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QUEUES IN SERIES IN LIGHT TRAFFIC

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We study light traffic approximations for queues in series with renewal arrivals and i.i.d. service time vectors. Formulae for limits of functions of the waiting time at different stations based on single customer effect are obtained for two approaches: dilation and thinning of the arrival process. Interdeparture times from a station possess a one-dependence property in light traffic. This paper complements previous studies of Daley and Rolski and also Asmussen's approach to light traffic limits applied to the cases considered.

1. Introduction. A series of queues is the simplest example of a queueing network. In most cases, the performance characteristics in such queues are analytically intractable. One possible approach proposed recently for overcoming this problem is to study systems in extreme situations; that is, in light and heavy traffic conditions. Although this approach does not give explicit formulae, interpolation formulae are possible in some cases as suggested in Reiman and Simon (1988) or Whitt (1989).

We assume a renewal input at the first station and that the vectors consisting of each customer's service times at all stations are independent and identically distributed. Moreover, we assume that the service times are independent of the input but allow dependence, for each customer, of service times at different stations. All customers obtain service in the same order, and an unlimited queue may form in front of each server. If the input is Poisson, light traffic results were given by Wolff (1982), Pinedo and Wolff (1982) and Greenberg and Wolff (1988). Reiman and Simon (1989) studied open Markovian networks of queues in light traffic conditions and from their paper one can also get results for queues in series. However, the techniques used in these papers are not applicable in a non-Markovian situation.

This paper arises out of work on light traffic in single-server queues by Daley and Rolski (1984, 1991, 1993), Asmussen (1991) and Błaszczyszyn (1990), using, in particular, methods developed in Daley and Rolski (1992) for many-server queues. As in these papers, we study stationary waiting times at a station when the light traffic conditions are defined in terms of either a γ-dilation or a π-thinning scheme. We note that there is an essential difference between such characteristics and, for example, the work load at a station, because the asymptotic behavior of the work load in light traffic does not depend on the fine structure of the input process, but is a consequence of

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Campbell's type formulae [see, e.g., recent papers by Baccelli and Brémaud (1991) or Sigman (1991) for such results]. A new feature studied in this paper concerns interdeparture times from a node in light traffic conditions: These are interarrival times at the next station. In light traffic we find asymptotic properties of a marginal interdeparture time from a station, and also show that the sequence of interdeparture times is asymptotically one-dependent. Our asymptotic results are based on the single-customer effect that has been observed in other situations, namely, that the waiting time of any customer at the mth station is similar in light traffic to the waiting time of a customer that follows a customer that has found the system empty.

As in previous papers, we conclude from our analysis that light traffic limits for queues in series reflect consequences of any local clustering or clumping behavior of the arrival process. For example, this paper considers renewal arrival processes for which

$$\Pr\{T \le t\} = c_A t^{\alpha} + o(t^{\alpha}), \qquad t \downarrow 0, \tag{1.1}$$

for some strictly positive constant c_A and nonnegative α. Using the definition introduced in Daley and Rolski (1991), we say that the interarrival time d.f. belongs to the class $\mathscr{S}_\alpha = \mathscr{S}_\alpha(c_A)$. Notice that a Poisson arrival process at rate λ belongs to $\mathscr{S}_1(\lambda)$. In Sections 4 and 5 we assume $\alpha > 0$.

2. Preliminaries. The following idea of equivalence of two sequences $\{X(\gamma)\}$ and $\{Y(\gamma)\}$ of nonnegative random variables that are degenerate at zero for $\gamma \to \infty$ comes from Asmussen (1991); see also Daley and Rolski (1992).

DEFINITION 2.1. Two families of nonnegative random variables $\{X(\gamma)\}$ and $\{Y(\gamma)\}$ are asymptotically conditionally equivalent (ACE) when, in the limit $\gamma \to \infty$.:

(a) $\Pr\{X(\gamma) \ne 0\} \to 0$, $\Pr\{Y(\gamma) \ne 0\} \to 0$, $\Pr\{X(\gamma) \ne 0\}/\Pr\{Y(\gamma) \ne 0\} \to 1$.
(b) $\|\Pr\{X(\gamma) \in \cdot | X(\gamma) \ne 0\} - \Pr\{Y(\gamma) \in \cdot | Y(\gamma) \ne 0\}\| \to 0$.

Suppose that a model is parameterized by $\gamma \ge 0$ and has a characteristic of interest $Y(\gamma)$ for which $\Pr\{Y(\gamma) > 0\} \to 0$ for $\gamma \to \infty$. The following technique was successfully applied to many-server queues in Daley and Rolski (1992) to characterize a family $\{Y(\gamma)\}$ in "light traffic conditions." It is supposed that there exists a probability space $(\overline{\Omega}, \overline{\mathscr{F}}, \overline{\Pr})$, and families $\{\overline{X}(\gamma)\}, \{\overline{Y}(\gamma)\}$ such that for each $\gamma \ge 0$, $\overline{X}(\gamma) \le \overline{Y}(\gamma)$ a.s., where $\overline{Y}(\gamma) =_{\mathrm{d}} Y(\gamma)$ and a family of nonnegative random variables $\overline{U}(\gamma)$ is such that $\overline{X}(\gamma) = \overline{Y}(\gamma)$ on $\{\overline{U}(\gamma) = 0\}$. The following proposition is extracted from the proof of Theorem 4 from Daley and Rolski (1992).

PROPOSITION 2.2. *If for each γ the random variable $\overline{U}(\gamma)$ is independent of $\overline{X}(\gamma)$ and if, as $\gamma \to \infty$,*

$$\overline{\Pr}\{\overline{U}(\gamma) = 0\} \to 1 \quad \text{and} \quad \frac{\overline{\Pr}\{\overline{X}(\gamma) > 0\}}{\overline{\Pr}\{\overline{Y}(\gamma) > 0\}} \to 1,$$

then the families $\{\overline{X}(\gamma)\}$ and $\{\overline{Y}(\gamma)\}$ are ACE.

The following lemma, proved by Daley and Rolski (1992), is frequently used in the paper.

LEMMA 2.3 (Multivariate Abelian lemma). *Let* $f\colon \mathbb{R}_+^j \to \mathbb{R}_+$ *be a componentwise nonincreasing function with* $f(\mathbf{0})$ *finite and* $f(\mathbf{t}) \downarrow 0$ *for* $\max_{1\le i\le j} t_i \to \infty$. *If the d.f. A is in* $\mathscr{S}_\alpha\,(c_A)$ *for some* $0 < \alpha < \infty$, *then as* $\gamma \to \infty$,

$$\gamma^{j\alpha}\int\underset{\mathbb{R}_+^j}{\cdots}\int f(\gamma\mathbf{t})\,A(dt_1)\cdots A(dt_j) \to (\alpha c_A)^j\int\underset{\mathbb{R}_+^j}{\cdots}\int (t_1\cdots t_j)^{\alpha-1} f(\mathbf{t})\,dt_1\cdots dt_j \tag{2.1}$$

whenever the integral on the rhs is finite.

3. Single-server queues in series. We consider a series of $m \ge 2$ single-server queues GI/G/1 $\to \cdots \to$ G/1 in which the arrivals occur at the epochs of a point process, specifically, the interarrival times $\{T_n\colon n = 0, \pm 1,\dots\}$ are assumed to constitute a renewal process, independent of the service times $\{\mathbf{S}_n\} \equiv \{(S_n^i,\ i = 1,\dots,m)\colon n = 0, \pm 1,\dots\}$ that are assumed to form a sequence of independent identically distributed (i.i.d) nonnegative random vectors. Here S_n^i denotes the service time of the nth customer at the ith station. Recall that we do not require independence between the service times of any given customer at different stations. Customers proceed from one station to the next in sequence and are served at each station in order of arrival. Denote the waiting time of the nth customer at the ith station by W_n^i. For the first station it satisfies the recurrence relationship

$$W_{n+1}^1 = \left(W_n^1 + S_n^1 - T_n\right)_+, \tag{3.1}$$

and for other stations we have

$$W_{n+1}^i = \left(\min\left(R_n^i, P_n^i - T_n\right)\right)_+, \tag{3.2}$$

where

$$P_n^i = W_n^1 + S_n^1 + \sum_{k=2}^{i}\left(W_n^k + S_n^k - S_{n+1}^{k-1}\right), \tag{3.3}$$

$$R_n^i = \begin{cases} P_n^1, & i = 1,\\ \min\limits_{2\le j\le i}\left\{\sum\limits_{k=j}^{i}\left(W_n^k + S_n^k - S_{n+1}^{k-1}\right)\right\}, & i = 2,\dots,m. \end{cases} \tag{3.4}$$

Then clearly (3.1) may be written in the form (3.2). For two station queues the foregoing relationship was given by Niu (1980), whereas longer series of queues were described in a similar manner by Masterson and Sherman (1963). Relations (3.1) and (3.2) can be established by noting that the input at the ith station equals the output from the $(i-1)$st station. Thus if T_n^i denotes the time between departures of the nth and $(n+1)$st customers from the $(i-1)$st station, then

$$W^i_{n+1} = \left(W^i_n + S^i_n - T^i_n\right)_+, \tag{3.5}$$

$$T^{i+1}_n = S^i_{n+1} + \left(T^i_n - W^i_n - S^i_n\right)_+, \tag{3.6}$$

where for consistency with (3.1) we put $T^1_n = T_n$.

In the paper we assume that

$$0 < \max_{1\le i\le m} \mathrm{E}S^i < \mathrm{E}T < \infty. \tag{3.7}$$

Here and in the sequel S^i denotes the generic service time r.v. at the ith station and T denotes a r.v. with the marginal distribution of any T_n. It is known [see, e.g., Loynes (1962)] that condition (3.7) ensures stability at all stations. It means that there exists a stationary sequence $\{(W^1_n,\dots,W^m_n),\ n\in\mathbb{Z}\}$ satisfying (3.1)–(3.4). According to the previously introduced convention we write $(W^1,\dots,W^m)$ for a random vector distributed like any member of the stationary sequence $\{(W^1_n,\dots,W^m_n),\ n\in\mathbb{Z}\}$. The process can be described briefly as a stable GI/G/1 $\to \cdots \to$ G/1 queueing system.

We now construct a lower bound for W^i. It is typical for our approach to choose a suitable probability space for proofs. We remark that this does not influence our results because they are distributional only. Suppose that for each $i = 1,\dots,m$ the sequence $\{V^i_n\}$ satisfies

$$V^i_{n+1} = \left(V^i_n + \sum_{k=1}^{i} S^k_n - T_n\right)_+. \tag{3.8}$$

Observe that $\{V^i_n\}$ can be regarded as the total waiting time in the series system restricted to the first i nodes, with a modified rule that allows only one customer to be present at any of the stations at any given moment. It is clear that such a discipline can only increase waiting times so

$$\sum_{k=1}^{i} W^k_n \le V^i_n \quad \text{a.s.} \tag{3.9}$$

LEMMA 3.1. *Let $\sum_{k=1}^m \mathrm{E}S^k < \mathrm{E}T$. There exists a probability space $(\Omega,\mathscr{F},\Pr)$ supporting the stationary sequence of random elements $\{\mathbf{V}_n, \mathbf{W}_n, \mathbf{S}_n, T_n\}$ such that for each $i = 1,\dots,m$,*

$$\left(\min\left(\underline{R}^i_n, \underline{P}^i_n - T_n\right)\right)_+ \le W^i_n, \tag{3.10}$$

where

$$\underline{P}^i_n = S^1_n + \sum_{k=2}^{i}\left(S^k_n - S^{k-1}_{n+1}\right), \tag{3.11}$$

$$\underline{R}^i_n = \begin{cases} \underline{P}^1_n, & i = 1,\\ \min\limits_{2\le j\le i}\left\{\sum\limits_{k=j}^{i}\left(S^k_n - S^{k-1}_{n+1}\right)\right\}, & i = 2,\dots,m; \end{cases} \tag{3.12}$$

furthermore,

$$W_n^i = \left(\min\left(\underline{R}_n^i, \underline{P}_n^i - T_n\right)\right)_+ \quad on \ \left\{V_n^i = 0\right\} \tag{3.13}$$

and for each n and i, and all $k \geq n$, the random elements V_n^i, T_k and $\mathbf{S}_k$ are independent.

PROOF. By standard arguments we construct a stationary sequence $\{T_n, \mathbf{S}_n, (W_n^1, \dots, W_n^i, V_n^1, \dots, V_n^m): n \in \mathbb{Z}\}$ satisfying (3.2)–(3.4) and (3.8) [see, e.g., Baccelli and Brémaud (1987) or Rolski (1981)]. Then by construction of V_n^i and by assumption for the other elements, V_n^i is independent of $(T_k, \mathbf{S}_k, k \geq n)$. Now by (3.9) we have (3.13). □

REMARK. Note that the lhs of (3.10) is the waiting time of a customer that follows one finding the system empty. Thus we call it a *single-customer representation.*

We need the most general version of Lemma 3.1 only in Section 5. Elsewhere it suffices that $n = 1$, in which case we omit the subscript n and write $\hat{\mathbf{S}}$ for $\mathbf{S}_2$. It is convenient to denote the partial sum of consecutive terms $a^i, \dots, a^j$ of a sequence $\{a^k\}$ of reals by $a^{[i,j]}$. Thus (3.10)–(3.12) can be rewritten as $\underline{P}_n^i = S_n^{[1,i]} - S_{n+1}^{[1,i-1]}$ and $\underline{R}_n^i = \min_{2 \leq j \leq i}\{S_n^{[j,i]} - S_{n+1}^{[j-1,i-1]}\}$. Similar notation can be used for subscripts.

4. Waiting time in light traffic. Except for Section 6 the rest of this paper considers light traffic conditions defined via γ-dilation discussed in Daley and Rolski (1984, 1991). It means that each interarrival time is multiplied by some $\gamma > 0$ and we take the limit as $\gamma \to \infty$. We write $(W^1(\gamma), \dots, W^m(\gamma))$ and $V^m(\gamma)$ for appropriate waiting times when the input is rescaled by γ; that is, each interarrival time is multiplied by this constant. We consider renewal arrival processes for which the interarrival time d.f. belongs to $\mathscr{S}_\alpha(c_A)$ [see (1.1)] for some strictly positive constant α.

When $\alpha > 0$ the interarrival times are a.s. strictly positive, in which case, in view of (3.9), it follows from the result (2.3a) of Daley and Rolski (1991) that the vectors $(W^1(\gamma), \dots, W^m(\gamma))$ decrease stochastically to $(0, \dots, 0)$. [A trickier part of checking this assertion concerns monotonicity; this is done in Błaszczyszyn (1990).] The case $\alpha = 0$ (i.e., when $\Pr\{T = 0\} > 0$) is different: the single-customer effect fails (see Corollary 2 for definition) and there is a positive waiting time vector that, however, remains difficult to compute except the case $m = 2$; see also Błaszczyszyn (1990).

In this section we show that for queues in series the families $\{W^i(\gamma)\}$ and $\{(\min(\underline{R}^i, \underline{P}^i - \gamma T))_+\}$ are ACE. For convenience we introduce the notation $p^i = (\underline{P}^i)_+$ and $r^i = (\underline{R}^i)_+$.

The results of this section are based on the following theorem.

THEOREM 4.1. *In a $GI/G/1 \to \cdots \to G/1$ queueing system for which the interarrival time d.f. is in $\mathscr{S}_\alpha(c_A)$ for some $0 < \alpha < \infty$,*

$$(4.1)\quad \begin{aligned}&\lim_{\gamma\to\infty} \gamma^\alpha \mathrm{E} g(W^1(\gamma),\dots,W^m(\gamma))\\ &\quad = c_A \sum_{i=1}^m \mathrm{E}\left[\int_0^{r^i} (p^i - t)^\alpha g(0,\dots,0,dt,r^{i+1},\dots,r^m)\right],\end{aligned}$$

where $g\colon \mathbb{R}^m \to \mathbb{R}$ is a coordinatewise monotonic, nondecreasing function for which $g(0,\dots,0) = 0$, provided that for some $\gamma_0 > 0$ and each $i = 1,\dots,m$,

$$(4.2)\quad \mathrm{E}\left[\int_0^{\bar r^i} (\bar p^i - t)^\alpha g(0,\dots,0,dt,\bar r^{i+1},\dots,\bar r^m)\right] < \infty,$$

where

$$(4.3)\quad \bar P^i = \bar P^i(\gamma_0) = W^1(\gamma_0) + S^1 + \sum_{k=2}^{i} \left(W^k(\gamma_0) S^k - \hat S^{k-1}\right),$$

$$(4.4)\quad \begin{aligned}\bar R^i &= \bar R^i(\gamma_0)\\ &= \begin{cases} \bar P^1, & i = 1,\\ \min\limits_{2\le j\le i} \left\{ \sum\limits_{k=j}^{i} \left(W^k(\gamma_0) + S^k - \hat S^{k-1}\right)\right\}, & i = 2,\dots,m,\end{cases}\end{aligned}$$

and $\bar p^i = (\bar P^i)_+$, $\bar r^i = (\bar R^i)_+$.

Before the proof we give the following lemma, which states the result for single-customer representation in light traffic.

LEMMA 4.2. *If the interarrival time d.f. is in $\mathscr{S}_\alpha(c_A)$ for some $0 < \alpha < \infty$, then*

$$(4.5)\quad \begin{aligned}&\lim_{\gamma\to\infty} \gamma^\alpha \mathrm{E} g\left((\min(\underline{R}^1, \underline{P}^1 - \gamma T))_+,\dots,(\min(\underline{R}^m,\underline{P}^m - \gamma T))_+\right)\\ &\quad = c_A \sum_{i=1}^m \mathrm{E}\left[\int_0^{r^i} (p^i - t)^\alpha g(0,\dots,0,dt,r^{i+1},\dots,r^m)\right],\end{aligned}$$

where $g\colon \mathbb{R}^m \to \mathbb{R}$ is a coordinatewise monotonic, nondecreasing function for which $g(0,\dots,0) = 0$ and such that the integrals in the rhs of (4.5) *exist.*

PROOF. By an Abelian lemma [see, e.g., below (2.1) or Daley and Rolski (1992)],

$$(4.6)\quad \begin{aligned}&\lim_{\gamma\to\infty} \gamma^\alpha \mathrm{E}\left[g\left((\min(\underline{R}^1,\underline{P}^1 - \gamma T))_+,\dots,(\min(\underline{R}^m,\underline{P}^m - \gamma T))_+\right)\right]\\ &\quad = \alpha c_A \int_0^\infty t^{\alpha-1} \mathrm{E} g\left((\min(\underline{R}^1,\underline{P}^1 - t))_+,\dots,\right.\\ &\qquad\qquad \left.(\min(\underline{R}^m,\underline{P}^m - t))_+\right) dt.\end{aligned}$$

To work out the rhs of (4.6), observe that by (3.11)–(3.12), $0 = \underline{P}^1 - \underline{R}^1 \leq \cdots \leq \underline{P}^m - \underline{R}^m$ and, for $i = 1, \dots, m-1$, $\underline{P}^i \leq \underline{P}^{i+1} - \underline{R}^{i+1}$. Hence the right-hand side of (4.6) can be rewritten as

$$\alpha c_A \sum_{i=1}^{m} \mathrm{E}\left[\int_{\underline{P}^i - \underline{R}^i}^{\underline{P}^{i+1} - \underline{R}^{i+1}} t^{\alpha-1} g\big(0, \dots, 0, (\underline{P}^i - t)_+, r^{i+1}, \dots, r^m\big)\, dt\right],$$

where $\underline{P}^{m+1} - \underline{R}^{m+1} = \infty$. Integration by parts yields the rhs of (4.5). □

PROOF OF THEOREM 4.1. Fix $\gamma_0 > 0$. We have for $\gamma > \gamma_0$,

$$\begin{aligned} &\limsup_{\gamma\to\infty} \gamma^\alpha \mathrm{E}\big[g(W^1(\gamma), \dots, W^m(\gamma))\big] \\ (4.7)\qquad &\leq \lim_{\gamma\to\infty} \gamma^\alpha \mathrm{E} g\Big(\big(\min(\bar{r}^1, \bar{p}^1 - \gamma T)\big)_+, \dots, \\ &\qquad\qquad (\min(\bar{r}^m, \bar{p}^m - \gamma T))_+\Big). \end{aligned}$$

Much as in Lemma 4.2, apply the Abelian lemma [see, e.g., below (2.1) or Daley and Rolski (1992)] to the rhs of (4.7) to show that it equals

$$(4.8)\quad \alpha c_A \int_0^\infty t^{\alpha-1} \mathrm{E} g\Big(\big(\min(\bar{r}^1, \bar{p}^1 - t)\big)_+, \dots, (\min(\bar{r}^m, \bar{p}^m - t))_+\Big)\, dt.$$

By a similar argument to that of the proof of Lemma 4.2, (4.8) equals

$$(4.9)\qquad c_A \sum_{i=1}^{m} \mathrm{E}\left[\int_0^{\bar{r}^i} (\bar{p}^i - t)^\alpha g(0, \dots, 0, dt, \bar{r}^{i+1}, \dots, \bar{r}^m)\right].$$

A monotone convergence argument applied to the limit $\gamma_0 \to 0$ in (4.9), provided (4.2) holds, coupled with the result of Lemma 4.2 and (3.10), proves the theorem. □

We need the following integral of the beta density:

$$\mathrm{B}(x_1, x_2, a, b) = \int_{x_1}^{x_2} (1-x)^{a-1} x^{b-1}\, dx.$$

COROLLARY 1. *Let* $x \geq 0$. *If* $\mathrm{E}(S^k)^{\alpha+1} < \infty$, $k = 1, \dots, m$, *then*

$$(4.10)\qquad \lim_{\gamma\to\infty} \gamma^\alpha \Pr\{W^m(\gamma) > x\} = c_A \mathrm{E}\big[(p^m - x)^\alpha; \underline{R}^m > x\big]$$

and if $\mathrm{E}(S^k)^{\alpha+\beta+1} < \infty$, $k = 1, \dots, m$, *then*

$$(4.11)\quad \lim_{\gamma\to\infty} \gamma^\alpha \mathrm{E}(W^m(\gamma))^\beta = \beta c_A \mathrm{E}\left[(p^m)^{\alpha+\beta} \mathrm{B}\left(0, \frac{r^m}{p^m}, \alpha+1, \beta\right)\right].$$

PROOF. Equations (4.10) and (4.11) follow from (4.1) after substituting $g(\mathbf{x}) = f(x_m)$ with an appropriate function f. All we need to do is to check that the moment conditions on **S** ensure finiteness in (4.2). Observe from (3.11)–(3.12), (3.9) and (4.3)–(4.4) that

$$\mathrm{E}\left[\int_0^{\bar{r}^m} (\bar{p}^m - t)^\alpha f(dt)\right] \le \mathrm{E}(p^m + V^m(\gamma_0))^\alpha f(r^m + V^m(\gamma_0)),$$

so for $f(y) = \mathrm{I}(y > x)$, finiteness is ensured when $\mathrm{E}(p^m + V^m(\gamma_0))^\alpha < \infty$, for which it is enough by Kiefer and Wolfowitz (1956) that $\mathrm{E}(S^k)^{\alpha+1} < \infty$, $k = 1, 2, \dots, i$. Similarly for $f(y) = y^\beta$, by Hölder's inequality, it suffices that $\mathrm{E}(p^m + V^m(\gamma_0))^{\alpha+\beta} < \infty$ and $\mathrm{E}(r^m + V^m(\gamma_0))^{\alpha+\beta} < \infty$, for which $\mathrm{E}(S^k)^{\alpha+\beta+1} < \infty$, $k = 1, \dots, i$, is enough. □

The proof of Lemma 4.2, with Corollary 1 and Theorem 4.1, yields the following result.

COROLLARY 2. *If* $\mathrm{E}(S^k)^{\alpha+1} < \infty$, $k = 1, \dots, m$, *then the families of r.v's.* $\{(\min(\underline{R}^m, \underline{P}^m - \gamma T))_+\}$ *and* $\{W^m(\gamma)\}$ *are ACE, so we conclude that the single-customer effect holds.*

As a result of Theorem 4.1 in its general version we have, for example, the following fact that can be justified in detail in a similar manner.

COROLLARY 3. *If* $\mathrm{E}(S^i)^{\alpha+3} < \infty$ $(i = 1, \dots, l \le m)$, *then for* $k < l$,

$$\operatorname{cov}(W^k(\gamma), W^l(\gamma)) = \gamma^{-\alpha} c_A \mathrm{e}(k, l) + o(\gamma^{-\alpha}),$$

where

$$\mathrm{e}(k, l) = \mathrm{E}\left[r^l \int_0^{r^k} (p^k - t)^\alpha \, dt\right] = \frac{1}{\alpha + 1} \mathrm{E}\left[r^l (p^k)^{\alpha+1} - r^l (p^k - r^k)^{\alpha+1}\right].$$

5. Interdeparture times in light traffic. In this section we study the stationary ergodic sequence $\{T_i^m, i \in \mathbb{Z}\}$ of interarrival times at the mth station [or, equivalently, interdeparture times from the $(m-1)$th station] in light traffic. More specifically we are going to study the expected value

$$\mathrm{E} g(T_1^m(\gamma), \dots, T_n^m(\gamma)) \quad \text{when } \gamma \to \infty, \tag{5.1}$$

for a coordinatewise nonincreasing function g. From this we prove as the main corollary of this section that the arrival process at the mth station is in light traffic asymptotically one-dependent. This one-dependence property can be defined as follows.

DEFINITION 5.1. A family of stationary sequences $\{X_n(\gamma): n \in \mathbb{Z}\}$, $\gamma \ge 0$, is said to be asymptotically one-dependent (AOD) if for each pair of monotone nonincreasing indicator functions $g_j\colon \mathbb{R}^{n_j} \to \mathbb{R}_+$, $j = 1,2$, such that $g_j(x)\downarrow 0$ when $\max_{1\le i\le n_j} x_i \to \infty$ and $d \ge 2$,

$$\lim_{\gamma\to\infty} \frac{\mathrm{E}g_1(X_1(\gamma),\dots,X_{n_1}(\gamma))g_2(X_{n_1+d}(\gamma),\dots,X_{n_1+d+n_2}(\gamma))}{\mathrm{E}g_1(X_1(\gamma),\dots,X_{n_1}(\gamma))\mathrm{E}g_2(X_{n_1+d}(\gamma),\dots,X_{n_1+d+n_2}(\gamma))} \to 1. \tag{5.2}$$

As before, we consider here only the case when $\Pr\{T = 0\} = 0$. We can prove by (3.1)–(3.6) that

$$T_l^m = W_l^m + S_l^m - \min(R_l^m, P_l^m - T_l). \tag{5.3}$$

We note that, in general, the departure process is not a renewal one. Moreover, although for the first station $T_l^1(\gamma) = \gamma T_l$, it is not true that interarrival times at the mth station satisfy $T_l^m(\gamma) = \gamma T_l^m$; however, it is true that for $\gamma \to \infty$,

$$T_l^m(\gamma) - \gamma T_l \to \sum_{k=1}^{m-1} (S_{l+1}^k - S_l^k)$$

from which we obtain

$$\gamma^{-1}T_l^m(\gamma) \to T_l, \qquad \gamma \to \infty.$$

To see this, it suffices to apply the sample path argument to (5.3), keeping in mind that $W_l^k(\gamma) \to 0$.

The time T_l^m that elapses between departures from the $(m-1)$th station of the lth and $(l+1)$th customer is fully determined by $W_1 = \{W_1^i\colon i = 1,\dots,m-1\}$, $(T_k\colon k = 1,\dots,l)$ and $(S_k^i\colon i = 1,\dots,m-1,\ k = 1,\dots,l+1)$; that is, by the waiting times of the first customer in all $m-1$ stations and by characteristics brought by the first $l+1$ customers. With this in mind we use the representation with explicit dependence on $\mathbf{W}_1$ and $(T_k\colon k = 1,\dots,l)$:

$$T_l^m = \mathscr{T}_l^m(\omega)(\mathbf{W}_1, T_1,\dots,T_l), \tag{5.4}$$

where $\mathscr{T}_l^m(\cdot)(\mathbf{w}, t_1,\dots,t_l)$ is a $\sigma(S_k^i\colon\ i = 1,\dots,m-1;\ k = 1,\dots,l+1)$-measurable function. The main idea of this is to have $T_1,\dots,T_l$ independent of other arguments, both explicit and implicit, of the representation $\mathscr{T}_l^m$.

THEOREM 5.2. *Let $g\colon \mathbb{R}_+^n \to \mathbb{R}_+$ be a coordinatewise nonincreasing function with $g(0,\dots,0)$ finite and such that $g(\mathbf{t})\downarrow 0$ when $\max_{1\le i\le n} t_i \to \infty$. In a GI/G/1 $\to \cdots \to$ G/1 queueing system for which the interarrival time d.f. is in $\mathscr{S}_\alpha(c_A)$ for some $0 < \alpha < \infty$,*

$$\lim_{\gamma\to\infty} \gamma^{\alpha n}\mathrm{E}g(T_1^m(\gamma),\dots,T_n^m(\gamma))$$
$$= (\alpha c_A)^n \int\cdots\int_{\mathbb{R}_+^n} \mathrm{E}g(\mathscr{T}_l^m(\mathbf{0}, t_1,\dots,t_l)\colon l = 1,\dots,n) \prod_{k=1}^n t_k^{\alpha-1}\,dt_k \tag{5.5}$$

provided that for some $\gamma_0 > 0$,

$$(5.6)\quad \int \underset{\mathbb{R}_+^n}{\cdots} \int \mathrm{E}g(\mathscr{T}_l^m(\mathbf{W}_1(\gamma_0), t_1, \dots, t_l) \colon l = 1, \dots, n) \prod_{k=1}^n t_k^{\alpha-1}\, dt_k < \infty.$$

PROOF. Observe first by (5.3) and (3.3)–(3.6), if $\mathbf{w} \geq \mathbf{w}'$ componentwise, then

$$(5.7)\qquad \mathscr{T}_l^m(\mathbf{w}, t_1, \dots, t_l) \leq \mathscr{T}_l^m(\mathbf{w}', t_1, \dots, t_l) \quad \text{a.s.}$$

The random function $\mathscr{T}_l^m$ is also monotone nondecreasing with respect to the other part of its explicit arguments and

$$(5.8)\qquad \mathscr{T}_l^m(\mathbf{w}, t_1, \dots, t_l) \nearrow \infty \quad \text{a.s. when } t_l \to \infty.$$

Now by the monotonicity of g and (5.7) we have

$$\begin{aligned} &\liminf_{\gamma\to\infty} \gamma^{\alpha n} \mathrm{E}g(T_1^m(\gamma), \dots, T_n^m(\gamma)) \\ (5.9)\qquad &\geq \lim_{\gamma\to\infty} \gamma^{\alpha n} \int \underset{\mathbb{R}_+^n}{\cdots} \int \mathrm{E}g(\mathscr{T}_l^m(\mathbf{0}, \gamma t_1, \dots, \gamma t_l) \colon \\ &\qquad\qquad l = 1, \dots, n) \prod_{k=1}^n A(dt_k), \end{aligned}$$

which in turn, by (5.8) and the multivariate Abelian lemma [see, e.g., below (2.1)] equals the rhs of (5.5), provided (5.6) holds.

On the other hand, we fix $\gamma_0 > 0$ and consider $\gamma \geq \gamma_0$. By (5.7),

$$\begin{aligned} &\limsup_{\gamma\to\infty} \gamma^{\alpha n} \mathrm{E}g(T_1^m(\gamma), \dots, T_n^m(\gamma)) \\ (5.10)\qquad &\leq \lim_{\gamma\to\infty} \gamma^{\alpha n} \int \underset{\mathbb{R}_+^n}{\cdots} \int \mathrm{E}g(\mathscr{T}_l^m(\mathbf{W}_1(\gamma_0), \gamma t_1, \dots, \gamma t_l) \colon \\ &\qquad\qquad l = 1, \dots, n) \prod_{k=1}^n A(dt_k). \end{aligned}$$

Now again we apply the Abelian lemma to the rhs of (5.10) and then let $\gamma \to \infty$. This, provided (5.6) holds, combined with (5.9) gives (5.5). □

REMARK. It is not easy to verify when condition (5.6) is satisfied. However, in view of (3.3)–(3.4), (3.10)–(3.12), (3.9) and (5.3), $T_l^m \geq S_l^m - \min(\underline{R}_l^m, \underline{P}_l^m - T_l) - V_l^{m-1}$, and by (3.8),

$$V_l^{m-1} = \max\left(0, V_{l-1}^{m-1} + S_{l-1}^{[1,m-1]} - T_{l-1}\right) = \cdots \leq V_1^{m-1} + \mathscr{S}_{[1,l-1]}^{[1,m-1]}.$$

So it suffices to check the finiteness of the integral

$$\int \underset{\mathbb{R}_+^n}{\cdots} \int \mathrm{E}\Big[g\Big(S_l^m - \min(\underline{R}_l^m, \underline{P}_l^m - t_l) - \Big(V_1^{m-1}(\gamma_0) + S_{[1,l-1]}^{[1,m-1]}\Big) \colon l = 1, \dots, n\Big)\Big] \prod_{l=1}^n t^{\alpha-1}\, dt_l.$$

Applying (5.3) to (5.5) with $n = 1$ and $g(t) = \mathrm{I}(t \le x)$, after some easy transformations including integration by parts, we obtain the following result. The assumption on the moments of $\mathbf{S}$ by Kiefer and Wolfowitz (1955, 1956) yields (5.6).

COROLLARY 4. *If* $\mathrm{E}(S^i)^{\alpha+1} < \infty$ *for* $i = 1, \dots, m$, *then*

$$\lim_{\gamma\to\infty} \gamma^{\alpha} \Pr\{T^m(\gamma) \le x\} = c_A \mathrm{E}\big[(x - S^m + \underline{P}^m)^{\alpha}; (x \ge S^m - \underline{R}^m)\big].$$

We now study the structure of the departure process.

COROLLARY 5. *In a* $GI/G/1 \to \cdots \to G/1$ *queueing system for which the interarrival time d.f. is in* $\mathscr{S}_\alpha(c_A)$ *for some* $0 < \alpha < \infty$ *and the service times have finite moments of all orders, the arrival process* $\{T_n^m(\gamma): n \in \mathbb{Z}\}$ *at the mth station is AOD when* $\gamma \to \infty$.

PROOF. Take g_1, g_2 as in Definition 5.1. For $\{T_l^m: 1 \le l \le n_1\}$ we use representation (5.4), whereas for $n_1 + d \le l \le n_1 + d + n_2$ we may write also $T_l^m = \bar{\mathscr{T}}_l^m(\omega)(\mathbf{W}_{n_1+d}, T_{n_1+d}, \dots, T_{n_1+d+n_2})$, where $\bar{\mathscr{T}}_l^m(\cdot)(\mathbf{w}, t_{n_1+d}, \dots, t_{n_1+d+n_2})$ is a $\sigma(S_k^i: i = 1, \dots, m-1;\ k = n_1 + d, \dots, l+1)$-measurable function, and put $\mathbf{W}_{n_1+d} = \{W_{n_1+d}^i: i = 1, \dots, m-1\}$. We prove that

$$\begin{aligned}
\lim_{\gamma\to\infty} \gamma^{\alpha(n_1+n_2)} &\mathrm{E} g_1\big(T_1(\gamma), \dots, T_{n_1}(\gamma)\big) g_2\big(T_{n_1+d}(\gamma), \dots, T_{n_1+d+n_2}(\gamma)\big) \\
&= (\alpha c_A)^{n_1+n_2} \int \underset{\mathbb{R}_+^{n_1+n_2}}{\cdots} \int \mathrm{E} g_1(\mathscr{T}_l^m(\mathbf{0}, t_1, \dots, t_l): \\
&\qquad\qquad l = 1, \dots, n) \prod_{k=1}^{n_1} t_k^{\alpha-1}\, dt_k \\
&\quad \times g_2\big(\bar{\mathscr{T}}_l^m(\mathbf{0}, t_{n_1+d}, \dots, t_l): l = n_1 \\
&\qquad\qquad + d, \dots, n_1 + d + n_2\big) \prod_{k=n_1+d}^{n_1+d+n_2} t_k^{\alpha-1}\, dt_k.
\end{aligned} \tag{5.11}$$

Now, because $\{\mathscr{T}_l^m,\ l = 1, \dots, n_1\}$ and $\{\bar{\mathscr{T}}_l^m,\ l = n_1 + d, \dots, n_1 + d + n_2\}$ are independent for $d \ge 2$, we can factor the rhs of (5.11) and conclude by Theorem 5.2 that the numerator and denominator of (5.2) are of the same order of magnitude with the same limit constant. To obtain (5.11), we cannot apply Theorem 5.2 directly, but, much as in the proof of (5.5), we observe that in (5.11) the lhs is not less than the rhs and that the lhs of (5.11) is not greater than

$$\begin{aligned}
(\alpha c_A)^{n_1+n_2} &\int \underset{\mathbb{R}_+^{n_1+n_2}}{\cdots} \int \mathrm{E} g_1(\mathscr{T}_l^m(\mathbf{W}_1(\gamma_0), t_1, \dots, t_l): l = 1, \dots, n) \\
&\times g_2\big(\mathscr{T}_l^m(\mathbf{W}_1(\gamma_0), t_1, \dots, t_{n_1}, \gamma_0 T_{n_1+1}, \dots, \gamma_0 T_{n_1+d-1}, t_{n_1+d}, \dots, t_l): \\
&\qquad l = n_1 + d, \dots, n_1 + d + n_2\big) \prod_{k=1}^{n_1} t_k^{\alpha-1}\, dt_k \prod_{k=n_1+d}^{n_1+d+n_2} t_k^{\alpha-1}\, dt_k.
\end{aligned} \tag{5.12}$$

Letting $\gamma \to \infty$ in (5.12), provided it is finite, we obtain again the rhs of (5.11). To ensure finiteness of (5.12), as well as (5.6) for g_1 and g_2 separately, it suffices to have finite moments of $\mathbf{S}$ of order $\alpha(n_1 + n_2) + 1$. For a formal proof we have to use the Remark after Theorem 5.2 and the theorem on moments of Kiefer and Wolfowitz (1955, 1956). □

6. Light traffic via thinning. Another possible approach to light traffic limits proceeds via a thinning operation [called π-thinning in Daley and Rolski (1991); see also Daley and Rolski (1992) and Asmussen (1991)]. By π-thinning we mean that arrivals of customers into the system can be regarded as the result of subjecting the process of potential arrivals to independent thinning with common retention probability π. We approach light traffic conditions by allowing $\pi \to 0$. In this section we discuss briefly the consequence of taking such limits for GI/G/1 $\to \cdots \to$ G/1 systems.

Note first that the intervals of a renewal process subject to independent thinning can be represented as

$$T_n(\pi) = \sum_{i=1}^{\nu_n} T_{in} \tag{6.1}$$

where $\{T_{in}: i = 1, 2, \dots; n = 1, 2, \dots\}$ is a doubly infinitely indexed sequence of i.i.d r.v.'s, each $T_{in} =_d T$ for some generic interarrival time r.v. T of the unthinned potential arrival process, and $\{\nu_i\}$ is a sequence of i.i.d. geometric r.v.'s with

$$\Pr\{\nu_n = r\}(1 - \pi)^{r-1}, \qquad r = 1, 2, \dots .$$

Then for each π, $(T_n(\pi),\ n \in \mathbb{Z})$ is a sequence of i.i.d. r.v.'s with the common distribution function $A^{(\pi)}$ and for each n the family of sequences $\{(T_n(\pi)),\ 0 < \pi < 1\}$ is a stochastically monotone $\mathbb{R}_+^\infty$-valued random process, because $T(\pi) \le_d T(\pi')$, when $\pi' < \pi$. Moreover, by a standard argument we can assume that it is a sample path monotone decreasing random process [take $\Omega = [0,1]^{\mathbb{Z}}$, $\Pr = \cdots \otimes dx \otimes dx \otimes \cdots$ and $T_n(\pi)(\omega) = (A^{(\pi)})^{-1}(\omega_i)$, where $\omega = (\dots, \omega_0, \omega_1, \dots)$ and the inverse function A^{-1} is defined as in Stoyan (1983), Section 1.2]. Note that, irrespective of whether or not $T > 0$ a.s., each $T_n(\pi) \to \infty$ a.s. as $\pi \to 0$. The result is that the presented argument justifies the existence of a probability space $(\Omega, \mathscr{F}, \Pr)$ (we do not introduce separate notation here although formally this may be a new space), a sequence $\{\mathbf{S}_n\}$ of independent and identically distributed random vectors, all distributed as the generic service times vector $\mathbf{S}$, which are independent of the process $\{(T_n(\pi)),\ 0 < \pi < 1\}$. Again as in Section 4 we can prove that the stationary waiting time $(W^m(\pi))$ at the mth station decreases stochastically to 0 regardless of whether $T > 0$ a.s. or not.

Let A denote the distribution of T. We want to point out that in this section any variable changing its values with the retention probability π is marked by (π).

THEOREM 6.1. *In a $GI/G/1 \to \cdots \to G/1$ queueing system in which the arrival process is subject to π-thinning, for a coordinatewise nondecreasing function $g\colon \mathbb{R}_+^m \to \mathbb{R}$ with $g(0,\dots,0) = 0$,*

$$\begin{aligned}&\lim_{\pi\to 0} \pi^{-1}\mathrm{E}g(W^1(\pi),\dots,W^m(\pi))\\ &\quad= \sum_{i=1}^{m} \mathrm{E}\left[\int_0^{r^i} H(p^i - t)g(0,\dots,0,dt,r^{i+1},\dots,r^m)\right],\end{aligned}\tag{6.2}$$

provided that for some π' in $0 < \pi' \le 1$ and each $i = 1,\dots,m$,

$$\mathrm{E}\left[\int_0^{\bar{r}^i} H(\bar{p}^i - t)g(0,\dots,0,dt,\bar{r}^{i+1},\dots,\bar{r}^m)\right] < \infty,\tag{6.3}$$

where $H = \sum_{i=1}^{\infty} A^{i^*}$ is the zero-deleted renewal function,

$$\bar{P}^i = \bar{P}^i(\pi') = W^1(\pi') + S^1 + \sum_{k=2}^{i}\left(W^k(\pi')S^k - \hat{S}^{k-1}\right),$$

$$\bar{R}^i = \bar{R}^i(\pi') = \begin{cases}\bar{P}^1, & i = 1,\\ \min\limits_{2\le j\le i}\left\{\sum\limits_{k=j}^{i}\left(W^k(\pi') + S^k - \hat{S}^{k-1}\right)\right\}, & i = 2,\dots,m,\end{cases}$$

$\bar{p}^i = (\bar{P}^i)_+$ and $\bar{r}^i = (\bar{R}^i)_+$.

PROOF. Much as in Section 4 we have for $\pi < \pi'$,

$$\left(\min\left(\underline{R}^i, \underline{P}^i - T(\pi)\right)\right)_+ \le W^i(\pi) \le \left(\min\left(\bar{R}^i(\pi'), \bar{P}^i(\pi') - T(\pi)\right)\right)_+.$$

For the lower bound (i.e., for the single-customer representation) we have

$$\begin{aligned}&\lim_{\pi\to 0}\pi^{-1}\mathrm{E}g\left(\left(\min\left(\underline{R}^1, \underline{P}^1 - T(\pi)\right)\right)_+,\dots,\left(\min\left(\underline{R}^m, \underline{P}^m - T(\pi)\right)\right)_+\right)\\ &\quad= \lim_{\pi\to 0}\sum_{n=1}^{\infty}(1-\pi)^{n-1}\int_0^{\infty}\mathrm{E}g\left(\left(\min\left(\underline{R}^1, \underline{P}^1 - t\right)\right)_+,\dots,\right.\\ &\qquad\qquad\left.\left(\min\left(\underline{R}^m, \underline{P}^m - t\right)\right)_+\right)A^{n^*}(dt)\\ &\quad= \int_0^{\infty}\mathrm{E}g\left(\left(\min\left(\underline{R}^1, \underline{P}^1 - t\right)\right)_+,\dots,\left(\min\left(\underline{R}^m, \underline{P}^m - t\right)\right)_+\right)H(dt)\end{aligned}$$

and for an upper bound,

$$\begin{aligned}&\limsup_{\pi\to 0}\pi^{-1}\mathrm{E}g\left(\left(\min\left(\bar{R}^i(\pi'), \bar{P}^i(\pi') - T(\pi)\right)\right)_+\right)\\ &\quad= \int_0^{\infty}\mathrm{E}g\left(\min\left(\bar{R}^1, \bar{P}^1 - t\right),\dots,\min\left(\bar{R}^m, \bar{P}^m - t\right)\right)H(dt).\end{aligned}$$

Now arguing as in the proof of Theorem 4.1, we let $\pi' \to 0$ to obtain

$$\begin{aligned}&\lim_{\pi\to 0}\pi^{-1}\mathrm{E}g(W^1(\pi),\dots,W^m(\pi))\\ &\quad= \int_0^{\infty}\mathrm{E}g\left(\left(\min\left(\underline{R}^1, \underline{P}^1 - t\right)_+\right),\dots,\left(\min\left(\underline{R}^m, \underline{P}^m - t\right)\right)_+\right)H(dt).\end{aligned}$$

This completes the proof because the same decomposition and integration techniques as in the analogous Theorem 4.1 may be used to state (6.2), provided (6.3) holds. □

As an analogue of Corollary 3, the following result is established in a straightforward manner. Note that the conditions are analogous to those of the case $\alpha = 1$ in that corollary.

COROLLARY 6. *If* $\mathrm{E}[(S^k)^2] < \infty$, $k = 1, \dots, m$, *then the families of r.v.'s* $\{\min(\underline{R}^m, \underline{P}^m - T^\pi)_+\}$ *and* $\{W^m(\pi)\}$ *are ACE and the single-customer effect holds.*

Other results analogous to those of Section 4 are omitted apart from the following three concerning the departure-time process.

THEOREM 6.2. *Let* $g\colon \mathbb{R}_+^n \to \mathbb{R}_+$ *be a coordinatewise nonincreasing function with* $g(0, \dots, 0)$ *finite and such that* $g(\mathbf{t}) \downarrow 0$ *when* $\max_{1 \le i \le n} t_i \to \infty$. *Let* $\mathscr{T}_l^m$ *be as in* (5.4). *Then*

$$\begin{aligned}
&\lim_{\pi \to 0} \pi^{-n} \mathrm{E} g(T_1^m(\pi), \dots, T_n^m(\pi)) \\
&\qquad = \int \underset{\mathbb{R}_+^n}{\cdots} \int \mathrm{E} g(\mathscr{T}_l^m(\mathbf{0}, t_1, \dots, t_l)\colon \\
&\qquad\qquad l = 1, \dots, n) \prod_{k=1}^n H(dt_k)
\end{aligned} \tag{6.4}$$

provided that for some $\pi' > 0$,

$$\int \underset{\mathbb{R}_+^n}{\cdots} \int \mathrm{E} g(\mathscr{T}_l^m(\mathbf{W}_1(\pi'), t_1, \dots, t_l)\colon l = 1, \dots, n) \prod_{k=1}^n H(dt_k) < \infty. \tag{6.5}$$

PROOF. By monotonicity of g we have

$$\begin{aligned}
&\liminf_{\pi \to 0} \pi^{-n} \mathrm{E} g(T_1^m(\pi), \dots, T_n^m(\pi)) \\
&\quad \ge \lim_{\pi \to 0} \sum_{\substack{1 \le i_k \le \infty \\ 1 \le k \le n}} (1 - \pi)^{i_k} \int \underset{\mathbb{R}_+^n}{\cdots} \int \mathrm{E} g(\mathscr{T}_l^m(\mathbf{0}, t_1, \dots, t_l)\colon \\
&\qquad\qquad l = 1, \dots, n) \prod_{k=1}^n A^{i_k *}(dt_k),
\end{aligned}$$

which equals the rhs of (6.4). On the other hand, for $\pi \le \pi'$ we have

$$\begin{aligned}
&\limsup_{\pi \to 0} \pi^{-n} \mathrm{E} g(T_1^m(\pi), \dots, T_n^m(\pi)) \\
&\quad \le \int \underset{\mathbb{R}_+^n}{\cdots} \int \mathrm{E} g(\mathscr{T}_l^m(\mathbf{W}_1(\pi'), t_1, \dots, t_l)\colon l = 1, \dots, n) \prod_{k=1}^n H(dt_k) < \infty.
\end{aligned}$$

Now we let $\pi' \to 0$ and obtain, provided (6.5) holds, the rhs of (6.4). This completes the proof. □

COROLLARY 7. *If* $E(S^i)^2 < \infty$ *for* $i = 1, \dots, m$, *then*

$$\lim_{\pi \to 0} \pi^{-1} \Pr\{T^m(\pi) \le x\} = E\left[H(x - S^m + \underline{P}^m);\ x \ge S^m - \underline{R}^m\right].$$

COROLLARY 8. *In a* $GI/G/1 \to \cdots \to G/1$ *queueing system in which the service times have finite moments of all orders, the arrival process* $\{T_n^m(\pi): n \in \mathbb{Z}\}$ *at the mth station is AOD when* $\pi \to 0$.

7. Concluding remarks. It is useful to point out a few conclusions we can draw from the foregoing results. In particular, we have given the order of magnitude in light traffic of various performance characteristics that are functions of the stationary waiting times or the interarrival times at nodes of a $\mathrm{GI/G/1} \to \cdots \to \mathrm{G/1}$ series queueing system.

One problem that remains concerns the range of validity of the "single-customer effect" in a G/G/1 queue (i.e., not necessarily renewal input). There are examples of GI/G/1 queues where the effect breaks down [see, e.g., Daley and Rolski (1984) and Asmussen (1991)]. Some work on the nonrenewal input case is given in Daley and Rolski (1991), but we are still far from a complete understanding of the problem. As another example, consider a node in a series of queues with independent service times and regard it as a G/GI/1 queue in its own right. Interdeparture times $T^m(\gamma)$ from the $(m-1)$st node are interarrival times at the mth node. Corollary 5.1 yields, for $x \ge 0$,

$$\gamma^\alpha A_\gamma^m(x) \to c_A E\left[(x - S^m + \underline{P}^m)^\alpha;\ x \ge S^m - \underline{R}^m\right],$$

where $A_\gamma^m(x) = \Pr\{T^m(\gamma) \le x\}$. If we suppose that the single-customer effect holds, then, neglecting mathematical rigor, we would write, for a nondecreasing function g such that $g(0) = 0$,

$$\gamma^\alpha E g(W^m(\gamma)) = \gamma^\alpha \int_0^\infty E g(S^m - x) A_\gamma^m(dx)$$

$$\to c_A \int_0^\infty E\left[g(S^m - x)(dx - S^m + \underline{P}^m)^\alpha;\ x \ge S^m - \underline{R}^m\right],$$

the "justification" here being Theorem 4.1. However, we have not as yet been able to justify the preceding steps without exploiting the detail of the particular queueing structure.

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