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PROBLEMS IN CERTAIN TWO-FACTOR TERM STRUCTURE MODELS

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The formal solution to a two-factor option pricing model in which a short-term rate and a bond yield are taken as instrumental variables is shown to explode. There are no real-valued solutions to the diffusion equations written down for the long and short rate by Brennan and Schwartz.

The Brennan and Schwartz model is an early and well known two-factor bond option pricing model. Its main idea is to take the short-term rate, which is an instantaneous rate, and the yield of a fixed bond as the instrumental variables that drive the model, and it is meant to supplement models that work from the short-term rate alone. It is therefore thought to be an improvement for pricing certain financial instruments in which the cobehavior of two different bonds is central, for example, an option to exchange a long bond for a short bond. A variant is frequently used in mortgage pricing, where the long rate determines when homeowners refinance their mortgages.

Several versions of the Brennan and Schwartz model exist. All take the fixed bond to be a "console" that pays out a coupon of $1 per unit time forever and never repays principal. The console has a stationarity that makes it easier to deal with than a bond that matures and repays principal. It also has the following simple relationship between yield and price going for it: The price of the console is the reciprocal of its yield. The price of a bond is the value of its cash flows, discounting at a rate equal to its yield. For the console, the discounted value of the cash flows is $\int_0^\infty \exp(-lt)\,dt$, which is $1/l$.

Starting from true or original processes for the prices of a collection of assets that are defined as the solutions to stochastic differential equations, the theory of contingent claims pricing, as presented in [5] (Section 5.8), requires that there be an absolutely continuous change of measure under which the (instantaneous) expected return on all investments is equal to the short-term rate. This measure is called the equivalent martingale measure because it is equivalent and under it the discounted price of any financial instrument that makes no payouts is a martingale. For an instrument like the console, which makes payouts, one has to add in the discounted value of the payouts, and this price plus dividend process forms a martingale.

The results of this paper could be stated and proved without reference to the equivalent martingale measure.

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It is easy to use the condition that the instantaneous expected return equals the short-term rate to derive a constraint on the change of measure required by the Brennan and Schwartz model. This is the tack followed in [1] or [2] and the result is that if the console yield is assumed to satisfy a stochastic differential equation of the form

$$dl(t) = \zeta(r,l)\,dt + \sigma_l(r,l)\,dB_2(t),$$

where $r = r(t)$ is the short-term rate process (as yet unspecified) and B_2 is a standard Brownian motion, then under the equivalent martingale measure, l will satisfy the same equation with ζ given by

$$\zeta(r,l) = \sigma_l^2(r,l)/l^2 + l(l-r). \tag{1}$$

The console price process, $Z(t) = 1/l(t)$, is more intuitive:

$$dZ = (r(t)Z - 1)\,dt - Z^2\sigma_l(r,l)\,dB_2. \tag{2}$$

The drift term expresses the fact that the console price must grow instantaneously at the risk-free rate and decrease by its payouts.

By Girsanov's theorem we expect that we have specified the likelihood ratio of the equivalent measure when we have specified the drift of the short-term rate process, but two things have been observed to go wrong. In [3], Cheng reports a nonobvious example in which an equivalent martingale measure does not exist. In [4] (Section 5) the authors give an example of how blithely changing the drift coefficients of the stochastic differential equation defining the rate process (as before) can lead to subtle problems: The result is a perfectly well behaved process, but it is not equivalent to the original (true) process, and so arbitrages exist.

We will show that the proposed solutions to the Brennan and Schwartz model, as given, for example, in [1] and [2], fail in the most dramatic fashion. The console rate explodes; that is, it reaches infinity in a finite amount of time with positive probability. In particular, no real-valued solutions will exist for the diffusion equations written down for the short-term rate and the console rate in [1] or [2]. This shows that the candidate measure is not equivalent to the original, since the process did not explode under the original measure. From a practitioner's viewpoint, we regard the fact that the proposed solution explodes as more serious than that it is technically wrong. The problems reported in [4] would probably not cause a problem when applied to the pricing of standard contracts.

This does not show that no equivalent martingale measure exists, only that the candidates in [1] and [2] are not it.

As before, r is the short-term rate, l the console rate and $Z = 1/l$ is the console price. The dynamics of the console price are given by (2) and the short rate dynamics under the martingale measure will be given by

$$dr = \mu(r,l)\,dt + \sigma_r(r,l)\,dB_1, \tag{3}$$

where B_1 and B_2 are jointly a Brownian motion with correlation $\rho \neq 1$. $P^{(r_0, Z_0)}\{A\} = P\{A|Z(0) = Z_0, r(0) = r_0\}$; that is, it denotes the process started from the point (r_0, Z_0). Furthermore, we specify that $\sigma_r(r, l) = \eta_r(r)$ and $\sigma_l(r, l) = \eta_l(l)$, which are specifications that are eventually made in [1] and [2]. Set $\eta_Z = Z^2\eta_l(l)$. Taking account of (1) and of the familiar relation between differential equations and diffusion processes yields the usual formulation, that of equation (8) of [1]. We assume that $|\eta_r(r)| < a + b|r|$ for some constants a and b, and also that η_r satisfies a Lipshitz condition of order 1 in the interval (ll, ∞) for all $ll > 0$. This assures that the r process is well defined, at least until it hits 0, and it will be seen later that problems occur before the processes hit 0. We assume that similar conditions are satisfied by $\eta_l(l)$. We will refer to the case where $\eta_r(r) = \eta_1 r$ and $\eta_l(l) = \eta_2 l$ as the lognormal case, although neither r nor l need be lognormally distributed in this case. The only other interesting cases are $\eta_r(r) = \eta_1\sqrt{r}$ and $\eta_r(r) = \eta_1$, and similarly for η_l.

Let $X = rZ = r/l$. Then X satisfies

$$dX = \left(r(X-1) + Z\mu(r,l) - \frac{X^2}{r^2}\rho\eta_r\eta_Z\right)dt + Z\eta_r\,dB_1 + r\eta_Z\,dB_2 \tag{4}$$
$$= \mu_X(r,X)\,dt + Z\eta_r\,dB_1 + r\eta_Z\,dB_2.$$

In the lognormal case, X is lognormal and that allows slightly stronger conclusions to be made.

We will show that with positive probability, X hits 0 when r is not zero. At that point $l = r/X$ has exploded.

PROPOSITION 1. *Let $\mu(r,l)$ be as in* (3). *If*

(i) $\mu(r,l) = \alpha + \beta(1-r)$ *or*
(ii) $\mu(r,l) = \alpha + \beta r(l-r)$ *and* $\beta < 1$ *or*
(iii) $\mu(r,l) = \alpha + \beta r\log(l/r)$,

the model explodes; that is, either r or l reaches ∞ in a finite amount of time with positive probability. In the log normal case, if

(iv) $\mu(r,l) = \beta r(l-r)$ *and* $\beta = 1$,

the model explodes.

Case (i) appears as equation (9) of [1]. Case (iii) is equation (13) of [2].

PROOF. We consider case (iii); cases (i) and (ii) are similar. The drifts of the X and r processes are given by

$$\mu(r,l) = \alpha + \beta r\log(1/X),$$
$$\mu_X(r,X) = r(X-1) + \alpha X/r + \beta X\log(1/X) - \rho(X^2/r^2)\eta_l(l)\eta_r(r).$$

Note that when r is large and X is small, the first summand defining μ_X is dominant and negative. Let C be a cup with one side on the r axis: $C = \{(r, X): 0 < X < X_0, r_0 < r\}$. If X_0 is chosen sufficiently small and r_0 is chosen sufficiently large, then in C we will always have $\mu > \delta > 0$ and

$$\mu_X < -\delta < 0 \tag{5}$$

for some δ. We will always assume our cups are chosen so that this is satisfied and we will assume $X_0 < 1/4$.

With these definitions and observations out of the way we can give a heuristic outline of the proof.

If the process stays in the cup for a long time, the negative drift of the X process will cause it to have a negative expected value, which is clearly incompatible with X being positive. If it leaves quickly, we will show, under the assumption that the r process does not explode, that it must exit through the sides of the cup with high probability. Again, because of the negative drift, exiting via $X = X_0$ must be balanced by exiting through $X = 0$. When that happens, the process has exploded.

Therefore, we will first attempt to show that if we start the process at a sufficiently large value of r, then we can be quite confident that it will not exit the cup through the bottom. Let $T_a = \inf\{t: r(t) < a\}$ and let

$$t_C = \inf\{t: (r(t), X(t)) \notin C\}. \tag{6}$$

Let the process r^* satisfy

$$dr^* = \mu(r(t \wedge t_C), l(t \wedge t_C))\,dt + \eta_r(r^*(t))\,dB_1.$$

The processes r and r^* agree up until they leave C, but the r^* always has positive drift. Let T_C^* be the stopping time for the r^* process. It is pretty clear that

$$\lim_{r \to \infty} P^{(r,X)}\{t_C = T_{r_0} < t\} \le \lim_{r \to \infty} P^{(r,X)}\{T_{r_0}^* < t\} = 0.$$

Now we observe that if the short-term rate r does not explode, it must take a long time to hit high levels. Let $\tau_a = \inf\{t: r(t) > a\}$. If r has a finite-valued maximum up to time t, $\lim_{a \to \infty} P^{(r_0, X)}\{\tau_a < t\} = 0$ for all X, r_0 and t.

Let $1/2 > \epsilon > 0$. First choose r_b and X_0 so that $\sup\{\mu_X(r, X): r > r_b, X < X_0\} < -100$. Next, choose $r_b < r_0 < r_1 < r_2$ and $t > 1/100\epsilon$ so that $P^{(r_1, X_0/2)}\{T_{r_0} < t\} < \epsilon/2$ and $P^{(r_1, X_0/2)}\{\tau_{r_2} < t\} < \epsilon/2$, which is possible by the foregoing statement. This forces τ_{r_2} to be large with high probability starting from $(r_1, X_0/2)$. In particular, if ω is any (random) time,

$$\begin{aligned} P^{(r_1, X_0/2)}\{\tau_{r_2} = \omega\} &\le P^{(r_1, X_0/2)}\{\tau_{r_2} = \omega, \tau_{r_2} < t\} + P^{(r_1, X_0/2)}\{\tau_{r_2} = \omega, \omega \ge t\} \\ &\le P^{(r_1, X_0/2)}\{\tau_{r_2} < t\} + P^{(r_1, X_0/2)}\{\omega \ge t\} \qquad (7) \\ &\le \epsilon/2 + E(\omega)/t \end{aligned}$$

and similarly for T_{r_0}. Let t_C be as in (6). We have

$$EX(t_C) - X(0) = E\int_0^{t_C} \mu_X(r(t), X(t))\, dt. \tag{8}$$

Equation (8) follows easily if X is kept away from 0 and r is bounded above, because in this case all coefficients of the equations defining X are actually bounded and, in general, it follows by bounded convergence on the left-hand side and monotone convergence on the right, taking account of (5). In particular, $EX(t_C) < X(0)$. We consider two alternatives.

ALTERNATIVE 1: $Et_C > 1/200$. Then by (8), $EX(t_C) < -1/2 + X_0/2 < 0$. Because its expectation is less than zero, X must also be less than zero with positive probability.

ALTERNATIVE 2: $Et_C \leq 1/200$. By (7), $P^{(r_1, X_0/2)}\{t_C = T_{r_0} \text{ or } t_C = \tau_{r_2}\} < 2\epsilon$. Therefore, the process exits through the upper or lower boundary only with small probability. If it only exits through $X = X_0$, then $EX(t_C) > X_0 * (1 - 2\epsilon) > X_0/2$ and (8) is violated again. Therefore $X(t_C) = 0$ must hold with positive probability. If r and l are lognormal and Case 4 prevails, the process satisfies

$$dr = r^2(1 - X)/X\,dt + \eta_1 r\,dB_1,$$
$$dX = -\rho X\,dt + \eta_3 X\,dB_3,$$

where dB_1 and dB_3 are correlated Brownian motions. Let $H = 1/r$. In terms of H and x the process satisfies

$$dH = \left(-(1 - X)/X - H\eta_1^2\right) dt - \eta_1 H\,dB_1.$$

We make an absolutely continuous change of measure under which the process is distributed as the solution to

$$dH = (-(1 - X)/X)\,dt - \eta_1 H\,dB_1,$$
$$dX = -\lambda X\,dt + \eta_3 X\,dB_3,$$

where λ is large. Then, similar to but more simply than before, because the expected drift for H is negative and large in magnitude it must hit zero with positive probability. By the absolute continuity of the measure the original process must also hit zero with positive probability. When H hits zero, r has exploded. □

Conclusion. This argument shows that the formal solutions to a two-factor bond option model in [1] and [2] are not correct. For better or worse, it does not show that the problem itself is improperly formulated. It is possible that a martingale measure does exist and that the problem with [1] and [2] is a naive handling of the drift of the short-term rate.

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