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ON THE DISTRIBUTION OF THE INTEGRAL OF THE ABSOLUTE VALUE OF THE BROWNIAN MOTION

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Kac has considered the integral of the absolute value of a Brownian motion process and determined the Laplace–Stieltjes transform of the distribution function of this integral. In this paper explicit expressions are given for the distribution and the moments of this integral.

1. Introduction. In 1946 Kac [5] proved that if $\xi_1, \xi_2, \dots, \xi_n, \dots$ are independent and identically distributed random variables for which $\mathbf{E}\{\xi_n\} = 0$ and $\mathbf{E}\{\xi_n^2\} = 1$ and if $\zeta_n = \xi_1 + \cdots + \xi_n$ for $n \geq 1$, then the limit distribution function

(1) $$\lim_{n \to \infty} \mathbf{P}\left\{\frac{|\zeta_1| + |\zeta_2| + \cdots + |\zeta_n|}{n^{3/2}} \leq x\right\} = H(x)$$

exists and if $s \geq 0$, then

(2) $$\Psi(s) = \int_0^\infty e^{-sx}\, dH(x)$$

can be expressed in the following form:

(3) $$\Psi(s) = \sum_{j=1}^{\infty} C_j e^{-2^{-1/3} a'_j s^{2/3}},$$

where

(4) $$C_j = \frac{1 + 3\int_0^{a'_j} Ai(-u)\, du}{3a'_j Ai(-a'_j)},$$

(5) $$Ai(z) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + tz\right) dt$$

is the Airy integral, and $z = -a'_j$, $j = 1, 2, \dots,$ are the zeros of $Ai'(z)$, arranged so that $0 < a'_1 < a'_2 < \cdots < a'_j < \cdots$. In a note, Kac [6] describes his motivation to find $\Psi(s)$ and Schwinger's contribution to the solution.

Since the process $\{\zeta_{[nt]}/\sqrt{n},\ 0 \leq t \leq 1\}$ converges weakly to the standard Brownian motion process $\{\xi(t),\ 0 \leq t \leq 1\}$, and since the integral

(6) $$\sigma = \int_0^1 |\xi(t)|\, dt$$

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is a continuous functional on the process $\{\xi(t),\ 0 \le t \le 1\}$, (1) implies that

$$\mathbf{P}\{\sigma \le x\} = H(x) \tag{7}$$

also holds.

The distribution function $H(x)$ naturally appears in order statistics and in reliability theory, but until now we have had very little information about its properties. The aim of this paper is to fill this gap.

2. The distribution of σ. Denote by $g(x)$ the stable density function whose Laplace transform is given by

$$\int_0^\infty e^{-sx} g(x)\, dx = e^{-s^{2/3}} \tag{8}$$

for $\mathrm{Re}(s) \ge 0$. By inverting (8) we obtain that

$$g(x) = \frac{2^{4/3} x^{-7/3}}{\sqrt{27\pi}} e^{-4/(27x^2)} U(1/6, 4/3, 4/(27x^2)) \tag{9}$$

for $x > 0$, where

$$U(1/6, 4/3, x) = \frac{1}{\Gamma(1/6)} \int_0^\infty e^{-tx} t^{-5/6} (1+t)^{1/6}\, dt \tag{10}$$

is a confluent hypergeometric function.

THEOREM 1. *If $x > 0$, then*

$$\frac{dH(x)}{dx} = \frac{\sqrt{3}}{x\sqrt{\pi}} \sum_{j=1}^{\infty} C_j e^{-v_j} v_j^{2/3} U(1/6, 4/3, v_j), \tag{11}$$

where C_j is given by (4),

$$v_j = 2(a'_j)^3/(27x^2) \tag{12}$$

and $z = -a'_j$, $j = 1, 2, \ldots,$ are the zeros of $Ai'(z)$ arranged so that $0 < a'_1 < a'_2 < \cdots$.

PROOF. By (3) we obtain that

$$h(x) = \frac{dH(x)}{dx} = \sum_{j=1}^{\infty} C_j g\left(x 2^{1/2} (a'_j)^{-3/2}\right) 2^{1/2} (a'_j)^{-3/2} \tag{13}$$

for $x > 0$, where $g(x)$ is given by (9). □

The theory of the stable distributions is thoroughly covered by Zolotarev [16]. For the definitions and properties of the confluent hypergeometric function and the Airy integral we refer to Slater [12], Miller [7] and Abramowitz and Stegun [1]. The first 50 zeros of $Ai'(z)$ and $Ai(-a'_j)$, $j = 1, 2, \dots, 50$, can be found in Miller ([7], page 43) for eight decimals. See also Abramowitz and Stegun ([1], page 478) for the first 10 zeros of $Ai'(z)$ and ([1], page 450) for the asymptotic series of the zeros.

If $x > 0$, then

$$(14)\quad \begin{aligned} Ai(-x) \sim \frac{x^{-1/4}}{\sqrt{\pi}} \Bigg[& \sin\left(\frac{2x^{3/2}}{3} + \frac{\pi}{4}\right) \sum_{j=0}^{\infty} (-1)^j c_{2j} \left(\frac{3}{2x^{3/2}}\right)^{2j} \\ & - \cos\left(\frac{2x^{3/2}}{3} + \frac{\pi}{4}\right) \sum_{j=0}^{\infty} (-1)^j c_{2j+1} \left(\frac{3}{2x^{3/2}}\right)^{2j+1} \Bigg] \end{aligned}$$

as $x \to \infty$, where $c_0 = 1$ and

$$(15)\quad c_j = \frac{\Gamma(3j + 1/2)}{54^j j! \Gamma(j + 1/2)} = \frac{(2j+1)(2j+3)\cdots(6j-1)}{216^j j!}$$

for $j = 1, 2, \dots$. We have

$$(16)\quad c_j = \frac{1}{2}\left(j - 1 + \frac{5}{36j}\right) c_{j-1}$$

for $j \geq 1$. (See Abramowitz and Stegun [1], page 448.) Also we can prove that if $x > 0$ and $x \to \infty$, then

$$(17)\quad \begin{aligned} \int_0^x Ai(-u)\, du & \\ \sim \frac{2}{3} - \frac{x^{-3/4}}{\sqrt{\pi}} \Bigg[& \cos\left(\frac{2x^{3/2}}{3} + \frac{\pi}{4}\right) \sum_{j=0}^{\infty} (-1)^j h_{2j} \left(\frac{3}{2x^{3/2}}\right)^{2j} \\ & + \sin\left(\frac{2x^{3/2}}{3} + \frac{\pi}{4}\right) \sum_{j=0}^{\infty} (-1)^j h_{2j+1} \left(\frac{3}{2x^{3/2}}\right)^{2j+1} \Bigg], \end{aligned}$$

where $h_0 = 1$ and

$$(18)\quad h_j = c_j + \frac{(2j-1)}{2} h_{j-1}$$

for $j \geq 1$. By using the above formulas we can calculate C_j for $j \geq 0$. Table 1 contains C_j for $j \leq 10$.

TABLE 1
The constants C_j

j	a'_j	$Ai(-a'_j)$	$\int_0^{a'_j} Ai(-u)\,du$	C_j
1	1.01879297	0.53565666	0.47573996	1.48257073
2	3.24819758	−0.41901548	0.70083344	−0.75983287
3	4.82009921	0.38040647	0.65122852	0.53695654
4	6.16330736	−0.35790794	0.67580399	−0.45747262
5	7.37217726	0.34230124	0.66048490	0.39382448
6	8.48848673	−0.33047623	0.67119622	−0.35809002
7	9.53544905	0.32102229	0.66316740	0.32553770
8	10.52766040	−0.31318539	0.66947361	−0.30414730
9	11.47505663	0.30651729	0.66435092	0.28365033
10	12.38478837	−0.30073083	0.66861921	−0.26901755

We note that

$$a'_j \sim \left(\frac{3\pi(4j-3)}{8}\right)^{2/3} \tag{19}$$

and

$$C_j \sim (-1)^{j-1}(2/(3j))^{1/2} \tag{20}$$

as $j \to \infty$. If in calculating $h(x)$ by (11) we stop at the nth term, then the magnitude of the error is about

$$R_n(x) \sim e^{-\pi^2 n^2/(6x^2)}\sqrt{n\pi}\,/x^2. \tag{21}$$

If $n = 10$ and $x = 2$, the error is about 2×10^{-18}. Table 2 and Figure 1 contain the density function $h(x)$ for $0 < x < 2$. The calculations have been made by utilizing the program Mathematica of Wolfram Research [15]. In this program $Ai(x)$ and $U(a, b, x)$ are built-in functions.

TABLE 2
The density function $h(x)$

x	$h(x)$	x	$h(x)$	x	$h(x)$	x	$h(x)$
0.05	0.000000	0.55	1.092241	1.05	0.277296	1.55	0.038464
0.10	0.016130	0.60	0.960699	1.10	0.235011	1.60	0.030330
0.15	0.558418	0.65	0.846447	1.15	0.197797	1.65	0.023741
0.20	1.447401	0.70	0.746032	1.20	0.165309	1.70	0.018446
0.25	1.884812	0.75	0.656629	1.25	0.137176	1.75	0.014226
0.30	1.930870	0.80	0.576235	1.30	0.113017	1.80	0.010890
0.35	1.798081	0.85	0.503549	1.35	0.092441	1.85	0.008275
0.40	1.609401	0.90	0.437764	1.40	0.075064	1.90	0.006241
0.45	1.418331	0.95	0.378363	1.45	0.060510	1.95	0.004673
0.50	1.244365	1.00	0.324978	1.50	0.048421	2.00	0.003472

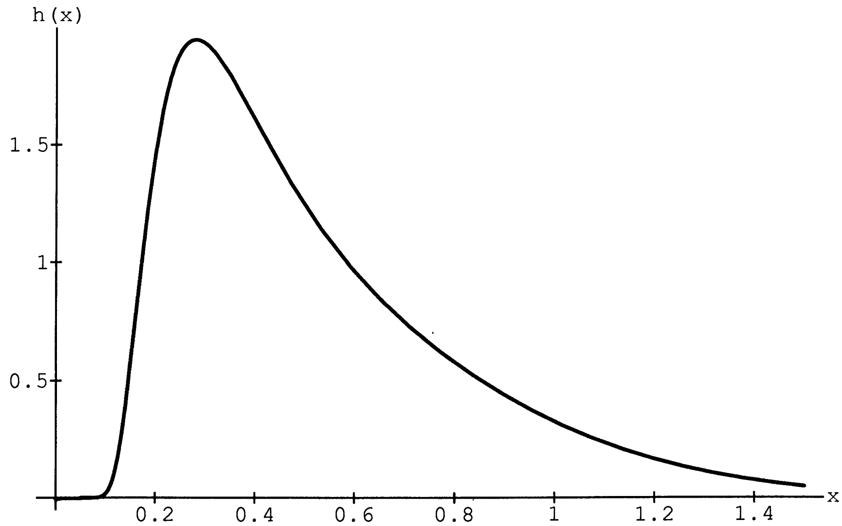

FIG. 1. *The density function* $h(x)$.

3. The moments of σ. The moments

$$\mu_r = \int_0^\infty x^r \, dH(x) \tag{22}$$

for $r = 0, 1, 2, \ldots,$ are determined by the following theorem.

THEOREM 2. *We have*

$$\mu_r = \frac{L_r r!}{2^{r/2}\Gamma((3r+2)/2)} \tag{23}$$

for $r = 0, 1, \ldots,$ *where* $L_0 = 1$ *and*

$$L_n = \beta_n + \sum_{j=1}^{n} \frac{(6j+1)}{(6j-1)} \alpha_j L_{n-j} \tag{24}$$

for $n \geq 1$, *where* $\alpha_0 = 1$,

$$\alpha_j = \frac{\Gamma(3j+1/2)}{(36)^j j! \Gamma(j+1/2)} = \frac{(2j+1)(2j+3)\cdots(6j-1)}{(144)^j j!} \tag{25}$$

for $j = 1, 2, \ldots,$ $\beta_0 = 1$ *and*

$$\beta_k = \alpha_k + \frac{3(2k-1)}{4}\beta_{k-1} \tag{26}$$

for $k = 1, 2, \ldots$.

PROOF. Cifarelli [3] proved that if $\{\xi(u), 0 \le u \le 1\}$ is a standard Brownian motion, and σ is defined by (6), then

$$\int_0^\infty e^{-zt}\mathbf{E}\left\{e^{-\sigma\sqrt{2t^3}}\,\middle|\,\sqrt{t}\,\xi(1) = x\right\}\varphi\left(\frac{x}{\sqrt{t}}\right)\frac{dt}{\sqrt{t}} = -\frac{Ai(z+\sqrt{2}|x|)}{\sqrt{2}\,Ai'(z)} \tag{27}$$

for $z > 0$, where

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \tag{28}$$

and $Ai(z)$ is the Airy integral defined by (5). If we integrate (27) over $x \in (-\infty, \infty)$, we obtain that

$$\int_0^\infty e^{-zt}\Psi\left(\sqrt{2t^3}\right)dt = \frac{3\int_0^z Ai(u)\,du - 1}{3Ai'(z)} \tag{29}$$

for $z > 0$. By analytical continuation we can extend (29) for $\mathrm{Re}(z) \ge 0$. Hence if $s > 0$,

$$\Psi(s) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{zs^{2/3}2^{-1/3}}\frac{\left[3\int_0^z Ai(u)\,du - 1\right]}{3Ai'(z)}\,dz. \tag{30}$$

In the integrand the denominator has zeros only on the negative real axis. The roots of $Ai'(z) = 0$ are $z = -a'_j$, $j = 1, 2, \dots,$ where $0 < a'_1 < a'_2 < \cdots < a'_j < \cdots$. Since $Ai''(z) = zAi(z)$, by using the theorem of residues, (30) yields Kac's formula (3).

The moments μ_r, $r = 0, 1, \dots,$ defined by (22), exist for all $r = 0, 1, 2, \dots$ and

$$\mu_r < \sqrt{8}\left(\frac{r}{e}\right)^{r/2} \tag{31}$$

if $r \ge 1$. This follows from the fact that

$$\int_0^1 |\xi(u)|\,du \le \max_{0 \le u \le 1} |\xi(u)| \tag{32}$$

and

$$\mathbf{E}\left\{\left[\max_{0\le u\le 1}|\xi(u)|\right]^r\right\} = \frac{2^{(r+3)/2}\Gamma((r+1)/2)}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^r} \tag{33}$$

for $r = 1, 2, \dots$. Thus

$$\Psi(s) = \sum_{r=0}^{\infty}(-1)^r\mu_r s^r/r! \tag{34}$$

and the series is convergent on the whole complex plane. If we put (34) into

(29), then by term integration we obtain the following asymptotic series:

$$z\int_0^\infty e^{-zt}\Psi\left(\sqrt{2t^3}\right)dt \sim \sum_{r=0}^{\infty}(-1)^r L_r z^{-3r/2} \tag{35}$$

as $z \to \infty$, where L_r is defined by (23).

If $z \to \infty$, then in (29),

$$Ai'(z) \sim -\frac{z^{1/4}e^{-2z^{3/2}/3}}{2\sqrt{\pi}}\sum_{j=0}^{\infty}(-1)^{j+1}\frac{(6j+1)}{(6j-1)}\alpha_j z^{-3j/2}, \tag{36}$$

where α_j is defined by (25). We can calculate α_j by the recurrence formula

$$\alpha_j = \frac{3}{4}\left(j-1+\frac{5}{36j}\right)\alpha_{j-1} \tag{37}$$

for $j = 1, 2, \dots$; $\alpha_0 = 1$. Furthermore, if $z \to \infty$, then

$$\int_0^z Ai(u)\,du \sim \frac{1}{3} - \frac{z^{-3/4}e^{-2z^{3/2}/3}}{2\sqrt{\pi}}\sum_{k=0}^{\infty}(-1)^k\beta_k z^{-3k/2}, \tag{38}$$

where $\beta_0 = 1$ and β_k, $k = 1, 2, \dots$, are determined by (26). By (29), (35), (36) and (38) we obtain that

$$\left(\sum_{r=0}^{\infty}(-1)^r L_r z^{-3r/2}\right)\left(\sum_{j=0}^{\infty}(-1)^{j+1}\frac{(6j+1)}{(6j-1)}\alpha_j z^{-3j/2}\right) \sim \sum_{k=0}^{\infty}(-1)^k\beta_k z^{-3k/2}. \tag{39}$$

By comparing the coefficients of $z^{-3n/2}$ on both sides of (39), we obtain that

$$\sum_{j=0}^{n}\frac{(6j+1)}{(6j-1)}\alpha_j L_{n-j} + \beta_n = 0 \tag{40}$$

for $n = 0, 1, 2, \dots$. This proves (24). Table 3 contains L_r and μ_r for $r \le 10$. □

The limit behavior of μ_r is given by the following theorem.

THEOREM 3. *We have*

$$\mu_r \sim \sqrt{2}\left(\frac{r}{3e}\right)^{r/2} \tag{41}$$

as $r \to \infty$.

PROOF. First we note that by (37) it follows that

$$\alpha_j < \left(\tfrac{3}{4}\right)^j j! \tag{42}$$

TABLE 3
The moments of σ

r	L_r	μ_r	μ_r
0	1	1	1
1	1	$\frac{4}{3}\frac{1}{\sqrt{2\pi}}$	0.5319230405
2	$\frac{9}{4}$	$\frac{3}{8}$	0.375
3	$\frac{263}{32}$	$\frac{263}{315}\frac{1}{\sqrt{2\pi}}$	0.3330851420
4	$\frac{2709}{64}$	$\frac{903}{2560}$	0.352734375
5	$\frac{578487}{2048}$	$\frac{2119}{1980}\frac{1}{\sqrt{2\pi}}$	0.4269488344
6	$\frac{2370249}{1024}$	$\frac{37623}{65536}$	0.5740814209
7	$\frac{1472890279}{65536}$	$\frac{11074363}{5250960}\frac{1}{\sqrt{2\pi}}$	0.8413759825
8	$\frac{1032772671}{4096}$	$\frac{114752519}{86507520}$	1.3265033953
9	$\frac{26915124080747}{8388608}$	$\frac{3845017725821}{688400856000}\frac{1}{\sqrt{2\pi}}$	2.2282658808
10	$\frac{11968136957889}{262144}$	$\frac{189970427903}{47982827760}$	3.9591328227

if $j \geq 1$ and by (25),

$$a(j) = \left(\frac{4}{3}\right)^j \frac{\alpha_j}{(2j-1)!!} = \frac{(6j)!j!}{(3j)!(2j)!(2j)!}\left(\frac{1}{216}\right)^j \tag{43}$$

for $j \geq 1$, and

$$1 + \sum_{j=1}^{\infty} a(j) = F(1/6, 5/6; 1/2; 1/2) = \sqrt{6}/2, \tag{44}$$

where $F(a, b; c; z)$ is the Gauss hypergeometric function. See Bailey [2] and Abramowitz and Stegun [1].

By (26) we obtain that

$$\left(\frac{4}{3}\right)^k \frac{\beta_k}{(2k-1)!!} = 1 + \sum_{j=1}^{k}\left(\frac{4}{3}\right)^j \frac{\alpha_j}{(2j-1)!!}. \tag{45}$$

Hence by (44),

$$\lim_{k\to\infty}\left(\frac{4}{3}\right)^k \frac{\beta_k}{(2k-1)!!} = \frac{\sqrt{6}}{2} \tag{46}$$

or

$$\beta_k \sim \sqrt{2}\left(\frac{k}{3e}\right)^{k/2} \tag{47}$$

as $k \to \infty$.

Now we shall prove that the limit

$$\lim_{r\to\infty}\left(\frac{4}{3}\right)^r \frac{L_r}{(2r-1)!!} = L \tag{48}$$

exists and

$$\sqrt{3/2} \le L \le 4/3. \tag{49}$$

It is proved in Takács [13] that if $D_0 = 1$, $D_1 = 1/4$ and

$$D_r = \frac{(3r-2)}{4} D_{r-1} - \frac{1}{2}\sum_{i=1}^{r-1} D_i D_{r-i} \tag{50}$$

for $r \geq 2$, then $(4/3)^r D_r/(r-1)!$, $r = 2, 3, \ldots$, is a decreasing sequence of positive real numbers and

$$\sum_{r=0}^{\infty} (-1)^r D_r z^{-3r/2} \sim -\frac{Ai(z) z^{1/2}}{Ai'(z)} \tag{51}$$

as $z \to \infty$. By (29) and (35),

$$\sum_{r=0}^{\infty} (-1)^r L_r z^{-(3r+2)/2} \sim \frac{3\int_0^z Ai(u)\,du - 1}{3Ai'(z)} \tag{52}$$

as $z \to \infty$. Hence

$$-\frac{1}{2}\sum_{r=0}^{\infty} (-1)^r L_r (3r+2) z^{-(3r+4)/2} \sim \frac{Ai(z)}{Ai'(z)} - \frac{\left[3\int_0^z Ai(u)\,du - 1\right]}{3Ai'(z)} \frac{zAi(z)}{Ai'(z)} \tag{53}$$

as $z \to \infty$. Here we used that $Ai''(z) = zAi(z)$. Accordingly,

$$-\frac{1}{2}\sum_{r=0}^{\infty} (-1)^r L_r (3r+2) z^{-(3r+3)/2} \sim \left(\sum_{r=0}^{\infty} (-1)^r D_r z^{-3r/2}\right)\left(\sum_{r=1}^{\infty} (-1)^r L_r z^{-3r/2}\right) \tag{54}$$

as $z \to \infty$. Consequently,

$$\frac{(3r-1)}{2} L_{r-1} = \sum_{j=0}^{r-1} D_j L_{r-j} \tag{55}$$

for $r \geq 1$ or

$$L_r = \frac{(3r-1)}{2} L_{r-1} - \sum_{j=1}^{r-1} D_j L_{r-j} \tag{56}$$

for $r \geq 2$. By (56),

$$L_r \leq \frac{(3r-1)}{2} L_{r-1} - \frac{1}{4} L_{r-1} = \frac{3(2r-1)}{4} L_{r-1} \tag{57}$$

for $r \geq 2$ or

$$\left(\frac{4}{3}\right)^r \frac{L_r}{(2r-1)!!} \leq \left(\frac{4}{3}\right)^{r-1} \frac{L_{r-1}}{(2r-3)!!} \tag{58}$$

for $r \geq 2$. Accordingly, the limit

$$\lim_{r\to\infty} \left(\frac{4}{3}\right)^r \frac{L_r}{(2r-1)!!} = L \tag{59}$$

exists and $L \leq 4/3$. On the other hand, by (24) $L_n \geq \beta_n$ for $n \geq 1$ and therefore by (46), $L \geq \sqrt{6}/2$. Now we shall prove that in (24),

$$\lim_{n\to\infty} \left(\frac{4}{3}\right)^n \frac{1}{(2n-1)!!} \sum_{j=1}^{n} \frac{(6j+1)}{(6j-1)} \alpha_j L_{n-j} = 0 \tag{60}$$

and this implies that $L = \sqrt{6}/2$. By (58),

$$\left(\frac{4}{3}\right)^r \frac{L_r}{(2r-1)!!} \leq \frac{4}{3} \tag{61}$$

for $r \geq 1$. If we use (42) and (61), then we obtain that

$$\begin{aligned} 0 &\leq \left(\frac{4}{3}\right)^n \frac{1}{(2n-1)!!} \sum_{j=1}^{n} \frac{(6j+1)}{(6j-1)} \alpha_j L_{n-j} \\ &\leq \frac{28}{15} \sum_{j=1}^{n} 2^j j! \frac{n!(2n-2j)!}{(2n)!(n-j)!} \to 0 \end{aligned} \tag{62}$$

as $n \to \infty$. This proves that $L = \sqrt{6}/2$.

Finally, by (23),

$$\mu_r = \frac{L_r r!}{2^{r/2}\Gamma((3r+2)/2)} \sim \sqrt{2}\left(\frac{r}{3e}\right)^{r/2} \tag{63}$$

as $r \to \infty$. □

4. Expansions in Laguerre series. As an alternative, we can express the distribution function $H(x)$ and the density function $h(x) = H'(x)$ by Laguerre series. The generalized Laguerre polynomials,

$$L_n^{(\alpha)}(x) = \sum_{j=0}^{n} (-1)^j \binom{n+\alpha}{n-j} \frac{x^j}{j!}, \tag{64}$$

defined for $n = 0, 1, 2, \dots$ and $\alpha > -1$, are orthogonal on the interval $0 \le x < \infty$ with respect to the weight function

$$g_\alpha(x) = e^{-x} x^\alpha / \Gamma(\alpha + 1). \tag{65}$$

We write

$$G_\alpha(x) = \int_0^x g_\alpha(u)\, du \tag{66}$$

for $x \ge 0$.

By using the results of Uspensky [14] and Nasarow [8] (Sansone [10], Chapter IV) we can prove that

$$H(x) = G_a(bx) + a \sum_{n=1}^{\infty} \frac{c_n}{n} g_{a+1}(bx) L_{n-1}^{(a)}(bx) \tag{67}$$

and

$$h(x) = g_a(bx) b \sum_{n=0}^{\infty} c_n L_n^{(a-1)}(bx) \tag{68}$$

for $x \ge 0$, where $a > 0$, $b > 0$ and

$$c_n \binom{n+a-1}{n} = \sum_{r=0}^{n} \frac{(-1)^r}{r!} \binom{n+a-1}{n-r} b^r \mu_r \tag{69}$$

for $n = 0, 1, 2, \dots$. The moments μ_r, $r \ge 0$, are defined by (23).

We have $c_0 = 1$, and if we choose

$$a = \frac{64}{27\pi - 64} = 3.073524225046 \tag{70}$$

and

$$b = \frac{48\sqrt{2\pi}}{27\pi - 64} = 5.778137043948, \tag{71}$$

then $c_1 = c_2 = 0$. In this case the coefficients c_n, $1 \le n \le 10$, are given in Table 4.

Even if we add only a few terms in the expansions (67) and (68), we can obtain good approximations for $H(x)$ and $h(x)$.

Finally, it should be mentioned that the analogous problem of finding the distribution of the integral of the absolute value of the Brownian bridge was solved in 1975 by Cifarelli [3] and independently in 1982 by Shepp [11]. In this case the density function was determined by numerical integration by Rice [9] and the explicit form of the distribution function by Johnson and Killeen [4].

TABLE 4
The coefficients c_n

n	c_n	n	c_n
1	−0.000000000000	6	−0.020663009369
2	−0.000000000000	7	−0.013188716669
3	−0.011584328208	8	−0.007518446236
4	−0.027178915962	9	−0.003542017611
5	−0.027729182039	10	−0.000784228414

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