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THE EXPECTED NUMBER OF LEVEL CROSSINGS FOR STATIONARY, HARMONISABLE, SYMMETRIC, STABLE PROCESSES

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Let $\{X(t),\ t \geq 0\}$ be a harmonisable, symmetric, α-stable stochastic process and let $C_u(T)$ be the number of times that X crosses the level u during the time interval $[0, T]$. Our main result is the precise numerical value of $C := \lim_{u \to \infty} u^{\alpha} EC_u(T)$. By way of examples, including an explicit evaluation of EC_u for a stationary process and a combination of analytic and Monte Carlo techniques for some others, we show that the asymptotic approximation $EC_u \sim Cu^{-\alpha}$ is remarkably accurate, even for quite low values of the level u. This formula therefore serves, for all practical purposes, as a "Rice formula" for harmonisable stable processes, and should be as important in the applications of harmonisable stable processes as the original Rice formula was for their Gaussian counterparts.

We also have upper and lower bounds for EC_u that hold for all u and that, unlike previous results in the area, also hold for all α and are of the correct order of magnitude for large u.

1. Introduction. We shall be interested in stationary, harmonisable, symmetric, α-stable ($S\alpha S$) stochastic processes $\{X(t),\ t \geq 0\}$. These processes have generated considerable interest over the past few years, primarily as a family of structurally Gaussian-like processes that provide good models for long-tailed processes. A good introduction to these processes is the review article by Weron (1984), with a more comprehensive and up-to-date treatment [Samorodnitsky and Taqqu (1992)] currently in preparation.

Let $C_u(t)$ denote the number of crossings of the level u by such a process in the time interval $[0, T]$. [For a formal definition of level crossings see, for example, Cramér and Leadbetter (1967). We are implicitly assuming that X is regular enough for $C_u(T)$ to be well defined.] Our primary interest lies in calculating $EC_u(T)$.

Since X is assumed stationary, it is immediate that $EC_u(T) = TEC_u(1)$, and so from now on we shall only study $C_u := C_u(1)$.

The problem of having good information about EC_u is of major importance in terms of applying $S\alpha S$ processes in real life problems. There is probably

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no result as fundamental to the application of Gaussian processes as the famous Rice formula,

$$EC_u = \frac{1}{\pi}\left(\frac{-R''(0)}{R(0)}\right)^{1/2} \exp\left(\frac{-u^2}{2R(0)}\right),$$

[Rice (1945); see Cramér and Leadbetter (1967) for its final version and history] which gives an explicit form for EC_u when X is a mean zero, stationary, Gaussian process with covariance function $R(t)$. Without an analogous result for stable processes, many modelling applications of these processes cannot even begin.

In light of the fact that, with rare exceptions, the explicit form of stable densities is unknown, it seems extremely unlikely that it will be possible to find a closed form expression for EC_u. Thus bounds and approximations are the order of the day.

A first step in this direction was taken in Marcus (1989), and since his methodology is also the starting point of our own calculations, we shall now describe his approach and results in some detail. We need to start with some definitions.

A real-valued stochastic process $\{X(t),\ t \geq 0\}$ is stationary, harmonisable and $S\alpha S$ iff it has the integral representation

$$X(t) = \operatorname{Re}\left\{\int_{-\infty}^{\infty} e^{it\lambda} Z(d\lambda)\right\}, \tag{1.1}$$

where Z is a complex, $S\alpha S$ random measure on $(\Re, \mathscr{B})$ with a finite control measure F: that is, if $A_1, \dots, A_n$ are n disjoint sets in $\Re$ and $\theta_1, \dots, \theta_n$ are complex numbers, then

$$\begin{aligned} &E \exp\left\{i \sum_{k=1}^{n} \theta_k Z(A_k)\right\} \\ &\quad = \exp\left\{\int_{-\infty}^{\infty}\int_{S_2}\left|\sum_{k=1}^{n}(s_1 \operatorname{Re}(\theta_k) + s_2 \operatorname{Im}(\theta_k)) 1_{A_k}(\lambda)\right|^{\alpha} F(d\lambda)\, ds\right\}, \end{aligned} \tag{1.2}$$

where S_2 is the unit circle. We shall call F the *spectral distribution function* of X.

Details on (1.2) can be found, for example, in Cambanis (1983). Although the spectral representation (1.1) will have only a minor role to play in the rest of the paper, it is useful for setting up and justifying our terminology, all of which becomes standard in the case $\alpha = 2$, when the preceding description is of a mean zero, stationary, Gaussian process.

What is more important for us is that stationary, harmonisable $S\alpha S$ processes can also be represented via the infinite sum

$$X(t) = \left(C_\alpha b_\alpha^{-1} \lambda_0\right)^{1/\alpha} \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha}\left(G_k^{(1)} \cos(t\Lambda_k) + G_k^{(2)} \sin(t\Lambda_k)\right). \tag{1.3}$$

The $\{G_k^{(i)}\}_k$, $i = 1, 2$, are independent sequences of i.i.d. standard normal variables. $\{\Gamma_k\}_k$ is a sequence of arrival times of a unit rate Poisson process, so that $\{\Gamma_{k+1} - \Gamma_k\}_k$ is a sequence of i.i.d. standard exponentials. $\{\Lambda_k\}_k$ is a sequence of i.i.d. random variables with distribution function $F(\lambda)/\lambda_0$, where $\lambda_0 := F(\infty)$. The four sequences are independent of one another. Finally, the constants are given by

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x \, dx\right)^{-1}$$

$$= \begin{cases} (1-\alpha)(\Gamma(2-\alpha)\cos(\pi\alpha/2))^{-1}, & \text{if } \alpha \neq 1, \\ 2/\pi, & \text{if } \alpha = 1, \end{cases} \tag{1.4}$$

$$b_\alpha = 2^{\alpha/2}\Gamma(1+\alpha/2). \tag{1.5}$$

This representation is due, initially, to LePage (1980) and LePage, Woodroofe and Zinn (1981), with extensions and fine tuning due, among others, to Marcus and Pisier (1981). [Note that while (1.2) and (1.3) are consistent, and (1.2) is consistent with the corresponding representation in Marcus (1989), this is not true of (1.3) and its analogue in Marcus' paper, where the two representations differ by a multiplicative factor. This does not affect Marcus' results, since these are all stated in terms of nonexplicit constants. It is important for us, however.]

The representation (1.3) is crucial for the study of EC_u. Note that if we condition on the $\{\Gamma_k\}$ and $\{\Lambda_k\}$ sequences, then what remains is a mean zero, stationary, Gaussian process with covariance function

$$R(t) = (\lambda_0 C_\alpha / b_\alpha)^{2/\alpha} \sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha} \cos(t\Lambda_k).$$

Rice's formula gives us the precise form of the level crossing rate for the conditional Gaussian process as

$$E\{C_u | \{\Gamma_k\}, \{\Lambda_k\}\} = \frac{1}{\pi}\left(\frac{\sum_{k=1}^\infty \Lambda_k^2 \Gamma_k^{-2/\alpha}}{\sum_{k=1}^\infty \Gamma_k^{-2/\alpha}}\right)^{1/2} \exp\left\{\frac{-u^2}{2\gamma_\alpha^2 \lambda_0^{2/\alpha} \sum_{k=1}^\infty \Gamma_k^{-2/\alpha}}\right\}, \tag{1.6}$$

where $\gamma_\alpha := (C_\alpha / b_\alpha)^{1/\alpha}$.

There does not seem to be any way of explicitly evaluating the remaining expectation in (1.6). There are, however, a number of paths that lead to useful results.

The first obvious path, which seems at first one of desperation, is to look at the behaviour of EC_u as $u \to \infty$. Given that more is known about the tails of stable random variables than about the central parts of their distributions, it is natural to hope that something can be done for this case. In the following section we shall show that, for all $\alpha \in (0,2)$,

$$\lim_{u \to \infty} u^\alpha EC_u = \frac{\lambda_1 C_\alpha}{\pi}, \tag{1.7}$$

where, in general, the spectral moments λ_β, $\beta \geq 0$, are given by

$$\lambda_\beta = \int_0^\infty \lambda^\beta F(d\lambda). \tag{1.8}$$

Because of the explicit form of the right-hand side of (1.7), this asymptotic formula is the main result of this paper.

In Sections 3 and 4 we shall show, by way of an example, that (1.7) is of far more than theoretical interest. Section 3 will consider a very special case of a sub-Gaussian process for which it is actually possible to calculate EC_u explicitly, and *en passant*, will establish a similar asymptotic formula for a class of nonharmonisable stable processes. The example we consider, which is the only one for which EC_u is known exactly for all u, will show that the asymptotic formula inherent in (1.6), namely,

$$EC_u \sim \frac{\lambda_1 C_\alpha}{\pi} u^{-\alpha},$$

provides a remarkably good approximation to EC_u once u is of the order of magnitude of the highest quartile of the distribution of $X(t)$.

This theme is continued in Section 4, where we note that although in other cases we cannot explicitly calculate EC_u, it is easy to evaluate numerically via simulation over the random variables in (1.6). We shall collect a number of typical such evaluations that reinforce the example of Section 3, to show that, in general, the asymptotic result gives a very good approximation at surprisingly low levels u. It is this fact, far more than the intrinsically interesting theoretical nature of (1.7), that makes this result so important.

Forerunners to (1.7) can be found in Marcus (1989), who studied necessary and sufficient conditions for the finiteness of EC_u by deriving upper and lower bounds of the form Ku^β. However, from our point of view, Marcus' results are incomplete, for the exponents β appearing in the upper and lower bounds do not always agree. Neither do his results provide both upper and lower bounds to EC_u for all values of the stable index α. We shall remedy this situation in Section 5, where simple extensions of the arguments used to derive (1.7) are used to provide exact bounds for EC_u for all u and all α. It should be noted, however, that with one exception the constants in both our bounds and Marcus' bounds are not given explicitly. (The exception is in our general lower bound.)

Finally, in Section 6 we look at the problem of evaluating numerical bounds for EC_u, and present some results, but without proofs. The results here are rather disappointing because the gap between the upper and lower bounds is large. Nevertheless, as will be seen from the graphs of Sections 3 and 4, the disappointment is more than made up for by the fact than the asymptotic approximation inherent in (1.7) is surprisingly good for surprisingly small values of u.

2. The asymptotic formula. Throughout the remainder of this paper we shall assume that EC_u is finite for all u. The following result, which places conditions on the spectral moments that guarantee this finiteness, is part of Theorem 1 of Marcus (1989):

2.1. PROPOSITION. *The following conditions on the spectral moments are necessary and sufficient for the finiteness of EC_u for all u:*

$$\begin{aligned} \lambda_1 &< \infty, \quad \text{if } \alpha < 1, \\ (\lambda \log \lambda)_\delta &< \infty, \quad \text{if } \alpha = 1, \\ \lambda_\alpha &< \infty, \quad \text{if } \alpha > 1, \end{aligned}$$

where

$$(\lambda \log \lambda)_\delta := \int_\delta^\infty \lambda \log\left(\frac{\lambda}{\delta}\right) F(d\lambda) \tag{2.1}$$

and δ is the unique solution of

$$\delta(2\lambda_0 - F(\delta)) = \int_\delta^\infty \lambda F(d\lambda) \tag{2.2}$$

when $\lambda_1 < \infty$. When $\lambda_1 = \infty$, we set $(\lambda \log \lambda)_\delta = \infty$.

The following is the main result of this section:

2.2. THEOREM. *Assume $EC_0 < \infty$. Then*

$$\lim_{u \to \infty} u^\alpha E C_u = \frac{\lambda_1 C_\alpha}{\pi}.$$

PROOF. Set

$$S_u^1 = \left(\frac{\Lambda_1^2 \Gamma_1^{-2/\alpha}}{\sum_{k=1}^\infty \Gamma_k^{-2/\alpha}} \right)^{1/2} \exp\left\{ \frac{-u^2}{2\gamma_\alpha^2 \lambda_0^{2/\alpha} \sum_{k=1}^\infty \Gamma_k^{-2/\alpha}} \right\},$$

$$S_u^2 = \left(\frac{\sum_{k=2}^\infty \Lambda_k^2 \Gamma_k^{-2/\alpha}}{\sum_{k=1}^\infty \Gamma_k^{-2/\alpha}} \right)^{1/2} \exp\left\{ \frac{-u^2}{2\gamma_\alpha^2 \lambda_0^{2/\alpha} \sum_{k=1}^\infty \Gamma_k^{-2/\alpha}} \right\}.$$

Then, by (1.6), and noting that all the random variables appearing there are positive,

$$\frac{ES_u^1}{\pi} \le EC_u \le \frac{E(S_u^1 + S_u^2)}{\pi}.$$

The proof of the Theorem will now follow from Lemmas 2.3 and 2.4. □

2.3. LEMMA. *Assume $EC_0 < \infty$, and let S_u^1 be as defined before. Then*

$$\lim_{u \to \infty} u^\alpha E S_u^1 = \lambda_1 C_\alpha.$$

2.4. LEMMA. *Assume $EC_0 < \infty$, and let S_u^2 be as defined before. Then*

$$\lim_{u \to \infty} u^{\alpha} E S_u^2 = 0.$$

We shall prove both of these results in a moment, but first we need to know something about the distribution of $\sum_{k=1}^{\infty} \Lambda_k^2 \Gamma_k^{-2/\alpha}$. It is well known [e.g., LePage (1980)] that $\sum_{k=1}^{\infty} \Lambda_k^2 \Gamma_k^{-2/\alpha}$ converges with probability 1 iff $E\Lambda^{\alpha} < \infty$. For later use, we note that in the case of convergence, the limit variable, S say, has a so-called $S_{\alpha/2}(\sigma, 1, 0)$ stable distribution with scaling parameter σ given by $\sigma^{\alpha/2} = \lambda_{\alpha}/(\lambda_0 C_{\alpha/2})$; that is, the distribution of the random variable S has Laplace transform

$$Ee^{-\theta S} = \exp\left\{ \frac{-\theta^{\alpha/2}\lambda_{\alpha}}{\lambda_0 C_{\alpha/2} \cos(\pi\alpha/4)} \right\}. \tag{2.3}$$

The other fact that we need to start the proofs of Lemmas 2.3 and 2.4 is contained in the following result.

2.5. LEMMA . *Let X be a positive, $\alpha/2$ strictly stable random variable with scale parameter σ, so that it has Laplace transform $E\exp\{-\theta X\} = \exp(-\sigma^{\alpha/2}\theta^{\alpha/2}/\cos(\pi\alpha/4))$. Then*

$$\lim_{u \to \infty} u^{\alpha} E\{\exp(-u^2/X)\} = \sigma^{\alpha/2} C_{\alpha/2} \Gamma\left(1 + \tfrac{1}{2}\alpha\right), \tag{2.4}$$

where C_{α} is defined by (1.4).

PROOF. Note that

$$\begin{aligned} \lim_{u \to \infty} u^{\alpha} E\{\exp(-u^2/X)\} &= \lim_{u \to \infty} u^{\alpha} \int_0^1 P\{\exp(-u^2/X) > \lambda\}\, d\lambda \\ &= \lim_{u \to \infty} u^{2+\alpha} \int_0^{\infty} P(X > x^{-1}) \exp(-xu^2)\, dx \\ &= \lim_{u \to \infty} u^{1+\alpha/2} \int_0^{\infty} P(X > x^{-1}) \exp(-xu)\, dx, \end{aligned} \tag{2.5}$$

where the last two lines follow from simple changes of variable.

We now use a Tauberian theorem. In particular, we turn to the $x \to 0$ version of Theorem XIII.5.4 of Feller (1971). Using the fact that

$$u(x) = P\{X > x^{-1}\} \sim \sigma^{\alpha/2} C_{\alpha/2} x^{\alpha/2},$$

as $x \to 0$ [e.g., Feller (1971)], (2.5) and the Tauberian theorem give us that

$$\lim_{u \to \infty} u^{\alpha} E\{\exp(-u^2/X)\} = \sigma^{\alpha/2} C_{a/2} \Gamma\left(1 + \tfrac{1}{2}\alpha\right),$$

as required. □

We can now turn to the proof of Lemma 2.3.

PROOF OF LEMMA 2.3. Note that by the definition of S_u^1 we have

$$ES_u^1 \le E|\Lambda_1| E\left\{\exp\left\{\frac{-u^2}{2\gamma_\alpha^2 \lambda_0^{2/\alpha} \sum_{k=1}^\infty \Gamma_k^{-2/\alpha}}\right\}\right\}. \tag{2.6}$$

On the other hand,

$$\begin{aligned} ES_u^1 &= E|\Lambda_1| E\left\{\exp\left\{\frac{-u^2}{2\gamma_\alpha^2 \lambda_0^{2/\alpha} \sum_{k=1}^\infty \Gamma_k^{-2/\alpha}}\right\}\right\} \\ &\quad - E|\Lambda_1| E\left\{\left[1 - \frac{\Gamma_1^{-1/\alpha}}{\left(\sum_{k=1}^\infty \Gamma_k^{-2/\alpha}\right)^{1/2}}\right] \exp\left\{\frac{-u^2}{2\gamma_\alpha^2 \lambda_0^{2/\alpha} \sum_{k=1}^\infty \Gamma_k^{-2/\alpha}}\right\}\right\}. \end{aligned} \tag{2.7}$$

By the triangle inequality, the second term here is bounded, in absolute value, by

$$\left(\frac{\lambda_1}{\lambda_0}\right) E\left\{\left[\frac{\left(\sum_{k=2}^\infty \Gamma_k^{-2/\alpha}\right)^{1/2}}{\left(\sum_{k=1}^\infty \Gamma_k^{-2/\alpha}\right)^{1/2}}\right] \exp\left\{\frac{-u^2}{2\gamma_\alpha^2 \lambda_0^{2/\alpha} \sum_{k=1}^\infty \Gamma_k^{-2/\alpha}}\right\}\right\}.$$

By Lemma 2.4 (which, admittedly, we have still to prove), with $\Lambda_k \equiv 1$, this expression is $o(u^{-\alpha})$ as $u \to \infty$. Thus, combining (2.6) and (2.7), we have that

$$\lim_{u\to\infty} u^\alpha ES_u^1 = \left(\frac{\lambda_1}{\lambda_0}\right) \lim_{u\to\infty} u^\alpha E\left\{\exp\left\{\frac{-u^2}{2\gamma_\alpha^2 \lambda_0^{2/\alpha} \sum_{k=1}^\infty \Gamma_k^{-2/\alpha}}\right\}\right\}.$$

The distribution of $X := \sum_{k=1}^\infty \Gamma_k^{-2/\alpha}$ is, by (2.3), that of a positive, $\alpha/2$ strictly stable random variable with scale parameter $\sigma = C_{\alpha/2}^{-2/\alpha}$. Thus we can apply Lemma 2.5 to show that

$$\begin{aligned} \lim_{u\to\infty} u^\alpha ES_u^1 &= \left(\frac{\lambda_1}{\lambda_0}\right)\left(\sqrt{2}\,\gamma_\alpha \lambda_0^{1/\alpha}\right)^\alpha \lim_{u\to\infty} u^\alpha E\left\{\exp\left\{\frac{-u^2}{\sum_{k=1}^\infty \Gamma_k^{-2/\alpha}}\right\}\right\} \\ &= \lambda_1\left(\sqrt{2}\,\gamma_\alpha\right)^\alpha \lim_{u\to\infty} u^\alpha E\{\exp(-u^2/X)\} \\ &= \lambda_1\left(\sqrt{2}\,\gamma_\alpha\right)^\alpha C_{\alpha/2}^{-1} C_{\alpha/2} \Gamma(1+\alpha/2) \\ &= \lambda_1 C_\alpha, \end{aligned}$$

on applying the definition of γ_α. This completes the proof, modulo proving Lemma 2.4. □

Before we turn to the proof of the remaining Lemma 2.4, we need the following two lemmas. In the bound of Lemma 2.6, and in all that follows, we shall let C denote a generic constant that may change from line to line.

2.6. LEMMA. *Let X be a positive, $\alpha/2$ strictly stable random variable. Then there exists a finite (α-dependent) constant C such that*

$$E\{X^{-\beta/2} \exp(-u^2/X)\} \le Cu^{-(\alpha+\beta)}$$

for all positive u.

PROOF. Since X is a $\alpha/2$ stable random variable, its density can be bounded by $Cx^{-(1+\alpha/2)}$ for some C. Thus we can bound the expectation in the statement of the lemma as

$$\begin{aligned} E\{X^{-\beta/2}\exp(-u^2/X)\} &\le C\int_0^\infty x^{-\beta/2}\exp(-u^2/x)x^{-(1+\alpha/2)}\,dx \\ &= C\int_0^\infty x^{-(\alpha+\beta+2)/2}\exp(-u^2/x)\,dx \\ &= Cu^{-(\alpha+\beta)}\int_0^\infty y^{-(\alpha+\beta+2)/2}\exp(-1/y)\,dy, \end{aligned}$$

where the last line follows from a simple change of variable.

Since the integral here is clearly finite, the proof is complete. □

2.7. LEMMA. *Assume that* $\lambda_{2\alpha} < \infty$. *Then*

$$E\left(\sum_{k=K}^{\infty} \Lambda_k^2 |\Gamma_k^{-1/\alpha} - k^{1/\alpha}|\right) < \infty$$

holds for all $K \ge 2$ *and all* $\alpha < 1$.

PROOF. An application of the triangle inequality gives

$$\begin{aligned} &E\left(\sum_{k=K}^{\infty} \Lambda_k^2 \left|\Gamma_k^{-1/\alpha} - k^{-1/\alpha}\right|\right)^{\alpha} \\ &\qquad \le \frac{\lambda_{2\alpha}}{\lambda_0}\sum_{k=K}^{\infty} E|\Gamma_k^{-1/\alpha} - k^{-1/\alpha}|^{\alpha}. \end{aligned}$$

It now clearly suffices to show that, for each k large enough, $E|\Gamma_k^{-1/\alpha} - k^{-1/\alpha}|^{\alpha}$ is bounded by the term of a summable series.

To this end, note that

$$\begin{aligned} &E|\Gamma_k^{-1/\alpha} - k^{-1/\alpha}|^{\alpha} \\ &\quad \le CE\left\{\left(|\Gamma_k - k|\left(\Gamma_k^{-(1+2/\alpha)} + k^{-(1+2/\alpha)}\right)\right)^{\alpha/2}\right\} \\ &\quad \le C\left(E|\Gamma_k - k|^2\right)^{\alpha/4}\left(E\left(\Gamma_k^{-(1+2/\alpha)} + k^{-(1+2/\alpha)}\right)^{2\alpha/(4-\alpha)}\right)^{(4-\alpha)/4} \\ &\quad \le C\left(E|\Gamma_k - k|^2\right)^{\alpha/4}\left(E\Gamma_k^{-(4+2\alpha)/(4-\alpha)} + k^{-(4+2\alpha)/(4-\alpha)}\right)^{(4-\alpha)/4}; \end{aligned}$$

the last two lines follow from the Cauchy–Schwarz and triangle inequalities, respectively.

Now use the fact that Γ_k has a gamma distribution, to check that $E|\Gamma_k - k|^2 = k$ and $E\Gamma_k^{-(4+2\alpha)/(4-\alpha)} = O(k^{-(4+2\alpha)/(4-\alpha)})$ for large k. Substituting these bounds in the preceding inequality completes the proof. □

PROOF OF LEMMA 2.4. We shall have to divide the proof into three separate cases, $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$. Throughout the proof we shall write X for the sum $\sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha}$. Recall that X is an $\alpha/2$ stable random variable, and note that from the definition of S_u^2 and the triangle inequality we have that, for any $K \geq 2$ (with the first sum taken identically zero if $K = 2$),

$$ES_u^2 \leq \sum_{k=2}^{K-1} E\{\Lambda_k \Gamma_k^{-1/\alpha} X^{-1/2} \exp(-u^2/AX)\} \tag{2.8}$$
$$+ E\left\{\left(\sum_{k=K}^{\infty} \Lambda_k^2 \Gamma_k^{-2/\alpha}\right)^{1/2} X^{-1/2} \exp(-u^2/AX)\right\},$$

where $A := 2\gamma_\alpha^2 \lambda_0^{2/\alpha}$.

CASE 1: $\alpha > 1$. From (2.8) and two separate applications of Hölder's inequality we have that

$$ES_u^2 \leq C\frac{\lambda_1}{\lambda_0}\left(E\{X^{-\beta/2(\beta-1)} \exp(-u^2/AX)\}\right)^{(\beta-1)/\beta} \sum_{k=2}^{K-1} \left(E\{\Gamma_k^{-\beta/\alpha}\}\right)^{1/\beta}$$
$$+ \left[E\left(\sum_{k=K}^{\infty} \Lambda_k^2 \left|\Gamma_k^{-2/\alpha} - k^{2/\alpha}\right|\right)^{\alpha/2}\right]^{1/\alpha} \tag{2.9}$$
$$\times \left[E\left(X^{-\alpha/2(\alpha-1)} \exp(-u^2/AX)\right)\right]^{(\alpha-1)/\alpha}$$
$$+ E\left(\sum_{k=K}^{\infty} \Lambda_k^2 k^{-2/\alpha}\right)^{1/2} E\left(X^{-1/2} \exp(-u^2/AX)\right),$$

where, for ease of writing, we have set $3\alpha/2 = \beta$.

Now fix $\varepsilon > 0$. By Lemma 2.6, the first term in (2.9) can be bounded by a constant factor times

$$Ku^{-(\alpha+\beta/(\beta-1))(\beta-1)/\beta} = Ku^{-(\alpha+1-\alpha/\beta)}. \tag{2.10}$$

As regards the second term, recall that, by Lemma 2.7, $\sum_{k=K}^{\infty} \Lambda_k^2 |\Gamma_k^{-2/\alpha} - \mathrm{k}^{-2/\alpha}|$ has a finite moment of order $\alpha/2$, so that by monotone convergence we can make $E(\sum_{k=K}^{\infty} \Lambda_k^2 |\Gamma_k^{-2/\alpha} - k^{2/\alpha}|)^{\alpha/2}$ less than ε for large enough K. Applying Lemma 2.6 once again, we therefore obtain that the second term is, for large enough K, less than a constant multiple of

$$\varepsilon u^{-(\alpha+\alpha/(\alpha-1))(\alpha-1)/\alpha} = \varepsilon u^{-\alpha}. \tag{2.11}$$

Finally, the last term is bounded by a constant factor times

$$Ku^{-(\alpha+1)}. \tag{2.12}$$

Combining (2.9)–(2.12), recalling that $\beta/\alpha = 3/2 > 1$ and going to the limit proves the lemma for this case.

CASE 2: $\alpha < 1$. We start, as before, with (2.8). To bound the summation from 2 to K, it will clearly suffice to show that

$$E\{\Gamma_k^{-1/\alpha} X^{-1/2} \exp(-u^2/X)\} = o(u^{-\alpha}). \tag{2.13}$$

To see this, choose $p \in (k^{-1}, 1)$ and break up the expectation according to the events $\Gamma_k^{-1/\alpha} \le X^{p/2}$ or $\Gamma_k^{-1/\alpha} > X^{p/2}$. In the first case we have

$$\begin{aligned} E\Big\{\Gamma_k^{-1/\alpha} X^{-1/2} e^{-u^2/X} 1_{\Gamma_k^{-1/\alpha} \le X^{p/2}}\Big\} &\le E\{X^{-(1-p)/2} e^{-u^2/X}\} \\ &\le C u^{-(\alpha+1-p)} \\ &= o(u^{-\alpha}), \end{aligned}$$

where the second line follows from Lemma 2.6.

In the second case, recall that $\Gamma_k^{-1/\alpha} \le (\sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha})^{1/2} = X^{1/2}$ to see that

$$\begin{aligned} E\Big\{\Gamma_k^{-1/\alpha} X^{-1/2} e^{-u^2/X} 1_{\Gamma_k^{-1/\alpha} > X^{p/2}}\Big\} &\le E\{\exp(-u^2 \Gamma_k^{2/p\alpha})\} \\ &= \frac{1}{\Gamma(k)} \int_0^{\infty} x^{k-1} \exp(-(x + u^2 x^{2/p\alpha})) \\ &= C \int_0^{\infty} y^{-1+kp\alpha/2} e^{-y^{p\alpha/2}} e^{-u^2 y}\, dy, \end{aligned}$$

where the second line uses the fact that Γ_k has a gamma distribution with k degrees of freedom.

Appealing again to the Tauberian theorem [Theorem XIII.5.4 of Feller (1971)], it is straightforward to check that the preceding expression is $O(-u^{pk\alpha}) = o(u^{\alpha})$, which completes the proof of (2.13).

We now turn to the expectation in (2.9) involving the summation over $k \ge K$. By the triangle inequality, it will suffice to show that there exists a K large enough so that

$$E\left\{\sum_{k=K}^{\infty} \Gamma_k^{-1/\alpha} X^{-1/2} e^{-u^2/X}\right\} = o(u^{-\alpha}). \tag{2.14}$$

Taking our lead from the previous case, fix $p \in (\alpha, 1)$ and split the integration into the regions $\sum_{k=K}^{\infty} \Gamma_k^{-1/\alpha} < (\sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha})^{p/2}$ and $\sum_{k=K}^{\infty} \Gamma_k^{-1/\alpha} \ge (\sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha})^{p/2}$.

The same argument as before easily gives that the first term is $o(u^{-\alpha})$. The second term can be bounded by

$$\begin{aligned} &E\left\{\frac{\sum_{k=K}^{\infty} \Gamma_k^{-1/\alpha}}{\left(\sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha}\right)^{1/2}} \exp\left(\frac{-u^2}{\left(\sum_{k=K}^{\infty} \Gamma_k^{-1/\alpha}\right)^{2/p}}\right)\right\} \\ &\quad \le \left[E\left(\frac{\sum_{k=K}^{\infty} \Gamma_k^{-1/\alpha}}{\left(\sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha}\right)^{1/2}}\right)^{\beta}\right]^{1/\beta} \left[E \exp\left(\frac{-\beta u^2}{\left(\sum_{k=K}^{\infty} \Gamma_k^{-1/\alpha}\right)^{2/p}}\right)\right]^{(\beta-1)/\beta}, \end{aligned} \tag{2.15}$$

where $\beta > 1$.

Consider the exponential term here. For ease of writing, set $V = \sum_{k=K}^{\infty} \Gamma_k^{-1/\alpha}$. Since $\alpha < 1$, it is easy to check that, for large enough K, $EV < \infty$, and so $P\{V > v\} = o(v^{-1})$ as $v \to \infty$. Since

$$\begin{aligned} E\{e^{-u^2/V^{2/p}}\} &= \int_0^\infty P\{e^{-u^2/V^{2/p}} > v\}\, dv \\ &= u^2 \int_0^\infty P\{V > w^{-p/2}\} e^{-wu^2}\, dw \\ &\le Cu^2 \int_0^\infty w^{p/2} e^{-wu^2}\, dw = C_1 u^{-p}. \end{aligned}$$

By choosing β large enough so that $p(\beta - 1)/\beta > \alpha$ [cf. (2.15)], we shall have completed the proof of (2.14), once we can show that, at least for large enough K,

$$E\left(\frac{\sum_{k=K}^{\infty} \Gamma_k^{-1/\alpha}}{\left(\sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha}\right)^{1/2}} \right)^{\beta} < \infty.$$

Apply the Cauchy-Schwarz inequality to the ratio to see that all we need show is that $E(\sum_{k=K}^{\infty} \Gamma_k^{-1/\alpha})^{2\beta}$ is finite for large enough K. Since we can always take 2β to be integral, we do so, and then expand the sum. Using the fact that $E\Gamma_k^{-\gamma} = O(k^{-\gamma})$ for large k then establishes the required finiteness. This completes this part of the proof.

CASE 3: $\alpha = 1$. This case is particularly easy. Start as before with (2.8). The first term there can be handled exactly as in the case $\alpha > 1$, since the first part of the argument will also work for $\alpha = 1$. Thus, we need only show that for given $\varepsilon > 0$ we can chose K large enough so that

$$E\left\{ \left(\sum_{k=K}^{\infty} \Lambda_k^2 \Gamma_k^{-2/\alpha} \right)^{1/2} X^{-1/2} \exp(-u^2/X) \right\} \le \varepsilon C u^{-1}. \tag{2.16}$$

The proof of this hinges on the elementary fact that there exists a u-independent constant C such that

$$\sup_{x>0} x^{-1/2} e^{-u^2/x} = Cu^{-1}. \tag{2.17}$$

It follows from this that (2.16) can be bounded above by

$$Cu^{-1} E\left(\sum_{k=K}^{\infty} \Lambda_k^2 \Gamma_k^{-2} \right)^{1/2}.$$

But Lemma 3 of Marcus (1989) and monotone convergence make the expectation as small as we like for large enough K, so we are done. □

3. A sub-Gaussian example. In this section we shall look at sub-Gaussian $S\alpha S$ processes, with a twofold aim. First, we shall consider a very specific example in which the process is also harmonisable and in which we shall actually be able to derive an explicit expression for EC_u. Beyond the intrinsic interest of such a result will be the fact that it will give us an idea of how good the asymptotics of the previous section are, as well as providing a good test case against which to check the simulations of the following section.

Second, we shall show that for general sub-Gaussian processes, which, with the exception of the once case just mentioned, are never harmonisable, an asymptotic formula for EC_u akin to that of the previous section can be derived.

Recall that a $S\alpha S$ process is called *sub-Gaussian* if it can be written in the form

$$X(t) = A^{1/2}G(t), \tag{3.1}$$

where G is a Gaussian process and A is a positive stable random variable independent of G. We shall assume that it has a $S_{\alpha/2}((\cos(\pi\alpha/4))^{2/\alpha}, 1, 0)$ distribution with Laplace transform $Ee^{-\theta A} = \exp(-\theta^{\alpha/2})$.

[One should note that, somewhat unfortunately, the term "sub-Gaussian" has been used in closely related literature with two quite distinct meanings. Aside from the definition we have just given, the term has also been used to describe processes whose increments have a moment generating function that is dominated by that of a Gaussian process with identical incremental variance function; cf., Jain and Marcus (1978). Both uses of the term seem too well entrenched to change.]

Now suppose that G is stationary, with zero mean and covariance function $R_G(t) = EG(t+s)G(s)$. Then, obviously, X is also stationary. If we denote the distribution function of A by $F_{\alpha/2}$, then it is immediate from (3.1) and Rice's formula that for a stationary sub-Gaussian process of the preceding form,

$$EC_u = \frac{1}{\pi}\left(\frac{\lambda_2^G}{\lambda_0^G}\right)^{1/2} \int_0^\infty \exp(-u^2/2\lambda_0^G a)F_{\alpha/2}(da), \tag{3.2}$$

where $\lambda_0^G = R_G(0)$ and $\lambda_2^G = -R''_G(0)$ are, respectively, the variance and second spectral moments of G.

Since $F_{\alpha/2}$ is not in general known, (3.2) will not in general enable us to obtain an explicit formula for EC_u. However, there is one special case of interest in which it is known: $\alpha = 1$. In this case, $F_{1/2}$ has the density

$$f_{1/2}(a) = \frac{e^{-1/4a}}{2\sqrt{\pi}a^{3/2}}. \tag{3.3}$$

Substituting this into (3.2) and performing the integration leads us to a proposition.

3.1. Proposition. *Let X be a stationary, 1-stable, sub-Gaussian process with the representation* (3.1), *where A has the density* (3.3). *Then, in the preceding notation,*

$$EC_u = \frac{\sqrt{\lambda_2^G}}{\pi\sqrt{\lambda_0^G + 2u^2}}. \tag{3.4}$$

In order to use this result for comparison purposes, we need to ensure that X is also harmonisable. There is only one choice of Gaussian process G that will guarantee this, and it is given by

$$G(t) = \sigma(G_1 \cos \omega t + G_2 \sin \omega t), \tag{3.5}$$

where G_1 and G_2 are independent standard normal variables. Since $R_G(t) = \sigma^2 \cos \omega t$, we have $\lambda_0^G = \sigma^2$, $\lambda_2^G = \sigma^2\omega^2$, and so a simple calculation, using the explicit form of the 1-stable (Cauchy) density, gives us

$$EC_u = \frac{\omega}{\pi\sqrt{1 + 2u^2/\sigma^2}}. \tag{3.6}$$

Comparing (3.5) with the "spectral representation" (1.1) it is not hard to see that the $S\alpha S$ process X has a spectral distribution function given by

$$F(d\lambda) = \frac{\sigma\pi}{2^{3/2}}\delta_\omega(d\lambda), \tag{3.7}$$

where δ_ω is the measure putting unit mass at the point ω.

Since it follows from this that, in the notation of the previous section, $\lambda_1 = \omega\sigma\pi/2^{3/2}$ and $C_\alpha = c_1 = 2/\pi$, it follows from Theorem 2.2 that

$$\lim_{u\to\infty} uEC_u = \frac{\lambda_1 C_\alpha}{\pi} = \frac{\sigma\omega}{\sqrt{2}\,\pi}, \tag{3.8}$$

which is, of course, exactly what one expects from (3.6).

To use (3.6) to see how well the asymptotic formula of the previous section approximates the truth, consider the graphs of Figure 1. The full line in the left-hand box of this figure is (3.6). The dash-dot-dash line is the approximation indicated by the asymptotic (3.8); that is, the function $u^{-1}C\omega/\sqrt{2}\,\pi$. The fit is remarkable. The dashed line is the result of a Monte Carlo evaluation of EC_u of the kind described in the following section.

In the right-hand box of Figure 1 we have plotted the ratio of the asymptotic to true formulae for EC_u. What is interesting is not that the ratio converges to unity, but that it does so very quickly.

We now turn to the problem of deriving an asymptotic formula for EC_u for general sub-Gaussian $S\alpha S$ processes. We shall prove the following theorem.

3.2. THEOREM. *For a general sub-Gaussian $S\alpha S$ process of the form* (3.1) *we have*

$$\lim_{u\to\infty} u^\alpha EC_u = \frac{2^{\alpha/2}\Gamma(1+\alpha/2)\left(\lambda_2^G\right)^{1/2}}{\pi\Gamma(1-\alpha/2)\left(\lambda_0^G\right)^{(1-\alpha)/2}}, \tag{3.9}$$

where $\lambda_0^G = \sigma^2$ and λ_2^G are the total mass and second spectral moment of the Gaussian process G of (3.1).

PROOF. We start with (3.2). To evaluate this, note first the standard result that for the $S\frac{1}{2}\alpha S$ random variable A,

$$\lim_{a\to\infty} a^{\alpha/2}P\{A > a\} = \lim_{u\to\infty} a^{\alpha/2}\left(1 - F_{\alpha/2}(a)\right) = \left(\Gamma(1-\alpha/2)\right)^{-1}. \tag{3.10}$$

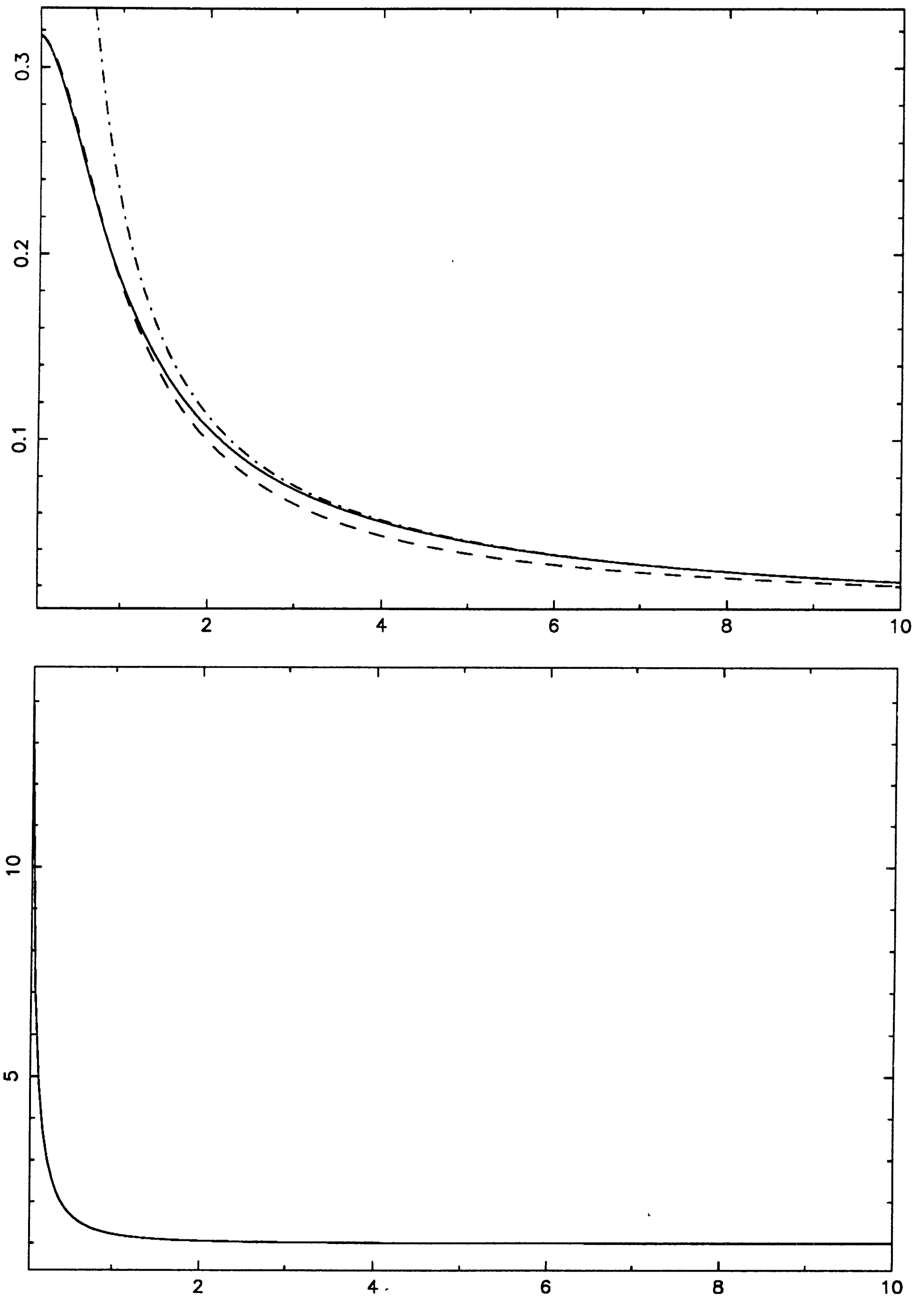


FIG. 1. *EC_u for the stationary, harmonisable S1S process. Upper graphs: exact graph of EC_u (full line); asymptotic approximation (dash-dot-dash); Monte Carlo evaluation $C_u^{5000,200}$ (broken line). Lower graph: ratio of asymptotic to exact formulae.*

It therefore follows that

$$\begin{aligned}
\lim_{v\to\infty} v^{\alpha/2}\int_0^\infty \exp\left(\frac{-v}{a}\right)F_{\alpha/2}(da) &= \lim_{v\to\infty} v^{\alpha/2}E(e^{-v/A}) \\
&= \lim_{v\to\infty} v^{\alpha/2}\int_0^\infty P\{e^{-v/A} > x\}\,dx \\
&= \lim_{v\to\infty}\int_0^\infty v^{\alpha/2}P\{A > v/t\}e^{-t}\,dt \\
&= \int_0^\infty \lim_{v\to\infty} v^{\alpha/2}P\{A > v/t\}e^{-t}\,dt \\
&= \frac{1}{\Gamma(1-\alpha/2)}\int_0^\infty t^{\alpha/2}e^{-t}\,dt \\
&= \frac{\Gamma(1+\alpha/2)}{\Gamma(1-\alpha/2)},
\end{aligned}$$

where the interchange of limit and integration is justified by dominated convergence in view of the fact that $v^{\alpha/2}P\{A > v/t\}$ is bounded by $ct^{\alpha/2}$ for some finite c.

Substituting the preceding equation into (3.2) now proves the theorem. □

4. Monte Carlo evaluation of EC_u. Consider once again the basic conditional expectation of the number of level crossings that has been behind all we have done so far, namely,

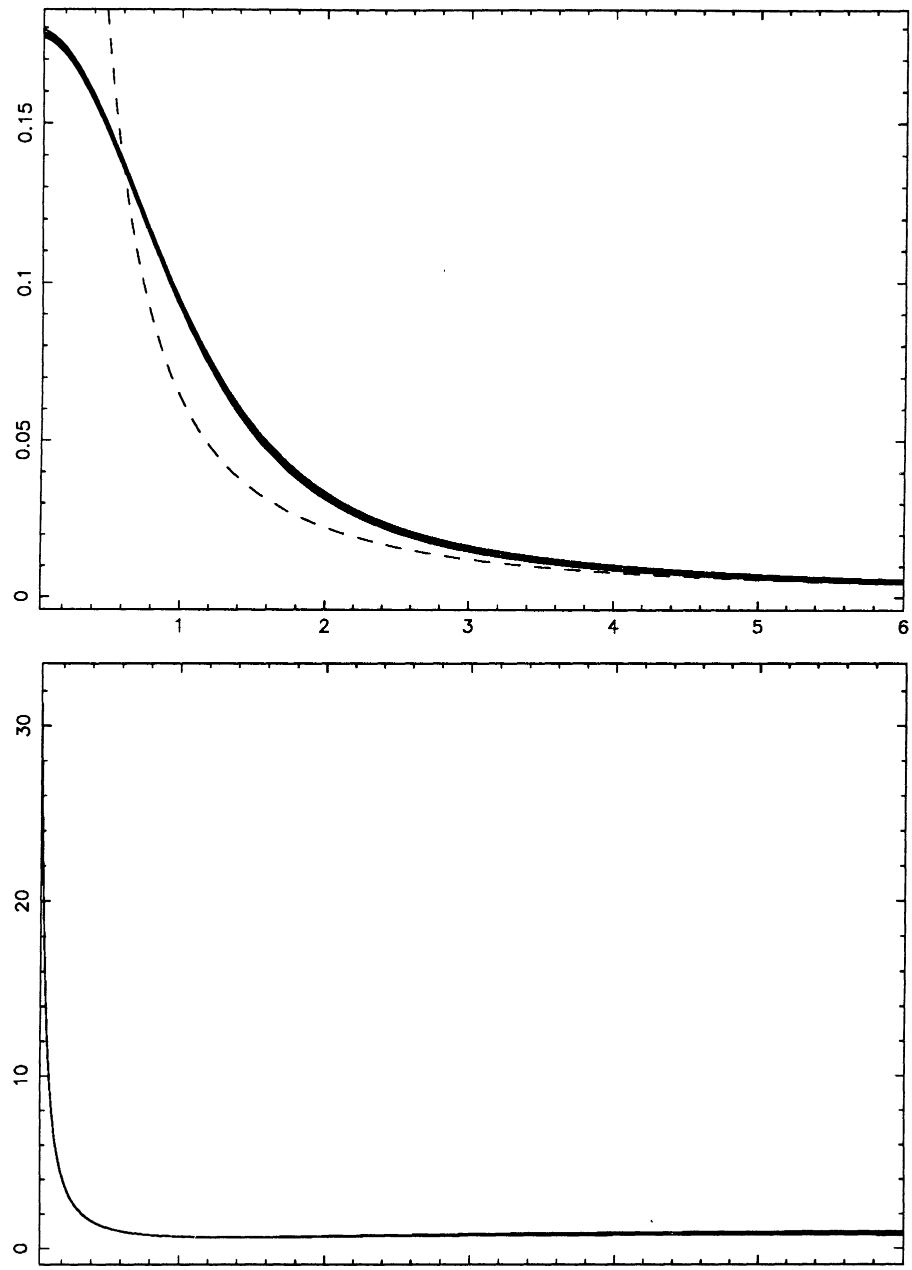
$$(4.1)\quad E\{C_u|\{\Gamma_k\},\{\Lambda_k\}\} = \frac{1}{\pi}\left(\frac{\sum_{k=1}^\infty \Lambda_k^2\Gamma_k^{-2/\alpha}}{\sum_{k=1}^\infty \Gamma_k^{-2/\alpha}}\right)^{1/2}\exp\left\{\frac{-u^2}{2\gamma_\alpha^2\lambda_0^{2/\alpha}\sum_{k=1}^\infty\Gamma_k^{-2/\alpha}}\right\}.$$

Despite the complicated form of this expression, it is particularly simple to evaluate via Monte Carlo methods. While the variables Γ_k are dependent, the fact that Γ_k can be written as $E_1 + \cdots + E_k$, where $E_1, E_2, \dots$ is a sequence of independent, parameter 1, exponential variables, and that these are independent of the i.i.d. Λ_i, means that the following expression should give a good estimate of EC_u for large N_1 and N_2:

$$(4.2)\quad C_u^{N_1,N_2} := \frac{1}{N_1\pi}\sum_{i=1}^{N_1}\left[\left(\frac{\sum_{k=1}^{N_2}\Lambda_{ik}^2\Gamma_{ik}^{-2/\alpha}}{\sum_{k=1}^{N_2}\Gamma_{ik}^{-2/\alpha}}\right)^{1/2}\exp\left\{\frac{-u^2}{2\gamma_\alpha^2\lambda_0^{2/\alpha}\sum_{k=1}^{N_2}\Gamma_{ik}^{-2/\alpha}}\right\}\right],$$

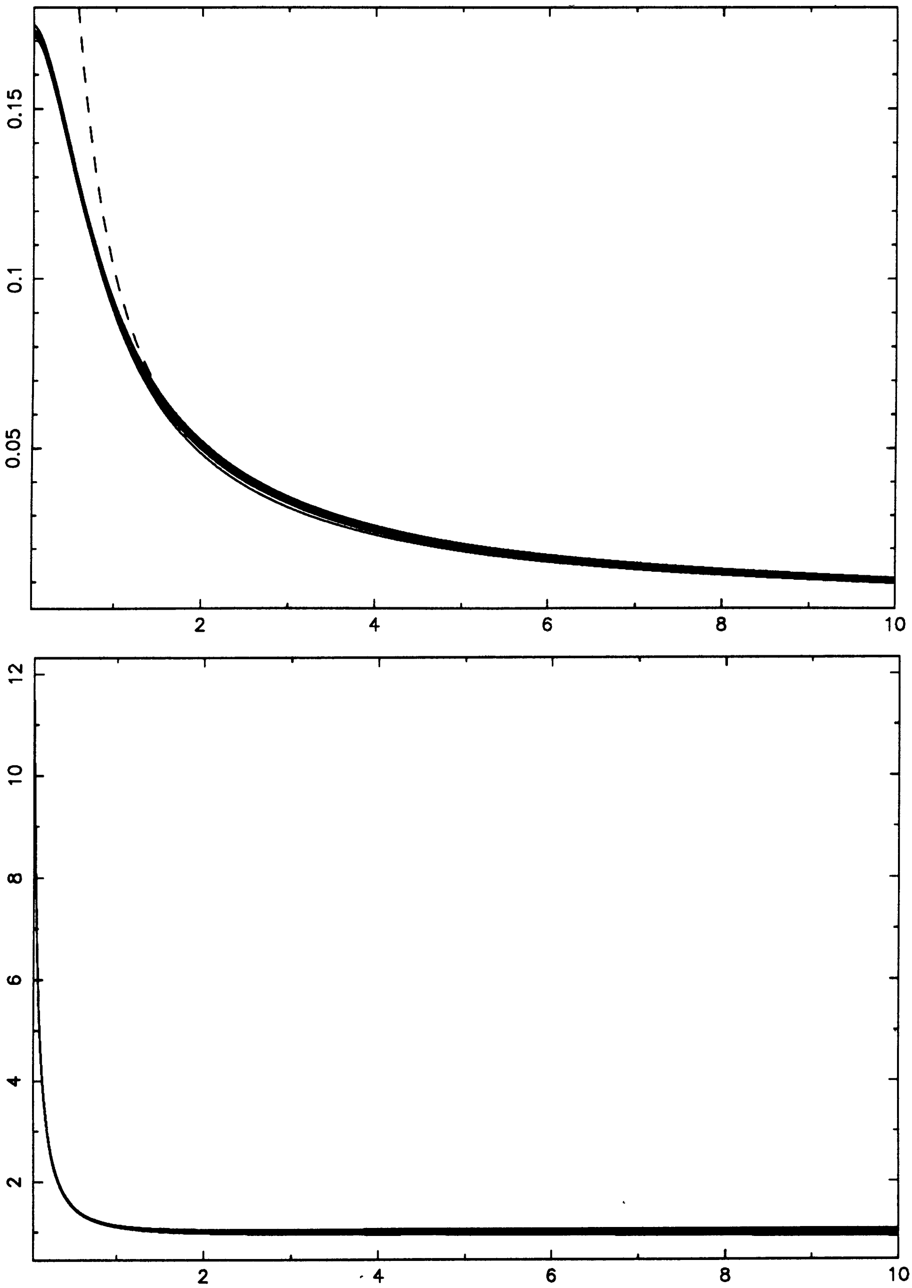
where $\{\Lambda_{ik}\}_{i=1,\dots,N_1,k=1,\dots,N_2}$ is a double sequence of i.i.d. random variables with distribution function $F(\lambda)/\lambda_0$, and $\Gamma_{ik} = E_{i1} + \cdots + E_{ik}$, where $\{E_{ik}\}_{i=1,\dots,N_1,k=1,\dots,N_2}$ is a double sequence of i.i.d. exponential variables with parameter 1.

In Figure 2(a)–(c) are the results of three such Monte Carlo evaluations. The stable parameter α ranges from 0.5 to 1.5, and the spectral distribution



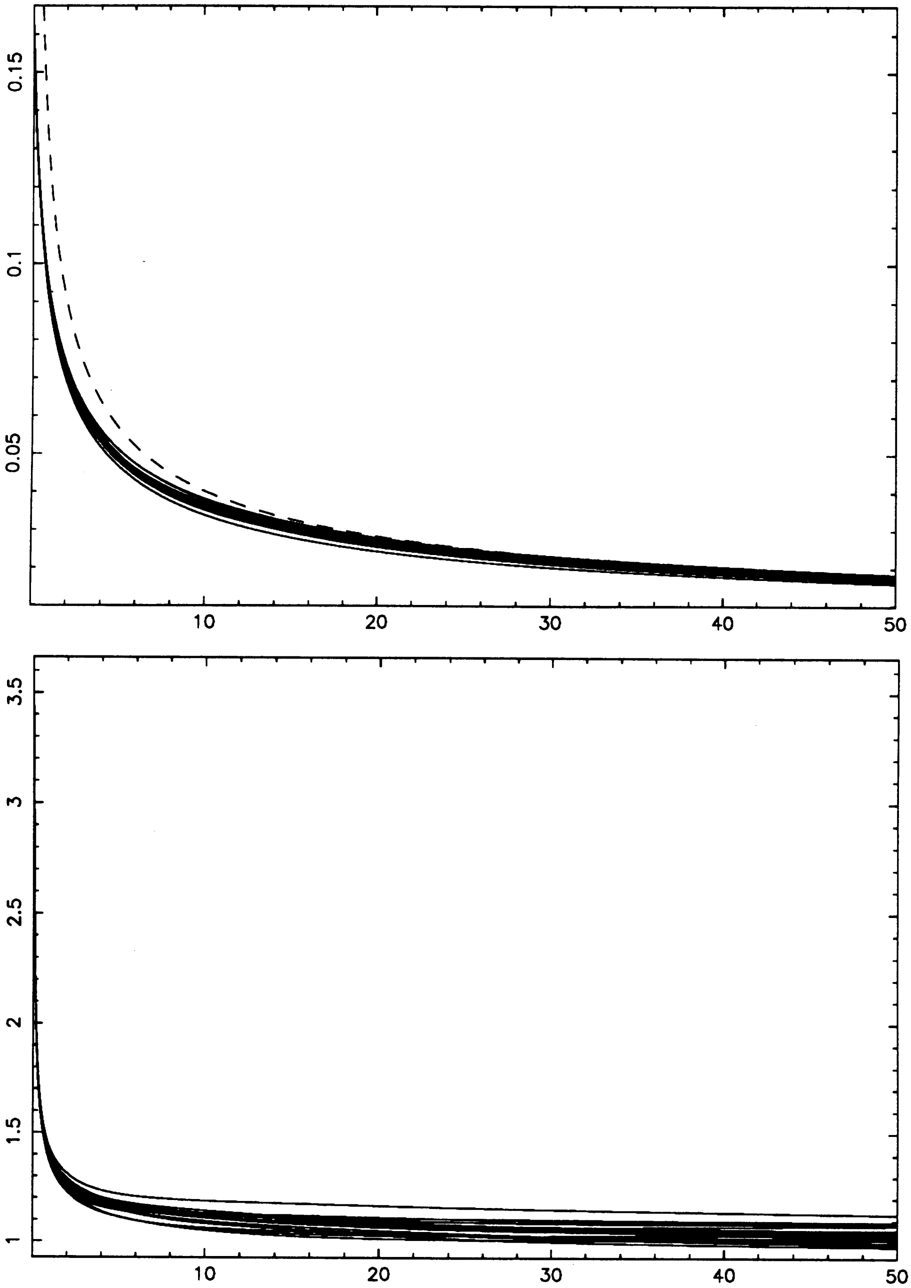
(a)

FIG. 2. (*a*) $\alpha = 0.5$; (*b*) $\alpha = 1.0$; (*c*) $\alpha = 1.5$. *Upper graphs*: 20 *Monte Carlo evaluations* $C_u^{5000,200}$ *of* EC_u (*full lines*) *and asymptotic formula* (*broken line*). *Lower graphs*: *ratio of Monte Carlo and asymptotic formulae.*



(b)

FIG. 2. *Continued.*



(c)

FIG. 2. *Continued.*

F is taken uniform on $[0,1]$ (i.e., the process is narrow band). The simulation parameters are $N_1 = 5000$ and $N_2 = 200$. Each graph shows the results of 20 evaluations of $C^{5000,200}$, so as to give an idea of the error of simulation at $N_1 = 5000$. Increasing the value of N_2 beyond 200 made no visible change to the results. Each set of evaluations took about 5 min on an IBM RS6000 RISC station, with no attempt made to write optimal Fortran.

The graphs compare the approximation inherent in Theorem 2.2, namely, $C_u \sim u^{-1}\lambda_1 C_\alpha/\pi$ to $C_u^{5000,200}$. The left-hand graphs show the functions themselves; the right-hand graphs show their ratio. The horizontal axes are different in each case because for smaller α the univariate distribution of the process has a heavier tail. Note that throughout the Monte Carlo evaluations start from the same point, since neither (4.1) nor (4.2) depends on α when $u = 0$. What is, of course, most impressive is how quickly the asymptotic expression for EC_u becomes accurate.

5. General bounds on EC_u. Our aim in this section is to prove the following result, in which the notation is identical to that of the previous sections.

5.1. THEOREM. *Let c denote a generic constant, dependent only on α, that may change from line to line. Then*

$$EC_u \le \begin{cases} c\lambda_1\{u^{-\alpha} + \lambda_0^{-1+1/\alpha}u^{-1}\}, & \text{if } 0 < \alpha < 1, \\ c(\lambda_1 + (\lambda \log \lambda)_\delta)u^{-1}, & \text{if } \alpha = 1, \\ c(\lambda_1 + \lambda_\alpha)u^{-\alpha} & \\ \quad + c\left(\lambda_\alpha^{2/\alpha}\lambda_0^{1-1/\alpha} + \lambda_1\lambda_0^{1/\alpha}\right)u^{-(\alpha+1)}, & \text{if } 1 < \alpha < 2. \end{cases} \tag{5.1}$$

Let X have Laplace transform $\exp(-\theta^{\alpha/2})$. *Fix $d > 0$ and set*

$$d_\alpha^{\alpha/2} = 2^{\alpha/2}\Gamma(1 - \alpha/2)\gamma_\alpha^\alpha, \tag{5.2}$$

$$\begin{aligned} K_0(\alpha) &= \inf_{x>1} x^{\alpha/2}\left(1 - F_{\alpha/2}(x)\right) \\ &= \inf_{x>1} x^{\alpha/2} P\{X > x\}, \end{aligned} \tag{5.3}$$

$$K_1(\alpha, d) = \int_0^d x^{\alpha/2}e^{-x}\,dx. \tag{5.4}$$

Then for each $d > 0$ and for all $u > d_\alpha^{1/\alpha}\sqrt{d\lambda_0}$ we have

$$EC_u \ge \left(\frac{K_0(\alpha)K_1(\alpha, d)}{\pi}\right)\lambda_1 u^{-\alpha}. \tag{5.5}$$

Note that we cannot expect a result like (5.5) to be true for all $u > 0$, since the right-hand side diverges as $u \to 0$. We could, of course, replace it by a function of the form $c\lambda_1/(1 + u^{-\alpha})$, which would then hold for all u, given an appropriate choice of c.

Note also that while the constants appearing in the lower bound (5.5) are explicit [modulo the fact that both $K_0(\alpha)$ and $K_1(\alpha, d)$ require some computational effort to be calculated numerically], this is not the case for the upper bounds, in which the constants are not explicitly known. We shall return to this issue in the following section.

PROOF. We start with the lower bound (5.5). In the notation of the proof of Theorem 2.2 we have that, for all $u > 0$,

$$EC_u \geq \frac{\lambda_1}{\pi\lambda_0} E\{\exp(-u^2/Z)\}, \tag{5.6}$$

where Z is a positive stable random variable with Laplace transform $Ee^{-\theta Z} = \exp(-\theta^{\alpha/2}\sigma^{\alpha/2})$, and where

$$\begin{aligned} \sigma^{\alpha/2} &:= 2^{\alpha/2}\Gamma(1-\alpha/2)\gamma_\alpha^\alpha\lambda_0 \\ &= d_\alpha^{\alpha/2}\lambda_0 \end{aligned}$$

[cf. (1.4) and (2.3)].

Choose now $d > 0$ and consider the expectation in (5.6). With X as defined in the statement of the theorem and σ as before, we have

$$\begin{aligned} E(\exp(-u^2/Z)) &= \int_0^\infty P\{Z > u^2/x\} e^{-x}\,dx \\ &\geq \int_0^{u^2/\sigma} P\{Z > u^2/x\} e^{-x}\,dx \\ &= \int_0^{u^2/\sigma} P\{X > u^2\sigma^{-1}/x\} e^{-x}\,dx \\ &\geq K_0(\alpha) u^{-\alpha}\sigma^{\alpha/2} \int_0^{u^2/\sigma} x^{\alpha/2} e^{-x}\,dx \\ &\geq K_0(\alpha) u^{-\alpha}\sigma^{\alpha/2} \int_0^{d} x^{\alpha/2} e^{-x}\,dx \\ &= K_0(\alpha) K_1(\alpha, d) u^{-\alpha}\sigma^{\alpha/2}, \end{aligned}$$

where the second to the last line relies on the assumption that $u > d\sqrt{d_\alpha\lambda_0}$. Substituting into (5.6) establishes the required inequality (5.5).

We now turn to the proof of the upper bounds (5.1) and commence as in the proof of Theorem 2.2 by noting that

$$EC_u \leq \frac{ES_u^1}{\pi} + \frac{ES_u^2}{\pi},$$

where the S_u^i are defined there. Note that

$$ES_u^1 \leq \frac{\lambda_1}{\lambda_0} E\{\exp(-u^2/Z)\},$$

with Z as before. Lemma 2.6, with $\beta = 0$, then gives that $ES_u^1 \le c\lambda_1 u^{-\alpha}$, which is the common first term in all of the upper bounds of (5.1). The second terms come from differing bounds for ES_u^2, which we obtain with three somewhat different arguments.

CASE 1: $0 < \alpha < 1$. Given α, choose m_α such that $E\{\sum_{k=m_\alpha}^{\infty} \Gamma_k^{-1/\alpha}\} < \infty$. Use (2.8) with $K = m_\alpha$ and the argument just used to bound ES_u^1 to see that

$$ES_u^2 \le cm_\alpha \lambda_1 u^{-\alpha} + E\{W^{1/2} X^{-1/2} \exp(-u^2/2AX)\},$$

where $A = 2\gamma_\alpha^2 \lambda_0^{2/\alpha}$, $X = \sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha}$ and $W = \sum_{k=m_\alpha}^{\infty} \Lambda_k^2 \Gamma_k^{-1/\alpha}$.

To bound the expectation, use the fact that $\sup_{x \ge 0} x^{-1/2} \exp(-1/x)$ is finite to obtain the bound $c(\lambda_1/\lambda_0)\lambda_0^{1/\alpha} u^{-1}$, as required.

CASE 2: $1 < \alpha < 2$. We start with (2.9) and set $K = 2$ there. This gives us

$$\begin{aligned} ES_u^2 \le \lambda_0 \left[E\left(\sum_{k=2}^{\infty} \Lambda_k^2 \left| \Gamma_k^{-2/\alpha} - k^{-2/\alpha} \right| \right)^{\alpha/2} \right]^{1/\alpha} \\ \times \left[E\left(Y^{-\alpha/2(\alpha-1)} \exp(-u^2/Y) \right) \right]^{(\alpha-1)/\alpha} \\ + E\left(\sum_{k=2}^{\infty} \Lambda_k^2 k^{-2/\alpha} \right)^{1/2} E\left(X^{-1/2} e^{-u^2/AX} \right), \end{aligned}$$

where Y is a $S\frac{1}{2}\alpha S$ whose distribution is independent of the spectral parameters λ_i. By Lemmas 2.6 and 2.7 the first term here is bounded above by $c\lambda_\alpha u^{-\alpha}$, while the second term is bounded by $c(\lambda_\alpha^{2/\alpha}\lambda_0^{1-1/\alpha} + \lambda_1 \lambda_0^{1/\alpha}) u^{-(\alpha+1)}$. Putting all the parts together completes the proof for this case.

CASE 3: $\alpha = 1$. This time we start with (2.8), with $K = 2$, and apply (2.17) to see that

$$ES_u^2 \le c\lambda_0 u^{-1} E\left(\sum_{k=2}^{\infty} \Lambda_k^2 \Gamma_k^{-2} \right)^{1/2}.$$

But

$$\begin{aligned} E\left(\sum_{k=2}^{\infty} \Lambda_k^2 \Gamma_k^{-2} \right)^{1/2} &\le E\left(\Lambda_2 \Gamma_2^{-1} \right) + E\left(\sum_{k=3}^{\infty} \Lambda_k^2 \Gamma_k^{-2} \right)^{1/2} \\ &\le \frac{c_1 \lambda_1}{\lambda_0} + \frac{c_2 (\lambda \log \lambda)_\delta}{\lambda_0}, \end{aligned}$$

the factor of $(\lambda \log \lambda)_\delta$ coming from Lemma 3 of Marcus (1989). Collecting terms completes the proof. □

6. Some numerical bounds on EC_0. In this final section we shall present some results that give numerical bounds for EC_0. This was actually the starting point of our research, but the results that we have been able to obtain have been rather disappointing, and so we present them without their (rather lengthy) proofs. The problem is that the gaps between the upper and lower bounds are generally quite large.

Here is our result, that covers the zero level. The proofs involve basically the same techniques used by Marcus (1989) to get his bounds, with much more care taken to keep track of the constants. Details of the arguments involved can be found in Gadrich (1991).

6.1. THEOREM. *Let X be a stationary harmonisable $S\alpha S$ process satisfying the usual conditions. The following expressions provide lower bounds for EC_0.*

$\alpha < 1$:

$$\frac{\lambda_1}{\pi\lambda_0\Gamma^{1/\alpha}(1-\alpha/2)} \sup_{1/(1+\alpha)<t<1} \left\{ \left[\frac{\Gamma(1-((1-t)/\alpha t))}{\Gamma(1+((t-1)/2t))} \right]^{t/(t-1)} \times \frac{1}{\Gamma^{1/(t-1)}(1+((t-1)/\alpha))} \right\};$$

$$\alpha = 1: \quad \frac{(\lambda \log \lambda)_\delta}{\lambda_0 e \pi^{3/2}} \sup_{1<t<2} \left\{ \frac{K^{t/(t-1)}(t/(t-1))\Gamma^{1/(t-1)}((3-t)/2)}{\Gamma^{1/(t-1)}(2-t)} \right\},$$

where

$$K(s) := \int_0^1 \exp\left\{ -\sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=0}^{k-1} \frac{\exp\{-k\lambda^{-s}\}(k\lambda^{-s})^n}{n!} \right\} d\lambda;$$

$$\alpha > 1: \quad \frac{\lambda_\alpha^{1/\alpha}}{\pi\lambda_0^{1/\alpha}} \sup_{1<t<1+\alpha} \left\{ \left[\frac{\Gamma(1-((t-1)/\alpha t))}{\Gamma(1-((t-1)/2t))} \right]^{t/(t-1)} \times \left[\frac{\Gamma(1-((t-1)/2))}{\Gamma(1-((t-1)/\alpha))} \right]^{1/(t-1)} \right\}.$$

Upper bounds are as follows:

$\alpha < 1$:

$$\frac{\lambda_1 \sum_{n=1}^{\infty} n^{-1/\alpha}}{\lambda_0 \pi} \min_{k \geq 2/\alpha} \left\{ k^{1/\alpha} + \Gamma(1+1/\alpha) e^{1/\alpha} \left[2 + \sum_{n \geq k} \frac{1}{(\alpha n - 1)2^{n\alpha - 1}} \right] \right\},$$

$$\alpha = 1: \quad \frac{2\lambda_1}{\pi\lambda_0} + \frac{4e(\lambda \log \lambda)_\delta}{\pi\lambda_0} \left\{ 2 + \sum_{k=2}^{\infty} \frac{1}{(k-1)2^{k-1}} \right\},$$

$$\alpha > 1: \quad \frac{\lambda_1}{\lambda_0 \pi} + \frac{\lambda_\alpha^{1/\alpha}\Gamma(1+1/\alpha)\Gamma(1-1/\alpha)\Gamma^{1/\alpha}(1-\alpha/2)}{\pi^{3/2}\lambda_0^{1/\alpha}}.$$

Given the complicated form of the results, we presume it is clear why we do not wish to include the derivations.

Nevertheless, the results are of interest insofar as they are the only way we know of to provide general numerical bounds for EC_0. What is somewhat disappointing, however, is the gap between the upper and lower bounds for specific examples.

For example, if we take the example used in the Monte Carlo studies in Section 3 of this the paper, that is, the process with uniform spectral density on [0, 1], then what we obtain from the preceding bounds is that

$$\begin{aligned} 0.1421 \le EC_0 &\le 14.608, &\text{if } \alpha = 0.5,\\ 0.0116 \le EC_0 &\le 4.293, &\text{if } \alpha = 1.0,\\ 0.1728 \le EC_0 &\le 0.721, &\text{if } \alpha = 1.5. \end{aligned}$$

Given, from the Monte Carlo evaluation of Section 4, that the true value of EC_0 is around 0.175, these bounds, while correct, are not particularly encouraging.

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