{d+1} \left(\frac{|\dot{u}_i|}{\sqrt{u_i}}\right)^2 \sum_{i=1}^{d+1} u_i \geq \left(\sum_{i=1}^{d+1} |\dot{u}_i|\right)^2.$$

Also by (1.12)(i) and (2.9),

$$|[h_d(\xi)]_i| \geq \frac{\mu}{3}|\xi_i - [\xi_{0,d}]_i| - [\xi_{0,d}]_i \|\sigma\|_\infty \sum_{j=1}^{d+1} |\xi_j - [\xi_{0,d}]_j|.$$

Therefore, upon summing over i we have

$$(2.51) \qquad \sum_{i=1}^{d+1} |[h_d(u)]_i| \geq \left(\frac{\mu}{3} - \|\sigma\|_\infty\right) \sum_{j=1}^{d+1} |\xi_j - [\xi_{0,d}]_j|.$$

Hence, by (1.18), Lemma 2.1, (2.43), and (2.50) and (2.51),

$$V_d(G) = 2\int\left(\sum_{i=1}^{d+1}\frac{\dot{u}_i^2}{u_i}\right)dt$$

$$\geq 2\int\left(\sum_{i=1}^{d+1}|\dot{u}_i|\right)^2 dt \tag{2.52}$$

$$\geq 2\left(\frac{\mu}{3}-\|\sigma\|_\infty\right)\int\sum_{j=1}^{d+1}|u_j(t)-[\xi_{0,d}]_j|\sum_{i=1}^{d+1}|\dot{u}_i(t)|\,dt.$$

Now estimate the last integral from below by integrating only after the last time $t=\tau$ that $u(t)$ is at distance $r_0/2$ from $\xi_{0,d}$ in $\mathbf{P}[0,1]$. Thus by (2.49),

$$\sum_{j=1}^{d+1}|u_j(t)-[\xi_{0,d}]_j|\geq r_0/2,\quad \text{for all } t\geq\tau.$$

So, by (2.52),

$$V_d(G)\geq r_0\left(\frac{\mu}{3}-\|\sigma\|_\infty\right)\int_\tau^\infty\sum_{i=1}^{d+1}|\dot{u}_i(t)|\,dt. \tag{2.53}$$

But by (2.49), $\int_\tau^\infty\sum_{i=1}^{d+1}|\dot{u}_i|\,dt\geq\sum_i|u_i(\infty)-u_i(\tau)|\geq\rho(u(\infty),u(\tau))\geq r_0/2$. Therefore, by (2.53),

$$V_d(G)\geq\left(\frac{\mu}{3}-\|\sigma\|_\infty\right)\frac{r_0^2}{2}. \tag{2.54}$$

Hence, by (1.12)(i), condition (2.42) holds. Further, by (2.49), (1.17) and the estimate $\sum_{i=1}^{d+1}|\xi_i-\eta_i|\leq(d+1)^{1/2}|\xi-\eta|_{\mathbb{R}^d}$, condition (2.40)(v) also holds.

Finally, condition (2.37) is established for the infinite alleles model as follows. Let ξ_γ be a point on $\partial(G_{-\gamma,d})$ that asymptotically achieves the infimum in the problem for $V_d(G_{-\gamma,d})$. Since the closure of $\operatorname{int}(\pi_d G)$ is compact in $\mathbb{R}^d$, there is a limit point ξ^* of the set $\{\xi_\gamma\}_{\gamma>0}$. Since $V_d(G_{-\gamma,d})\leq V_d(G)$, for all $\gamma>0$, it follows by continuity that

$$V_d(\xi^*):=-2\left[\sum_{i=1}^{d+1}\left(\sigma_i(\xi_i^*-(\xi_{0,d})_i)+\mu\log(\xi_i^*/(\xi_{0,d})_i)\right)\right]\leq V_d(G).$$

Therefore the last inequality is in fact an equality. Hence, replacing ξ^* by ξ_γ and letting $\gamma\to 0$ we have that $V_d(\xi_{\gamma'})\to V_d(\xi^*)$ along a subsequence $\gamma'\to 0$. Thus (2.37)(ii) follows by monotonicity of $V_d(G_{-\gamma,d})$, $\gamma\downarrow 0$. Further, we claim that $\xi_\gamma\in(G^{-2\delta})^c$ for $\gamma=\gamma'$ sufficiently small, where δ is such that $d=d(\delta)$ in (2.2). Suppose not. Then $\xi_\gamma\in G^{-2\delta}$ and for all $\eta\in\pi_d(G^c)$, it would hold that $\rho(\xi_\gamma,\eta)>\delta$. In particular, for all $\eta\in(\pi_d G)^c\subset\mathbb{R}^d$, it would follow that $\rho(\xi_\gamma,\eta)>\delta$. Hence, letting $\gamma\to 0$ and noting by (2.46) that $\xi^*\notin\partial(\pi_d\mathbf{P}[0,1])$, it would follow that $\xi^* \notin \partial(\mathrm{int}(\pi_d G))$. This is a contradiction. Hence (2.37)(i) holds for $r(d) = r(d(\delta)) = 2\delta$.

To summarize the last argument we make the following general remark.

REMARK 2.1. Let $F \subset Q$. (a) If $\xi \in \pi_d F$, $\xi \notin \partial(\pi_d Q)$ and $\rho(\xi, \eta) \geq \rho_0 > 0$ for all $\eta \in (\pi_d F)^c \cap \pi_d Q$, then by the metric space embedding $\pi_d Q \subset \mathbb{R}^d$ it follows that $\xi \in \mathrm{int}(\pi_d F)$. (b) Let $\delta > 0$ and $d = d(\delta)$ as in (2.2) and let $G \subset Q$ be open. If $\xi \in \partial(\mathrm{int}(\pi_d G))$ and $\xi \notin \partial(\pi_d Q)$, then $\xi \in (G^{-\delta})^c$. Indeed, if on the contrary $\xi \in G^{-\delta} \cap \pi_d Q$, then it would follow that $\inf\{\rho(\xi, \eta): \eta \in (\pi_d G)^c \cap \pi_d Q\} > 0$, which by (a) would contradict the assumption $\xi \notin \mathrm{int}(\pi_d G)$.

2.5. *General result.* Our main result for the general setup and conditions of Sections 2.1–2.4 is summarized as follows.

THEOREM 2.1. *Assume in addition to the above conditions that $N\beta/\log N \to \infty$. Then*:

(i) $$\limsup_{N \to \infty} (\log \mathbf{E}_x T_e)/N\beta \leq V_\infty^+(D),$$

(ii) $$\liminf_{N \to \infty} (\log \mathbf{E}_x T_e)/N\beta \geq V_\infty^-(D),$$

for each $x \in D$, where V_∞^+ and V_∞^- are defined by (2.42).

REMARK. In Theorem 1.1 we have an explicit representation for V_∞. In Section 5 it is shown that (2.41) is consistent with (1.20) for the infinite alleles model and that $V_\infty^-(D)$ and $V_\infty^+(D)$ coincide with $V_\infty(D)$, when D is given by (1.19) and $d = 2^l - 1$, $l = 1, 2, 3, \ldots,$ in (2.2).

3. The finite-dimensional process. The groundwork for the exit problem in infinite dimensions is laid by establishing tightness arguments and upper and lower large deviation bounds for the finite-dimensional process $\{\pi_d X_n\}$. These bounds are designed to fit an approximation scheme to estimate the exit time for the original Markov chain in Section 4.

3.1. *Tightness.* Here the tightness results from the finite-dimensional theory are summarized. We note that for each $\gamma > 0$ the conditions (2.1), (2.3), (2.5)(i), (2.5)(ii), (2.12), (2.16)(i) and (2.26)(ii) imply corresponding conditions of Morrow and Sawyer (1989) for the process $\{\pi_d X_n\}$ in the domain $Q_{-\gamma, d} \subset \mathbb{R}^d$. Hence the following lemmas and propositions are proved as in corresponding results from that paper. Therefore we only sketch their proofs.

LEMMA 3.1. *Let $a_0 = 1/(2C_0)$ with C_0 in* (2.12)(*ii*). *Then*

$$H_d(N, \xi, u) \geq a_0 N|u - \beta\pi_d \mathbf{h}_N(\xi)|^2, \quad \textit{for all } \xi, \xi + u \in \pi_d Q.$$

PROOF. If $z \in \mathbb{R}^d$, then $H_d(N, \xi, u) \geq z(u - \beta\pi_d \mathbf{h}_N(\xi)) - [G_d(N, \xi, z) - \beta z \pi_d \mathbf{h}_N(\xi)]$. At $z = 0$ the term in square brackets as well as its derivative in z both vanish. Hence the result follows by applying Taylor's theorem to this term, using (2.12)(ii) and choosing $z = cN(u - \beta\pi_d \mathbf{h}_N(\xi))$ with $c = 1/C_0$. □

LEMMA 3.2. *Let $\gamma > 0$. There exist $\mathbf{\Delta}_{\gamma,d} > 0$ and big-O functions $O_{\gamma,d}$ with implied constants depending only on γ and d such that if $\mathbf{\Delta}_N$ are constants satisfying $\beta \leq \mathbf{\Delta}_N \leq \mathbf{\Delta}_{\gamma,d}$, then, uniformly for all $\xi, \xi + u \in Q_{-\gamma,d}$ and $|u| \leq \mathbf{\Delta}_N$,*

$$H_d(N, \xi, u) = \tfrac{1}{2}\big(u - \beta\pi_d \mathbf{h}_N(\xi), \Sigma_{N,d}^{-2}(\xi)(u - \beta\pi_d \mathbf{h}_N(\xi))\big)\big(1 + O_{\gamma,d}(\mathbf{\Delta}_N)\big)$$

as $N \to \infty$.

PROOF. By (2.13), (2.3) and (2.12)(i) it holds that $z_d(N, \xi, u)$ is defined and analytic in an open set containing the line segment from $\pi_d \mathbf{f}_N(\xi)$ to $\xi + u$ and that

$$\begin{aligned} \nabla_u z_d(N, \xi, u) = \nabla_u^2 H_d(N, \xi, u) &= \tilde{\Sigma}_{N,d,z}^{-2}(\xi) \\ &= \Sigma_{N,d}^{-2}(\xi) + O_{\gamma,d}(|z| + N\beta), \end{aligned} \tag{3.1}$$

for $z = z_d(N, \xi, u)$ whenever $|u| \leq \mathbf{\Delta}_N$. Therefore, since by (2.1) and (2.13)

$$H_d(N, \xi, \beta\pi_d \mathbf{h}_N(\xi)) = \nabla_u H_d(N, \xi, \beta\pi_d \mathbf{h}_N(\xi)) = \mathbf{0},$$

the result follows by Taylor's theorem, (3.1) and (2.12)(ii). □

Let $[u] = (u_n)_{n=0}^T$ be a discrete trajectory in $\pi_d Q$. Define

$$S_d(N, T, [u]) = \sum_{n=0}^{T-1} H_d(N, u_n, u_{n+1} - u_n). \tag{3.2}$$

PROPOSITION 3.1. *If $[u] \subset \pi_d Q$ is a trajectory such that $|u_n - \xi_{0,d}^{(N)}| \geq \alpha' > 0$ for all n, then*

$$S_d(N, T, [u]) \geq N\beta a_0 \kappa_0 \Big(\kappa_0 \beta T \alpha'^2 - \big|u_0 - \xi_{0,d}^{(N)}\big|^2\Big).$$

PROOF. The proposition follows from (2.16)(i) and Lemma 3.1 as in the proof of Proposition 3.1 of Morrow and Sawyer (1989). □

For any set $F \subset Q$, define

$$\begin{aligned} &A(N, F_{-\gamma,d}, \xi) \\ &\quad = \inf\big\{S_d(N, T, [u]) : [u] \subset \operatorname{int}(\pi_d Q),\ u_0 = \xi,\ u_T \notin F_{-\gamma,d}\big\}. \end{aligned} \tag{3.3}$$

In the following let G denote D^ε for some $|\varepsilon| \leq \varepsilon_0$. Assume $\{T_N\} = \{T_{N,\gamma,d,\varepsilon}\}_{N \geq 1}$

is a sequence of integers that satisfies the following:

(3.4)

(i) $\beta T_N \to \infty$;

(ii) $\beta^2 T_N \to 0$;

(iii) $(\beta T_N)^{3/2}/(\beta N)^{1/2} \to 0$;

(iv) $$\beta T_N \sup_{\xi \in G_{-\gamma/2,d}} \left(\left| \pi_d \mathbf{h}_N(\xi) - h_d(\xi) \right| + \left| N \Sigma^2_{N,d}(\xi) - \sigma^2_d(\xi) \right| \right) \to 0, \quad \text{as } N \to \infty.$$

Here we are simply making $T_N \to \infty$ such that βT_N goes to zero as slowly as we like. Later, in (3.7), we may impose further conditions on T_N such that βT_N goes to zero even more slowly.

PROPOSITION 3.2. (i) *If $\xi_N \to \xi_{0,d}$, then*

$$\lim_{N\to\infty} A(N, G_{-\gamma,d}, \xi_N)/N\beta = V_d(G_{-\gamma,d}),$$

and in general if $K \subset Q_{-\gamma,d}$ is compact, then there is a constant $c(K) > 0$ such that,

$$\liminf_{N\to\infty} A(N, Q_{-\gamma,d}, \xi)/N\beta \geq c(K), \quad \text{for all } \xi \in K.$$

(ii) *Let $\xi^* = u^*_\gamma(\infty) \in \partial G_{-\gamma,d}$ for a solution u^*_γ to the problem $V_d(G_{-\gamma,d})$. Here we fix $u^*_\gamma(t) = u^*_\gamma(\infty)$ at the first time t that u^*_γ exits $G_{-\gamma,d}$. Put $L_N = T_N\beta$ for T_N in* (3.4) *and let $v(\cdot)$ satisfy $v(0) = \xi^*$ and $\dot{v} = h_d(v)$. Set $\xi_N = v(L_N)$. Define $u_N(t)$, $0 \leq t \leq L_N$, by $u_N(0) = \xi_N$ and $\dot{u}_N = -h_d(u_N)$. Define the discrete trajectory $[u] = (u_n^{(N,\gamma,d)})$, $0 \leq n \leq T_N$, by $u_n = u_N(n\beta)$. Then $u_0 = \xi_N$, $|u_{n+1} - u_n| \leq C_{\gamma,d}\beta$, $u_{T_N} = \xi^* \in \partial G_{-\gamma,d}$ and*

$$\lim_{N\to\infty} S_d(N, T_N, [u])/N\beta = V_d(G_{-\gamma,d}).$$

Also, for any $T \leq T_N$,

$$S_d(N,T,[u])/N\beta = \frac{1}{2}\int_0^{T\beta} \left(\dot{u}_N - h_d(u_N), \sigma_d^{-2}(u_N)(\dot{u}_N - h_d(u_N)) \right) dt + o(1) \quad \text{as } N \to \infty,$$

with $o(1)$ uniform in $T \leq T_N$.

PROOF. The proof follows from (2.3), (2.5)(i), (2.5)(ii), (2.26)(ii), Lemmas 3.1 and 3.2 and Proposition 3.1 as in the proofs of Proposition 3.2 and Corollary 3.1 of Morrow and Sawyer (1989). In particular, the proof of Lemma 3.3 of that paper implies $c(K) > 0$. □

LEMMA 3.3. *Let $u(t)$ be a trajectory in $\pi_d Q$ satisfying $\dot{u} = h_d(u)$. Let κ_0 be the constant in* (2.16)(i) *and let L_N be defined as in Proposition* 3.2. *Then*

$$|u(L_N) - \xi_{0,d}^{(N)}| \le 2\exp(-\kappa_0 L_N) + o_d(1) \quad \text{as } N \to \infty.$$

PROOF. By Taylor's theorem and the differentiability of h_d,

$$u(t+\beta) = \pi_d \mathbf{f}_N(u(t)) + O_d\Big(\beta \sup_{\xi \in \pi_d Q} |h_d(\xi) - \pi_d \mathbf{h}_N(\xi)|\Big).$$

Hence by (2.5)(ii), (2.16)(i), and (3.4)(iv), putting $L_N = T_N \beta$, it holds that

$$|u(L_N) - \xi_{0,d}^{(N)}| \le (1 - \kappa_0 \beta)^{L_N/\beta} |u(0) - \xi_{0,d}^{(N)}| + o_d(1). \qquad \square$$

3.2. *Lemmas.* The following lemmas are more or less standard results in large deviation theory. However, special care is taken to trace the dependence of the estimates on the parameters γ and d. Recall the definition of the cumulant generating function $G_d(N, x, z)$ in (2.10) and its Legendre transform $H_d(N, x, u)$ in (2.13).

LEMMA 3.4. *There exist big-O functions O_d with implied constants depending only on d such that the following hold*:

(i) *uniformly for $\xi, \eta \in \pi_d Q$ and $|z| \le \zeta_d N$,*

$$\begin{aligned} &|G_d(N,\xi,z) - G_d(N,\eta,z)| \\ &\quad \le O_d\Big[|z|\beta(|\xi-\eta| + O_d(\beta)) + |z|^2(|\xi-\eta| + O_d(\beta))/N \\ &\qquad + \big(|z|^3 + |z|^2 N\beta\big)/N^2\Big]; \end{aligned}$$

(ii) *for all $x \in Q$ and $z \in \mathbb{R}^d$,*

$$|G_d(N,x,z)| \le \beta |z|\, |\pi_d \mathbf{h}_N(x)| + C_0 |z|^2/N,$$

with the constants ζ_d and C_0 of (2.12).

PROOF. By (2.12)(i) and (2.4),

$$\begin{aligned} &\nabla_z^2 G_d(N,\xi,z) - \nabla_z^2 G_d(N,\eta,z) \\ &\quad = \tilde{\Sigma}_{N,d,z}^2(\xi) - \tilde{\Sigma}_{N,d,z}^2(\eta) \\ &\quad = \theta[c_d|\xi-\eta| + O_d(\beta)]/N + O_d\big((|z| + N\beta)/N^2\big), \end{aligned}$$

where $|\theta| \le 1$. Thus, by Taylor's theorem,

$$\begin{aligned} &\nabla_z G_d(N,\xi,z) - \nabla_z G_d(N,\eta,z) \\ &\quad = \pi_d \mathbf{f}_N(\xi) - \xi - (\pi_d \mathbf{f}_N(\eta) - \eta) + \theta |z| [c_d |\xi - \eta| + O_d(\beta)]/N \\ &\qquad + O_d\big((|z|^2 + |z| N\beta)/N^2\big), \end{aligned}$$

where, by (2.1) and (2.16)(ii),

$$\pi_d \mathbf{f}_N(\xi) - \xi - (\pi_d \mathbf{f}_N(\eta) - \eta) = \beta[\pi_d \mathbf{h}_N(\xi) - \pi_d \mathbf{h}_N(\eta)]$$
$$= \theta\beta(K_d|\xi - \eta| + O_d(\beta)),$$

for some $|\theta| \le 1$. Hence part (i) follows by applying Taylor's theorem again, this time to the difference $G_d(N, \xi, z) - G_d(N, \eta, z)$, since $G_d(N, \xi, \mathbf{0}) = G_d(N, \eta, \mathbf{0}) = 0$. The proof of part (ii) follows from Taylor's theorem, (2.12)(ii) and the facts that $G_d(N, x, \mathbf{0}) = 0$, $\nabla_z G_d(N, x, \mathbf{0}) = \beta\pi_d \mathbf{h}_N(x)$ and $\nabla_z^2 G_d(N, x, z) = \tilde{\Sigma}^2_{N,d,z}(x)$. □

Lemma 3.5. *There exist big-O functions $O_{\gamma,d}$ and constants $\Delta_{\gamma,d} > 0$ such that the following hold*:

(i) *for all $x \in Q$ such that $\xi = \pi_d x \in Q_{-\gamma,d}$ and for all $|u| \le \Delta_{\gamma,d}$,*

$$|z_d(N, x, u)| \le O_{\gamma,d}(N(|u| + \beta)),$$

(ii) *for all $\xi, \eta \in Q_{-\gamma,d}$ and $|u| \le \Delta_{\gamma,d}$ such that both $\xi + u, \eta + u \in Q_{-\gamma,d}$,*

$$\frac{1}{N}|H_d(N, \xi, u) - H_d(N, \eta, u)| \le O_{\gamma,d}\big((|u| + \beta)^2[|\xi - \eta| + |u| + \beta]\big),$$

(iii) *for all $\xi \in Q_{-\gamma,d}$ and $|u_2 - u_1|, |u_1|, |u_2| \le \Delta_{\gamma,d}$ such that both $\xi = u_1$ and $\xi + u_2 \in Q_{-\gamma,d}$,*

$$\frac{1}{N}|H_d(N, \xi, u_2) - H_d(N, \xi, u_1)| \le O_{\gamma,d}\big(|u_2 - u_1|(\beta + \max\{|u_1|, |u_2|\})\big).$$

Proof. Part (i) of the lemma follows from (2.1), (2.3) and (2.12)(i) as in the proof of Lemma 2.3(i) of Morrow and Sawyer (1989). Part (ii) is proved by Lemmas 3.4 and 3.5(i) and the following estimate:

$$H_d(N, \xi, u) - H_d(N, \eta, u)$$
$$= \sup_z (zu - G_d(N, \xi, z)) - \sup_z (zu - G_d(N, \eta, z))$$
$$\le \sup\{G_d(N, \xi, z) - G_d(N, \eta, z) \colon |z| \le \max\{|z_d(N, \xi, u)|, |z_d(N, \eta, u)|\}\}.$$

To prove part (iii), note that there exists a line from $\xi + u_1$ to $\xi + u_2$ lying entirely in $Q_{-\gamma/2,d}$ if $\Delta_{\gamma,d}$ is small enough. Thus by Taylor's theorem,

$$|H_d(N, \xi, u_2) - H_d(N, \xi, u_1)| = |u_2 - u_1|\,|\nabla_u H_d(N, \xi, u)|$$

for u on a line between u_1 and u_2. However, by definition of H_d in (2.13), $\nabla_u H_d(N, \xi, u) = z_d(N, \xi, u)$. Hence part (iii) follows from Lemma 3.5(i). □

Notice, by Lemma 3.1, that Lemma 3.5(ii) is weaker than condition (2.28)(ii). This latter condition is used to establish (3.14) in our proof of the large deviation upper bound. It seems that Darden's [Darden (1983)] proof for the upper large deviation bound fails for this weaker estimate since this author implicitly uses an estimate of the form $G_d(N, x, z) \le$ const. $\beta|z|$, whereas the bound in Lemma 3.4(ii) is essentially best possible. Thus the note added in proof of Morrow and Sawyer (1989) needs (2.28)(ii) as well.

3.3. *Large deviation lower bound*. The approach to large deviation bounds in Ventsel' (1976) is extended for the process $\{\pi_d X_n\}$ along the lines of Darden (1983) (who treated the one-dimensional Wright–Fisher model). Part of the novelty of our lower bound lies in its dependence on the dimension d through condition (2.28)(i). This estimate is applied together with the further dimension-dependent conditions (2.29) and (2.40)(i)–(v) in proving an upper bound for the exit time via Proposition 4.1. In the proof of the lower bound we also give a novel method to handle the event that the path $(\boldsymbol{\xi}_n\colon 0 \le n \le T_N)$, once started near $\xi_{0,d}$, remains in $Q_{-\gamma,d}$.

Define

$$\theta_N(T) = 2\sqrt{C_0 T/N}, \tag{3.5}$$

where C_0/N is a bound on the maximum eigenvalue of $\tilde{\Sigma}^2_{N,d,z}(x)$ for x and z in (2.12)(ii). In the following we denote $\boldsymbol{\xi}_n = \pi_d X_n$, $\xi = \pi_d x$ and $G = D^\varepsilon$, and statements involving G hold uniformly for $|\varepsilon| \le \varepsilon_0$ unless otherwise stated.

Let $\gamma > 0$ and $d \ge 1$ and let $[u] = (u_n)_{n=0}^T$, $T = T_N$, $N \ge 3$, be the trajectory of Proposition 3.2. Let also $\eta_N \to \xi_{0,d}$. Define $[w] = (w_n)$ as follows. Define T_1 by $|u_n - \xi_{0,d}| \ge b_d$ for all $n \ge T_1 - 1$ and T_1 is the smallest integer with this property. Set $T_2 = [1/\beta]$, $w_0 = \eta_N$, $w_{T_2} = u_{T_1}$, $w_n = u_{n+T_1-T_2}$, for all $n \ge T_2$, and $w_{n+1} - w_n = (u_{T_1} - \eta_N)/T_2$, for all $0 \le n \le T_2 - 1$. Thus w_n is defined for $0 \le n \le T^*$ with $T^* = T - T_1 + T_2 \le T + T_2$. Notice, by (3.4)(i), that $T^* = T(1 + o(1))$, so (3.4) holds with T^* in place of T. Notice also that $\sup_{0 \le n \le T^*-1}|w_{n+1} - w_n| \le c\beta$ for some constant c depending only on γ and d.

PROPOSITION 3.3. *Let $[w] = (w_n)$ be as defined above. There are little-o functions $o_{\gamma,d}$ with implied constants depending only on γ and d, a sequence $\omega_d \to 0$ as $d \to \infty$ independent of γ and N, and $N_0 = N_0(\gamma, d) < \infty$ such that if $\pi_d x = \eta_N$ and $N \ge N_0$, then*

$$\mathbf{P}_x\Big(\max_{0 \le n \le T^*}|\boldsymbol{\xi}_n - w_n| \le \theta_N(T^*)\Big)$$
$$\ge \tfrac{1}{2}\exp\big\{-N\beta V_d(G_{-\gamma,d}) - o_{\gamma,d}(N\beta) - \omega_d N\beta\big\}.$$

PROOF. Let A be the event $\max_{0 \le n \le T^*}|\xi_n - w_n| \le \theta_N(T^*)$ and let I_n be the event $\boldsymbol{\xi}_n \in Q_{-\gamma/2,d}$. Define $I = \bigcap_{0 \le n \le T^*} I_n$. Since $T^*/N \to 0$ by (3.4)(iii) and therefore $\theta_N(T^*) \to 0$ by (3.5), it holds that $A \subset I$. In the following we denote for brevity $z(x, w) = z_d(N, x, w)$, $G(x, z) = G_d(N, x, z)$, $\Delta\boldsymbol{\xi}_n = \boldsymbol{\xi}_{n+1} - \boldsymbol{\xi}_n$ and $\Delta w_n = w_{n+1} - w_n$, for z_d and G_d in (2.10)–(2.13). Define

$$\mathbf{Z}_m = 1_{I_m} z(X_m, \Delta w_m)\,\Delta\boldsymbol{\xi}_m \quad \text{and} \quad \mathbf{G}_m = \log \mathbf{E}_{X_m}\exp(\mathbf{Z}_m).$$

These definitions make sense as $\sup_{0 \le m \le T^*-1} \Delta w_m < \gamma/4$ for large N. Define

$R_{[w]}(n) = \prod_{m=0}^{n} \exp[\mathbf{Z}_m - \mathbf{G}_m]$. Then $R_{[w]}(T^*)$ is a martingale relative to $\mathbf{P}$ and $\sigma(x, X_1, \dots, X_{T^*})$ with mean 1. Note further that

$$\text{on } I, \quad R_{[w]}(T^*) = \exp\left[\sum_{m=0}^{T^*-1} \left(z(X_m, \Delta w_m)\, \Delta \boldsymbol{\xi}_m - G(X_m, z(X_m, \Delta w_m))\right)\right].$$

Define the probability $\tilde{\mathbf{P}}$ on $\sigma(x, X_1, \dots, X_{T^*})$ by $d\tilde{\mathbf{P}}_x/d\mathbf{P}_x = R_{[w]}(T^*)$. Hence $\mathbf{P}_x(A) = \tilde{\mathbf{E}}_x[1_A R_{[w]}^{-1}(T^*)]$. Therefore for any event F it holds that

$$\mathbf{P}_x(A) \geq \tilde{\mathbf{P}}_x[A \cap F] \times \inf_{A \cap F} R_{[w]}^{-1}(T^*).$$

Now we want to estimate $\tilde{\mathbf{P}}(A^c)$ from above. Define $\tau = \inf\{n: \boldsymbol{\xi}_n \notin Q_{-\gamma/2, d}\}$. Since w_n is inside the closure of $Q_{-\gamma, d}$, it follows that, on A^c, the $\max_{0 \leq k \leq n} |\boldsymbol{\xi}_k - w_k|$ is bigger than $\theta(T^*)$ by time $n = \min(T^*, \tau)$. Hence

$$\tilde{\mathbf{P}}_x(A^c) = \tilde{\mathbf{P}}_x\left\{ \max_{0 \leq n \leq T^*} |\boldsymbol{\xi}_n - w_n| > \theta(T^*)\right\}$$
$$= \tilde{\mathbf{P}}_x\left\{ \max_{0 \leq n \leq \min(T^*, \tau)} |\boldsymbol{\xi}_n - w_n| > \theta(T^*)\right\}.$$

Thus

$$\tilde{\mathbf{P}}_x(A^c) = \tilde{\mathbf{P}}_x\left\{ \max_{0 \leq n \leq \min(T^*, \tau)} \left| \sum_{k=0}^{n-1} (\Delta \boldsymbol{\xi}_k - \Delta w_k) 1_{I_k} \right| > \theta(T^*)\right\},$$

since if $k \leq \tau$, then $1_{I_k} = 1$. But $(\sum_{k=0}^{n-1} (\Delta \boldsymbol{\xi}_k - \Delta w_k) 1_{I_k})$ is a $\tilde{\mathbf{P}}$-martingale relative to the filtration $(\mathbf{F}_n) = (\sigma(x, X_1, \dots, X_n))$. Indeed,

$$\tilde{\mathbf{E}}_{X_n = x}\{(\Delta \boldsymbol{\xi}_n - \Delta w_n) 1_{I_n}\} = \mathbf{0},$$

since if $\boldsymbol{\xi}_n = \xi \in Q_{-\gamma/2, d}$, then $\boldsymbol{\xi}_n + \Delta w_n \in G_{-\gamma/4, d}$ and thus, by (2.10), (2.13) and (2.14), this last expectation is equal to

$$\mathbf{E}_x(\Delta \boldsymbol{\xi} - \Delta w) \exp[z(x, \Delta w)\, \Delta \boldsymbol{\xi} - G(x, z(x, \Delta w))]$$
$$= \nabla_z G(x, z(x, \Delta w)) - \Delta w = \mathbf{0}.$$

Thus $U_n := [\sum_{k=0}^{n-1} (\Delta \boldsymbol{\xi}_k - \Delta w_k) 1_{I_k}]^2$ is a nonnegative submartingale relative to $\tilde{\mathbf{P}}$ and $(\mathbf{F}_n)$. Hence, by the L^2 submartingale inequality and the optional stopping theorem for nonnegative submartingales and by (2.12)(ii) and (3.5),

$$\tilde{\mathbf{P}}(A^c) \leq \theta_N(T^*)^{-2} \tilde{\mathbf{E}}\left[U^2_{\min(T^*, \tau)}\right]$$
$$\leq \theta_N(T^*)^{-2} \tilde{\mathbf{E}}\left[U^2_{T^*}\right]$$
$$\leq \theta(T^*)^{-2} \sum_{n=0}^{T^*-1} \sup_{\pi_d x \in Q_{-\gamma/2, d}} \tilde{\mathbf{E}}_x\left[(\Delta \boldsymbol{\xi}_k - \Delta w_k)^2 1_{I_k}\right]$$
$$\leq \theta(T^*)^{-2} C_0 T^*/N$$
$$= 1/4.$$

Next define

$$
(3.6)\quad
\begin{aligned}
\Omega &= 2 \sup \sum_{n=0}^{T^*-1} \big(z(x, \Delta w_n), \nabla_z^2 G(x, z(x, \Delta w_n)) z(x, \Delta w_n) \big), \\
F &= \left\{ \left| \sum_{n=0}^{T^*-1} z(X_n, \Delta w_n)(\Delta w_n - \Delta \xi_n) 1_{I_n} \right|^2 \le \Omega \right\},
\end{aligned}
$$

where the supremum is extended over all $x \in Q$ such that $\xi = \pi_d x \in Q_{-\gamma/2, d}$. Then since $(\sum_{k=0}^{n-1} z(X_k, \Delta w_k)(\Delta w_k - \Delta \xi_k) 1_{I_k})$ is a $\tilde{\mathbf{P}}$-martingale relative to $(\mathbf{F}_n)$, it follows by (2.11) that

$$
\tilde{\mathbf{E}} \left| \sum_{n=0}^{T^*-1} z(X_n, \Delta w_n)(\Delta w_n - \Delta \xi_n) 1_{I_n} \right|^2 \le \Omega / 2.
$$

Hence, by (3.6) and Chebyshev's inequality, we have $\tilde{\mathbf{P}}(F^c) \le 1/2$. Hence $\tilde{\mathbf{P}}(A \cap F) \ge 1/4$.

To finish the proof we need to estimate $\inf_{A \cap F} R_{[w]}^{-1}(T^*)$. By definition we have that

$$
\begin{aligned}
\text{on } I, \quad R_{[w]}^{-1}(T^*) &= \exp\{-S_d(N, T^*, [w])\} \\
&\quad \times \exp\Big\{ \sum [H(w_n, \Delta w_n) - H(\xi_n, \Delta w_n)] \\
&\qquad + \sum [H(\xi_n, \Delta w_n) - H(X_n, \Delta w_n)] \\
&\qquad + z(X_n, \Delta w_n)(\Delta w_n - \Delta \xi_n) \Big\} \\
&= \exp\{-S_d(N, T^*, [w]) + \mathrm{I} + \mathrm{II} + \mathrm{III}\},
\end{aligned}
$$

where the summations in I–III are all from $n = 0$ to $T^* - 1$. Then, by (3.2), Lemmas 3.1 and 3.5(ii), (3.4)(ii), (3.4)(iii) and (3.6) and by the fact that $A \subset I$, it follows that

$$
\text{on set } A, \quad |\mathrm{I}| \le O_{\gamma, d}\big(N\beta^2(\theta_N(T^*) + O_d(\beta))T^*\big) \le o_{\gamma, d}(N\beta).
$$

Also, by (3.6), Lemma 3.5(i) and (1.8)(ii),

$$
\text{on set } F \cap I, \quad |\mathrm{III}| \le \sqrt{\Omega} \le O_{\gamma, d}\big(N\beta^2 T^*\big)^{1/2} \le o_{\gamma, d}(N\beta).
$$

Finally, write

$$
\mathrm{II} = \left(\sum_{n=0}^{T_0 - 1} + \sum_{n=T_0}^{T^*-1} \right) \big(H(\xi_n, \Delta w_n) - H(X_n, \Delta w_n)\big) = \mathrm{II}_0 + \mathrm{II}_1.
$$

Then by definition of T_0 and by (2.28)(i),

$$
\begin{aligned}
|\mathrm{II}_0|/N &\le O_{\gamma, d}\big(T_0 \beta^3\big) + a_d \beta^2 T_0 \\
&\le O_{\gamma, d}\big(\beta^2\big) + a_d \beta \\
&\le a_d \beta(1 + o(1)), \quad \text{as } N \to \infty.
\end{aligned}
$$

Also, by Lemma 3.3 and the estimate $|\Delta u_n| = O_{\gamma,d}(\beta)$ from Proposition 3.2(ii), it holds that

$$T^* - T_0 = T - T_1 \le |\log(b_d + o(1))|/\kappa_0\beta, \quad \text{as } N \to \infty.$$

Hence by (2.28)(i) and Proposition 3.2,

$$|\mathrm{II}_1|/N \le O_{\gamma,d}\big(\beta^3(T - T_1)\big) + a_d\beta|\log(b_d + o(1))|/\kappa_0.$$

Further, by (3.4)(ii) and (2.40)(i), it obtains that

$$|\mathrm{II}_1|/N \le o_d(\beta) + \beta a_d|\log(b_d + o(1))|/\kappa_0 \le o_d(\beta) + \omega_d\beta.$$

Finally, by (1.17) and the last assertion of Proposition 3.2,

$$\begin{aligned} -S_d(N, T^*, [w]) &\le -\sum_{n=0}^{T-1} H(u_n, \Delta u_n) + \sum_{n=0}^{T_1-1} H(u_n, \Delta u_n) \\ &\le -N\beta\big(V_d(G_{-\gamma,d}) + o(1)\big) + N\beta[V_d(G) - I_d(G) + o(1)] \\ &\le -N\beta V_d(G_{-\gamma,d}) + \omega_d N\beta + o(N\beta), \quad \text{as } N \to \infty. \end{aligned}$$

Therefore, putting all these estimates together, the proof of Proposition 3.3 is complete. □

3.4. *Large deviation upper bound.* In this section the arguments of Ventsel' (1976) and Darden (1983) are extended via condition (2.28)(ii); see the comments at the end of Section 3.2. Let $\gamma > 0$ and $d \ge 1$ and define $\phi_x(s) = \phi(x, N, T_N, \gamma, d, s)$, for all $s > 0$, as the set of all discrete trajectories $[u] = (u_n)$, $0 \le n \le T_N$, with $u_0 = \xi = \pi_d x$ and $[u] \subset Q_{-2\gamma,d}$ such that $S_d(N, T_N[u]) \le N\beta s$. Let $T_N \to \infty$ satisfy (3.4) and choose $\alpha = \alpha_N \to 0$, such that the following hold:

$$(3.7) \qquad \begin{aligned} &\text{(i)} \ \sqrt{\alpha}\,\beta T_N \to 0; \\ &\text{(ii)} \ \alpha^{-5}\beta^2 T_N \to 0; \\ &\text{(iii)} \ T_N\alpha^{-3}|\log(\alpha)|\beta = o(N\beta); \\ &\text{(iv)} \ \log\big(\beta T_N/\alpha^3\big) = o(N\beta), \quad \text{as } N \to \infty. \end{aligned}$$

Notice that (3.7)(i)–(iv) are possible by the remarks after (3.4).

PROPOSITION 3.4. *For each $s < \infty$ there are integers T_N that satisfy* (3.4) *and* (3.7), *and there are little-o functions $o = o_{\gamma,d,s}$ and integers $N_0 = N_0(\gamma, d, s) < \infty$ such that, for any fixed time $T_0 \le T_N$ and any $N \ge N_0$,*

$$(3.8) \qquad \begin{aligned} &\sup \mathbf{P}_x\Big(\inf_{[u]\subset\phi_x(s)} \sup_{0\le n\le T_0} |\xi_n - u_n| > 3\alpha_N, \ \xi_n \in Q_{-\gamma,d}, 0 \le n \le T_0\Big) \\ &\quad \le \exp(-N\beta s + o(N\beta)), \quad \textit{as } N \to \infty, \end{aligned}$$

where the supremum on the probability is extended over all $x \in Q$ *such that* $\pi_d x \in Q_{-2\gamma, d}$.

PROOF. Define $\lambda = \lambda_N \to 0$ and integers Δt and $\overline{T}$ by

$$\lambda = \alpha^{5/2}, \qquad \Delta t = [\lambda/\beta] \quad \text{and} \quad \overline{T} = [T_N/\Delta t], \tag{3.9}$$

for T_N in (3.4). Define the integer $\overline{T}_0$ by the condition $\overline{T}_0 \Delta t \le T_0 \le (\overline{T}_0 + 1)\Delta t$, so that, by (3.9), $0 \le \overline{T}_0 \le \overline{T}$. Introduce the interpolation process (l_n), $0 \le n \le T_0$, by $l_0 = \xi_0$, $l_{j\Delta t} = \xi_{j\Delta t}$, for all $j = 0, \dots, \overline{T}_0$, and $l_n = \xi_{\overline{T}_0 \Delta t}$, for all $n = \overline{T}_0 \Delta t, \dots, T_0$, and l_n otherwise linear in n. Define sets A_j and I_j for all $j = 0, \dots, \overline{T}_0$ by

$$A_j = \left\{ |\xi_{(j+1)\Delta t} - \xi_{j\Delta t}| \le \alpha, \sup_{j\Delta t \le n \le (j+1)\Delta t} |\xi_n - \xi_{j\Delta t}| \le 2\alpha \right\}$$

and

$$I_j = \left\{ \xi_n \in Q_{-\gamma, d}, \text{for all } j\Delta t \le n \le (j+1)\Delta t \right\},$$

where for the case $j = \overline{T}_0$ we replace $(j+1)\Delta t$ by T_0.

Put also $A = \bigcap_{j=0}^{\overline{T}_0} A_j$ and $I = \bigcap_{j=0}^{\overline{T}_0} I_j$. If A holds then $\sup_{0 \le n \le T_0} |\xi_n - l_n| \le 3\alpha$, while on $A \cap I$, $[l] \subset Q_{-\gamma/2, d}$ for all $N \ge N_0$.

By the above definitions, the left-hand side of (3.8) is bounded by

$$\begin{aligned} &\mathbf{P}_x(A^c \cap I) + \mathbf{P}_x(A \cap I, S_d(N, T_0, [l]) > N\beta s) \\ &\quad \le \sum_{j=0}^{\overline{T}_0} \mathbf{P}_x(A_j^c \cap I_j) + \exp(-N\beta s(1 - o(1))) \\ &\qquad \times \mathbf{E}_x(1_{A \cap I} \exp[(1 - o(1)) S_d(N, T_0, [l])]), \end{aligned} \tag{3.10}$$

where the term $o(1)$ will be chosen to go to zero slower than $\alpha \to 0$ as $N \to \infty$. We now work on the right-hand side of this basic estimate. First, following Ventsel' (1976),

$$\begin{aligned} \mathbf{P}_x(A_j^c \cap I_j) &\le \mathbf{P}_x(|\xi_{(j+1)\Delta t} - \xi_{j\Delta t}| > \alpha) \\ &\quad + \mathbf{P}_x\left(|\xi_{(j+1)\Delta t} - \xi_{j\Delta t}| \le \alpha, \sup_{j\Delta t \le n \le (j+1)\Delta t} |\xi_n - \xi_{j\Delta t}| > 2\alpha\right). \end{aligned}$$

Given $X_{j\Delta t} = y$, with $\pi_d y = \eta$, introduce the stopping time τ_y as the first time $n \ge j\Delta t$ such that $|\xi_n - \eta| > 2\alpha$. Since the original process $\{X_n\}$ is strong Markov, it obtains that

$$\mathbf{P}_x(A_j^c \cap I_j) \le 2 \sup_{y \in Q} \sup_{0 \le m \le \Delta t} \mathbf{P}_y(|\xi_m - \eta| > \alpha). \tag{3.11}$$

The Chebyshev inequality is applied to each component of $\xi_n - \eta$ to estimate this last probability. By successive conditioning relative to the filtration $\sigma(y, X_1, \dots, X_n)$, we have

$$\begin{aligned}\sup_{y\in Q}\sup_{0\le m\le \Delta t}&\mathbf{P}_y\big([\xi_m-\eta]_k>\alpha/\sqrt{d}\,\big)\\ &\le \sup_{y\in Q}\sup_{0\le m\le \Delta t}\exp\big(-z\alpha/\sqrt{d}+mG_d(N,x,z_k)\big),\end{aligned}\tag{3.12}$$

for $z_k = (0, \dots, z, \dots, 0)$ and a positive number z to be chosen below. Define $C_d = \sup_{N\ge 1}\sup_{x\in Q}|\pi_d \mathbf{h}_N(x)|$. This constant is finite by (2.16)(iii). Hence by Lemma 3.4(ii) the right-hand side of (3.12) is bounded by $\exp(-z\alpha/\sqrt{d} + \Delta t[C_d\beta z + C_0 z^2/N])$. Choose now $z = N\beta s\sqrt{d}/\alpha$. Then, by (3.9), (3.11) and (3.12),

$$\begin{aligned}\mathbf{P}_x(A_j^c\cap I_j)&\le d\exp\left(-N\beta s+\frac{\lambda}{\beta}\left(C_d\beta z+C_0\frac{z^2}{N}\right)\right)\\ &\le d\exp\big(-N\beta s+N\beta\big(\alpha\sqrt{\alpha}\,sC_d\sqrt{d}+\sqrt{\alpha}\,s^2dC_0\big)\big)\\ &\le \exp(-N\beta s+o(N\beta))\quad \text{as } N\to\infty.\end{aligned}\tag{3.13}$$

Therefore, since by (3.7)(iv) and (3.9), $\log \overline{T} = o(N\beta)$, it follows by (3.13) that the first term in (3.10) is also bounded by $\exp(-N\beta s + o(N\beta))$.

We pass to the expectation in the second term of (3.10). If $j\,\Delta t \le n < (j+1)\,\Delta t$, then $\Delta l_n = l_{n+1} - l_n$ is constant and $|\xi_n - \xi_{j\Delta t}| \le 2\alpha$ and $|\Delta l_n| \le \alpha/\Delta t$ on $A \cap I$. Thus we have by (2.28)(ii) that on $A \cap I$,

$$\begin{aligned}S_d(N,T_0,[l])&\le \Delta t\sum_{j=0}^{\overline{T}_0-1}H_d(N,\xi_{j\Delta t},\Delta l_{j\Delta t})\big(1+O_{\gamma,d}(\alpha)\big)\\ &\quad+O_{\gamma,d}\left(T_0N\left[\alpha\beta^2+\left(\frac{\alpha}{\Delta t}+\beta\right)^3\right]\right)\\ &\quad+\big(T_0-\overline{T}_0\,\Delta t\big)H_d\big(N,\xi_{\overline{T}_0\Delta t},\mathbf{0}\big).\end{aligned}\tag{3.14}$$

Further, by Lemma 3.2, (2.3), (2.16)(iii) and (3.9), the last term is bounded on $A \cap I$ by $O_{\gamma,d}(\Delta t\,\beta^2 N) = O_{\gamma,d}(\lambda\beta N)$. Hence, by (3.9) and (3.14) and by choosing $o(1) \to 0$ slower than $\alpha \to 0$, it follows that, on $A \cap I$,

$$\begin{aligned}(1-o(1))&S_d(N,T_0,[l])\\ &\le \Delta t\sum_{j=0}^{\overline{T}_0-1}H_d(N,\xi_{j\Delta t},\Delta l_{j\Delta t})\\ &\quad+O_{\gamma,d}\big(N\beta(\alpha\beta+\alpha^{-5}\beta^2)T_0\big)+O_{\gamma,d}(\lambda\beta N).\end{aligned}\tag{3.15}$$

By (3.7)(i), (3.7)(ii) and (3.9) the big-O terms on the right-hand side of (3.15) are $o(N\beta)$, so we study now the sum in (3.15) in case $\overline{T}_0 \ge 1$. Again by successive conditioning,

$$\mathbf{E}_x\left(1_{A\cap I}\exp\left(\Delta t\sum_{j=0}^{\overline{T}_0-1}H_d(N,\boldsymbol{\xi}_{j\Delta t},\Delta l_{j\Delta t})\right)\right)$$
(3.16)
$$\le\left[\sup_{x\in Q,\,\pi_d x\in Q_{-\gamma,d}}\mathbf{E}_x\big(1_{A_0\cap I_0}\exp(\Delta t H_d(N,\pi_d x,\Delta l_{\Delta t}))\big)\right]^{\overline{T}_0}.$$

To estimate the term inside the square brackets, the Legendre transform H_d is approximated by using its definition (2.13) as follows. First by the definition of G_d in (2.10) it holds that $\mathbf{E}_x(z\cdot(\boldsymbol{\xi}_1-\xi)-G_d(N,\xi,z))=1$. Thus, by successive conditioning, it also holds that, if $|z|\le\zeta_d N$, then $\mathbf{E}_x(\exp(z\cdot(\boldsymbol{\xi}_{\Delta t}-\xi)-\sum_{n=0}^{\Delta t-1}G_d(N,\boldsymbol{\xi}_n,z))=1$. Therefore, by Lemma 3.4(i),

$$\sup_{x\in Q,\,\pi_d x\in Q_{-\gamma,d}}\mathbf{E}_x\big(1_{A_0\cap I_0}\exp(z\cdot(\boldsymbol{\xi}_{\Delta t}-\xi)-\Delta t\,G_d(N,\xi,z))\big)$$
$$\le\mathbf{E}_x\left(\exp\left(z\cdot(\boldsymbol{\xi}_{\Delta t}-\xi)-\sum_{n=0}^{\Delta t-1}G_d(N,\boldsymbol{\xi}_n,z)\right)\right)$$
(3.17)
$$\times\exp\left\{\Delta t\sup_{\xi,\eta\in Q_{-\gamma,d},\,|\xi-\eta|\le 3\alpha}|G_d(N,\xi,z)-G_d(N,\eta,z)|\right\}$$
$$\le\exp\Big(O_d\Big(\Delta t\big\{|z|\beta(\alpha+\beta)+|z|^2(\alpha+\beta)/N+\big(|z|^3+|z|^2N\beta\big)/N^2\big\}\Big)\Big),$$
as $N\to\infty$.

For the remainder of the argument we follow Darden (1983). Let $k=k_N=\alpha_N^{-4}\to\infty$. Let $\{\xi^i\}_{i=1}^{k^d}$ be a collection of points in $Q_{-\gamma,d}$ such that for any $\xi\in Q_{-\gamma,d}$ there exists ξ^i with $|\xi^i-\xi|_{\mathbb{R}^d}\le$ const.$/k$. Define also k^d points $\{u^j\}\subset\mathbb{R}^d$ such that for any $u\in\mathbb{R}^d$ with $|u|\le\alpha/\Delta t$, there is a u^j with $|u-u^j|\le$ const. $\alpha/(k\,\Delta t)$. Define $z(i,j)=z_d(N,\xi^i,u^j)$ and

$$\text{(3.18)}\quad \Delta H:=\sup\Big[H_d(N,\xi,u)-\max_{i,j}\{z(i,j)u-G_d(N,\xi,z(i,j))\}\Big],$$

where the supremum is extended over all $\xi\in Q_{-\gamma,d}$ and $|u|\le\alpha/\Delta t$. By (3.18) and (2.13),

$$\Delta H\le H_d(N,\xi,u)-H_d(N,\xi^i,u^j)+|z(i,j)|\,|u-u^j|$$
(3.19)
$$+|G_d(N,\xi^i,z(i,j))-G_d(N,\xi,z(i,j))|$$
$$=\mathrm{I}+\mathrm{II}+\mathrm{III}.$$

By Lemma 3.5(ii) and (iii),

$$\mathrm{I}\le O_{\gamma,d}\left(N\left\{\left(\frac{\alpha}{\Delta t}+\beta\right)^2\left(\frac{1}{k}+\frac{\alpha}{\Delta t}+\beta\right)+\frac{\alpha}{k\,\Delta t}\left(\beta+\frac{\alpha}{\Delta t}\right)\right\}\right)$$
(3.20)
$$=O_{\gamma,d}\big(N\beta(\alpha^{-5}\beta^2+\alpha\beta)\big),$$

where we have used $k = \alpha^{-4}$ and $\lambda = \alpha^{5/2}$. Next, by Lemma 3.5(i),

$$\text{II} \le O_{\gamma,d}\left(N\left(\frac{\alpha}{\Delta t} + \beta\right)\frac{\alpha}{k\,\Delta t}\right) = O_{\gamma,d}(N\beta(\alpha\beta)). \tag{3.21}$$

Finally, by Lemma 3.4(i),

$$\begin{aligned} \text{III} &\le O_{\gamma,d}\left(N\left\{\left(\frac{\alpha}{\Delta t} + \beta\right)\beta\left(\frac{1}{k} + \beta\right)\right.\right. \\ &\qquad \left.\left. + \left(\frac{\alpha}{\Delta t} + \beta\right)^2\left(\frac{1}{k} + \beta\right) + \left(\frac{\alpha}{\Delta t} + \beta\right)^3\right\}\right) \\ &= O_{\gamma,d}(N\beta(\alpha^{-5}\beta^2 + \alpha\beta)). \end{aligned} \tag{3.22}$$

To finish the estimate of (3.10) by means of (3.15)–(3.22), we argue that, by (3.18),

$$\begin{aligned} &\mathbf{E}_x\left[1_{A_0\cap I_0}\exp(\Delta t H_d(N,\xi,\Delta l_{\Delta t}))\right] \\ &\quad \le \exp(\Delta t\,\Delta H)\mathbf{E}_x\left(1_{A_0\cap I_0}\exp\left[\Delta t \max_{i,j}\{z(i,j)\,\Delta l_{\Delta t}\right.\right. \\ &\qquad\qquad \left.\left. - G_d(N,\xi,z(i,j))\}\right]\right). \end{aligned} \tag{3.23}$$

Now estimate by Lemma 3.5(i) that $|z(i,j)| \le O_{\gamma,d}(N(\alpha/\Delta t + \beta))$, and therefore, by (3.7)(i), (3.7)(v) and (3.9) and by substituting $z = z(i,j)$, the right-hand side of (3.17) becomes $\exp(O_d(N\beta[\sqrt{\alpha}\,\beta + \alpha^{-2}\beta^2]))$. Also, it is estimated that $\exp(\max a_{ij}) \le \Sigma\Sigma_{i,j}\exp(a_{ij})$. Therefore, putting together (3.17)–(3.23) and these last two estimates, it follows that

$$\begin{aligned} &\mathbf{E}_x\left[1_{A_0\cap I_0}\exp(\Delta t H_d(N,\xi,\Delta l_{\Delta t}))\right] \\ &\qquad \le \alpha^{-8}\exp\left(O_{\gamma,d}\left(\Delta t N\beta(\alpha^{-5}\beta^2 + \sqrt{\alpha}\,\beta)\right)\right). \end{aligned} \tag{3.24}$$

Finally, raising both sides of (3.24) to the power $\overline{T}_0$ and using (3.7)(i)–(iii), we obtain from (3.24), (3.15) and (3.16) that

$$\begin{aligned} &\mathbf{E}_x(1_{A\cap I}\exp[(1 - o(1))S_d(N,T_0,[l])]) \\ &\quad \le \exp\left(O_{\gamma,d}\left(N\beta T_N(\alpha^{-5}\beta^2 + \sqrt{\alpha}\,\beta) + T_N\alpha^{-3}|\log(\alpha)|\beta + N\beta\alpha^{5/2}\right)\right) \\ &\quad = \exp(o(N\beta)). \end{aligned} \tag{3.25}$$

By (3.10), (3.13) and (3.25) the proof of the proposition is complete. □

4. Exit proofs. The arguments of Freidlin and Wentzell (1984) and Morrow and Sawyer (1989) are adapted to establish Theorem 1.1 using an approximation scheme and the conditions (2.29) and (2.40). This is complicated somewhat by the typical possibility that, for an initial point $x \in D$, $\pi_d x$ may lie on the boundary of $\pi_d Q$. The next lemma allows one to circumvent this problem.

LEMMA 4.1. *Let $1 > \delta > 0$ and put $d = d(\delta)$ as in* (2.2). *Choose $0 < t_0 < \delta$ so small that $\gamma(t_0, d)/[t_0 C_1 \delta] > 1$, which is possible by* (2.26)(i) *and* (2.40)(iv). *Put $\xi_n = \pi_d X_n$. Then there exists $\gamma_\delta > 0$ such that for some absolute constant C, some $\delta_0 > 0$, and all $0 < \delta < \delta_0$,*

$$\text{(i)} \qquad \lim_{N\to\infty} \sup_{x\in D} \mathbf{P}_x\big(\xi_{[t_0/\beta]} \notin Q_{-\gamma_\delta, d}\big) = 0,$$

and, for each fixed $x \in D$,

$$\text{(ii)} \quad \lim_{N\to\infty} \mathbf{P}_x\big(X_n \in D, n = 0, \dots, [t_0/\beta], \xi_{[t_0/\beta]} \in (D^{-r(x)})_{-\gamma_\delta, d}\big) = 1,$$

where $r(x)$ is defined in (2.26)(iii).

PROOF. The proof uses the discrete-time Gronwall inequality and the submartingale inequality. By the proof of Section 2.2 but with the $\mathbb{R}^d$ norm in place of the metric $\rho(\cdot,\cdot)$, and by setting $\dot{u}(t) = h_d(u)$, $u(0) = \pi_d x$ and $p_N(n\beta) = (\pi_d \mathbf{f}_N)^{\circ n}(\pi_d x)$, we have

$$|u(t) - p_N(t)| \le o(1) t (1 + K_d \beta)^{t/\beta} \tag{4.1}$$

for K_d in (2.16)(ii) and any $t = n\beta \le 1$.

Now

$$\begin{aligned} \xi_n - p_N(n\beta) &= \left[\xi_n - \xi_0 - \beta \sum_{k=1}^{n-1} \pi_d \mathbf{h}_N(X_k)\right] \\ &\quad + \beta \sum_{k=1}^{n-1} \left[\pi_d \mathbf{h}_N(X_k) - \pi_d \mathbf{h}_N(\xi_k)\right] \\ &\quad + \beta \sum_{k=1}^{n-1} \left[\pi_d \mathbf{h}_N(\xi_k) - \pi_d \mathbf{h}_N(p_N(k\beta))\right] \\ &= \text{I} + \text{II} + \text{III}, \end{aligned}$$

where I is a mean-zero martingale M_n and, by (2.29), II is bounded in $\mathbb{R}^d$-norm by $n\beta(C_1\delta + O_d(\beta))$, as $N \to \infty$. Therefore, by (2.16)(ii) and the discrete-time Gronwall inequality,

$$|\xi_n - p_N(n\beta)| \le \left[\max_{1\le k\le n} |M_k| + C_1 n\beta\delta + o_d(1)\right](1 + K_d\beta)^n, \tag{4.2}$$

for all $0 \le n \le t/\beta$. Now choose $t = ([t_0/\beta] + 1)\beta$, so that $t \ge t_0$ and $t = t_0 + O(\beta) \le \delta < 1$. We know by (2.26)(i) that $u(t) \in Q_{-\gamma(t_0, d), d}$. By (2.40)(iii) we choose $\gamma_\delta > 0$ such that $tC_1\delta - \gamma(t_0, d) < -3\gamma_\delta$. Hence, by (4.1), (4.2) and (2.40)(ii), it follows that

$$\text{if} \quad \xi_{[t_0/\beta]} \notin Q_{-\gamma_\delta, d}, \quad \text{then} \quad \max_{1\le k\le t/\beta} |M_k| \ge \gamma_\delta - o(1),$$

as long as δK_d is small enough and N is sufficiently large. Now simply apply the L^2 submartingale inequality, using (2.12)(ii) to obtain

$$\mathbf{P}_x\Big(\max_{1\le k\le t/\beta}|M_k|>\lambda\Big)\le dC_0t/\big[N\beta\lambda^2\big].$$

Since $N\beta\to\infty$, this completes the proof of the first assertion.

To prove the second assertion, let $t=t_0+O(\beta)$ be as above. By (2.26)(iii) it holds that $u(s)\in D^{-r(x)}$, for all $0\le s\le 1$, so by (2.26)(i) and Remark 2.1(a),

$$u(t)\in Q_{-\gamma,d}\cap\operatorname{int}\big(\pi_dD^{-r(x)}\big)=\big(D^{-r(x)}\big)_{-\gamma,d},\quad\text{with } \gamma=\gamma(t_0,d).$$

Therefore, also by (4.1), it holds that $p_N(t)\in(D^{-r(x)})_{-\gamma(t_0,d),d}$ if N is sufficiently large. In particular, by (4.2) as above, $\mathbf{P}_x(\boldsymbol{\xi}_{[t_0/\beta]}\notin(D^{-r(x)})_{-\gamma_\delta,d})\to 0$. But also by (4.1), if N is large, then $p_N(n\beta)\in D^{-r(x)/2}$ for all $n\le t/\beta$. Hence, if $\boldsymbol{\xi}_n\in(D^{-r(x)/2+C\delta})^c$ for some $0\le n\le[t_0/\beta]$ and for any fixed $C>(C_1+1)\sup_{0<\delta<1}\delta q(\delta)$ with $q(\delta)$ of (2.40)(v), then by (4.2), (2.40)(ii) and (2.40)(v),

$$\max_{1\le k\le t/\beta}|M_k|\ge\delta^2-o(1),$$

if δ is sufficiently small, since $t_0<\delta$. Since $\boldsymbol{\xi}_n\in D^{-a}$ implies $X_n\in D^{-a+\delta}$, the submartingale inequality yields statement (ii) if in addition δ is so small that $-r(x)/2+C\delta<0$. □

We now proceed to establish the basic ingredients needed to prove Theorem 2.1. Let $T=T_N^*$ and $\alpha=\alpha_N$ satisfy (3.4) and (3.7) such that Propositions 3.3 and 3.4 hold with T in place of T_N^* and T_N, respectively. This is possible since by its definition, T_N^* in Proposition 3.3 satisfies (3.4) and (3.7). Define $\theta=\theta_N(T)$ by (3.5) so that $\theta\to 0$ as $N\to\infty$ by (3.4)(i) and (3.4)(iii). Choose $\alpha'=\alpha'_N\to 0$ so slowly that, by (3.4),

$$\alpha'^2\beta T\to\infty,\qquad \alpha'/\alpha\to\infty\quad\text{and}\quad\alpha'/\theta\to\infty. \tag{4.3}$$

Define

$$B_{N,d}=\big\{\xi\in\mathbb{R}^d\colon|\xi-\xi_{0,d}^{(N)}|\le\alpha'\big\}.$$

PROPOSITION 4.1. *For each $\delta>1$ and $\varepsilon>0$ there exists $\gamma>0$ such that if $d=d(\delta)$, $r(d)$ is given by* (2.37)(i) *and $G=D^\varepsilon$, then*

$$\sup_{x\in D}\mathbf{P}_x(T_e>2T)\le 1-\exp\Big(-N\beta\big[V_d\big((G^{-r(d)})_{-\gamma,d}\big)+o_{\gamma,d}(1)+\omega_d\big]\Big),$$

where $\omega_{d(\delta)}\to 0$ as $\delta\to 0$ and $o_{\gamma,d}(1)\to 0$ as $N\to\infty$.

PROOF. Let $t_0<1$ and $\gamma_\delta>0$ be as in Lemma 4.1. Define a discrete trajectory $(v_n)_{n=0}^T$ as follows. Define $u(t)$ by $\dot u=h_d(u)$, with $u(0)=\pi_dx\in Q_{-\gamma_\delta,d}$. Then set $v_n=u(n\beta)$ until a time $n=T_1$ given by Lemma 3.3 such that $|v_n-\xi_{0,d}^{(N)}|\le b_d$. Next set $v_{n+1}-v_n=(v_{T_1}-\xi_{0,d}^{(N)})/T_2$ for all $T_1\le n\le T_1+T_2-1$, with $T_2=[1/\beta]$. Finally, if $n\ge T_1+T_2$, set $v_n=\xi_{0,d}^{(N)}$. Define the trajectory $(w_n)_{n=0}^T$ as in Proposition 3.3 for a starting point $\pi_dx\in B_{N,d}$

that asymptotically minimizes $\inf_{\pi_d x \in B_{N,d}} \mathbf{P}_x(T_e \le T)$. We have

$$
\begin{aligned}
\inf_{x \in D} \mathbf{P}_x(T_e \le 3T) \ge & \inf_{x \in D} \mathbf{P}_x\big(\boldsymbol{\xi}_{[t_0/\beta]} \in Q_{-\gamma_\delta, d}\big) \\
& \times \inf_{\pi_d x \in Q_{-\gamma,d}} \mathbf{P}_x\Big(\max_{0 \le n \le T} |\xi_n - v_n| \le \theta_N(T)\Big) \\
& \times \inf_{\pi_d x \in B_{N,d}} \mathbf{P}_x(T_e \le T) \\
= {} & A \times B \times C.
\end{aligned}
\tag{4.4}
$$

We follow the proof of Proposition 3.3 verbatim, with (v_n) in place of (w_n) and γ_δ in place of γ, until the estimate of $-S_d(N, T, [v])$. We write

$$
\begin{aligned}
-S_d(N,T,[v]) = & \sum_{n=T_1+T_2}^{T} 0 - \sum_{n=T_1}^{T_1+T_2-1} H_d\big(N, v_{T_1}, \big(\xi_{0,d}^{(N)} - v_{T_1}\big)/T_2\big) \\
& - \sum_{n=0}^{T_1-1} H_d(N, v_n, v_{n+1} - v_n).
\end{aligned}
\tag{4.5}
$$

Apply Lemma 3.5(iii) to the last sum on the right-hand side of (4.5), noting that $v_{n+1} - v_n = \pi_d \mathbf{f}_N(v_n) - v_n + o(\beta)$. Hence the contribution from this last sum is at most $O_d(N\beta^2)$. Apply Lemma 3.2 and (2.40)(iii) to estimate the other sum by

$$NT_2\beta^2 b_d^2(1 + K_d^2)/\lambda_d + o(N\beta) = \omega_d N\beta + o(N\beta).$$

Hence $B \ge \exp(-N\beta[\omega_d + o(1)])$. Also, by Lemma 4.1, $A \ge 1 - o(1)$. Finally, by (2.37)(i) and Proposition 3.3 there is a $\gamma > 0$ such that

$$C \ge \exp\Big(-N\beta\big[V_d\big((G^{-r(d)})_{-\gamma,d}\big) + o_{\gamma,d}(1) + \omega_d\big]\Big).$$

Putting together the estimates for A, B and C, the proof of Proposition 4.1 is complete by (4.4). □

Define

$$\tau_1 = \min\{n \ge 1 \colon \boldsymbol{\xi}_n \in B_{N,d} \text{ or } X_n \in D^c\}.$$

PROPOSITION 4.2. *Assume $N\beta/\log N \to \infty$. Let $0 < \varepsilon < \varepsilon_0$. There exists some $\delta(\varepsilon) > 0$ and little-o functions $o = o_{\gamma,d}$ such that if $0 < \delta < \delta(\varepsilon)$, $d = d(\delta)$ and $G = D^{-\varepsilon}$, then the following hold for all $\gamma > 0$:*

(i) $\inf_{\pi_d x \in B_{N,d}} \mathbf{P}_x(\tau_1 < T_e) \ge 1 - \exp\big(-N\beta\big[V_d(G_{-2\gamma,d}) + o(1)\big]\big)$;

(ii) *for any compact $K \subset Q_{-\gamma,d}$ there is a constant $c(K) > 0$ such that*

$$\sup_{\pi_d x \in K} \mathbf{P}_x(\tau_1 = T_e) = O_{\gamma,d}(\exp[-c(K)N\beta(1 + o(1))]);$$

(iii) *for each $x \in D$, $\lim_{N\to\infty} \mathbf{P}_x(\tau_1 < T_e) = 1$.*

REMARK. Besides the final proof of Theorem 2.1, the proof of Proposition 4.2 is the only place we use the condition $N\beta/\log N \to \infty$ instead of just $N\beta \to \infty$.

PROOF OF PROPOSITION 4.2. Assume in addition to (3.4) and (3.7) that

$$\log T = o(N\beta), \quad \text{as } N \to \infty. \tag{4.6}$$

This is possible by our assumption $N\beta \gg \log N$. Define, for all $\gamma > 0$ and $d \geq 1$,

$$\tau_{\gamma, d} = \min\{n \geq 1 \colon \xi_n \notin Q_{-\gamma, d}\}.$$

Work first on (i). Let $\pi_d x \in B_{N,d}$ and write

$$\begin{aligned} \mathbf{P}_x(\tau_1 = T_e) &\leq \mathbf{P}_x(\tau_1 = T_e \leq T, \tau_{\gamma,d} \leq T_e) + \mathbf{P}_x(\tau_1 = T_e \leq T, \tau_{\gamma,d} > T_e) \\ &\quad + \mathbf{P}_x(\tau_1 > T, \tau_{\gamma,d} \leq T) + \mathbf{P}_x(\tau_1 > T, \tau_{\gamma,d} > T) \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}. \end{aligned} \tag{4.7}$$

Estimate first that

$$\mathrm{I} + \mathrm{III} \leq 2\mathbf{P}_x(\tau_1 > \tau_{\gamma,d}, \tau_{\gamma,d} \leq T) \leq 2 \sum_{n=1}^{T} \mathbf{P}_x(\tau_1 > n, \tau_{\gamma,d} = n) = 2\sum p_n.$$

However, by Proposition 3.2,

$$\begin{aligned} p_n &\leq \mathbf{P}_x\Big(\inf_{[u] \in \phi_x(s)} \max_{0 \leq k \leq n} |\xi_k - u_k| > \alpha, \xi_k \in Q_{-\gamma/2,d}, k = 0, \dots, n\Big) \\ &\quad + \mathbf{P}(\xi_1 \notin Q_{-\gamma/2,d} | \xi_0 \in Q_{-\gamma,d}) \\ &= p_{n1} + p_{n2}, \end{aligned}$$

with $s = V_d(Q_{-2\gamma,d}) + o(1)$ as $N \to \infty$, since $\alpha = o(1)$ and, on the event for p_n, ξ_k is out of $Q_{-\gamma,d}$ by time n. Since, by (2.16)(iii), $\mathbf{E}(\xi_1 | \xi_0 \in Q_{-\gamma,d}) = \xi_0 + O_{\gamma,d}(\beta)$, it follows by a standard exponential inequality [see Morrow and Sawyer (1989), Lemma 2.1] that

$$p_{n2} \leq \exp[-C_{\gamma,d} N]$$

for some $C_{\gamma,d} > 0$. Also, by Proposition 3.4,

$$p_{n1} \leq \exp\big(-N\beta\big[V_d(Q_{-2\gamma,d}) + o(1)\big]\big).$$

Therefore, by (4.6),

$$\mathrm{I} + \mathrm{III} \leq 2 \sum_{n=1}^{T} p_n \leq \exp\big(-N\beta\big[V_d(Q_{-2\gamma,d}) + o(1)\big]\big), \quad \text{as } N \to \infty. \tag{4.8}$$

Next, write

$$\mathrm{II} = \sum_{n=1}^{T} \mathbf{P}_x(\tau_1 = T_e = n, \tau_{\gamma,d} > n) = \sum \mathrm{II}_n. \tag{4.9}$$

We claim that

$$\text{(4.10)}\quad \mathrm{II}_n \le \mathbf{P}_x\Big(\inf_{[u]\in\phi_x(s)} \max_{0\le k\le n} |\boldsymbol{\xi}_k - u_k| > \alpha,\ \boldsymbol{\xi}_k \in Q_{-\gamma,d},\ k = 0,\dots,n\Big),$$

for $s = V_d(G_{-2\gamma,d}) + o(1)$. Indeed, if X_k is out of D by time n, then $\boldsymbol{\xi}_k$ is out of $\pi_d(D^{-2\delta})$ by this time. But we shall see that the Euclidean distance between $\partial(G_{-2\gamma,d})$ and $\partial((D^{-2\delta})_{-\gamma,d})$ is positive if $0 < \delta < \delta(\varepsilon)$ and $0 < \gamma < \gamma(\delta)$ for some $\delta(\varepsilon) > 0$ and $\gamma(\delta) > 0$. If this is so, then by Proposition 3.2(i) and our choice of s, $[u]$ cannot get out of $G_{-2\gamma,d}$. Thus the L_∞ distance between $[\boldsymbol{\xi}]$ and $[u]$ will be a positive number, independent of N, on the event of II_n. Now if indeed by contrapositive argument, there does exist some point $\xi \in \partial(G_{-2\gamma,d}) \cap \partial((D^{-2\delta})_{-\gamma,d})$, then $\xi \in Q_{-\gamma,d}$, and also $\xi \in G^{2\delta}$. Further, by Remark 2.1(a) with $F = D^{-2\delta}$, it follows that $\xi \notin (\pi_d(D^{-5\delta}))$, since $\xi \notin \partial(\pi_d Q)$ and since $\inf\{\rho(\xi,\eta)\colon\ \eta \in (\pi_d(D^{-2\delta}))^c$ and $\xi \in \pi_d(D^{-5\delta})\} \ge \delta$. Thus $\xi \in G^{2\delta} \setminus D^{-5\delta} =$ empty set, if δ is sufficiently small, depending on ε. This is a contradiction to the existence of ξ. Hence the claim is established and (4.10) holds. Thus, by (4.9), (4.10), Proposition 3.4 and (4.6),

$$\text{(4.11)}\qquad \mathrm{II} \le \exp\big(-N\beta\big[V_d(G_{-2\gamma,d}) + o(1)\big]\big), \quad \text{as } N \to \infty.$$

Finally, we estimate that

$$\text{(4.12)}\qquad \begin{aligned} \mathrm{IV} &= \mathbf{P}_x(\tau_1 > T,\ \tau_{\gamma,d} > T)\\ &\le \mathbf{P}_x\Big(\inf_{[u]\in\phi_x(s)} \max_{0\le n\le T} |\boldsymbol{\xi}_n - u_n| \ge \alpha'/2,\ \boldsymbol{\xi}_n \in Q_{-\gamma,d},\ n = 0,\dots,T\Big),\end{aligned}$$

for any $s = s_0 < \infty$. Indeed, given s_0, if u_n were outside $A_{N,d} := \{\xi\colon |\xi - \xi_{0,d}^{(N)}| \le \alpha'/2\}$ for all $0 \le n \le T$, then, by Proposition 3.1 and our choice of α' in (4.3), it would follow that $S_d(N,T,[u]) > N\beta s_0$, which would contradict the condition $[u] \in \phi_x(s_0)$. However, if u_n is inside $A_{N,d}$ for some $0 \le n \le t$, then on the event $\tau_1 > T$ it would hold that $\max_{0\le n\le T}|\boldsymbol{\xi}_n - u_n| \ge \alpha'/2$. Since $\alpha'/2 > \alpha$ by (4.3), it follows that estimate (4.12) holds. Hence, by Proposition 3.4,

$$\text{(4.13)}\qquad \mathrm{IV} \le \exp\big(-N\beta[s_0 + o(1)]\big), \quad \text{as } N \to \infty,$$

for any $s_0 < \infty$. Putting together (4.7), (4.8), (4.11) and (4.13), we obtain statement (i) of the proposition. Part (ii) of the proposition follows in the same way by replacing s in (4.7) and (4.10) by $c(K)(1 + o(1))$ for $c(K)$ in Proposition 3.2(i). Part (iii) is proved by first applying Lemma 4.1(ii). We sketch the argument as follows. As in the proof of part (i), write out (4.7), except now it is understood that x is a fixed initial point in D and we define the stopping time $\sigma = \tau_{\gamma_\delta/4,d}$ with γ_δ from Lemma 4.1. Look at II in (4.7), with σ in place of $\tau_{\gamma,d}$. We have

$$\text{(4.14)}\qquad \begin{aligned} \mathrm{II} &= \mathbf{P}_x(\tau_1 = T_e \le T,\ \sigma > T_e)\\ &\le \mathbf{P}_x\big(X_n \notin D,\ \text{some } n = 0,\dots,[t_0/\beta],\ \text{or } \boldsymbol{\xi}_{[t_0/\beta]} \notin (D^{-r(x)})_{-\gamma_\delta,d}\big)\\ &\quad + \sup P_y\big(\tau_1 = T_e \le T - [t_0/\beta],\ \sigma > T_e\big),\end{aligned}$$

where the supremum is extended over all y such that $\pi_d y \in (D^{-r(x)})_{-\gamma_\delta, d}$. The first term is estimated by Lemma 4.1 to be $o(1)$. The second term is estimated by Proposition 3.4 and the argument for estimating II in part (i). Indeed, let $s = s_{\varepsilon, x, \delta}$ = the minimal action required for a trajectory to go from $(D^{-a})_{-\gamma_\delta, d}$ to $(D^{-a/2})_{-\gamma_\delta/2, d}$, for some $a = a_x > 0$. Then $s > 0$ by Proposition 3.2(i), since the Euclidean distance between the boundaries of these two sets is positive. So Proposition 3.4 gives that the second term on the right-hand side of (4.14) is bounded by $\exp(-N\beta[s + o(1)])$. The other estimates for part (iii) follow similar lines as this one, but using the ideas of the estimates of I, III and IV as above. Hence, for each fixed $x \in D$,

$$P_x(\tau_1 = T_e) \le o(1) + \exp(-N\beta[s + o(1)]), \quad \text{as } N \to \infty, \tag{4.15}$$

for some $s = s_{\varepsilon, x, \delta} > 0$. This concludes the proof of Proposition 4.2. □

PROOF OF THEOREM 2.1. By Propositions 4.1 and 4.2, Theorem 2.1 is proved along the lines of Morrow and Sawyer (1989) until at the last step the parameters γ, d and ε are sent to their limits. For the upper expectation bound we put $G = D^{\varepsilon - r(d(\delta))}$ for $r(d)$ in (2.37)(i). By the Markov property of the original chain, Proposition 4.1 and (4.6), we have that

$$\mathbf{P}_x(T_e \ge 4nT) \le \Big(1 - \exp\big(-N\beta\big[V_d(G_{-2\gamma, d}) + o(1)\big]\big)\Big)^n,$$

for all $n \ge 1$. Hence

$$\begin{aligned}
\mathbf{E}_x(T_e) &\le 4T \sum_{n=0}^{\infty} \mathbf{P}_x(T_e \ge 4nT) \\
&\le 4T \sum_{n=0}^{\infty} \Big(1 - \exp\big(-N\beta\big[V_d(G_{-2\gamma, d}) + o(1)\big]\big)\Big)^n \\
&= 4T \exp\big(N\beta\big[V_d(G_{-2\gamma, d}) + o(1)\big]\big) \\
&= \exp\big(N\beta\big[V_d(G_{-2\gamma, d}) + o(1)\big]\big), \quad \text{as } N \to \infty.
\end{aligned}$$

Thus $\limsup_{N \to \infty} \mathbf{E}_x(T_e)/N\beta \le V_d(G_{-2\gamma, d})$. Letting first, $\gamma \to 0$ and second, $\delta \to 0$, so $d(\delta) \to \infty$, and finally $\varepsilon \downarrow 0$, part (i) of the theorem follows by (2.37)(ii) and (2.41). To prove part (ii), define stopping times τ_k by induction as follows: $\tau_0 = 0$, and

$$\tau_{k+1} = \begin{cases} \min\{n \ge \tau_k + 1 \colon \boldsymbol{\xi}_n \in B_{N,d} \text{ or } X_n \in D^c\}, & \text{if } \tau_k < T_e, \\ T_e, & \text{if } \tau_k = T_e. \end{cases}$$

Then

$$T_e = \tau_1 + \sum_{k=1}^{\infty} (\tau_{k+1} - \tau_k) \ge 1 + \sum_{k=1}^{\infty} 1(\tau_k < T_e).$$

Thus by Proposition 4.2 and (2.42), if $G = D^{-\varepsilon}$ for some $0 < \varepsilon < \varepsilon_0$, then, for any given $x \in D$,

$$\mathbf{E}_x(T_e) \geq \mathbf{P}_x(\tau_1 < T_e) \sum_{k=1}^{\infty} \inf_{\pi_d y \in B_{N,d}} \mathbf{P}_y(\tau_1 < T_e)$$

$$\geq (1 - o_{\varepsilon, x, \delta}(1)) \sum_{k=1}^{\infty} \left(1 - \exp\left(-N\beta\left[V_d(G_{-\gamma, d}) + o(1)\right]\right)\right)^n$$

$$= \exp\left(N\beta\left[V_d(G_{-\gamma, d}) + o(1)\right]\right).$$

Hence, $\liminf_{N \to \infty} \mathbf{E}_x(T_e)/N\beta \geq V_d(G_{-\gamma, d})$. Therefore, by (2.37)(ii) and (2.41), letting first $\gamma \downarrow 0$, then $\delta \downarrow 0$ and finally $\varepsilon \downarrow 0$, we obtain part (ii) of the theorem. This completes the proof of Theorem 2.1. □

PROOF OF THEOREM 1.1. The proof of Theorem 1.1(i) follows from Theorem 2.1, the work of Section 2, and Theorem 5.1 in Section 5 on the characterization of V_∞ by (1.20). To prove Theorem 1.1(ii), obtain the estimate $\mathbf{P}_x\{T_e > \exp[N\beta(V_\infty + \omega_N)]\} \to 0$ by Theorem 1.1(i) and Chebyshev's inequality. Finally, the estimate $\mathbf{P}_x\{T_e < \exp[N\beta(V_\infty - \omega_N)]\} \to 0$ follows from Proposition 4.2(ii) and (iii) as in the proof of Lemma 5.2 of Morrow and Sawyer (1989). □

5. Characterization of V_∞. Let D be the open ball of radius r defined by (1.19). The purpose of this section is to prove the following facts about the large deviation exponents associated to the infinite alleles model.

THEOREM 5.1. *If $d = 2^l - 1$ in (1.15), then (1.20) holds and $V_\infty(D)$ is continuous as a function of the radius r of D.*

An outline of the proof is as follows. Define

$$(5.1)\quad J(y) = -2\left\{\int \sigma(t)[y(dt) - x_0(t)\,dt] + \mu \int \log[(dy_{\mathrm{ac}}/dt)/x_0(t)]\,dt\right\},$$

for all $y \in \mathbf{P}[0,1]$, where x_0 is defined by (1.13). It is first shown that $J(y) = \lim_{l \to \infty} J(\pi_l y)$, where π_l denotes π_d defined by (1.15) with $d = d(l) = 2^l - 1$. It is also proved that J is lower semicontinuous from $\mathbf{P}[0,1]$ to the extended real line. It is proved that (1.20) is consistent with (2.41) in the following sense:

$$(5.2)(\mathrm{i})\quad \lim_{\varepsilon \downarrow 0} \liminf_{l \to \infty} V_{d(l)}(D^{-\varepsilon}) = \liminf_{\varepsilon \downarrow 0} \{J(y) : y \notin D^{-\varepsilon}\} = V_\infty^-(D)$$

and

$$(5.2)(\mathrm{ii})\quad \lim_{\varepsilon \downarrow 0} \limsup_{l \to \infty} V_{d(l)}(D^{+\varepsilon}) = \liminf_{\varepsilon \downarrow 0} \{J(y) : y \notin D^{+\varepsilon}\} = V_\infty^+(D).$$

Finally, using (5.2)(i) and (5.2)(ii) and the lower semicontinuity of J, it is proved that V_∞^+ and V_∞^- coincide.

We proceed to establish the above outline. As in (1.15), let $\pi_l y$ denote the projection of $y \in \mathbf{P}[0,1]$ onto the conditional expectation of y given the algebra generated by the dyadic intervals of level l in the Lebesgue unit interval. With a slight abuse of notation, $\pi_l y$ actually denotes the density of this projection relative to Lebesgue measure, so that if A is a dyadic subinterval of level l, then $\int_A (\pi_l y)(t)\,dt = y(A)$.

PROPOSITION 5.1. *Let* $\Psi\colon (0,\infty) \to \mathbb{R}$ *be continuous, concave up and strictly decreasing with* $\Psi(u)\downarrow -\infty$ *as* $u \uparrow \infty$, $\Psi(u)\uparrow\infty$ *as* $u\downarrow 0$, $\Psi(0)=\infty$ *and* $\lim_{u\to\infty}\Psi(u)/u = 0$. *For each* $y \in \mathbf{P}[0,1]$, *write* $y = y_{\mathrm{ac}} + y_{\mathrm{s}}$ *(Lebesgue decomposition), where* y_{ac} *and* y_{s} *denote, respectively, the absolutely continuous part and the singular part of* y. *Then*,

$$\sup_l \int_0^1 \Psi(\pi_l y)\,dt = \lim_{l\to\infty}\int_0^1 \Psi(\pi_l y)\,dt = \int_0^1 \Psi(dy_{\mathrm{ac}}/dt)\,dt,$$

for each $y \in \mathbf{P}[0,1]$.

PROOF. Let $\varepsilon_0, B_0 > 0$ such that $\Psi(u) \ge -u$, for all $u \ge B_0$, and $\Psi(\varepsilon_0) = 0$. Let $\mathbf{F}_l$ be the finite sigma field generated by the dyadic intervals of $[0,1]$ of level l. Then $\pi_l y = \mathbf{E}(\pi_{l+1} y | \mathbf{F}_l)$. Hence, by Jensen's inequality, the integrals $\int \Psi(\pi_l y)$ are nondecreasing. Note that $\int\Psi(\pi_l y) = \infty$ if and only if $\pi_l y = 0$ on some dyadic interval. Here and below when convenient we denote for brevity $\int_0^1(\cdot\,\cdot)\,dt$ by $\int(\cdot\,\cdot)$.

(a) Assume first $y = y_{\mathrm{ac}}$ and denote $y' = dy/dt$. Then either

$$\text{(i)} \int_{\{y'\le\varepsilon_0\}} \Psi(y') < \infty \quad \text{or} \quad \text{(ii)} \int_{\{y'\le\varepsilon_0\}} \Psi(y') = \infty.$$

CASE a(i). Since $\int_{\{y'\ge B_0\}}\Psi(y') \ge \int -y' = -1$, it holds that $\Psi(y')$ is absolutely integrable, as is $\Psi(\pi_l y)$. Hence, by adjoining a last element $\Psi(y')$ to the sequence $\{\Psi(\pi_l y)\}_{l\ge 1}$, we obtain a submartingale with a last element: $\Psi(\pi_l y) \le \mathbf{E}(\Psi(\pi_{l+1} y)|\mathbf{F}_l)$ and $\Psi(\pi_l y) \le \mathbf{E}(\Psi(y')|\mathbf{F}_l)$. The uniform integrability of a submartingale with a last element is well known, but follows easily by writing

$$\int_{\{\Psi(\pi_l y) > b\}} \Psi(\pi_l y) \le \int_{\{\Psi(\pi_l y)>b\}} \Psi(y') = \mathrm{I} + \mathrm{II},$$

where this last sum corresponds to splitting up the last integral by the identity $1 = 1_{\{\Psi(y')\le a\}} + 1_{\{\Psi(y')>a\}}$. Estimate $\mathrm{I} \le a|\{\Psi(\pi_l y) > b\}| \le a\int_0^1\Psi(y')/b$ and $\mathrm{II} \le \int_{\{\Psi(y')>a\}}\Psi(y')$, and let first $b\to\infty$ and then $a\to\infty$ to obtain $\sup_l \int_{\{\Psi(\pi_l y)>b\}}\Psi(\pi_l y) \to 0$ as $b\to\infty$. Hence the submartingale is uniformly integrable. Further, since the fields $\mathbf{F}_l$ increase and generate the Borel field $\mathbf{B}$ on $[0,1]$, it follows from the continuity theorem for conditional expectations that $\pi_l y \to E(y'|\mathbf{B}) = y'$ a.e. relative to Lebesgue measure. Hence because the submartingale is a.e. convergent to $\Psi(y')$ and uniformly integrable, it is convergent in L^1. This completes the proof of Case a(i).

CASE a(ii). In this case $\int \Psi(y')$ makes sense and converges to $+\infty$ since

$$\int_{\{y' > \varepsilon_0\}} |\Psi(y')| = \left(\int_{\{y' > B_0\}} + \int_{\{\varepsilon_0 < y' \le B_0\}}\right) |\Psi(y')| \le |\Psi(B_0)| + 1.$$

Note that the same estimate holds with $\boldsymbol{\pi}_l y$ in place of y'. Next,

$$\Psi(\boldsymbol{\pi}_l y) 1_{\{\boldsymbol{\pi}_l y \le \varepsilon_0\}} \to \Psi(y') 1_{\{y' \le \varepsilon_0\}} \quad \text{a.e.}$$

Therefore since these are all nonnegative functions, we have by Fatou's lemma that

$$\infty = \int_{\{y' \le \varepsilon_0\}} \Psi(y') \le \liminf_{l \to \infty} \int_{\{\boldsymbol{\pi}_l y \le \varepsilon_0\}} \Psi(\boldsymbol{\pi}_l y).$$

Hence the proposition is verified in Case a(ii). This completes the proof in case $y = y_{\mathrm{ac}}$.

(b) We now pass to the case when $y = y_s$, that is, y has no absolutely continuous part. Then it is well known [see Billingsley (1979), Example 35.10] that $\boldsymbol{\pi}_l y \to 0$ a.e. relative to Lebesgue measure. Now let $\varepsilon > 0$ and write

$$\begin{aligned}\int \Psi(\boldsymbol{\pi}_l y) &= \left(\int_{\{\boldsymbol{\pi}_l y \le \varepsilon\}} + \int_{\{\varepsilon < \boldsymbol{\pi}_l y \le \varepsilon_0\}} + \int_{\{\varepsilon_0 < \boldsymbol{\pi}_l y \le B_0\}} + \int_{\{\boldsymbol{\pi}_l y > B_0\}}\right) \Psi(\boldsymbol{\pi}_l y) \\ &\ge \int_{\{\boldsymbol{\pi}_l y \le \varepsilon\}} \Psi(\boldsymbol{\pi}_l y) - 1 - |\Psi(B_0)| \\ &\ge \Psi(\varepsilon)/2 - 1 - |\Psi(B_0)|,\end{aligned}$$

for all $l \ge l(\varepsilon)$. By the same reasoning as in Case a(ii) we know $\int \Psi(\boldsymbol{\pi}_l y)$ is nondecreasing, so letting $\varepsilon \to 0$ it holds that the supremum over l is ∞. This completes the proof in the purely singular case.

(c) Consider finally the case $y = \theta y_1 + (1 - \theta) y_2$, where $y_1, y_2 \in \mathbf{P}[0, 1]$, and y_1 is absolutely continuous, y_2 is singular with respect to Lebesgue measure and $0 < \theta < 1$. We obtain first an upper bound. Since Ψ is decreasing, it holds that

$$\int \Psi(\boldsymbol{\pi}_l y) = \int \Psi(\theta \boldsymbol{\pi}_l y_1 + (1 - \theta) \boldsymbol{\pi}_l y_2) \le \int \Psi(\theta \boldsymbol{\pi}_l y_1). \tag{5.3}$$

To obtain a lower bound, let $\varepsilon > 0$ be small and let $B \ge B_0/(1 - \theta)$. Then

$$\begin{aligned}\int \Psi(\boldsymbol{\pi}_l y) \ge &\int_{\{\boldsymbol{\pi}_l y_2 \le \varepsilon\}} \Psi(\theta \boldsymbol{\pi}_l y_1 + (1 - \theta)\varepsilon) \\ &+ \int_{\{\varepsilon < \boldsymbol{\pi}_l y_2 \le B\}} \Psi(\theta \boldsymbol{\pi}_l y_1 + (1 - \theta) B) \\ &+ \int_{\{\boldsymbol{\pi}_l y_2 > B\}} \Psi(\theta \boldsymbol{\pi}_l y_1 + (1 - \theta) \boldsymbol{\pi}_l y_2) \\ = &\ \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned} \tag{5.4}$$

Estimate first that

$$\text{III} \geq -\Big(\sup_{u \geq B(1-\theta)} -\Psi(u)/u\Big)\int_{\{\pi_l y_2 > B\}}[\theta\pi_l y_1 + (1-\theta)\pi_l y_2] \tag{5.5}$$
$$= o(1) \quad \text{as } B \to \infty,$$

independent of l and ε. We assume now that $\int\Psi(\theta y_1') < \infty$, so in particular $\int\Psi(\theta y_1' + (1-\theta)\varepsilon) < \infty$. Then, by the same argument used to establish Case a(i), $\Psi(\theta\pi_l y_1 + (1-\theta)\varepsilon) \to \Psi(\theta y_1' + (1-\theta)\varepsilon)$ in L^1. Hence because $1_{\{\pi_l y_2 \leq \varepsilon\}} \to 1$ a.e. as $l \to \infty$, we have

$$\lim_{l\to\infty} \text{I} = \int\Psi(\theta y_1' + (1-\theta)\varepsilon). \tag{5.6}$$

Also, by the same reasoning, except using now $1_{\{\varepsilon < \pi_l y_2 < B\}} \to 0$ a.e. as $l \to \infty$, we have

$$\lim_{l\to\infty} \text{II} = 0. \tag{5.7}$$

Therefore, letting first $l \to \infty$ and then $B \to \infty$ it follows by (5.4)–(5.7) that $\lim_{l\to\infty}\int\Psi(\pi_l y) = \liminf_{l\to\infty}\int\Psi(\pi_l y) \geq \int_0^1\Psi(\theta y_1' + (1-\theta)\varepsilon)$ for every $\varepsilon > 0$. Hence, by the dominated convergence theorem, $\lim_{l\to\infty}\int\Psi(\pi_l y) \geq \int\Psi(\theta y_1')$. Since by the upper bound (5.3) and the argument of Case a(i) with $\theta y_1'$ in place of y', we have also $\lim_{l\to\infty}\int\Psi(\pi_l y) \leq \int\Psi(\theta y_1')$, the proof is complete under the assumption that this last integral converges.

Suppose finally that $\int\Psi(\theta y_1')$ diverges. We know that $\pi_l y_1$ converges a.e. to y_1' and that $1_{\{\pi_l y_2 \leq \varepsilon\}} \to 1$ a.e. for each $\varepsilon > 0$. Hence if $\varepsilon < \varepsilon_0$, then, by Fatou's lemma,

$$\liminf_{l\to\infty} \text{I} \geq \liminf_{l\to\infty}\int_{\{\pi_l y_2 \leq \varepsilon,\, \pi_l y_1 < \varepsilon_0\}} \Psi(\theta\pi_l y_1 + (1-\theta)\varepsilon)\,dt - |\Psi(B_0)| - 1$$
$$\geq \int_{\{y_1' < \varepsilon_0\}} \Psi(\theta y_1' + (1-\theta)\varepsilon)\,dt - |\Psi(B_0)| - 1,$$

since $\liminf_{l\to\infty} 1_{\{\pi_l y_1 < \varepsilon_0\}} \geq 1_{\{y_1' < \varepsilon_0\}}$. Therefore, since by monotone convergence,

$$\int_{\{y_1' < \varepsilon_0\}} \Psi(\theta y_1' + (1-\theta)\varepsilon) \to +\infty \quad \text{as } \varepsilon \to 0,$$

it obtains that $\lim_{l\to\infty} \text{I} = \infty$. Thus because II is bounded below by $-|\Psi(B_0)| - 1$, the proof of the proposition is complete. □

We will now prove that the functional $I(y) := \int\Psi(y_{\text{ac}}')$ is lower semicontinuous from $(\mathbf{P}[0,1], \rho(\cdot,\cdot))$ to the extended real line. Unfortunately, the functionals $I_l(y) := \int\Psi(\pi_l y)$ are not lower semicontinuous in this setup since the maps π_l are not continuous. Thus we cannot immediately conclude that $I(y)$ is lower semicontinuous. However, the maps π_l are bounded on the class of nonnegative measures that are absolutely continuous with respect to Lebesgue measure with densities uniformly bounded by b for some $b < \infty$. It is essentially this fact that is used to prove the following.

LEMMA 5.1. *Let Ψ be as in Proposition* 5.1. *Then $I(y) = \int \Psi(y'_{\text{ac}})$ is lower semicontinuous from* $(\mathbf{P}[0,1], \rho(\cdot, \cdot))$ *to the extended real line.*

PROOF. Let y_m converge weakly to y in $\mathbf{P}[0,1]$. Assume $\liminf_m I(y_m) < \infty$. We must show $I(y) \le \liminf_m I(y_m)$. For the proof, we pick first a subsequence (y_{m^*}) of the sequence (y_m) along which $\lim I(y_{m^*}) = \liminf I(y_m)$. For convenience, write now (y_n) for the subsequence (y_{m^*}). Let $y_n = y_{n,\text{ac}} + y_{n,\text{s}}$ be the Lebesgue decomposition of y_n into its absolutely continuous and singular parts, respectively. Let $f_n(t) = y'_{n,\text{ac}}(t)$ be the density of the absolutely continuous part, so $f_n(t) \ge 0$ and $\int f_n \le 1$.

For each $0 < b < \infty$, we define a sequence of absolutely continuous subprobability measures $z_n = z_{n,\text{ac}}$ with densities relative to Lebesgue measure given by $z'_n = f_n 1_{\{f_n \le b\}}$. Then some subsequence (z_{n^*}) of (z_n) converges weakly to a subprobability z: $\int \varphi z_{n^*}(dt) \to \int \varphi z(dt)$ for all continuous functions φ on $[0,1]$. Let $z = z_{\text{ac}} + z_{\text{s}}$ be the Lebesgue decomposition of z into its absolutely continuous and singular parts, respectively. It is clear that $z_{\text{s}} = 0$, since otherwise there would be a Lebesgue measurable set E with $|E| = 0$ but with $z_{\text{s}}(E) > 0$. Indeed, if the latter would hold, then, since for each $\varepsilon > 0$, E is enclosed in a countable union of open intervals (a_i, b_i) with $\Sigma(b_i - a_i) < \varepsilon$, it would hold for some finite N that, $z_{\text{s}}(\bigcup_{1 \le i \le N}(a_i, b_i)) \ge z_{\text{s}}(E)/2 > 0$. But choosing then $0 \le \varphi \le 1$ to be a continuous function approximating the indicator function of the set $\bigcup_{1 \le i \le N}(a_i, b_i)$ with support of φ in a set of measure at most 2ε, it would hold that $2\varepsilon b \ge \lim \int_{\{f_{n^*} \le b\}} \varphi f_{n^*}(t)\,dt \ge z_{\text{s}}(E)/2$. This is a contradiction since $\varepsilon > 0$ is arbitrary. Thus $z_{\text{s}} = 0$. Note therefore that $z_{\text{ac}} \le y_{\text{ac}}$ since

$$y_n = 1_{\{f_n \le b\}} f_n + y_n - 1_{\{f_n \le b\}} f_n \to z_{\text{ac}} + y - z_{\text{ac}},$$

where $y - z_{\text{ac}} \ge 0$.

Now fix l and a dyadic interval A of level l with interior A^0, and let $\varphi_\varepsilon(t)$ be continuous and supported on A^0 such that as $\varepsilon \to 0$, $\varphi_\varepsilon(t) \uparrow 1_{A^0}(t)$. Then,

$$\int_{\{f_{n^*} \le b\}} \varphi_\varepsilon f_{n^*} \to \int z'_{\text{ac}} \varphi_\varepsilon\,dt + \int \varphi_\varepsilon z_{\text{s}}(dt) \quad \text{as } n^* \to \infty.$$

Thus by the uniform bound on the density $f_{n^*} 1_{\{f_{n^*} \le b\}}$ of z_{n^*} and by the integrability of z'_{ac}, it follows after letting $\varepsilon \to 0$ that

$$\boldsymbol{\pi}_l z_{n^*} \to \boldsymbol{\pi}_l z_{\text{ac}} \quad \text{a.e. } t, \text{ as } n^* \to \infty. \tag{5.8}$$

Estimate now that

$$\begin{aligned} \liminf_{n^*} \int \Psi(f_{n^*} 1_{\{f_{n^*} \le b\}}) &\ge \lim_l \liminf_{n^*} \int \Psi(\boldsymbol{\pi}_l z_{n^*}) \\ &\ge \lim_l \int \Psi(\boldsymbol{\pi}_l z_{\text{ac}}) \\ &= \int \Psi(z'_{\text{ac}}) \ge \int \Psi(y'_{\text{ac}}). \end{aligned} \tag{5.9}$$

Indeed the first inequality holds by Proposition 5.1 since the integrals $\int \Psi(\boldsymbol{\pi}_l z_{n^*})$ are nondecreasing in l with limit $\int \Psi(f_{n^*} 1_{\{f_{n^*} \leq b\}})$. The second inequality follows by (5.8) and Fatou's lemma since $\Psi(\boldsymbol{\pi}_l z_{n^*}) - \Psi(b) \geq 0$. The equality in (5.9) follows by Proposition 5.1. The last inequality in (5.9) holds because Ψ is decreasing and because $z_{\mathrm{ac}} \leq y_{\mathrm{ac}}$, so that $z'_{\mathrm{ac}} \leq y'_{\mathrm{ac}}$.

Finally, we use first the fact that, by construction, $\liminf I(y_m) = \lim I(y_n)$, since $(y_n) = (y_{m^*})$, and second the facts that $\Psi(u)/u \to 0$, as $u \to \infty$, and $\int f_{n^*} \leq 1$ to obtain

$$(5.10)\qquad \begin{aligned} \liminf I(y_m) &= \lim I(y_{n^*}) = \lim \int \Psi(y'_{n^*, \mathrm{ac}}) \\ &= \lim_{b \to \infty} \liminf_{n^*} \int \Psi(f_{n^*} 1_{\{f_{n^*} \leq b\}}). \end{aligned}$$

Therefore by (5.9) and (5.10), $\liminf I(y_m) \geq \int \Psi(y'_{\mathrm{ac}}) = I(y)$. This proves the lemma. □

We now go back to the infinite alleles model. By applying Lemma 5.1 with $\Psi = -\log$, it follows by (5.1) that J is lower semicontinuous on $\mathbf{P}[0,1]$. First set $G = D^{-\varepsilon}$. Let y achieve the infimum in the problem $\inf\{J(y): y \notin G^{+2\delta}\}$, where $\delta = 1/[2(d(l) + 1)]$. The probability y exists by lower semicontinuity of J and compactness of $\mathbf{P}[0,1]$. Then $\boldsymbol{\pi}_l y \notin (\operatorname{int}(\boldsymbol{\pi}_l G))$. Hence, by (2.46), (5.1) and (1.13),

$$V_d(G) \leq J(\boldsymbol{\pi}_l y) + -2\left\{\int \sigma(t)[x_0(t) - \xi_{0,d}(t)]\,dt + \mu \int \log[x_0(t)/\xi_{0,d}(t)]\,dt\right\},$$

for $d = d(l)$. Therefore by (1.13) and the definition of $\xi_{0,d}(t) = \mu/[\alpha_{0,d} - \pi_d \sigma(t)]$,

$$\liminf_{l \to \infty} V_{d(l)}(G) \leq \liminf_{l \to \infty} J(\boldsymbol{\pi}_l y).$$

But $J(\boldsymbol{\pi}_l y) \to J(y)$ as $l \to \infty$ by Proposition 5.1 and the fact that by the uniform continuity of $\sigma(t)$,

$$\sup_{y \in \mathbf{P}[0,1]} \left| \int \sigma(t)[\boldsymbol{\pi}_l y(t)\,dt - y(dt)] \right| \to 0 \quad \text{as } l \to \infty.$$

Thus letting first $l \to \infty$ (so also $\delta \to 0$) and then $\varepsilon \downarrow 0$, obtain that $V_\infty^-(D) \leq \lim_{\varepsilon \downarrow 0} \inf\{J(y): y \notin D^{-\varepsilon}\}$, where $V_\infty^-(D)$ is defined in (2.41). Next, let $\xi \in \partial(\operatorname{int}(\boldsymbol{\pi}_l(G))$ be minimizing for $V_d(G)$, so, by (2.46), $\xi \notin \partial(\boldsymbol{\pi}_l \mathbf{P}[0,1])$. Then by Remark 2.1(b), ξ also belongs to $(G^{-\delta})^c$. Hence,

$$\inf\left\{J(y): y \in (G^{-\delta})^c\right\} \leq V_d(G) + 2\left\{\int \sigma(t)[x_0(t) - \xi_{0,d}(t)]\,dt + \mu \int \log[x_0(t)/\xi_{0,d}(t)]\,dt\right\},$$

for $d = d(l)$. Thus letting first $l \to \infty$ and then $\varepsilon \downarrow 0$ obtain also $\lim_{\varepsilon \downarrow 0} \inf\{J(y): y \notin D^{-\varepsilon}\} \le V_\infty^-(D)$. Hence (5.2)(i) holds. By the same arguments with instead $G = D^{+\varepsilon}$ and with $\limsup_l$ in place of $\liminf_l$, obtain also (5.2)(ii).

We are now ready to complete the proof of Theorem 5.1. By (5.2) we need only the following lemma.

LEMMA 5.2. *Let $D = D(r)$ be given by* (1.19). *Then $V(r) = \inf\{J(y): y \notin D(r)\}$ is a continuous function of r.*

PROOF. We must prove $V(r-) = V(r) = V(r+)$. Let $J(y_n) = V(r - 1/n)$ for some $y_n \in \mathbf{P}[0,1]$. Let y be a weak limit of the y_n. Then $\rho(y, x_0) \ge r$ and by the lower semicontinuity of $J(\cdot)$, $V(r-) = \liminf_n J(y_n) \ge J(y) \ge V(r)$.

To complete the proof we need to show $V(r+) = V(r)$. Note that this is trivially true if $V(r) = \infty$. So assume $V(r) < \infty$. We know that $V(r) = \inf_y J(y)$ is achieved for some y with $\rho(y, x_0) = r$. To see this, use the fact just proved that $V(r-) = V(r)$ as follows. Write $V(r-) = V_\infty^-(D(r)) = \lim_{\varepsilon \downarrow 0} \liminf_l J(y_{l,\varepsilon})$ with $y_{l,\varepsilon} \in \partial(\mathrm{int}(\pi_l D(r-\varepsilon)))$, so that by (2.2) and Remark 2.1(b), $y_{l,\varepsilon} \in D(r - \varepsilon + 2\delta) \setminus D(r - \varepsilon - \delta)$, where $\delta \to 0$ as $l \to \infty$. Then take y_ε to be a weak limit of $y_{l,\varepsilon}$ along a sequence $l \to \infty$, and finally take y to be a weak limit of y_ε as $\varepsilon \to 0$. Thus we obtain by lower semicontinuity of J that $J(y) \le \liminf_\varepsilon J(y_\varepsilon) \le \lim_{\varepsilon \downarrow 0} \liminf_l J(y_{l,\varepsilon}) = V(r)$. Therefore, since $\rho(y, x_0) = r$, we have $J(y) = V(r)$.

In the following we fix $y \in \mathbf{P}[0,1]$ with $\rho(y, x_0) = r$ and $J(y) = V(r)$. We decompose this probability measure by $y = \theta y_1 + (1-\theta) y_2$, where y_1 and y_2 are, respectively, absolutely continuous and singular probabilities with respect to Lebesgue measure. Since $V(r) = J(y) < \infty$, we have that $\theta > 0$ by Proposition 5.1. The strategy is to find a nice measure y_α with $\rho(y_\alpha, x_0) > r$ for all $\alpha < 1$, so that $y_\alpha \to y$ as $\alpha \uparrow 1$ with also $J(y_\alpha) \to J(y)$. We construct $y_\alpha = \alpha y + (1-\alpha)\delta_t$ for an appropriate number $t = t_0(y)$. Then $\rho(y, y_\alpha) = (1-\alpha)\rho(\delta_t, y) \le (1-\alpha) \to 0$, as $\alpha \uparrow 1$. Further, it is clear that the absolutely continuous part of y_α is $\theta\alpha y_1 \to \theta y_1$ so that by (5.1), $J(y_\alpha) \to J(y)$ since $\log(\alpha) \to 0$. It remains to show that there exists a $t = t_0$ so that $\rho(y_\alpha, \delta_t) > r$ for all $\alpha < 1$. We recall that the set Λ of Lipschitz functions φ on $[0,1]$ with both $\|\varphi\|_\infty \le 1$ and $\|\varphi\|_{\mathrm{Lip}} \le 1$ is compact in $\mathbf{C}[0,1]$ by the Arzela–Ascoli theorem. Therefore there exists $\varphi_y \in \Lambda$ such that $\int \varphi_y \, d[y - x_0] = r$. Therefore,

$$\begin{aligned}\rho(\alpha y + (1-\alpha)\delta_t, x_0) &\ge \alpha \int \varphi_y \, d[y - x_0] + (1-\alpha)\int \varphi_y [\delta_t - dx_0] \\ &= \alpha r + (1-\alpha)\int \varphi_y (dy - dx_0 + \delta_t - dy) \\ &= r + (1-\alpha)\int \varphi_y [\delta_t - dy].\end{aligned}$$

We now choose $t = t_0$ so that $\varphi_y(t_0) = \sup\{\varphi_y(t): t \in [0,1]\}$, which exists by continuity of φ_y. We claim $\int \varphi_y [\delta_{t_0} - dy] > 0$. Indeed, $\int \varphi_y [\delta_{t_0} - dy] =$

$\theta\int\varphi_y[\delta_{t_0} - dy_1] + (1-\theta)\int\varphi_y[\delta_{t_0} - dy_2] \geq \theta\int\varphi_y[\delta_{t_0} - dy_1]$ by the choice of t_0. Thus since $\theta > 0$, it suffices to show $\int\varphi_y[\delta_{t_0} - dy_1] > 0$. Suppose on the contrary that $\int\varphi_y[\delta_{t_0} - dy_1] = \varphi_y(t_0) - \int\varphi_y[dy_1/dt]\,dt = 0$. Then since $dy_1/dt > 0$ a.e. [because $J(y) < \infty$], it would follow that $\varphi_y(t) = \varphi_y(t_0)$ a.e. Indeed, if $\varphi_y(t) < \varphi_y(t_0)$ on a set E of positive Lebesgue measure, $|E| > 0$, then $(\varphi_y(t_0) - \varphi_y(t))(dy_1/dt) > 0$ a.e. on E, so that $\int_E \varphi_y(dy_1/dt) < \varphi_y(t_0)y_1(E)$ and therefore $\int_0^1\varphi_y[dy_1/dt]\,dt < \varphi_y(t_0)$, contrary to hypothesis. But if $\varphi_y(t) = \varphi_y(t_0)$ a.e., then $\varphi_y(t) = \varphi_y(t_0)$ for all $t \in [0,1]$ by continuity. In this case, $\int\varphi_y d[y - x_0] = 0 < r$, a contradiction. Therefore the hypothesis fails and $\int\varphi_y[\delta_{t_0} - dy_1] > 0$. Hence $\rho(y_\alpha, \delta_{t_0}) > r$ for all $\alpha < 1$. Therefore $V(r+) \leq \lim_{\alpha\uparrow 1} J(y_\alpha) = J(y) = V(r)$. Since on the other hand $V(r+) \geq V(r)$, the proof of the lemma is complete. □

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