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# PRODUCTION CONTROL IN A FAILURE-PRONE MANUFACTURING SYSTEM: DIFFUSION APPROXIMATION AND ASYMPTOTIC OPTIMALITY

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We consider a problem of controlling the production rate of a single machine, single product, stochastic manufacturing system in order to minimize the total discounted inventory/backlog costs. The demand has two components: one is deterministic with constant rate $d$ and the other is stochastic with random demand batches. Under heavy loading (or heavy traffic) conditions, that is, when the average production capacity is close to the average demand, the control problem is approximated by a singular stochastic control problem. The approximate problem can be solved explicitly. The solution is then interpreted in terms of the actual manufacturing system and a control policy for this system is derived. We prove that the resulting policy is nearly optimal under the heavy traffic condition. This policy is characterized by a single critical level $z_0$. The commodity should be produced only when inventory is less than or equal to $z_0$: The production rate is maximal if the inventory is less than $z_0$ and equal to the deterministic component $d$ of the demand rate if the inventory is equal to $z_0$.

**1. Introduction.** We study a manufacturing system $S$ consisting of a single machine producing a single part type. The machine is unreliable. It has two states: up and down. If it is up, then its production rate can be adjusted to be any value between zero and the maximum production rate $r$. On the other hand, if the machine is down, then the production rate is zero. The durations of up and down periods are sequences of i.i.d. random variables with arbitrary distributions. The demand facing the manufacturing system has two components: a deterministic one with a constant rate $d$ and a

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stochastic component with random demand batches. The objective is to control the production rate during the up periods so that the total discounted inventory/backlog costs can be kept as low as possible.

Production systems with unreliable machines were studied by Kimemia and Gershwin [21]. A system similar to ours but with a constant demand rate and exponential distribution for machine up and down periods was studied by Akella and Kumar [1] and Bielecki and Kumar [3]. They showed that the optimal control policy is of a threshold type. Furthermore, they succeeded in deriving a closed form solution for the threshold value known also as hedging point. Unfortunately, more complex systems cannot be solved in closed form. We therefore turn to the aid of the asymptotic approach based on the idea of diffusion approximation. So far, it has been applied mainly to various queueing systems operating under heavy traffic conditions. The essential features of this approach can be described as follows. Based on the existing heavy traffic theorems for queueing models, the control problem for a given physical system is approximated by a limiting optimal control problem involving Brownian motion. This control problem is easier to analyze and in many cases is explicitly solvable. The solution is then interpreted in the original terms and a certain control policy is derived for the original system.

There are a number of papers in which the controlled diffusions have been used as models for real systems. The literature on this subject can be classified according to the level of rigor in the justification of their use.

In works of the first type, real processes are replaced by diffusion processes and then the resulting problems of controlled diffusions are solved. The optimal policies obtained are then interpreted in terms of the original system. Justification of the procedure employed is based usually on mere intuition or simulation. Earliest examples of such works are those of Harrison and Wein [15, 16] and Wein [36], who analyzed queueing network scheduling problems (references in [15], [16] and [36] contain the list of literature on this subject).

Works of the second type such as [25], [27] and [28] begin with a sequence of systems whose limit is a controlled diffusion problem. The traffic intensities of the systems in the sequence converge to the critical intensity of 1. Justification of the use of controlled diffusions is provided by proving convergence of the resulting sequence of value functions to that of the limiting system. This convergence together with the solution of the limiting problem enables one to construct a sequence of asymptotically optimal policies, defined to be those for which the difference between the associated cost and the value function converges to zero as traffic intensity approaches its critical value.

The primary advantage of the first stream of research is that considerable insight can be derived from the closed form solution of the limiting problem, whereas the main weakness is that a rigorous justification of convergence is not obtained. The second approach, on the other hand, provides rigorous justification of convergence, but a closed form solution is not obtained. Instead, a Markov chain approximation technique is used [26] to obtain numerical results, and therefore it is difficult to obtain insight from this approach.

The current paper combines these two complementary approaches in that insight from the solution of the singular control problem is obtained and rigorous justification is provided. Although the system considered in this paper is much simpler than those investigated in [15], [16], [36] and [25], [27], [28], it is of considerable value in studying failure-prone manufacturing systems. Furthermore, the explicit solution of the limiting problem can be obtained for this model. In addition, using the results in [1], one has an opportunity to compare the policy provided by the diffusion approximation with the optimal one.

The main purpose of studying diffusion approximations is to develop a reasonable policy for a given real system with traffic intensity close to 1. However, there are several issues that one faces in such procedures. First, if one imbeds a given system into a sequence of systems in heavy traffic, then an asymptotically optimal policy corresponding to the given system depends on the limiting diffusion model whose parameters are not uniquely determined by those of the original system (usually drift has an arbitrary value). An appropriate way to address this problem would be to obtain an asymptotically optimal policy that is expressed only in terms of the system's parameters. Second, asymptotic analyses of systems in heavy traffic usually employ the idea of rescaling of the state space. The rescaling procedure introduces a difference of several orders of magnitude between the cost functional of the original system and that of the representation of the original system in the sequence. Therefore, it is difficult to estimate the absolute error of the constructed control for the original system even if one has asymptotic optimality for the sequence of systems.

In this paper we obtain asymptotically optimal policies for the sequence of manufacturing systems in the manner suggested; that is, we express the policies only in terms of the parameters of the original system. We also suggest using the relative error rather than the absolute error to evaluate the quality of approximation, because the relative errors for the real system and for its representation in the sequence do coincide.

The paper has the following structure. We start with the formulation of the optimal control problem for the original system $S$ (Section 2). It is assumed that the average production capacity exceeds the average demand. Traffic intensity is defined as the ratio of these two quantities. For the system $S$, this ratio is assumed to be close to 1; that is, it operates under the heavy traffic condition. In order to perform an asymptotic analysis, we construct a sequence $\{S_\varepsilon\}$ of systems similar to the original one, but with its parameters being functions of $\varepsilon$. The original system $S$ is then identified with $S_{\varepsilon_0}$ for some $\varepsilon_0$. The small parameter $\varepsilon$ indexes the traffic intensity, so that it converges to 1 as $\varepsilon \to 0$.

We formulate the main results of the paper in Section 3. The optimal control problem for $S_\varepsilon$ is related to that of the limiting system $S_W$, which is a singular controlled Brownian motion with properly defined drift and variance. In the case considered, the general nature of the solution to the problem is similar to that of the other papers on singular control [14], [17]–[20]. There exists a level $z^*$, such that the optimal process $X$ is a Brownian motion on $(-\infty, z^*]$ with reflection at the point $z^*$. The optimal control functional $L^*$ is the one that induces the reflection at $z^*$; that is, it coincides with the local time of $X$ at $z^*$. We provide an explicit formula for the optimal threshold level $z^*$.

It is proved that as $\varepsilon \to 0$, the lim inf of the cost functions for $\{S_\varepsilon\}$ is bounded below by the optimal cost for the limiting system $S_W$. Using an explicit expression for the optimal level $z^*$, we construct a threshold control policy for $S_\varepsilon$ such that the corresponding values for the cost functionals converge to this lower bound as $\varepsilon$ approaches zero. In other words, these control policies are proved to be asymptotically optimal. From these, a nearly optimal control policy for the original system $S$ is constructed.

In Section 4, we prove a theorem concerning an application of the asymptotic results to the original system $S$. Because the system $S$ is identified with $S_{\varepsilon_0}$ for some $\varepsilon_0$, we use a threshold control policy for $S$. In our problem, we have an explicit formula for the threshold value. Using it, we show that this value can be expressed in terms of the original system parameters. Furthermore we compare numerically the optimal threshold values found in [1] with the corresponding ones provided by our approach. Our comparison shows that if two of the parameters, namely, $\delta$ and $\alpha$ (see Section 7), are close to zero, then relative error is small. The parameter $\delta$ is associated with the traffic intensity of $S$. When $\delta$ is small, the system is in heavy traffic. The parameter $\alpha$, on the other hand, is the proportion of time the machine is up in the long run. The smaller is the value of $\alpha$, the "further" is the original system from the deterministic one. In this case the Brownian motion is a good approximation for the underlying stochastic process.

The limiting system $S_W$ and the corresponding singular control problem are treated in Section 5. Sections 6 and 7 are devoted to the proofs of Theorems 2 and 3 stated in Section 3.

**2. System description.** We consider a failure-prone manufacturing system $S$ producing a single part type. Let $\{\alpha_i\}_{i \geq 1}$ and $\{\beta_i\}_{i \geq 1}$ be two sequences of i.i.d. random variables representing successive up and down periods for the machine. Another two sequences of i.i.d. random variables $\{\xi_i\}_{i \geq 1}$ and $\{\eta_i\}_{i \geq 1}$ represent, respectively, successive interarrival times of demand batches and their sizes. We assume that all random variables are defined on a complete probability space $(\Omega, \mathscr{F}, P)$ and have a moment of the order $2 + q$, $q > 0$. Let us introduce the parameters

$$E\alpha_1 = a^{-1}, \qquad E\beta_1 = b^{-1}, \qquad E\xi_1 = \lambda^{-1}, \qquad E\eta_1 = \mu^{-1},$$

$$\operatorname{Var} \alpha_1 = \sigma_1^2, \quad \operatorname{Var} \beta_1 = \sigma_2^2, \quad \operatorname{Var} \xi_1 = \sigma_3^2, \quad \operatorname{Var} \eta_i = \sigma_4^2.$$

The surplus $X(t)$ is defined as the difference between the cumulative production and demand up to time $t$. A positive surplus means excessive inventory, whereas a negative surplus corresponds to a backlog. To describe the dynamics of $X = (X(t),\ t \geq 0)$, we define the processes

$$G(t) = \sum_{i \geq 0} I\left( \sum_{j=0}^{i} (\alpha_j + \beta_j) \leq t < \sum_{j=0}^{i} (\alpha_j + \beta_j) + \alpha_{i+1} \right), \tag{2.1}$$

$$A(t) = \max\{j \geq 0: \xi_1 + \cdots + \xi_j \leq t\}, \tag{2.2}$$

where $\alpha_0 = \beta_0 = 0$ and $I(F)$ is the indicator function of the set $F$. The process $G(t)$ is the "indicator" process for the machine state: $G(t) = 1$ when the machine is up and $= 0$ when the machine is down. The renewal process $A(t)$ describes the cumulative number of arrivals of the demand batches.

Given the initial inventory level $x$ and production rate $p(s)$, the surplus process can be represented in the form

$$X(t) = x + \int_0^t G(s)p(s)\,ds - td - \sum_{i=1}^{A(t)} \eta_i. \tag{2.3}$$

The production rate $p(s)$ in (2.3) is the control variable and the following definition specifies the restrictions it must satisfy.

DEFINITION 1. A real-valued process $(p(t),\ t \geq 0)$ is called an *admissible policy*, if the following conditions hold:

(i) the process $p(t)$ is adapted to the filtration $\mathscr{F}_t = \sigma\{G(s), H(s),\ s \leq t\}$, where $H(s) = \sum_{i=1}^{A(s)} \eta_i$.

(ii) $0 \leq p(t) \leq r$.

We use $\mathscr{P}$ to denote the set of admissible controls. It should be noted that the dynamics of $X(t)$ described by (2.3) allows us to omit a formal requirement on $p(t)$ to be equal to zero whenever the machine is down.

Let $C^+$ be the unit cost of holding excess inventory and $C^-$ be the unit cost of backlog per unit time. Set $h(x) = C^+x^+ + C^-x^-$, where $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$. The objective is to minimize the expected discounted cost

$$J(x, p) = E\int_0^\infty e^{-\rho t} h(X(t))\,dt \tag{2.4}$$

over all admissible policies $p \in \mathscr{P}$, where $\rho > 0$ is the given discount rate. We denote the corresponding value function by

$$v(x) = \inf_{p \in \mathscr{P}} J(x, p). \tag{2.5}$$

A closed form solution for the optimal control problem of (2.3)–(2.5) is very difficult if not impossible to obtain. In view of this, we find an approximate solution by means of an asymptotic analysis.

We assume that the system $S$ is operating in heavy traffic; that is, the parameter

$$\delta = \left(rb(a+b)^{-1} - d - \lambda/\mu\right)(d+\lambda/\mu)^{-1} \tag{2.6}$$

is close to zero. Note that $rb/(a+b)$ is the expected long run production per unit time when the maximum production rate $r$ is used and $d + \lambda/\mu$ is the expected demand per unit time. This condition requires the average demand to be close to the system capacity and the *traffic intensity*, defined as $rb(a+b)^{-1}/(d+\lambda/\mu) = 1 + \delta$, to be close to 1.

Consider a family of systems $\{S_\varepsilon\}$, $\varepsilon \to 0$, indexed by parameter $\varepsilon$ belonging to a countable set $\mathscr{E}$. (Setting $\varepsilon = n^{-1/2}$, one gets the conventional notation for heavy traffic limit theorems.) More precisely, each system is described by its maximal production rate $r_\varepsilon$, discount rate $\rho_\varepsilon$ and families of i.i.d. random variables $\{\alpha_i^\varepsilon\}$, $\{\beta_i^\varepsilon\}$, $\{\xi_i^\varepsilon\}$, $\{\eta_i^\varepsilon\}$ with corresponding mean and variance parameters indexed by $\varepsilon$ as well. Our main assumption on the family $\{S_\varepsilon\}$ is the following: There exist constants $\hat{a}, \hat{b}, \hat{\mu}, \hat{\lambda}, \hat{d}, \hat{r}, \hat{\sigma}_i^2$, $i = 1,2,3,4$; $c > 0$, $\gamma > 0$ and $q > 0$ such that

$$\begin{aligned} &a_\varepsilon \to \hat{a}, \quad b_\varepsilon \to \hat{b}, \quad \mu_\varepsilon \to \hat{\mu}, \quad \lambda_\varepsilon \to \hat{\lambda}, \quad d_\varepsilon \to \hat{d}, \quad r_\varepsilon \to \hat{r}, \\ &\sigma_{1\varepsilon}^2 \to \hat{\sigma}_1^2, \quad \sigma_{2\varepsilon}^2 \to \hat{\sigma}_2^2, \quad \sigma_{3\varepsilon}^2 \to \hat{\sigma}_3^2, \quad \sigma_{4\varepsilon}^2 \to \hat{\sigma}_4^2, \quad \varepsilon \to 0; \end{aligned} \tag{2.7}$$

$$c_\varepsilon \equiv \varepsilon^{-1}\left(\frac{r_\varepsilon b_\varepsilon}{a_\varepsilon + b_\varepsilon} - d_\varepsilon - \frac{\lambda_\varepsilon}{\mu_\varepsilon}\right) \to c, \qquad \varepsilon \to 0; \tag{2.8}$$

$$\gamma_\varepsilon \equiv \rho_\varepsilon \varepsilon^{-2} \to \gamma, \qquad \varepsilon \to 0. \tag{2.9}$$

$$\begin{aligned} &\sup_\varepsilon E(\alpha_1^\varepsilon)^{2+q} \le \infty, \qquad \sup_\varepsilon E(\beta_1^\varepsilon)^{2+q} \le \infty, \\ &\sup_\varepsilon E(\xi_1^\varepsilon)^{2+q} \le \infty, \qquad \sup_\varepsilon E(\eta_1^\varepsilon)^{2+q} \le \infty. \end{aligned} \tag{2.10}$$

Note that according to (2.8), the traffic intensity of $S_\varepsilon$ converges to 1 in the order of $\varepsilon$.

Let $p^\varepsilon(\cdot)$ be an admissible policy for the system $S_\varepsilon$ and $X^\varepsilon(\cdot)$ be the corresponding solution to (2.1)–(2.3). For each initial position $x$ in (2.3) and each $\varepsilon > 0$, let $J^\varepsilon$ be the following cost functional associated with $p^\varepsilon(\cdot)$:

$$\begin{aligned} J^\varepsilon(x, p^\varepsilon(\cdot)) = \varepsilon^3 E\int_0^\infty \exp(-\rho_\varepsilon t)\big(C^+(X^\varepsilon(t))^+ \\ + C^-(X^\varepsilon(t))^-\big)\,dt. \end{aligned} \tag{2.11}$$

Define also

$$v^\varepsilon(x) = \inf_{p \in \mathscr{P}_\varepsilon} J^\varepsilon(x, p(\cdot)), \tag{2.12}$$

where $\mathscr{P}_\varepsilon$ is the set of admissible controls for the system $S_\varepsilon$.

Our original system dynamics is identified with that of $S_{\varepsilon_0}$ for some fixed $\varepsilon_0 \in \mathscr{E}$, that is,

$$a_{\varepsilon_0} = a, \quad b_{\varepsilon_0} = b, \quad \mu_{\varepsilon_0} = \mu, \quad \lambda_{\varepsilon_0} = \lambda, \quad d_{\varepsilon_0} = d, \quad r_{\varepsilon_0} = r,$$
$$\rho_{\varepsilon_0} = \rho, \quad \sigma^2_{i\varepsilon_0} = \sigma_i^2, \qquad i = 1,2,3,4. \tag{2.13}$$

From (2.6) and the definition (2.8) of $c_\varepsilon$, we have

$$\delta \equiv \delta_{\varepsilon_0} = c_{\varepsilon_0}\varepsilon_0\left(d_{\varepsilon_0} + \lambda_{\varepsilon_0}/\mu_{\varepsilon_0}\right)^{-1}. \tag{2.14}$$

For any admissible control $p$ ($p \equiv p^{\varepsilon_0}$),

$$J(x,p) \equiv \varepsilon_0^{-3} J^{\varepsilon_0}(x, p^{\varepsilon_0}) \tag{2.15}$$

and, consequently,

$$v(x) \equiv \varepsilon_0^{-3} v^{\varepsilon_0}(x). \tag{2.16}$$

Owing to (2.13), (2.15), and (2.16), we will refer to $S_{\varepsilon_0}$ as a system that corresponds to $S$, while noting that they are not identical systems. Our goal is to draw conclusions for the original system $S$ by using (2.15) and (2.16) and a solution of the limiting optimal control problem. For this we need the following definitions of asymptotically optimal controls for the sequence $\{S_\varepsilon\}$ and a nearly optimal control for $S$; see Section 4 on motivation for these definitions.

DEFINITION 2. A sequence of controls $\hat{p}^\varepsilon$, $\varepsilon \in \mathscr{E}$, is said to be *asymptotically optimal in the sense of absolute error* for the sequence $\{S_\varepsilon\}$ if

$$\lim_{\varepsilon \to 0} \left| v^\varepsilon(x\varepsilon^{-1}) - J^\varepsilon(x\varepsilon^{-1}, \hat{p}^\varepsilon) \right| = 0$$

and *asymptotically optimal in the sense of relative error* if

$$\lim_{\varepsilon \to 0} \left| \frac{v^\varepsilon(x\varepsilon^{-1}) - J^\varepsilon(x\varepsilon^{-1}, \hat{p}^\varepsilon)}{v^\varepsilon(x\varepsilon^{-1})} \right| = 0. \tag{2.17}$$

DEFINITION 3. Let $S_{\varepsilon_0}$ corresponds to $S$ and $p^{\varepsilon_0}$ be a policy for $S_{\varepsilon_0}$. We call a policy $p$ for $S$ an image of $p^{\varepsilon_0}$ if

$$\left| \frac{J(x,p) - v(x)}{v(x)} \right| \equiv \left| \frac{v^{\varepsilon_0}(x) - J^{\varepsilon_0}(x, p^{\varepsilon_0})}{v^{\varepsilon_0}(x)} \right|.$$

If $p^{\varepsilon_0}$ is an element of the sequence of asymptotically optimal policies, then following the usual practice in the literature (e.g., [25], [27], [28]), its image is loosely referred to as a *nearly or approximately optimal policy* for $S$.

**3. Summary of results.** As a limiting problem for the family of systems $\{S_\varepsilon\}$ as $\varepsilon \to 0$, we consider a system $S_W$ that is a singular controlled Brownian motion with drift $c$ equal to the right-hand side of (2.8) and variance

$$\sigma^2 = \hat{r}^2\left(\hat{a}^3\hat{b}(\hat{a}+\hat{b})^{-3}\hat{\sigma}_1^2 + \hat{a}\hat{b}^3(\hat{a}+\hat{b})^{-3}\hat{\sigma}_2^2\right) + \hat{\lambda}^3\hat{\mu}^{-2}\hat{\sigma}_3^2 + \hat{\lambda}\hat{\sigma}_4^2, \tag{3.1}$$

where $\hat{r}, \hat{a}, \hat{b}, \ldots$ are given by (2.7). In this problem, we start with a process $(W(t),\ t \geq 0)$, $W(0) = 0$, defined on $(\Omega, \mathscr{F}, P)$, which is a $(c, \sigma^2)$ Brownian motion with respect to a family of $\sigma$-fields $(\mathscr{F}_t,\ t \geq 0)$. A control policy is defined to be a nonnegative, right-continuous nondecreasing process $L = (L(t),\ t \geq 0)$ adapted to the filtration $\mathscr{F}_t$ such that $L(0) \geq 0$. We say that $L$ is an *admissible policy* if

$$E\int_0^\infty e^{-\gamma t}\,dL(t) < \infty. \tag{3.2}$$

The dynamics of the system are given by

$$X(t) = x + W(t) - L(t), \tag{3.3}$$

where $x$ is the initial state. With each control functional $L$, we associate the cost

$$J_x(L) = E\int_0^\infty e^{-\gamma t}h(X(t))\,dt, \tag{3.4}$$

where the cost function $h(x)$ is defined in the Section 2. The objective is to find $V(x)$ and $L^*$ such that

$$V(x) = J_x(L^*) = \inf_L J_x(L). \tag{3.5}$$

Applying techniques similar to the ones developed in [14] and [17]–[20], we prove the following result.

THEOREM 1. *The optimal control policy for the problem* (3.2)–(3.5) *is given by*

$$L^*(t) = \sup_{s\leq t}\left[x + W_s - z^*\right]^+, \qquad t \geq 0, \tag{3.6}$$

*where*

$$z^* = \frac{\sigma^2}{c + \sqrt{c^2 + 2\sigma^2\gamma}}\ln\left(\frac{C^+ + C^-}{C^+}\right). \tag{3.7}$$

We can describe the optimal control $L^*$ and the associated process $X^*$ as follows. If $x > z^*$, take $L^*(0) = x - z^*$ [so that $X^*(0) = z^*$]; otherwise $L^*(0) = 0$. After time 0, $L^*$ increases by the minimal amount sufficient to achieve $X^*(t) \leq z^*$. Under this policy, $X^*$ is a $(c, \sigma^2)$ Brownian motion on $(-\infty, z^*)$ with reflecting barrier at $z^*$. Thus $L^* - L^*(0)$ is the local time of $X^*$ at $z^*$. Process $L^*$ may have a jump at $t = 0$, but it is continuous and singular on $(0, \infty)$. The latter means that the set of time points where $L^*$ increases has zero Lebesgue measure.

In the next set of results, we specify the relation between the structure of the optimal control for the limiting system and asymptotically optimal controls for the sequence $\{S_\varepsilon\}$. Before we do that, however, we provide a heuristic explanation that reveals the connection between our original system $S$ and the singular stochastic control problem (3.2)–(3.5). In (3.3), we can view $W(t)$ as $W_1(t) - W_2(t)$, where $W_2(t)$ is a Brownian motion approximation of the demand process in the system $S$ and $W_1(t)$ is a Brownian motion approximation of the maximum possible cumulative production process. Finally, $L(t)$ stands for the cumulative underproduction; that is, the difference between the maximal cumulative production and actual cumulative production associated with a feasible policy. Because $W_1(t)$ and $W_2(t)$ are known, finding $L^*(t)$ satisfying (3.5) corresponds to finding a nearly optimal production policy for $S$.

We can now state the main results of the paper.

THEOREM 2. *For any $x$ and any admissible control $p^\varepsilon$,*

$$V(x) \le \liminf_{\varepsilon\to 0} J^\varepsilon(x\varepsilon^{-1}, p^\varepsilon).$$

Put

$$\begin{aligned}\sigma_\varepsilon^2 = {} & r_\varepsilon^2\left(a_\varepsilon^3 b_\varepsilon(a_\varepsilon + b_\varepsilon)^{-3}\sigma_{1\varepsilon}^2 + a_\varepsilon b_\varepsilon^3(a_\varepsilon + b_\varepsilon)^{-3}\sigma_{2\varepsilon}^2\right)\\ & + \lambda_\varepsilon^3\mu_\varepsilon^{-2}\sigma_{3\varepsilon}^2 + \lambda_\varepsilon\sigma_{4\epsilon}^2\end{aligned} \tag{3.8}$$

and let $z_\varepsilon^*$ be defined as

$$z_\varepsilon^* = \frac{\sigma_\varepsilon^2}{c_\varepsilon + \sqrt{c_\varepsilon^2 + 2\sigma_\varepsilon^2\gamma_\varepsilon}}\ln\left(\frac{C^+ + C^-}{C^+}\right), \tag{3.9}$$

where $c_\varepsilon$ and $\gamma_\varepsilon$ are given by (2.8) and (2.9), respectively. Consider the policy

$$\hat p^\varepsilon(t) = r_\varepsilon I\left(X^\varepsilon(t) < \varepsilon^{-1}z_\varepsilon^*\right) + d_\varepsilon I\left(X^\varepsilon(t) = \varepsilon^{-1}z_\varepsilon^*\right). \tag{3.10}$$

THEOREM 3. *For any $x$,*

$$V(x) = \lim_{\varepsilon\to 0} J^\varepsilon(x\varepsilon^{-1}, \hat p^\varepsilon(\cdot)).$$

COROLLARY 1. *Let $v^\varepsilon$ be defined by* (2.12). *Theorems* 2 *and* 3 *imply*

$$\lim_{\varepsilon\to 0} v^\varepsilon(x\varepsilon^{-1}) = V(x).$$

COROLLARY 2. *The sequence of control policies $\hat p^\varepsilon$ is asymptotically optimal in the sense of both absolute and relative errors; that is,*

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left|v^\varepsilon(x\varepsilon^{-1}) - J^\varepsilon(x\varepsilon^{-1}, \hat p^\varepsilon)\right| = 0,\\ &\lim_{\varepsilon\to 0}\left|\frac{v^\varepsilon(x\varepsilon^{-1}) - J^\varepsilon(x\varepsilon^{-1}, \hat p^\varepsilon)}{v^\varepsilon(x\varepsilon^{-1})}\right| = 0.\end{aligned} \tag{3.11}$$

THEOREM 4. *Let*

$$z_0 = \frac{\bar{\sigma}^2}{\delta(d + \lambda/\mu) + \sqrt{\delta^2(d + \lambda/\mu)^2 + 2\bar{\sigma}^2 \rho}} \ln\left(\frac{C^+ + C^-}{C^-}\right), \tag{3.12}$$

*where* $\bar{\sigma}^2 = r^2(a^3 b(a+b)^{-3}\sigma_1^2 + b^3 a(a+b)^{-3}\sigma_2^2) + \lambda^3\mu^{-2}\sigma_3^2 + \lambda\sigma_4^2$. *Then*

$$\hat{p}(t) = rI(X(t) < z_0) + dI(X(t) = z_0) \tag{3.13}$$

*is an image of* $\hat{p}^{\varepsilon_0}$ *given by* (3.10).

## 4. Proof of Theorem 4 and discussion.

PROOF OF THEOREM 4. The proof of Theorem 4 clarifies how asymptotic analysis is applied to a real system. This is the most important practical issue of this paper. Therefore, we prove Theorem 4 prior to Theorems 1, 2 and 3, which are more technical in nature.

Recall that the original system $S$ is related to $S_{\varepsilon_0}$ for some $\varepsilon_0$. Asymptotically optimal control policy given by (3.10) is characterized by the level $z_0 = \varepsilon_0^{-1} z_{\varepsilon_0}^*$, where $z_{\varepsilon_0}^*$ is defined by (3.8) and (3.9) with $\varepsilon = \varepsilon_0$. Taking into account that $\rho_{\varepsilon_0} = \gamma_{\varepsilon_0}\varepsilon_0^2$, we can write

$$z_0 = \frac{\sigma_{\varepsilon_0}^2}{c_{\varepsilon_0}\varepsilon_0 + \sqrt{c_{\varepsilon_0}^2 \varepsilon_0^2 + 2\sigma_{\varepsilon_0}^2 \rho_{\varepsilon_0}}} \ln\left(\frac{C^+ + C^-}{C^-}\right). \tag{4.1}$$

Note that both $\varepsilon_0$ and $c_{\varepsilon_0}$ are nonuniquely determined parameters, because there exist infinitely many sequences $\{S_\varepsilon\}$ in which the original system $S$ can be imbedded with $\varepsilon_0$ indicating its position in the sequence and being an arbitrary index. In our problem, nevertheless, the expression for $z_0$ appears to be a function of $c_{\varepsilon_0}\varepsilon_0$, which is equal to $\delta(d + \lambda/\mu)$ in accordance with (2.14); that is, it is expressed in terms of the original system parameters. Using (3.8) with $\varepsilon = \varepsilon_0$ and (2.13) in (4.1), we derive (3.12). As a result, the control policy defined by (3.13) is suggested for the given system $S$. In accordance with (2.15) and (2.16), the relative and absolute errors of using $\hat{p}$ can be expressed as

$$\begin{aligned} \left|\frac{J(x,\hat{p}) - v(x)}{v(x)}\right| &\equiv \left|\frac{v^{\varepsilon_0}(x) - J^{\varepsilon_0}(x, \hat{p}^{\varepsilon_0})}{v^{\varepsilon_0}(x)}\right|, \\ |J(x,\hat{p}) - v(x)| &= \varepsilon_0^{-3}|v^{\varepsilon_0}(x) - J^{\varepsilon_0}(x,\hat{p}^{\varepsilon_0})|, \end{aligned} \tag{4.2}$$

that is, $\hat{p}$ is an image of $\hat{p}^{\varepsilon_0}$ given by (3.10). This completes the proof of Theorem 4. □

Certain remarks are in order at this point.

From Corollary 2 we know that the right-hand side of (4.2) is small when $\varepsilon_0$ is small. An important open question is that of finding an estimate for the relative error in (4.2) without solving the original control problem. Mathematically this issue is related to finding the rate of convergence in (3.11). Error estimates for constructed controls have been found in Sethi and Zhang [32] and Sethi, Zhang and Zhou [33] in another context of piecewise deterministic manufacturing systems. In this paper, the limiting problem is obtained by replacing fast-changing Markov processes by their average; see also Lehoczky et al. [29]. To our knowledge these kinds of estimates have not been obtained in models approximated by controlled diffusions.

Furthermore, in contrast to the case considered here, the corresponding limiting system in [32] is uniquely determined by the original system associated with a known $\varepsilon_0$ indicating the frequency of the fast process. Therefore, the error bound expressed in terms of $\varepsilon_0$ is adequate. In our problem, on the other hand, it seems to be reasonable to seek an error bound in (4.2) as a function of $c_{\varepsilon_0}\varepsilon_0$ or, equivalently, $\delta$. This would enable one to express the relative error in terms of the original system parameters.

Next, we turn to the numerical comparison of the optimal threshold values found in Akella and Kumar [1] with the corresponding ones provided by our approach. Recall that the system considered in [1] is the same as the one described in Section 2 with additional restrictions: (1) Up and down periods ($\alpha_1$ and $\beta_1$ in our notation) are exponentially distributed, and (2) the demand process is deterministic with a constant rate $d$.

For this system, the optimal production policy, which minimizes the expected discounted cost (2.4), can be explicitly computed. The optimal policy is characterized by a critical number $z_0^*$ (known as a hedging point inventory level), whose closed form expression is known. One can write the optimal policy in the form of (3.13) with $z_0$ replaced by $z_0^*$.

We test our method on those systems studied in [1] that operate in heavy traffic and compare answers; that is, we compare $z_0$ defined by (3.12) and the optimal inventory level $z_0^*$. To this end, we introduce four independent parameters of the system, namely:

1. Traffic parameter $\delta$ defined by (2.6).
2. Demand rate $d$.
3. Mean cycle duration (total duration of mean up and down periods) $k = a^{-1} + b^{-1}$.
4. Proportion of up period with respect to the cycle duration $\alpha = a^{-1}/k$.

All other parameters except the cost coefficients can be expressed in terms of $\delta$, $d$, $k$, $\alpha$ and $\rho$. The parameter $\alpha$ takes values in the interval $[0, 1]$. If $\alpha \to 1$, then the machine is reliable and the system becomes deterministic. If $\alpha \to 0$, then the machine is unreliable and the system is far from deterministic. Analysis of the numerical data shows that the approximation is good (i.e., the relative error $\Delta = |z_0 - z_0^*|z_0^{-1}$ is small), if the discount factor $\rho$ is of the order of $\delta^2$ and both $\delta$ and $\alpha$ are small. We present the graphs of $z^*$ and $z_0^*$ as functions of $\alpha$ for $\delta = 0.1$, $k = 1$, $d = 1$, and $\rho = 0.01$ in Figure 1 and a graph of corresponding relative error in Figure 2. Recall that the traffic intensity is equal to $1 + \delta$.

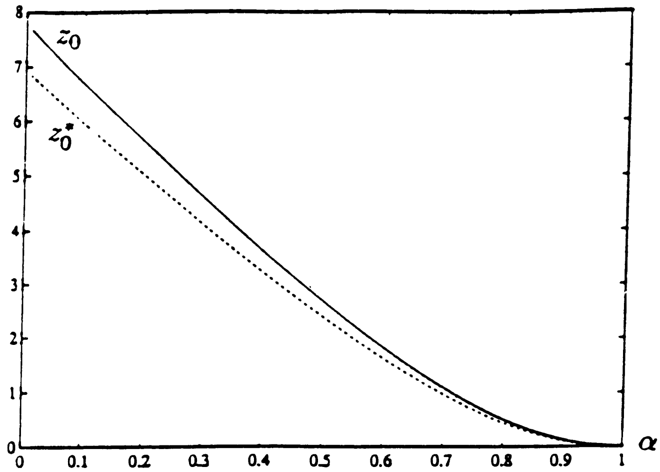


FIG. 1.

**5. Singular control problem for Brownian motion: Proof of Theorem 1.** In this section we study the singular control problem described by (3.2)–(3.5) and prove Theorem 1. The proof is done in two steps. First, we derive the lower bound of the cost functional $J_x(L)$ associated with any admissible control $L$. Let

$$\Gamma = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + c\frac{\partial}{\partial x}.$$

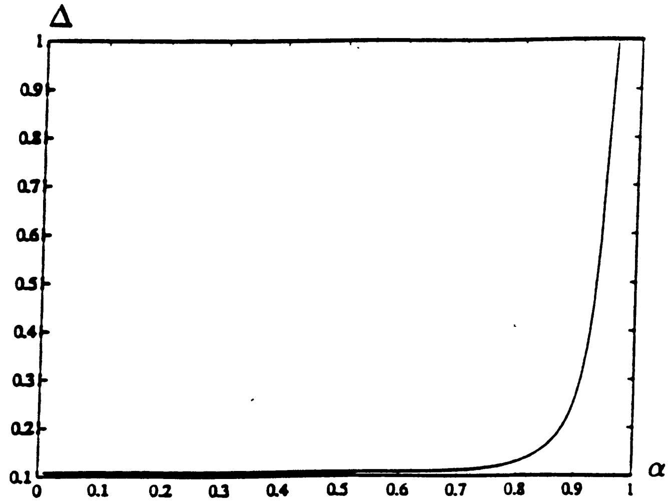


FIG. 2.

We show that if $f \in C^2$ satisfies

$$\Gamma f(x) - \gamma f(x) + h(x) \geq 0, \tag{5.1}$$

$$f'(x) \leq 0 \tag{5.2}$$

and the linear growth condition

$$|f(x)| \leq K(1 + |x|), \qquad K = \text{const.}, \tag{5.3}$$

then $f(x) \leq J_x(L)$ for any $x$ and $L$.

In the second step we show that the cost functional associated with $L^*$, defined by (3.6) and (3.7), satisfies (5.1)–(5.3).

STEP 1. Denote $\Delta L(t) = L(t) - L(t-)$, assuming that $L(0-) = 0$; that is, $\Delta L(0) = L(0)$. The continuous part $\lambda(t)$ of $L(t)$ is defined as $\lambda(t) = \sum_{s \leq t} \Delta L(s)$, $t \geq 0$. For arbitrary $f \in C^2$, denote $\Delta f(Z)(t) = f(Z(t)) - f(Z(t-))$ with the convention $Z(0-) = X(0)$.

LEMMA 5.1. *Let $f \in \overline{C}^2$, where $\overline{C}^2$ is the set of functions $h$ such that $h \in C^1$ and the second derivative of $h$ is continuous everywhere except at a finite number of points, where it has left and right limits. Then*

$$E\left[e^{-\gamma T} f(X_T)\right] = f(x) + E\left[\int_0^T e^{-\gamma t}(\Gamma f - \gamma f)(X_t)\,dt\right] - E\left[\int_0^T e^{-\gamma t} f'(X_t)\,d\lambda_t\right] + E\left[\sum_{0 \leq t \leq T} e^{-\gamma t} \Delta f(X)_t\right].$$

PROOF. The proof of Lemma 5.1 can be found in [35]. □

LEMMA 5.2. *If $f \in C^2$ satisfies* (5.1)–(5.3), *then for any $x$ and any admissible control $L$, we have*

$$f(x) \leq J_x(L).$$

PROOF. Define $K_t = \int_0^t e^{-\gamma s} h(X_s)\,ds$. It follows from Lemma 5.1 that

$$\begin{aligned} E\left[K_T + e^{-\gamma T} f(X(T))\right] &= f(x) + E\left\{\int_0^T e^{-\gamma t}[\Gamma f - \gamma f + h](X(t))\,dt\right\} \\ &\quad - E\left\{\int_0^T e^{-\gamma t} f'(X(t))\,d\lambda(t)\right\} \\ &\quad + E\left\{\sum_{0 \leq t \leq T} e^{-\gamma t} \Delta f(X(t))\right\}. \end{aligned} \tag{5.4}$$

If $f$ satisfies (5.1)–(5.3), then formulas (5.1) and (5.2) imply that the second and the third terms in (5.4) are nonnegative. Because

$$\Delta f(X(t)) = f(X(t)) - f(X(t-)) = \int_{X(t)+\Delta L(t)}^{X(t)} f'(y)\,dy,$$

formula (5.2) yields nonnegativity of the preceding expression. Thus,

$$E\left[K_T + e^{-\gamma T} f(X(T))\right] \ge f(x). \tag{5.5}$$

On the other hand, using (3.3) and (5.3), we can write $|f(X(T))| \le K(1 + |x| + |W(T)| + L(T))$. From (3.2) one can derive the existence of a sequence $T_k$, $k \ge 1$, such that $T_k \to \infty$ as $k \to \infty$ and $\lim_{k\to\infty} e^{-\gamma T_k} EL(T_k) = 0$. Because $E|W_T| \le (\frac{1}{2}\sigma^2 T)^{1/2}$, we conclude $\lim_{k\to\infty} e^{-\gamma T_k} Ef(X(T_k)) = 0$. Setting $T = T_k$ and taking $k \to \infty$ in (5.5), we have $J_x(L) \ge f(x)$. Because $L$ and $x$ are arbitrary, Lemma 5.2 is proved. □

STEP 2. Consider the control functional

$$L(t) = \sup_{s \le t}\left[x + W(s) - z\right]^+, \qquad t \ge 0. \tag{5.6}$$

Because $L(t) \le |x - z| + \sup_{s\le t}|W(s)|$ and $E\sup_{s\le t}|W(s)| \le (\frac{1}{2}\sigma^2 t)^{1/2}$, we have

$$E\int_0^\infty e^{-\gamma t}\,dL_t = \lim_{T\to\infty}\left\{EL_T e^{-\gamma T} + \gamma\int_0^T e^{-\gamma t} EL_t\,dt\right\} \le \text{const.}$$

Thus, $L$ is an admissible control. It is known that the process $X(t) = x + W(t) - L(t)$ is a Brownian motion with values in $(-\infty, z]$ reflected at the barrier $z$ [11]. The following lemma characterizes the cost function associated with the control $L$.

LEMMA 5.3. *Suppose that $f \in \overline{C}^2$ satisfies* (5.3) *and*

$$\Gamma f(x) - \gamma f(x) + h(x) = 0, \qquad x \le z, \tag{5.7}$$

$$f'(x) = 0, \qquad x \ge z. \tag{5.8}$$

*Then $f(x) = E\{\int_0^\infty e^{-\gamma t} h(X(t))\,dt\}$.*

PROOF. Note that (5.7) and (5.8) imply that $f$ is twice continuously differentiable on $[-\infty, z]$ and on $[z, \infty)$ with $z$ being the only possible point of discontinuity for $f''$. If $x \in (-\infty, z]$, then $\Delta L_0 = 0$. In this case (5.4) yields

$$\begin{aligned} &E\left[K_T + e^{-\gamma T} f(X(T))\right] \\ &\quad = f(x) + E\left\{\int_0^T (\Gamma f - \gamma f + h)(X(t))\,dt\right\} \\ &\qquad - E\int_0^T e^{-\gamma t} f'(X(t))\,dL(t). \end{aligned} \tag{5.9}$$

Because $X(t) \in (-\infty, z]$, the second term on the right-hand side of (5.9) vanishes due to (5.7). Next, according to (5.6), $L$ increases only when $X = z$. Therefore, the third term on the right-hand side of (5.9) also vanishes due to (5.8). Letting $T \to \infty$ and using the linear growth condition of $f$, one can show that $EK_\infty = f(x)$ as desired.

If $x > z$, then the value of $E(K_\infty)$ is the same as it is for the process with the initial state $z$. From the first part of the proof, we know that $E(K_\infty) = f(z)$ and $f(x) = f(z)$ in accordance with (5.8). □

To determine the optimal threshold value $z^*$ we use the "principle of smooth fit" (see [17] and [18]). To this end, we have to find $z^*$ such that the solution $f$ of (5.7) and (5.8) is twice continuously differentiable and satisfies (5.1)–(5.3).

LEMMA 5.4. *The optimal threshold value* $z^*$ *is given by* (3.7).

PROOF. Denote $k(x) = f(z - x)$. We have

$$\tfrac{1}{2}\sigma^2 k''(x) - ck'(x) - \gamma k(x) + h_z(x) = 0, \qquad x \geq 0, \tag{5.10}$$

where $h_z(x) = h(z - x)$, $x \geq 0$. Because $f$ is twice continuously differentiable and $f(x) = \text{const}$ for $x \geq z$, one gets the boundary conditions $k'(0) = k''(0) = 0$. Differentiating (5.10) and putting $g(x) = k'(x)$, we conclude that $g$ is the solution of the following Cauchy problem:

$$\begin{gathered} \tfrac{1}{2}\sigma^2 g''(x) - cg'(x) - \gamma g(x) + h'_z(x) = 0, \\ g(0) = g'(0) = 0, \end{gathered} \tag{5.11}$$

where $h'_z(x) = -C^+ I(x < z) + C^- I(x \geq z)$. A general solution of the homogeneous part of the preceding equation is given by

$$g_0(x) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

with $\lambda_1 = (\sigma^2)^{-1}(c + \sqrt{c^2 + 2\sigma^2\gamma})$ and $\lambda_2 = (\sigma^2)^{-1}(c - \sqrt{c^2 + 2\sigma^2\gamma})$. Using Green's formula [5], we can find the solution of the Cauchy problem (5.11) for $x \leq z$:

$$g(x) = \frac{C^+}{\lambda_2 - \lambda_1}\left\{\lambda_2^{-1}(e^{\lambda_2 x} - 1) - \lambda_1^{-1}(e^{\lambda_1 x} - 1)\right\}. \tag{5.12}$$

Next we extend the solution for $x \geq z$, preserving the continuity of $g$ and $g'$. For $x \geq z$, it has the form

$$g(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \hat{g}(x), \tag{5.13}$$

where

$$\hat{g}(x) = -\frac{C^-}{\lambda_1\lambda_2} - \frac{C^-}{\lambda_2(\lambda_2 - \lambda_1)} e^{\lambda_2(x-z)} + \frac{C^-}{\lambda_1(\lambda_2 - \lambda_1)} e^{\lambda_1(x-z)}$$

is the solution of the nonhomogeneous equation (5.11) for $x \ge z$ subject to $\hat{g}(z) = \hat{g}(z) = 0$. Using the required continuity of $g$ and $g'$ at the point $x = z$ we can find the equations for the constants $C_1$ and $C_2$ in (5.13); that is,

$$C_1 e^{\lambda_1 z} + C_2 e^{\lambda_2 z} = C^+(\lambda_2 - \lambda_1)^{-1}\{\lambda_2^{-1}(e^{\lambda_2 z} - 1) - \lambda_1^{-1}(e^{\lambda_1 z} - 1)\},$$
$$C_1\lambda_1 e^{\lambda_1 z} + C_2\lambda_2 e^{\lambda_2 z} = C^+(\lambda_2 - \lambda_1)^{-1}\{e^{\lambda_2 z} - e^{\lambda_1 z}\}.$$

Therefore,

$$C_1(z) = -\frac{C^+}{\lambda_1(\lambda_2 - \lambda_1)}(1 - e^{-\lambda_1 z}), \qquad C_2(z) = \frac{C^+}{\lambda_2(\lambda_2 - \lambda_1)}(1 - e^{-\lambda_2 z}).$$

Thus, (5.13) yields the following expression for $g$ on $[z, \infty)$:

$$\begin{aligned} g(x) = {} & -C^-\lambda_1^{-1}\lambda_2^{-1} - C^-\lambda_2^{-1}(\lambda_2 - \lambda_1)^{-1}e^{-\lambda_2 z}e^{\lambda_2 x} \\ & + C^-\lambda_1^{-1}(\lambda_2 - \lambda_1)^{-1}e^{-\lambda_1 z}e^{\lambda_1 x} \\ & - C^+\lambda_1^{-1}(\lambda_2 - \lambda_1)^{-1}(1 - e^{-\lambda_1 z})e^{\lambda_1 x} \\ & + C^+\lambda_2^{-1}(\lambda_2 - \lambda_1)^{-1}(1 - e^{-\lambda_2 z})e^{\lambda_2 x}. \end{aligned} \tag{5.14}$$

The linear growth requirement on $k(x)$ results in boundedness of its derivative $g(x)$ given by (5.14). Because $\lambda_1 > 0$ and $\lambda_2 < 0$, the coefficient in front of $e^{\lambda_1 x}$ in (5.14) must be equal to zero. Therefore,

$$C^-\lambda_1^{-1}(\lambda_2 - \lambda_1)^{-1}e^{-\lambda_1 z} = C^+\lambda_1^{-1}(\lambda_2 - \lambda_1)^{-1}(1 - e^{-\lambda_1 z}).$$

The solution $z^*$ of the foregoing equation is given by (3.7). Note that for $z = z^*$, the function $g$ in (5.14) is decreasing and $g(x) \to -C^-\lambda_1^{-1}\lambda_2^{-1} \ge 0$ as $x \to +\infty$. Consequently, $g(x) \ge 0$ for $x > z^*$. It is easy to see from (5.12) that $g(x) \ge 0$ for $x \in [0, z^*]$. Set

$$k(0) = \gamma^{-1}h_{z^*}(0) = C^+z^*\gamma^{-1},$$

$$k(x) = \begin{cases} k(0) + \int_0^x g(t)\,dt, & x \ge 0, \\ k(0), & x \le 0. \end{cases}$$

One can conclude that $f(x) = k(z^* - x)$ satisfies (5.7), (5.8) and the linear growth condition. According to Lemma 5.3, $f$ is the cost function corresponding to the control $L^*$ defined by (3.6). Moreover, $f$ is twice continuously differentiable, $f'(x) \le 0$ for all $x$ and $\Gamma f(x) - \gamma f(x) + h(x) = -\gamma f(z^*) + h(x) \ge -\gamma f(z^*) + h(z^*) = 0$ for $x \ge z^*$. Consequently, conditions (5.1)–(5.3) are satisfied. By virtue of Lemma 5.2, $L^*$ is the optimal control. □

**6. Rescaled inventory process: Proof of Theorem 2.** We introduce the rescaled inventory process $Y^\varepsilon(t) = \varepsilon X^\varepsilon(\varepsilon^{-2}t)$. Using (2.3) we rewrite $Y^\varepsilon$ as

$$Y^\varepsilon(t) = x + W^\varepsilon(t) - L^\varepsilon(t), \tag{6.1}$$

where

$$L^\varepsilon(t) = \varepsilon\int_0^{\varepsilon^{-2}t} G^\varepsilon(u)(r_\varepsilon - p^\varepsilon(u))\,du, \tag{6.2}$$

$$W^\varepsilon(t) = c_\varepsilon t + r_\varepsilon M^\varepsilon(t) - N^\varepsilon(t), \tag{6.3}$$

$$M^\varepsilon(t) = \varepsilon\left\{\int_0^{\varepsilon^{-2}t} G^\varepsilon(s)\,ds - \frac{b_\varepsilon}{a_\varepsilon + b_\varepsilon}\varepsilon^{-2}t\right\}, \tag{6.4}$$

$$N^\varepsilon(t) = \varepsilon\left\{\sum_{i=1}^{A^\varepsilon(\varepsilon^{-2}t)} \eta_i^\varepsilon - \frac{\lambda_\varepsilon}{\mu_\varepsilon}\varepsilon^{-2}t\right\}. \tag{6.5}$$

The process $M^\varepsilon(\cdot)$ is the centered and rescaled cumulative up time of the machine, whereas $N^\varepsilon(\cdot)$ is the centered and rescaled stochastic part of the demand.

Note that the expression (6.2) establishes a one-to-one correspondence between $p^\varepsilon$ and $L^\varepsilon$. Set

$$J_x^\varepsilon(L^\varepsilon) = J^\varepsilon(x\varepsilon^{-1}, p^\varepsilon).$$

After simple transformations, we have

$$J_x^\varepsilon(L^\varepsilon) = E\int_0^\infty e^{-\gamma_\varepsilon t} h(Y^\varepsilon(t))\,dt. \tag{6.6}$$

Our goal is to compare the sequence $J_x^\varepsilon(L^\varepsilon)$, $\varepsilon \in \mathscr{E}$, with the optimal cost $V(x) = J_x(L^*)$ of the singular control problem and to show that

$$\liminf_{\varepsilon\to 0} J_x^\varepsilon(L^\varepsilon) \geq J_x(L^*). \tag{6.7}$$

We need a number of auxiliary results, which are formulated as lemmas.

LEMMA 6.1 ([24], Appendix I). *Let $\{\xi_i^\varepsilon\}_{i\geq 1}$ be a sequence of i.i.d. random variables with $E\xi_1^\varepsilon = \lambda_\varepsilon^{-1}$, $\operatorname{Var}\xi_1^\varepsilon = \sigma_\varepsilon^2$ and*

$$A^\varepsilon(t) = \max\{j\colon \xi_1^\varepsilon + \cdots + \xi_j^\varepsilon \leq t\}.$$

*Suppose that $\lambda_\varepsilon \to \lambda$, $\sigma_\varepsilon^2 \to \sigma^2$ as $\varepsilon \to 0$ and $\sup_\varepsilon E(\xi_1^\varepsilon)^{2+q} < \infty$ for some $q > 0$. Then there exists a constant $K$ such that for any $t \geq 0$, we have*

$$\sup_\varepsilon\left(E\left(\sup_{s\leq t}\left|\varepsilon\left(A^\varepsilon(\varepsilon^{-2}s) - \lambda_\varepsilon\varepsilon^{-2}s\right)\right|\right)^2\right)^{1/2} \leq K + K\sqrt{t},$$

$$\sup_\varepsilon\left(E\left(\varepsilon^2 A^\varepsilon(\varepsilon^{-2}t)\right)^2\right)^{1/2} \leq K + Kt.$$

LEMMA 6.2. *There exists a constant $\overline{K}$ such that*

$$\sup_\varepsilon\left(E\left(\sup_{s\leq t}|W^\varepsilon(s)|\right)^2\right)^{1/2} \leq \overline{K} + \overline{K}t.$$

LEMMA 6.3. *Let*

$$v_1 = \hat{\lambda}^3 \hat{\mu}^{-2} \hat{\sigma}_3^2 + \hat{\lambda} \hat{\sigma}_4^2, \tag{6.8}$$

$$v_2 = \hat{a}^3 \hat{b} (\hat{a} + \hat{b})^{-3} \hat{\sigma}_1^2 + \hat{b}^3 \hat{a} (\hat{a} + \hat{b})^{-3} \hat{\sigma}_2^2. \tag{6.9}$$

*Under the preceding assumptions and the definitions, we have*

$$N^\varepsilon \to_d N, \qquad M^\varepsilon \to_d M, \qquad \varepsilon \to 0,$$

*where $N$ and $M$ are independent Wiener processes with variations $v_1$ and $v_2$, respectively, and the symbol $\to_d$ stands for convergence in distribution.*

COROLLARY. $W^\varepsilon \to_d W$ *as* $\varepsilon \to 0$, *where $W$ is a Wiener process with drift $c$ and variance $\sigma^2$ defined by* (2.8) *and* (3.1), *respectively.*

The proofs of Lemmas 6.2 and 6.3 are given in the Appendix.

LEMMA 6.4 [23]. *Let $\{\alpha_i^\varepsilon\}_{i \ge 1}$ be a sequence of nonnegative i.i.d. random variables with* $\sup_\varepsilon E(\alpha_1^\varepsilon)^{2+q} < \infty$ *for some $q > 0$. Then,*

$$\text{P-}\lim_{\varepsilon \to 0} \varepsilon \max_{i \le \varepsilon^{-2}} \alpha_i^\varepsilon = 0,$$

*where the notation* P-lim *stands for limit in probability.*

We now proceed with the proof of (6.7). One can assume without any loss of generality that the sequence $J_x^\varepsilon(L^\varepsilon)$, $\varepsilon \in \mathscr{E}$, converges as $\varepsilon \to 0$ to a finite limit; otherwise the left-hand side of (6.7) is equal to $+\infty$ and (6.7) is valid. Thus, for some constant $C$,

$$J_x^\varepsilon(L^\varepsilon) = E \int_0^\infty e^{-\gamma_\varepsilon t} h(Y^\varepsilon(t))\, dt \le C. \tag{6.10}$$

LEMMA 6.5. *For each $T > 0$ there exists a constant $C(T)$ such that for each $\varepsilon \in \mathscr{E}$,*

$$E\{L^\varepsilon(T)\} \le C(T). \tag{6.11}$$

PROOF. It follows from (6.1) that

$$\int_0^\infty e^{-\gamma_\varepsilon t} EL^\varepsilon(t)\, dt \le \int_0^\infty e^{-\gamma_\varepsilon t} \{|x| + E|W^\varepsilon(t)| + E|Y^\varepsilon(t)|\}\, dt. \tag{6.12}$$

Lemma 6.2 implies $E|W^\varepsilon(t)| \le \overline{K} + \overline{K}t$. Besides, $|x| \le (\min(C^+, C^-))^{-1} h(x)$ for any $x$; consequently, (6.10) and (6.12) imply

$$\int_0^\infty e^{-\gamma_\varepsilon t} EL^\varepsilon(t)\, dt \le \overline{C}_\varepsilon \tag{6.13}$$

with $\overline{C}_\varepsilon = |x| \gamma_\varepsilon^{-1} + C(\min(C^+, C^-))^{-1} + \overline{K}(\gamma_\varepsilon^{-1} + \gamma_\varepsilon^{-2})$. Because $L^\varepsilon(T) \le L^\varepsilon(u)$ for $u \ge T$, we have

$$\int_T^\infty EL^\varepsilon(T) e^{-\gamma_\varepsilon t}\, dt \le \int_0^\infty e^{-\gamma_\varepsilon u} EL^\varepsilon(u)\, du \le \overline{C}_\varepsilon$$

and

$$EL^\varepsilon(T) \le \overline{C}_\varepsilon \gamma_\varepsilon^{-1} e^{\gamma_\varepsilon T}. \tag{6.14}$$

Because the sequence $\{\gamma_\varepsilon, \ \varepsilon \in \mathscr{E}\}$ converges, the right-hand side of (6.14) is bounded and we get the statement of Lemma 6.5. □

Let $V^+$ be the space of nondecreasing right continuous functions $V^+$ with Skorohod $M_1$ metric on it [34]. Consider the process $L^\varepsilon$ as a random element with values in $V^+$. It is known that convergence of $m_\varepsilon$ to $m$ in $V^+$ is equivalent to convergence of $\rho_N(m_\varepsilon, m)$ as $\varepsilon \to 0$ for each $N > 0$, where

$$\begin{aligned}\rho_N(m_1, m_2) = \inf\{\eta \colon m_1(x-\eta) - \eta \le m_2(x) \le m_1(x+\eta) + \eta,\\ m_2(x-\eta) - \eta \le m_1(x) \le m_2(x+\eta) + \eta\\ \text{for all } x \in [\eta, N-\eta]\}.\end{aligned}$$

An equivalent description of the convergence of $m_\varepsilon$ to $m$ in $V^+$ would be the convergence of $m_\varepsilon(x)$ to $m(x)$ for every $x > 0$ which is a point of continuity of $m$ (see also [31]). Using the classical Helly's theorem, one can show that a set $\mathscr{K} \in V^+$ is relatively compact if for each $N > 0$ there exists $K_N$ such that for any $m \in \mathscr{K}$,

$$m(N) \le K_N. \tag{6.15}$$

Let $\delta > 0$ and

$$\mathscr{K} = \{m \in V^+ \colon m(N) \le \delta^{-1} 2^{N+1} C(N), N = 1, 2, \dots\},$$

where the constants $C(N)$ are given in the previous lemma. By virtue of (6.15), the set $\mathscr{K}$ is compact in $V^+$. Chebyshev's inequality and (6.11) imply that $P\{L^\varepsilon(\cdot) \notin \mathscr{K}\} \le \delta$. Therefore, the sequence $L^\varepsilon(\cdot)$ is tight [4].

Consider $M^\varepsilon$ and $N^\varepsilon$, $\varepsilon \in \mathscr{E}$, defined by (6.4) and (6.5), respectively, as sequences of elements from the space $D = D[0, \infty)$ with the Skorohod $J_1$ metric on it [34]. Define a triple $U^\varepsilon = (M^\varepsilon, N^\varepsilon, L^\varepsilon)$ as a random element with values in $\mathscr{U}$, where

$$\mathscr{U} = D \times D \times V^+.$$

Lemma 6.3 implies that $M^\varepsilon(\cdot)$ and $N^\varepsilon(\cdot)$, $\varepsilon \in \mathscr{E}$, are tight. Therefore the sequence $U^\varepsilon \in \mathscr{U}$ is tight because each of its components is tight. Without loss of generality we may assume that $U^\varepsilon$ converges in distribution. By the Skorohod representation theorem ([10], Section 3, Theorem 1.8), there exists a sequence $\hat{U}^\varepsilon$, $\varepsilon \in \mathscr{E}$, on another probability space such that $\hat{U}^\varepsilon$ has the same distribution as $U^\varepsilon$ and $\hat{U}^\varepsilon$ converges almost surely. Let $(\hat{M}, \hat{N}, L)$ be its limit. Obviously $(\hat{M}, \hat{N})$ has the same distribution as $(M, N)$ in Lemma 6.3. With some abuse of notation, we identify $(\hat{M}, \hat{N})$ with $(M, N)$ and $\hat{U}^\varepsilon$ with $U^\varepsilon$, thus assuming

$$\lim_{\varepsilon \to 0} (M^\varepsilon, N^\varepsilon, L^\varepsilon) = (M, N, L), \qquad \text{P-a.s.} \tag{6.16}$$

Because $(M, N)$ have continuous trajectories, almost sure convergence of $(M^\varepsilon, N^\varepsilon)$ to $(M, N)$ in the $J_1$ topology is equivalent to uniform convergence on all finite intervals. Convergence of $L^\varepsilon$ to $L$ in $V^+$ coincides with convergence of $L^\varepsilon(s)$ to $L(s)$ for each $s$ that is a point of continuity of $L$ (see [31]).

Let $W(t) = ct + rM(t) - N(t)$ and $Y(t) = x + W(t) - L(t)$. In view of convergence of $M^\varepsilon$, $N^\varepsilon$ and $L^\varepsilon$, we can use (6.1) to get

$$Y(t) = \lim_{\varepsilon\to 0} Y^\varepsilon(t), \qquad \text{P-a.s.},$$

for each $t$ that is a point of continuity of $L$. Consequently, there exists a countable set $\Lambda = \Lambda(\omega)$ such that

$$h(Y^\varepsilon(t)) \to h(Y(t)), \qquad \varepsilon \to 0, \text{P-a.s. for each } t \notin \Lambda. \tag{6.17}$$

Fatou's lemma and (6.17) imply

$$\begin{aligned}\int_0^\infty e^{-\gamma t} h(Y(t))\,dt &= \int_0^\infty e^{-\gamma t} \lim_\varepsilon e^{-(\gamma_\varepsilon-\gamma)t} h(Y^\varepsilon(t))\,dt \\ &\le \liminf_\varepsilon \int_0^\infty e^{-\gamma_\varepsilon t} h(Y^\varepsilon(t))\,dt.\end{aligned} \tag{6.18}$$

Taking expectation in (6.18) and applying Fatou's lemma once again, we obtain

$$E\int_0^\infty e^{-\gamma t} h(Y(t))\,dt \le \liminf_\varepsilon E\int_0^\infty e^{-\gamma_\varepsilon t} h(Y^\varepsilon(t))\,dt.$$

The preceding inequality can be rewritten as $J_x(L) \le \liminf_\varepsilon J_x^\varepsilon(L^\varepsilon)$. In order to complete the last step of the proof, namely, to show that

$$J_x(L^*) \le J_x(L), \tag{6.19}$$

we need to apply the results of Section 5. To this end, we have to identify the filtration $\mathscr{F}_t$ such that $W$ is a Brownian motion with respect to $\mathscr{F}_t$ and $L(t)$ is $\mathscr{F}_t$-measurable. We define $\mathscr{F}_t$ as $\sigma(M(s), N(s), L(s),\ s \le t)$. Obviously $L(t)$ is $\mathscr{F}_t$-measurable.

LEMMA 6.6. *The process $W$ is a Brownian motion with respect to* $(\mathscr{F}_t, t \ge 0)$.

PROOF. Let $F$ and $G$ be bounded functions of $3k$ arguments and a single argument, respectively. Let

$$0 \le s_1 < s_1 < \cdots < s_k \le t < u$$

and $P\{L(s_i) \ne L(s_i -)\} = 0$, $i = 1, 2, \dots, k$. We need to show

$$\begin{aligned}&E\{F(M(s_1),\dots,M(s_k),N(s_1),\dots,N(s_k),L(s_1),\dots,\\ &\qquad\qquad L(s_k))G(W(u) - W(t))\}\\ &= E\{F(M(s_1),\dots,M(s_k),N(s_1),\dots,N(s_k),L(s_1),\dots,L(s_k))\}\\ &\quad\times E\{G(W(u) - W(t))\}.\end{aligned} \tag{6.20}$$

Define

$$l_1^\varepsilon(t) = \sum_{i=1}^{A^\varepsilon(t)+1} \xi_i^\varepsilon - t, \qquad l_2^\varepsilon(t) = \sum_{i=1}^{D^\varepsilon(t)+1} (\alpha_i^\varepsilon + \beta_i^\varepsilon) - t,$$

where

$$D^\varepsilon(t) = \max\{j\colon \alpha_1^\varepsilon + \beta_1^\varepsilon + \cdots + \alpha_j^\varepsilon + \beta_j^\varepsilon \le t\}.$$

For any $t > 0$, the following estimates hold:

$$\begin{aligned} \sup_{s\le t} |l_1^\varepsilon(s)| &\le \max_{i\le A^\varepsilon(t)+1} \xi_i^\varepsilon, \\ \sup_{s\le t} |l_2^\varepsilon(s)| &\le \max_{i\le D^\varepsilon(t)+1} (\alpha_i^\varepsilon + \beta_i^\varepsilon). \end{aligned} \tag{6.21}$$

Using the elementary renewal theorem (see Chung [8], Section 5.5), (2.7), (2.10) and (6.16), one can conclude

$$\text{P-}\lim_{\varepsilon\to 0} \varepsilon^2 A^\varepsilon(\varepsilon^{-2}t) = \hat{\lambda}t, \qquad \text{P-}\lim_{\varepsilon\to 0} \varepsilon^2 D^\varepsilon(\varepsilon^{-2}t) = \hat{a}\hat{b}(\hat{a}+\hat{b})^{-1}t,$$

where P-lim stands for the limit in probability. Together with (6.21) and Lemma 6.4 this yields

$$\text{P-}\lim_{\varepsilon\to 0} \sup_{s\le t} \varepsilon l_1^\varepsilon(\varepsilon^{-2}s) \to 0, \qquad \text{P-}\lim_{\varepsilon\to 0} \sup_{s\le t} \varepsilon l_2^\varepsilon(\varepsilon^{-2}s) \to 0. \tag{6.22}$$

Choosing a subsequence, one can get a.s. convergence in the foregoing formula. In view of (6.16) and (6.22), we can write

$$\begin{aligned} N(u) - N(t) = \lim_{\varepsilon\to 0} \big[ & N^\varepsilon\big(u + \varepsilon^2 l_1^\varepsilon(\varepsilon^{-2}t)\big) \\ & - N^\varepsilon\big(t + \varepsilon^2 l_1^\varepsilon(\varepsilon^{-2}t)\big)\big], \qquad \text{P-a.s.}, \end{aligned} \tag{6.23}$$

and

$$\begin{aligned} M(u) - M(t) = \lim_{\varepsilon\to 0} \big[ & M^\varepsilon\big(u + \varepsilon^2 l_2^\varepsilon(\varepsilon^{-2}t)\big) \\ & - M^\varepsilon\big(t + \varepsilon^2 l_2^\varepsilon(\varepsilon^{-2}t)\big)\big], \qquad \text{P-a.s.} \end{aligned} \tag{6.24}$$

Denote

$$\begin{aligned} \Delta^\varepsilon\binom{u}{t} &= r_\varepsilon\big[M^\varepsilon\big(u + \varepsilon^2 l_2^\varepsilon(\varepsilon^{-2}t)\big) - M^\varepsilon\big(t + \varepsilon^2 l_2^\varepsilon(\varepsilon^{-2}t)\big)\big] \\ &\quad - \big[N^\varepsilon\big(u + \varepsilon^2 l_1^\varepsilon(\varepsilon^{-2}t)\big) - N^\varepsilon\big(t + \varepsilon^2 l_1^\varepsilon(\varepsilon^{-2}t)\big)\big] + c_\varepsilon[u-t]. \end{aligned}$$

Using (6.4) and (6.5), we derive

$$\begin{aligned} \Delta^\varepsilon\binom{u}{t} &= r_\varepsilon \varepsilon \int_0^\infty I\big(\varepsilon^{-2}t + l_2(\varepsilon^{-2}t) < s \\ &\qquad\qquad \le \varepsilon^{-2}u + l_2(\varepsilon^{-2}t)\big) G^\varepsilon(s)\, ds \\ &\quad - \varepsilon \sum_{i=1}^\infty \eta_i^\varepsilon I\big\{A^\varepsilon\big(\varepsilon^{-2}t + l_1^\varepsilon(\varepsilon^{-2}t)\big) < i \\ &\qquad\qquad \le A^\varepsilon\big(\varepsilon^{-2}u + l_1^\varepsilon(\varepsilon^{-2}t)\big)\big\}. \end{aligned} \tag{6.25}$$

Fix $\varepsilon$. Let $S = A^\varepsilon(\varepsilon^{-2}t + l_1^\varepsilon(\varepsilon^{-2}t)) \equiv A^\varepsilon(\varepsilon^{-2}t) + 1$. If one considers a sequence of i.i.d. random variables $\{\xi_i^\varepsilon\}_{i\ge 1}$, then $S$ is a stopping time with respect to the filtration $\sigma\{\xi_i^\varepsilon,\ i = 1,\dots,k\}$, $k \ge 1$. Therefore, $\sigma\{\xi_i^\varepsilon,\ i \le S\}$ is independent of $\sigma\{\xi_{S+i},\ i = 1,2\dots\}$ (see [7], Lemma 2, Chapter 5.3). Obviously $S$ is a stopping time with respect to the sequence $\{\xi_i^\varepsilon, \eta_i^\varepsilon\}_{i\ge 1}$. Similarly, $\mathscr{H}_S \equiv \sigma\{\xi_i^\varepsilon, \eta_i^\varepsilon,\ i \le S\}$ and $\mathscr{H}^S \equiv \sigma\{\xi_{S+i}^\varepsilon, \eta_{S+i}^\varepsilon,\ i = 1,2\dots\}$ are independent.

Let $T = D^\varepsilon(\varepsilon^{-2}t + l_2(\varepsilon^{-2}t)) \equiv D^\varepsilon(\varepsilon^{-2}t) + 1$. In the same manner, $\mathscr{G}_T \equiv \sigma\{\alpha_i^\varepsilon, \beta_i^\varepsilon,\ i \le T\}$ and $\mathscr{G}^T \equiv \sigma\{\alpha_{T+i}^\varepsilon, \beta_{T+i}^\varepsilon,\ i = 1,2\dots\}$ are independent. Because $\{\alpha_i^\varepsilon, \beta_i^\varepsilon\}_{i\ge 1}$ and $\{\xi_i^\varepsilon, \eta_i^\varepsilon\}_{i\ge 1}$ are independent sequences, the $\sigma$-fields $\mathscr{H}_S$, $\mathscr{H}^S$, $\mathscr{G}_T$ and $\mathscr{G}^T$ are mutually independent. Therefore, $\mathscr{H}_S \vee \mathscr{G}_T$ and $\mathscr{H}^S \vee \mathscr{G}^T$ are independent. Using (6.25) and the definition of the process $G^\varepsilon$ [see (2.1)] one can conclude that $\Delta^\varepsilon\binom{u}{t}$ is $\mathscr{H}^S \vee \mathscr{G}^T$-measurable. Obviously

$$\mathscr{F}_t^\varepsilon \equiv \sigma\{M^\varepsilon(s), N^\varepsilon(s),\ s \le t\} \subseteq \mathscr{H}_S \vee \mathscr{G}_T.$$

Thus, $\Delta^\varepsilon\binom{u}{t}$ and $\mathscr{F}_t^\varepsilon$ are independent. It follows from the definition of the admissible policies that $L^\varepsilon(t)$ is $\mathscr{F}_t^\varepsilon$-measurable. Consequently,

$$\begin{aligned}
&E\Big\{F(M^\varepsilon(s_1),\dots,M^\varepsilon(s_k),N^\varepsilon(s_1),\dots,N^\varepsilon(s_k),\\
&\qquad\qquad L^\varepsilon(s_1),\dots,L^\varepsilon(s_k))\\
&\quad\times G\big(r_\varepsilon\big[M^\varepsilon\big(u+\varepsilon^2 l_2^\varepsilon(\varepsilon^{-2}t)\big) - M^\varepsilon\big(t+\varepsilon^2 l_2^\varepsilon(\varepsilon^{-2}t)\big)\big]\\
&\qquad -\big[N^\varepsilon\big(u+\varepsilon^2 l_1^\varepsilon(\varepsilon^{-2}t)\big) - N^\varepsilon\big(t+\varepsilon^2 l_1^\varepsilon(\varepsilon^{-2}t)\big)\big]\\
&\qquad\qquad + c_\varepsilon[u-t]\big)\Big\}\\
&= EF(M^\varepsilon(s_1),\dots,M^\varepsilon(s_k),N^\varepsilon(s_1),\dots,N^\varepsilon(s_k),\\
&\qquad\qquad L^\varepsilon(s_1),\dots,L^\varepsilon(s_k))\\
&\quad\times EG\big(r_\varepsilon\big[M^\varepsilon\big(u+\varepsilon^2 l_2^\varepsilon(\varepsilon^{-2}t)\big) - M^\varepsilon\big(t+\varepsilon^2 l_2^\varepsilon(\varepsilon^{-2}t)\big)\big]\\
&\qquad -\big[N^\varepsilon\big(u+\varepsilon^2 l_1^\varepsilon(\varepsilon^{-2}t)\big) - N^\varepsilon\big(t+\varepsilon^2 l_1^\varepsilon(\varepsilon^{-2}t)\big)\big]\\
&\qquad\qquad + c_\varepsilon[u-t]\big).
\end{aligned}\tag{6.26}$$

Allowing $\varepsilon \to 0$ on both sides of (6.26) and using (6.3), (6.23) and (6.24), we derive equality (6.20). Applying monotone class arguments, one can extend (6.20) for any measurable functions $F$ and $G$, whence independence of $\mathscr{F}_t$ and $W(u) - W(t)$ follows. □

LEMMA 6.7. *The functional L is an admissible control*; *that is*,

$$E\int_0^\infty e^{-\gamma t}\, dL(t) < \infty.$$

PROOF. From (6.13), (6.16) and Fatou's lemma, we derive

$$\int_0^\infty e^{-\gamma t} EL(t)\, dt \le \overline{C}_\varepsilon \le \overline{C},$$

where $\overline{C}$ is a constant. Consequently there exists a sequence $t_k$, $k \geq 1$, such that $t_k \to \infty$ and $\lim_{k\to\infty} e^{-\gamma t_k} EL(t_k) = 0$. From this fact, it follows that

$$E\int_0^\infty e^{-\gamma t}\, dL(t) = \lim_{k\to\infty}\left\{EL(t_k)e^{-\gamma t_k} + \gamma\int_0^{t_k} EL(t)e^{-\gamma t}\, dt\right\} \leq \overline{C}_\gamma. \qquad \square$$

Lemmas 6.6 and 6.7 and the results of Section 5 yield (6.19). This completes the proof of Theorem 2.

**7. Asymptotically optimal control policy: Proof of Theorem 3.** In this section, we consider policies characterized by a single critical level. The commodity is produced only when inventory is less than or equal to this level, the production rate being maximal if the inventory is strictly less than this level and equal to the deterministic demand rate $d$ if the inventory is equal to this level. We show that by choosing the critical level properly, one can construct a sequence of policies whose associated costs approach the optimal cost of the singular control problem described in Section 5.

Let $z_\varepsilon^*$ be defined by (3.9). The associated control policy [see (3.10)] can be written in terms of the rescaled inventory process $Y^\varepsilon(t) = \varepsilon X^\varepsilon(\varepsilon^{-2}t)$ as

$$p^\varepsilon(\varepsilon^{-2}t) = r_\varepsilon I(Y^\varepsilon(t) \leq z_\varepsilon^*) + d_\varepsilon I(Y^\varepsilon(t) = z_\varepsilon^*). \tag{7.1}$$

Let $L^\varepsilon, W^\varepsilon, \dots$ be defined by (6.2)–(6.5) with the control $p^\varepsilon$ given by (7.1), and let $Y^\varepsilon$ be the corresponding solution to (6.1). Let $L^*$ be the optimal control for the limiting problem given by (3.6) and let $Y^*$ be the optimal process corresponding to the control $L^*$; that is,

$$Y^*(t) = x + W(t) - \sup_{s\leq t}\left[x + W(s) - z^*\right]^+, \tag{7.2}$$

where $W$ is a Brownian motion with drift $c$ and variance (3.1). Put $J_x^\varepsilon(L^\varepsilon) = J^\varepsilon(\varepsilon^{-1}x, p^\varepsilon)$. Our goal is to show that $J_x^\varepsilon(L^\varepsilon) \to J_x(L^*)$ as $\varepsilon \to 0$; that is,

$$\lim_{\varepsilon\to 0}\int_0^\infty e^{-\gamma_\varepsilon t} h(Y^\varepsilon(t))\, dt = E\int_0^\infty e^{-\gamma t} h(Y^*(t))\, dt. \tag{7.3}$$

The proof of (7.3) is based on two facts:

1. For any $t > 0$, $Y^\varepsilon(t) \to_d Y^*(t)$, $\varepsilon \to 0$.
2. There exists $K > 0$ such that

$$\sup_\varepsilon \left[E(Y^\varepsilon(t))^2\right]^{1/2} \leq x + Kt + K, \qquad t \geq 0.$$

Indeed, if fact 1 holds, then

$$h(Y^\varepsilon(t)) \to_d h(Y^*(t)), \qquad \varepsilon \to 0, t > 0, \tag{7.4}$$

by the continuous mapping theorem. Because $|h(x)| \leq \max(C^+, C^-)|x|$, statement 2 implies that $\sup_\varepsilon E[h(Y^\varepsilon(t))]^2 < \infty$ and that the family of random variables $h(Y^\varepsilon(t))$, $\varepsilon \in \mathscr{E}$, is uniformly integrable. Thus, we have from (7.4),

$$Eh(Y^\varepsilon(t)) \to Eh(Y^*(t)) \quad \text{as } \varepsilon \to 0 \text{ for any } t > 0. \tag{7.5}$$

Let $g^{\varepsilon}(t) = E\{e^{-(\gamma_\varepsilon - \gamma)t} h(Y^{\varepsilon}(t))\}$ and $g(t) = Eh(Y(t))$. Equality (7.5) implies pointwise convergence of the sequence of functions $g^{\varepsilon}(\cdot)$ to $g(\cdot)$ as $\varepsilon \to 0$. Using fact 2 we can write

$$[Eh(Y^{\varepsilon}(t))]^2 \le E[h(Y^{\varepsilon}(t))]^2 \le \max(C^+, C^-)^2 (x + Kt + K)^2.$$

Without loss of generality we can assume that $|\gamma_\varepsilon - \gamma| \le \gamma/4$. By Jensen's inequality we have

$$\int_0^\infty e^{-\gamma t} (g^{\varepsilon}(t))^2 \, dt \le \int_0^\infty e^{-\gamma t/2} \max(C^+, C^-)^2 (x + Kt + K)^2 \, dt \le \text{const}.$$

Therefore, the sequence $g^{\varepsilon}(\cdot)$ is uniformly integrable with respect to the measure $e^{-\gamma t}\, dt$ and

$$\int_0^\infty e^{-\gamma t} g^{\varepsilon}(t) \, dt \to \int_0^\infty e^{-\gamma t} g(t) \, dt,$$

whence (7.3) follows.

We first prove facts 1 and 2 for $x < z^*$, where $z^*$ is given by (3.7). It is easy to see that

$$z_\varepsilon^* \to z^* \quad \text{as } \varepsilon \to 0, \tag{7.6}$$

where $z_\varepsilon^*$ and $z^*$ are given by (3.9) and (3.7), respectively. Therefore, we may assume without loss of generality that $x < z_\varepsilon^*$ for all $\varepsilon \in \mathscr{E}$. In this case, the process $Y^\varepsilon$ does not exceed $z_\varepsilon^*$, and using (6.1) and (6.2), we can write

$$Y^{\varepsilon}(t) = x + W^{\varepsilon}(t) - L^{\varepsilon}(t), \tag{7.7}$$

$$L^{\varepsilon}(t) = (r_\varepsilon - d_\varepsilon)\varepsilon^{-1} \int_0^t G^{\varepsilon}(\varepsilon^{-2}s) I(Y^{\varepsilon}(s) = z_\varepsilon^*) \, ds. \tag{7.8}$$

For further analysis, we need to introduce the concepts of a Skorohod problem and a reflecting mapping [11], [13]. Denote by $D_0[0,\infty)$, the space of functions $u \in D[0,\infty)$ with $u(0) \in (-\infty, z]$.

DEFINITION 4. Given $u \in D_0[0,\infty)$, the *Skorohod problem in the region* $(-\infty, z]$ *with reflecting barrier at* $z$ consists in finding a pair of functions $(y, l)$, $y, l \in D[0,\infty)$ such that:

(i) $y(t) = u(t) - l(t)$, $t \ge 0$.
(ii) $y(t) \in (-\infty, z]$, $t \ge 0$.
(iii) $l(\cdot)$ is a nondecreasing function with $l(0) = 0$ and

$$\int_0^\infty I(y(t) \ne z) \, dl(t) = 0.$$

It is known that for any $u \in D_0[0,\infty)$, there exists a unique solution to the Skorohod problem, which can be written as

$$y(t) = u(t) - \sup_{s \le t} [u(s) - z]^+, \qquad l(t) = \sup_{s \le t} [u(s) - z]^+ \tag{7.9}$$

(see [11]). Thus, one can define a *reflecting mapping* $\Phi^z\colon D_0[0,\infty) \to D_0[0,\infty)$, setting $\Phi^z(u) = y$, where $y$ is defined by (7.9). We denote by $\Phi_t^z(u)$ the value of $\Phi^z(u)$ at time $t$.

Note that by virtue of (7.7) and (7.8), the pair $(Y^\varepsilon, L^\varepsilon)$ is a solution to the Skorohod problem for the process $x + W^\varepsilon(\cdot)$ in the region $(-\infty, z_\varepsilon^*]$. Indeed, $Y^\varepsilon(t) \in (-\infty, z_\varepsilon^*]$ and the first condition in Definition 4 is fulfilled. Because $r_\varepsilon > d_\varepsilon$, it holds that $L^\varepsilon$ is nondecreasing and it increases only when $Y^\varepsilon(t) = z_\varepsilon^*$. Consequently, we can write

$$Y^\varepsilon = \Phi^{z_\varepsilon^*}(x + W^\varepsilon). \tag{7.10}$$

It follows from (7.6) and (7.9) that

$$\sup_{s \le t} |\Phi_s^{z_\varepsilon^*}(x + W^\varepsilon - \Phi_s^{z^*})(x + W^\varepsilon)| \le |z_\varepsilon^* - z^*| \to 0 \text{ as } \varepsilon \to 0. \tag{7.11}$$

The mapping $\Phi^{z^*}$ is continuous in the uniform topology on $D[0,\infty)$ (see [11]), and according to the corollary to Lemma 6.3, $W^\varepsilon \to_d W$ as $\varepsilon \to 0$, where $W$ is a Brownian motion with drift $c$ and variance $\sigma^2$ given by (3.1). By the continuous mapping theorem, $\Phi^{z^*}(x + W^\varepsilon) \to_d Y^*$ as $\varepsilon \to 0$, where $Y^* = \Phi^{z^*}(x + W)$ is given by (7.2). This fact together with (7.11) implies $Y^\varepsilon \to_d Y^*$ as $\varepsilon \to 0$. Therefore, fact 1 is proved.

Because the reflection mapping $\Phi^{z_\varepsilon^*}$ satisfies the Lipschitz condition with coefficient 1 in the uniform topology in $D_0[0,\infty)$, we can use (7.10) and write $|Y^\varepsilon(t)| \le \sup_{s \le t} |Y^\varepsilon(t)| \le x + \sup_{s \le t} |W^\varepsilon|$. By Minkowski's inequality, $[E(Y^\varepsilon(t))^2]^{1/2} \le x + [E \sup_{s \le t} |W^\varepsilon(s)|^2]^{1/2}$. Applying Lemma 2, we derive fact 2.

Suppose $x \ge z^*$. According to the Corollary to Lemma 6.3, $W^\varepsilon$ defined by (6.3) converges weakly in $D[0,\infty)$ to the Brownian motion $W$. By the Skorohod representation theorem, this convergence can be realized on another probability space as almost sure convergence. Therefore, we can assume that $W^\varepsilon$ and $W$ are defined on the same probability space and for any $T > 0$,

$$\lim_{\varepsilon \to 0} \sup_{t \le T} |W^\varepsilon(t) - W(t)| = 0, \qquad \text{P-a.s.}$$

Because $L^*(t) \equiv \sup_{s \le t}[x + W(s) - z^*]^+ = x - z^* + \sup_{s \le t}[W(s)]^+$, relation (3.6) yields

$$Y^*(t) = \Phi_t^{z^*}(z^* + W). \tag{7.12}$$

To prove fact 1, it is sufficient to show that

$$\text{P-}\lim_{\varepsilon \to 0} |Y^\varepsilon(t) - Y^*(t)| = 0, \qquad t > 0. \tag{7.13}$$

Define $\tau^\varepsilon = \inf\{s \ge 0\colon Y^\varepsilon(s) \le z_\varepsilon^*\}$ and $\overline{Y}^\varepsilon(t) = Y^\varepsilon(t + \tau^\varepsilon)$, $t \ge 0$. Obviously,

$$\overline{Y}^\varepsilon(0) = Y^\varepsilon(\tau^\varepsilon) \le z_\varepsilon^* \tag{7.14}$$

and, similarly to (7.10),

$$\overline{Y}^\varepsilon(t) = \Phi_t^{z_\varepsilon^*}(\overline{Y}^\varepsilon(0) + W^\varepsilon(t + \tau^\varepsilon) - W^\varepsilon(\tau^\varepsilon)). \tag{7.15}$$

Put $Y^{*,\varepsilon}(t) = \Phi_t^{z_\varepsilon^*}(z_\varepsilon^* + W)$. For arbitrary $t > 0$ and $\delta > 0$, we can write

$$(7.16)\qquad \begin{aligned} &P(|Y^\varepsilon(t) - Y^*(t)| > \delta) \\ &\quad \le P\Big(\sup_{\tau^\varepsilon \le u \le \tau^\varepsilon + t} |Y^\varepsilon(u) - Y^*(u)| > \delta, \tau^\varepsilon \le t\Big) \\ &\qquad + P(\tau^\varepsilon > t) \\ &\quad \le P\Big(\sup_{0 \le u \le t} \big|\bar{Y}^\varepsilon(u) - Y^{*,\varepsilon}(u)\big| > \delta/3, \tau^\varepsilon \le t\Big) \\ &\qquad + P\Big(\sup_{0 \le u \le t} |Y^*(u + \tau^\varepsilon) - Y^*(u)| > \delta/3\Big) \\ &\qquad + P\Big(\sup_{0 \le u \le t} |Y^{*,\varepsilon}(u) - Y^*(u)| > \delta/3\Big) \\ &\qquad + P(\tau^\varepsilon > t). \end{aligned}$$

Note that $\tau^\varepsilon \le (x - z^*)(d_\varepsilon^{-1})\varepsilon$ and $d_\varepsilon \to d$ as $\varepsilon \to 0$. Hence, $P(\tau_\varepsilon > t) \to 0$, $\varepsilon \to 0$. Because $Y^*$ has continuous trajectories, the second term in the right-hand side of (7.16) approaches zero as $\varepsilon \to 0$. Obviously $\sup_{0 \le u \le t} |Y^{*,\varepsilon}(u) - Y^*(u)| \le 2|z_\varepsilon^* - z^*|$. Thus in view of (7.6), the third term converges to zero. Now we show that the first term also goes to zero as $\varepsilon \to 0$.

Using the Lipschitz property of $\Phi^{z_\varepsilon^*}$ and (7.12), (7.14) and (7.15), we can write

$$\begin{aligned} \sup_{0 \le u \le t} \big|\bar{Y}^\varepsilon(u) - Y^{*,\varepsilon}(u)\big| &\le z^* - Y^\varepsilon(\tau^\varepsilon) \\ &\quad + \sup_{0 \le u \le t} |W^\varepsilon(u + \tau^\varepsilon) - W^\varepsilon(\tau^\varepsilon) - W(u)|. \end{aligned}$$

On the set $\{\tau^\varepsilon \le t\}$, the estimate

$$(7.17)\qquad z_\varepsilon^* - Y^\varepsilon(\tau^\varepsilon) \le \varepsilon \max_{i \le A^\varepsilon(\varepsilon^{-2}t)} \eta_i^\varepsilon$$

is true, and we can write

$$\begin{aligned} &P\Big(\sup_{0 \le u \le t} \big|\bar{Y}^\varepsilon(u) - Y^{*,\varepsilon}(u)\big| > \delta/3, \tau^\varepsilon \le t\Big) \\ &\quad \le P\Big(\max_{i \le A^\varepsilon(\varepsilon^{-2}t)} \varepsilon\eta_i^\varepsilon > \delta/6\Big) \\ &\qquad + P\Big(\sup_{0 \le u \le t} |W^\varepsilon(u + \tau^\varepsilon) - W^\varepsilon(\tau^\varepsilon) - W(u)| > \delta/6\Big). \end{aligned}$$

Because $\tau^\varepsilon$ converges in probability to zero as $\varepsilon \to 0$ and $W^\varepsilon$ converges to $W$, which has continuous trajectories, the second term in the preceding inequality converges to zero as $\varepsilon \to 0$. For any $t \ge 0$, we have P-lim $\varepsilon^2 A^\varepsilon(\varepsilon^{-2}t) = \hat{\lambda}t$, and by Lemma 6.4, it follows, therefore, that

$$P\Big(\varepsilon \max_{i \le A^\varepsilon(\varepsilon^{-2}t)} \eta_i^\varepsilon > \delta/6\Big) \to 0, \qquad \varepsilon \to 0.$$

Allowing $\varepsilon$ to approach zero in (7.16), we derive (7.13).

Next we prove fact 2. Note that $Y^\varepsilon(t) \in [z_\varepsilon^*, x]$ if $\omega \in \{\tau^\varepsilon > t\}$ and on the set $\{\tau^\varepsilon \le t\}$, (7.15) and (7.17) imply $|Y^\varepsilon(t)| \le x + \varepsilon \max_{i \le A^\varepsilon(\varepsilon^{-2}t)} \eta_i^\varepsilon + 2\sup_{s \le t} |W^\varepsilon(s)|$. Consequently, the latter estimate is valid everywhere and

$$\text{(7.18)} \qquad \begin{aligned} \left[E(Y^\varepsilon(t))^2\right]^{1/2} &\le x + 2\left[E \sup_{s \le t} |W^\varepsilon(s)|^2\right]^{1/2} \\ &\quad + \left[E\left(\varepsilon \max_{i \le A^\varepsilon(\varepsilon^{-2}t)} \eta_i^\varepsilon\right)^2\right]^{1/2}. \end{aligned}$$

Applying Lemma 6.1 and using independence of $A^\varepsilon$ and $\{\eta_i^\varepsilon\}_{i \ge 1}$, we can write

$$\text{(7.19)} \qquad \begin{aligned} \left[E\left(\varepsilon \max_{i \le A^\varepsilon(\varepsilon^{-2}t)} \eta_i^\varepsilon\right)^2\right]^{1/2} &\le \left[\varepsilon^2 E \sum_{i=1}^{A^\varepsilon(\varepsilon^{-2}t)} (\eta_i^\varepsilon)^2\right]^{1/2} \\ &= (\mu_\varepsilon^{-2} + \sigma_{4\varepsilon}^2) E \varepsilon^2 A^\varepsilon(\varepsilon^{-2}t) \\ &\le Kt + K. \end{aligned}$$

By virtue of Lemma 6.3, (7.18) and (7.19), we derive statement 2.

## APPENDIX

Proof of Lemma 6.2. Consider the representation (6.3)–(6.5). By virtue of (3.8), the first term on the right-hand side of (6.3) is less than $K_1 t$, where $K_1$ is a constant. In view of the convergence of $r^\varepsilon$, we can choose $K_1$ such that $r_\varepsilon \le K_1$. Minkowski's inequality implies that

$$\text{(A.1)} \qquad \begin{aligned} E\left(\sup_{s \le t} |W^\varepsilon(s)|^2\right)^{1/2} &\le K_1 t + K_1\left[\left(E \sup_{s \le t} |M^\varepsilon(s)|^2\right)^{1/2} \right. \\ &\quad \left. + \left(E \sup_{s \le t} |N^\varepsilon(s)|^2\right)^{1/2}\right]. \end{aligned}$$

In the sequel we prove that there exists a constant $K_2$ such that

$$\text{(A.2)} \qquad \sup_\varepsilon \left(E \sup_{s \le t} |M^\varepsilon(s)|^2\right)^{1/2} \le K_2\sqrt{t} + K_2$$

and

$$\text{(A.3)} \qquad \sup_\varepsilon \left(E \sup_{s \le t} |N^\varepsilon(s)|^2\right)^{1/2} \le K_2\sqrt{t} + K_2.$$

One can see that (A.1)–(A.3) imply Lemma 6.2.

In order to prove (A.2) and (A.3), we introduce a renewal process $D^\varepsilon = (D^\varepsilon(t),\ t \ge 0)$ generated by the sequence $\{\alpha_i^\varepsilon + \beta_i^\varepsilon\}_{i \ge 1}$; that is,

$$\text{(A.4)} \qquad D^\varepsilon(t) = \max\{j: \alpha_1^\varepsilon + \beta_1^\varepsilon + \cdots + \alpha_j^\varepsilon + \beta_j^\varepsilon \le t\}.$$

One can write the estimate

$$\sup_{s\le t}\left|\int_0^s G^\varepsilon(u)\,du - \sum_{i=1}^{D^\varepsilon(s)+1}\alpha_i^\varepsilon\right| \le \max_{i\le D_t^\varepsilon+1}\alpha_i^\varepsilon \le \max_{i\le D_t^\varepsilon+1}\left(\alpha_i^\varepsilon+\beta_i^\varepsilon\right). \tag{A.5}$$

We will also need the identity

$$\begin{aligned}\sum_{i=1}^{D^\varepsilon(t)+1}\alpha_i^\varepsilon - \frac{b_\varepsilon}{a_\varepsilon+b_\varepsilon}t = {}& \frac{a_\varepsilon}{a_\varepsilon+b_\varepsilon}\sum_{i=1}^{D^\varepsilon(t)+1}\left(\alpha_i^\varepsilon - a_\varepsilon^{-1}\right)\\ &+\frac{b_\varepsilon}{a_\varepsilon+b_\varepsilon}\sum_{i=1}^{D^\varepsilon(t)+1}\left(\beta_i^\varepsilon - b_\varepsilon^{-1}\right)\\ &+\frac{b_\varepsilon}{a_\varepsilon+b_\varepsilon}\left\{\sum_{i=1}^{D^\varepsilon(t)+1}\left(\alpha_i^\varepsilon+\beta_i^\varepsilon\right)-t\right\}.\end{aligned} \tag{A.6}$$

Note that

$$\sup_{s\le t}\left|\sum_{i=1}^{D^\varepsilon(t)+1}\left(\alpha_i^\varepsilon+\beta_i^\varepsilon\right)-t\right| \le \max_{i\le D^\varepsilon(t)+1}\left(\alpha_i^\varepsilon+\beta_i^\varepsilon\right). \tag{A.7}$$

Define processes $M_1^\varepsilon$ and $M_2^\varepsilon$ as

$$M_1^\varepsilon(t) = \varepsilon\sum_{i=1}^{D^\varepsilon(\varepsilon^{-2}t)+1}\left(\alpha_i^\varepsilon - a_\varepsilon^{-1}\right),$$

$$M_2^\varepsilon(t) = \varepsilon\sum_{i=1}^{D^\varepsilon(\varepsilon^{-2}t)+1}\left(\beta_i^\varepsilon - b_\varepsilon^{-1}\right)$$

and let $\Delta M_i^n(s) = M_i^\varepsilon(s) - M_i^\varepsilon(s-)$, $i=1,2$. Then we can write

$$\begin{aligned}\varepsilon\max_{i\le D^\varepsilon(\varepsilon^{-2}t)+1}\left(\alpha_i^\varepsilon+\beta_i^\varepsilon\right) \le{}& |M_1^\varepsilon(0)| + |M_2^\varepsilon(0)| + \sup_{0<s\le t}|\Delta M_1^\varepsilon(s)|\\ &+\sup_{0<s\le t}|\Delta M_2^\varepsilon(s)| + \left(\frac{1}{a_\varepsilon}+\frac{1}{b_\varepsilon}\right)\varepsilon.\end{aligned}$$

For $M^\varepsilon$ defined by (6.4), this inequality together with (A.5)–(A.7) implies

$$\begin{aligned}\sup_{s\le t}|M^\varepsilon(s)| \le{}& \frac{a_\varepsilon}{a_\varepsilon+b_\varepsilon}\sup_{s\le t}|M_1^\varepsilon(s)|\\ &+\frac{b_\varepsilon}{a_\varepsilon+b_\varepsilon}\sup_{s\le t}|M_2^\varepsilon(s)| + \left(\frac{1}{b_\varepsilon}+\frac{2}{a_\varepsilon}\right)\varepsilon\\ &+\left(1+\frac{b_\varepsilon}{a_\varepsilon+b_\varepsilon}\right)\Big(|M_1^\varepsilon(0)| + |M_2^\varepsilon(0)|\\ &+\sup_{0<s\le t}|\Delta M_1^\varepsilon(s)| + \sup_{0<s\le t}\Delta|M_2^\varepsilon(s)|\Big).\end{aligned} \tag{A.8}$$

Define $\mathscr{J}_k^\varepsilon = \sigma\{\alpha_1^\varepsilon, \beta_1^\varepsilon, \dots, \alpha_k^\varepsilon, \beta_k^\varepsilon\}$. It is easy to see that $D^\varepsilon(\varepsilon^{-2}t)+1$ is a stopping time with respect to the family of $\sigma$-fields $(\mathscr{J}_k^\varepsilon)_{k\ge 1}$. Hence we can define $\mathscr{G}_t^\varepsilon = \mathscr{J}^\varepsilon_{D^\varepsilon(\varepsilon^{-2}t)+1}$, $t \ge 0$. In a way similar to the proof of Lemma 1 in [22], one can prove that the processes $M_1^\varepsilon$ and $M_2^\varepsilon$ are martingales with respect to the filtration $\mathscr{G}^\varepsilon$. Consequently, using the Burkholder–Handy inequality for martingales ([30], Chapter 1, Section 9), we can write

$$\text{(A.9)}\qquad \Big(E\sup_{s\le t}|M_i^\varepsilon(s)|^2\Big)^{1/2} \le \Big(E(M_i^\varepsilon(0))^2\Big)^{1/2} + C_2\big(E[M_i^\varepsilon]_t\big)^{1/2},$$

where $C_2$ is a constant and $[M_i^\varepsilon]_t = \Sigma_{0<s\le t}(\Delta M_i^\varepsilon(s))^2$ is the quadratic variation of the martingale $M_i^\varepsilon$, $i = 1,2$.

Because $\sup_{0<s\le t}(\Delta M_i^\varepsilon(s))^2 \le \Sigma_{0<s\le t}(\Delta M_i^\varepsilon(s))^2$, Minkowski's inequality, (A.8) and (A.9) imply

$$\text{(A.10)}\qquad \begin{aligned} \Big(E\sup_{s\le t}|M^\varepsilon(s)|^2\Big)^{1/2} &\le 2\Big(E(M_1^\varepsilon(0))^2\Big)^{1/2} \\ &\quad + \left(1+\frac{2b_\varepsilon}{a_\varepsilon+b_\varepsilon}\right)\Big(E(M_2^\varepsilon(0))^2\Big)^{1/2} \\ &\quad + \left(\frac{C_2 a_\varepsilon}{a_\varepsilon+b_\varepsilon}+1+\frac{b_\varepsilon}{a_\varepsilon+b_\varepsilon}\right)\big(E[M_1^\varepsilon]_t\big)^{1/2} \\ &\quad + \left(\frac{C_2 b_\varepsilon}{a_\varepsilon+b_\varepsilon}+1+\frac{b_\varepsilon}{a_\varepsilon+b_\varepsilon}\right)\big(E[M_2^\varepsilon]_t\big)^{1/2} \\ &\quad + \varepsilon\left(\frac{1}{b_\varepsilon}+\frac{2}{a_\varepsilon}\right). \end{aligned}$$

Because $\{i \le D^\varepsilon(\varepsilon^{-2}t)\} = \{\Sigma_{j=1}^i(\alpha_j^\varepsilon + \beta_j^\varepsilon) \le \varepsilon^{-2}t\}$ does not depend on $\alpha_{i+1}^\varepsilon$, we can write

$$\begin{aligned} E[M_1^\varepsilon]_t &= E\varepsilon^2 \sum_{i=1}^{D^\varepsilon(\varepsilon^{-2}t)} \big(\alpha_{i+1}^\varepsilon - a_\varepsilon^{-1}\big)^2 \\ &= \varepsilon^2 \sum_{i=1}^{\infty} E\big(\alpha_{i+1}^\varepsilon - a_\varepsilon^{-1}\big)^2 I(i \le D^\varepsilon(\varepsilon^{-2}t)) \\ &= \sigma_{1\varepsilon}^2 E\varepsilon^2 D^\varepsilon(\varepsilon^{-2}t). \end{aligned}$$

In addition, we also have $(E(M_1^\varepsilon(0))^2)^{1/2} = \varepsilon\sigma_{1\varepsilon}$. Similarly,

$$E[M_2^\varepsilon]_t = \sigma_{2\varepsilon}^2 E\varepsilon^2 D^\varepsilon(\varepsilon^{-2}t), \qquad \Big(E\big(M_2^\varepsilon(0)^2\big)\Big)^{1/2} = \varepsilon\sigma_{2\varepsilon}.$$

According to Lemma 6.1, $\sup_\varepsilon E\varepsilon^2 D^\varepsilon(\varepsilon^{-2}t) \le K + Kt$. Thus, it follows from (A.10) that

$$\begin{aligned}\left(E \sup |M^\varepsilon(s)|^2\right)^{1/2} &\le 2\sigma_{1\varepsilon}\varepsilon + \varepsilon\left(1 + \frac{2b_\varepsilon}{a_\varepsilon + b_\varepsilon}\right)\sigma_{2\varepsilon} + \varepsilon\left(b_\varepsilon^{-1} + 2a_\varepsilon^{-1}\right) \\ &\quad + \left[\sigma_{1\varepsilon}\left(\frac{C_2 a_\varepsilon}{a_\varepsilon + b_\varepsilon} + 1 + \frac{b_\varepsilon}{a_\varepsilon + b_\varepsilon}\right)\right. \\ &\quad \left. + \sigma_{2\varepsilon}\left(\frac{C_2 b_\varepsilon}{a_\varepsilon + b_\varepsilon} + 1 + \frac{b_\varepsilon}{a_\varepsilon + b_\varepsilon}\right)\right](Kt + K)^{1/2}.\end{aligned}$$

The foregoing inequality and (2.7) yield (A.2).

Now let us prove (A.3). We have

$$N^\varepsilon(t) = \varepsilon \sum_{i=1}^{A^\varepsilon(\varepsilon^{-2}t)} \left(\eta_i^\varepsilon - \frac{1}{\mu_\varepsilon}\right) + \frac{1}{\mu_\varepsilon}\varepsilon\left(A^\varepsilon(\varepsilon^{-2}t) - \lambda_\varepsilon \varepsilon^{-2}t\right). \tag{A.11}$$

According to Lemma 6.1, there exists a constant $K$ such that

$$\left(E \sup_{s \le t} \left|\varepsilon\left(A^\varepsilon(\varepsilon^{-2}t) - \lambda_\varepsilon \varepsilon^{-2}t\right)\right|^2\right)^{1/2} \le K + K\sqrt{t}. \tag{A.12}$$

The process

$$H^\varepsilon(t) = \varepsilon \sum_{i=1}^{A^\varepsilon(\varepsilon^{-2}t)} \left(\eta_i^\varepsilon - \mu_\varepsilon^{-1}\right), \qquad t \ge 0,$$

is a martingale with respect to the family of $\sigma$-fields $\mathscr{H}_t^\varepsilon = \sigma\{H^\varepsilon(u),\, u \le t\}$, $t \ge 0$. Applying the Burkholder–Handy inequality, we derive

$$\begin{aligned}E\left(\sup_{s \le t}\left|\varepsilon \sum_{i=1}^{A^\varepsilon(\varepsilon^{-2}t)} \left(\eta_i^\varepsilon - \mu_\varepsilon^{-1}\right)\right|^2\right)^{1/2} &\le C_2\left(E\varepsilon^2 \sum_{i=1}^{A^\varepsilon(\varepsilon^{-2}t)} \left(\eta_i^\varepsilon - \mu_\varepsilon^{-1}\right)^2\right)^{1/2} \\ &= C_2\sigma_{4\varepsilon}\left(E\varepsilon^2 A^\varepsilon(\varepsilon^{-2}t)\right)^{1/2}.\end{aligned}$$

By Lemma 6.1, $E\varepsilon^2 A^\varepsilon(\varepsilon^{-2}t) \le K + Kt$. Consequently, using (A.11) and (A.12), we can derive

$$\left(E \sup_{s \le t} |N^\varepsilon(s)|^2\right)^{1/2} \le \frac{1}{\mu_\varepsilon}\left(K + K\sqrt{t}\right) + C_2\sigma_4^\varepsilon(Kt + K)^{1/2}.$$

Because $\mu_\varepsilon \to \hat{\mu}$ and $\sigma_4^\varepsilon \to \hat{\sigma}_4$ as $\varepsilon \to 0$, the sequences $\mu_\varepsilon$ and $\sigma_{4\varepsilon}$, $\varepsilon \in \mathscr{E}$, are bounded. Consequently, there exists $K_2$ such that (A.3) holds. Lemma 6.2 is proved. □

PROOF OF LEMMA 6.3. Let $N^\varepsilon$ be the same as in (6.5) and show that

$$N^\varepsilon \to_d N, \qquad \varepsilon \to 0. \tag{A.13}$$

Denote

$$S^\varepsilon(t) = \varepsilon \sum_{i=1}^{[\varepsilon^{-2}t]} (\eta_1^\varepsilon - \mu^{-1}), \qquad \hat{A}^\varepsilon(t) = \varepsilon\big(A^\varepsilon(\varepsilon^{-2}t) - \lambda_\varepsilon \varepsilon^{-2}t\big),$$

$$\phi^\varepsilon(t) = \varepsilon^{-2}A^\varepsilon(\varepsilon^{-2}t).$$

Then from (5.5), we have

$$N^\varepsilon(t) = S^\varepsilon(\phi^\varepsilon(t)) + \frac{1}{\mu_\varepsilon}\hat{A}^\varepsilon(t). \tag{A.14}$$

Donsker's theorem and Theorem 17.3 [4] yield $S^\varepsilon \to_d S$ and $\hat{A}^\varepsilon \to_d \hat{A}$, $\varepsilon \to 0$, where $S$ is $(0, \sigma_4^2)$ and $\hat{A}$ is a $(0, \lambda^3\sigma_3^2)$ Brownian motion. Applying the law of large numbers for the renewal processes $A^\varepsilon(\cdot)$ we can write P-$\lim_\varepsilon \sup_{t \le T}|\phi^\varepsilon(t) - \lambda t| = 0$, $T \ge 0$. Because the processes $S^\varepsilon$ and $\hat{A}^\varepsilon$ are independent and $\phi^\varepsilon$ converges to a deterministic function $\phi(t) = \lambda t$, we have $(S^\varepsilon, \hat{A}^\varepsilon, \phi^\varepsilon) \to_d (S, \hat{A}, \phi)$, as $\varepsilon \to 0$. The required relation (A.13) follows from (A.14) by virtue of the continuous mapping theorem and we have the equality $N(t) = S(\lambda t) = (1/\mu)\hat{A}(t)$. Because $S$ and $\hat{A}$ are independent, we conclude that $N$ is a Brownian motion with zero drift and variance given by (6.8).

Now we show that

$$M^\varepsilon \to_d M, \qquad \varepsilon \to 0. \tag{A.15}$$

Denote

$$Z^\varepsilon(t) = \varepsilon\left\{\sum_{i=1}^{D^\varepsilon(\varepsilon^{-2}t)} \alpha_i^\varepsilon - \frac{b_\varepsilon}{a_\varepsilon + b_\varepsilon}\varepsilon^{-2}t\right\},$$

where the process $D^\varepsilon$ is defined by (A.4). It is easy to verify that

$$\sup_{t \le T}|M^\varepsilon(t) - Z^\varepsilon(t)| \le \varepsilon \max_{i \le D^\varepsilon(\varepsilon^{-2}T)+1} \alpha_i^\varepsilon. \tag{A.16}$$

Put $D = \hat{a}\hat{b}(\hat{a} + \hat{b})^{-1}$. Because

$$\text{P-}\lim_{\varepsilon \to 0} \sup_{s \le t}|\varepsilon^2 D^\varepsilon(\varepsilon^{-2}t) - Dt| = 0, \tag{A.17}$$

Lemma 6.4 implies that the right-hand side of (A.16) converges to zero as $\varepsilon \to 0$. Thus, we have P-$\lim_\varepsilon \sup_{t \le T}|M^\varepsilon(t) - Z^\varepsilon(t)| = 0$. To prove (A.15), it suffices to show that

$$Z^\varepsilon \to_d M, \qquad \varepsilon \to 0. \tag{A.18}$$

One can write the decomposition

$$\begin{aligned} Z^\varepsilon(t) = {} & \frac{a_\varepsilon}{a_\varepsilon + b_\varepsilon}\varepsilon \sum_{i=1}^{D^\varepsilon(\varepsilon^{-2}t)} (\alpha_i^\varepsilon - a_\varepsilon^{-1}) \\ & - \frac{b_\varepsilon}{a_\varepsilon + b_\varepsilon}\varepsilon \sum_{i=1}^{D^\varepsilon(\varepsilon^{-2}t)} (\beta_i^\varepsilon - b_\varepsilon^{-1}) + \Delta^\varepsilon(t), \end{aligned} \tag{A.19}$$

where $\Delta^\varepsilon(t) = b_\varepsilon(a_\varepsilon + b_\varepsilon)\varepsilon\{\sum_{i=1}^{D^\varepsilon(\varepsilon^{-2}t)}(\alpha_i^\varepsilon + \beta_i^\varepsilon) - t\}$. Applying Lemma 6.4 and

(A.17) to the estimate

$$\sup_{t\le T}|\Delta^{\varepsilon}(t)| \le \varepsilon \max_{i\le D^{\varepsilon}(\varepsilon^{-2}T)+1}(\alpha_i^{\varepsilon}+\beta_i^{\varepsilon}),$$

we get

$$\text{P-}\lim_{\varepsilon}\sup_{t\le T}|\Delta^{\varepsilon}(t)| = 0. \tag{A.20}$$

Denoting

$$R_1^{\varepsilon}(t) = \varepsilon\sum_{i=1}^{[\varepsilon^{-2}t]}(\alpha_i^{\varepsilon}-a_{\varepsilon}^{-1}), \qquad R_2^{\varepsilon}(t) = \varepsilon\sum_{i=1}^{[\varepsilon^{-2}t]}(\beta_i^{\varepsilon}-b_{\varepsilon}^{-1})$$

and applying Donsker's theorem, we derive $R_1^{\varepsilon}\to_d R_1$ and $R_2^{\varepsilon}\to_d R_2$ as $\varepsilon\to 0$, where $R_1$ and $R_2$ are $(0,\sigma_1^2)$ and $(0,\sigma_2^2)$ Brownian motions, respectively. It follows from (A.19) that

$$Z^{\varepsilon}(t) = \frac{a_{\varepsilon}}{a_{\varepsilon}+b_{\varepsilon}}R_1^{\varepsilon}(\varepsilon^2D^{\varepsilon}(\varepsilon^{-2}t)) - \frac{b_{\varepsilon}}{a_{\varepsilon}+b_{\varepsilon}}R_2^{\varepsilon}(\varepsilon^2D^{\varepsilon}(\varepsilon^{-2}t)) + \Delta^{\varepsilon}(t).$$

Because processes $R_1^{\varepsilon}$ and $R_2^{\varepsilon}$ are independent, $a_{\varepsilon}\to\hat{a}$ and $b_{\varepsilon}\to\hat{b}$, as $\varepsilon\to 0$, and (A.20) and (A.17) are satisfied, we can apply the continuous mapping theorem to the right-hand side of the preceding equality and get (A.18) with

$$M(t) = \frac{a}{a+b}R_1(Dt) - \frac{b}{a+b}R_2(Dt).$$

Taking into account independence of $R_1$ and $R_2$, we conclude that $M$ is a Brownian motion with zero drift and variance (6.9). □

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