The Annals of Applied Probability
1993, Vol. 3, No. 3, 874–880

ASYMPTOTIC PROPERTIES OF CENTERED SYSTEMATIC SAMPLING FOR PREDICTING INTEGRALS OF SPATIAL PROCESSES[1]

BY MICHAEL L. STEIN

University of Chicago

This paper studies the asymptotic mean squared error for predicting the integral of a weakly stationary spatial process over a unit cube based on a centered systematic sample. For processes whose spectral density decays sufficiently slowly at infinity, the asymptotic mean squared error takes a form similar to that obtained by letting the cube increase in size with the number of observations. However, if the spectral density decays faster than a certain critical rate, then the asymptotic mean squared error takes on a completely different form. By adjusting the weights given to observations near an edge of the cube, it is possible to obtain asymptotic results for the fixed cube that again resemble those for the increasing cube.

1. Introduction. Centered systematic sampling is a natural method for selecting observation sites for predicting the integral of a spatial process over a d-dimensional cube. A centered systematic sample, sometimes called a midpoint sample, is obtained by dividing the cube into an m^d grid of smaller cubes of side m^{-1} and placing observations at the center of each of these m^d cubes. The usual predictor of the integral over the cube is the unweighted average of the observations. This paper studies the asymptotic mean squared error of this predictor obtained by fixing the size of the cube and letting the observations get increasingly dense, as was done by Tubilla (1975), Schoenfelder (1982), Stein (1991) and, to a certain extent, Matheron (1965). Cressie (1991) calls this approach infill asymptotics. An alternative asymptotic regime is to fix the distance between neighboring observations and let the size of the cube grow with m, which, following Cressie (1991), I will call increasing-domain asymptotics. Quenouille (1949), Matérn (1986) and Iachan (1985) used this approach.

Using both theoretical and numerical results, I argue that the fixed-domain asymptotic approach provides more insight into the problem of predicting averages of spatial processes. In particular, fixed-domain asymptotics allows for distinctions in the rates of convergence depending on the high frequency behavior of the spectral density, or equivalently, the smoothness of the process, that the increasing-domain approach does not make. To

Received October 1992; revised January 1993.

[1]Research was supported in part by NSF Grant DMS-89-02267. Computations for this work were done using computer facilities supported in part by NSF Grants DMS-89-05292, DMS-87-03942, DMS-86-01732 and DMS-84-04941 awarded to the Department of Statistics at the University of Chicago, and by the University of Chicago Block Fund.

AMS 1991 *subject classifications*. Primary 62M40; secondary 62M20.

Key words and phrases. Spatial statistics, fixed-domain asymptotics, increasing-domain asymptotics, numerical integration.

summarize the main ideas, for simplicity, let us assume the process is weakly stationary and has a spectral density f satisfying $f(\omega)|\omega|^p \to C$, some positive constant, as $|\omega| \to \infty$, although the results in the next section hold more generally. For $p < 4$, Theorem 2 shows that the asymptotic mean squared error is some constant times m^{-p}, where the constant only depends on the behavior of $f(\omega)$ as $|\omega| \to \infty$ and takes on a form similar to that obtained using increasing-domain asymptotics. Furthermore, as a special case of results in Stein (1991), this predictor is asymptotically optimal relative to all linear predictors based on centered systematic samples, at least if the spectral density is strictly positive. However, for all $p > 4$, Theorem 1 shows that the asymptotic mean squared error is asymptotically m^{-4} times a constant depending on $f(\cdot)$ over all frequencies. In addition, this predictor is not asymptotically optimal. Increasing-domain asymptotics makes no such distinction based on the value of p. Section 3 provides some numerical support for the criticality of p in determining the mean squared error for moderate m.

Stein (1991) shows that by adjusting the weights given to observations near the edge of the cube, it is possible to obtain an asymptotic mean squared error of order m^{-p} for $p > 4$. Applying those results here allows us to extend Theorem 2 to larger p and to obtain an asymptotically optimal linear predictor. Thus, the fixed-domain asymptotics highlights the need to use a predictor other than the simple average when predicting integrals of sufficiently smooth spatial processes.

Predicting integrals of spatial processes is a common problem in geological applications, for which $Z(x)$ might represent the concentration of a mineral or the depth of a boundary between two types of soil at a place x [Journel and Huijbregts (1978)]. Observations on a regular grid are common in these settings [Journel and Huijbregts (1978), page 8]. The results in this paper can also be viewed as a Bayesian approach to error analysis for numerical integration, although the assumption that the function being integrated is a realization of a stationary stochastic process may not be compelling in this setting.

2. Main results. For a stationary random field $Z(\cdot)$ on $\mathbb{R}^d$ with spectral density $f(\cdot)$, consider using a centered systematic sample to predict $\int_S Z(x)\,dx$, where $S = [0,1]^d$. Specifically, let $J = (j_1, \dots, j_d)$, $c_J = ((j_1 - \frac{1}{2})/m, \dots, (j_d - \frac{1}{2})/m)$, $L(m) = \{1, 2, \dots, m\}^d$ and predict $\int_S Z(x)\,dx$ by $m^{-d}\sum_{L(m)} Z(c_J)$. Letting $\omega = (\omega_1, \dots, \omega_d)$, by straightforward calculation,

$$\operatorname{var}\left(\int_S Z(x)\,dx - \frac{1}{m^d}\sum_{J\in L(m)} Z(c_J)\right) = \frac{1}{m^{2d}}\int_{\mathbb{R}^d}\left\{\prod_{\alpha=1}^d \frac{\sin^2(\omega_\alpha/2)}{\sin^2(\omega_\alpha/2m)}\right\} \times f(\omega)\left\{1 - \prod_{\alpha=1}^d \operatorname{sinc}\left(\frac{\omega_\alpha}{2m}\right)\right\}^2 d\omega, \tag{2.1}$$

where $\mathrm{sinc}(t) = t^{-1}\sin t$. All of the predictors considered in this paper are unbiased for $\int_S Z(x)\,dx$, so there will be no need to distinguish between error variance and mean squared error. Hannan (1962) gives a similar expression for the mean squared error for the predictor of the average of a lattice process in two dimensions based on observations on a coarser lattice. Define

$$g_m(\omega; J) = \left\{ \prod_{\alpha=1}^{d} \frac{\sin^2(\omega_\alpha/2) 1\{|\omega_\alpha| \le m\pi\}}{m^2 \sin^2(\omega_\alpha/2m)} \right\} \times f(\omega + 2m\pi J) \left\{ 1 - \prod_{\alpha=1}^{d} \frac{2(-1)^{j_\alpha} m \sin(\omega_\alpha/2m)}{(\omega_\alpha + 2m\pi j_\alpha)} \right\}^2,$$

where $1\{\cdot\}$ is an indicator function. Writing the last integral in (2.1) as a sum of integrals over cubes of width $2m\pi$ and centers at $2m\pi J$, where J ranges over the integer lattice,

$$\operatorname{var}\left(\int_S Z(x)\,dx - \frac{1}{m^d} \sum_{J \in L(m)} Z(c_J) \right) = \int_{\mathbb{R}^d} g_m(\omega; 0)\,d\omega + \int_{\mathbb{R}^d} \sum\nolimits^* g_m(\omega; J)\,d\omega, \tag{2.2}$$

where Σ^* means to sum over all elements of the integer lattice except the origin. The value of p determines which of the two terms on the right-hand side of (2.2) dominates asymptotically. The following two results essentially follow by showing that the first term dominates if $p < 4$ and the second dominates if $p > 4$ and then approximating each term separately; see Section 4 for details of the proofs.

THEOREM 1. *If $f(\omega) = o(|\omega|^{-4})$ as $|\omega| \to \infty$, then*

$$\lim_{m \to \infty} m^4 \operatorname{var}\left(\int_S Z(x) - \frac{1}{m^d} \sum_{J \in L(m)} Z(c_J) \right) = \frac{1}{576} \int_{\mathbb{R}^d} \left\{ \prod_{\alpha=1}^{d} \mathrm{sinc}^2\left(\frac{\omega_\alpha}{2} \right) \right\} f(\omega) |\omega|^4 \,d\omega. \tag{2.3}$$

Tubilla [(1975), Theorem 5] gives a similar result under the much stronger condition that the covariance function is infinitely differentiable. In addition, he writes the limiting variance in terms of the covariance function rather than the spectral density, which yields a rather more complicated expression than given by (2.3).

A function $L(\cdot)$ on $[0, \infty)$ is said to be slowly varying at ∞ if for every $a > 0$, $L(ta)/L(t) \to 1$ as $t \to \infty$. A function $U(\cdot)$ is regularly varying with exponent p if $U(x)/x^p$ is slowly varying [Feller (1971), page 276].

THEOREM 2. *Assume there exists a monotonic function $\beta(t)$ and $d < p \le 4$ such that $\beta(\cdot)$ is regularly varying with exponent p and a function $\bar{f}(\omega)$ such that for fixed ν, $\omega \in \mathbb{R}^d$,*

$$\lim_{t \to \infty} \beta(t) f(\nu + t\omega) = \bar{f}(\omega). \tag{2.4}$$

Furthermore, assume $f(\omega)\beta(|\omega|)$ is bounded. If $\beta(t)t^{-4} \to 0$ as $t \to \infty$, then

$$\begin{aligned}\lim_{m \to \infty} \beta(m) \operatorname{var}\left(\int_S Z(x)\, dx - \frac{1}{m^d} \sum_{J \in L(m)} Z(c_J) \right) \\ = (2\pi)^{d-p} \textstyle\sum^* \bar{f}(J).\end{aligned} \tag{2.5}$$

If $\beta(t)t^{-4}$ is bounded away from 0 and ∞ as $t \to \infty$, then

$$\begin{aligned}&\operatorname{var}\left(\int_S Z(x)\, dx - \frac{1}{m^d} \sum_{J \in L(m)} Z(c_J) \right) \\ &= \frac{(2\pi)^{d-4}}{\beta(m)} \textstyle\sum^* \bar{f}(J) \\ &\quad + \frac{1}{576 m^4} \int_{\mathbb{R}^d} \left\{ \prod_{\alpha=1}^{d} \operatorname{sinc}^2\left(\frac{\omega_\alpha}{2} \right) \right\} f(\omega) |\omega|^4 \, d\omega \\ &\quad + o(m^{-4}).\end{aligned} \tag{2.6}$$

Note that because we must have $p > d$ for $f(\cdot)$ to be a density, Theorem 2 is vacuous for $d \ge 4$. By Propositions 2.1 and 3.1 of Stein (1991), the simple mean is an asymptotically optimal linear predictor under the conditions needed for (2.5) and the additional condition that $f(\cdot)$ is positive [although as noted in Stein (1991), this positivity condition may be unnecessary]. Indeed, (2.5) is in some regards a special case of Proposition 3.1 of Stein (1991). The advantage of (2.5) is that it gives a much simpler expression for the asymptotic variance than is possible under the more general setting considered in Stein (1991).

As an example of when (2.5) of Theorem 2 applies, consider $f(\omega) = (a^2 + |\omega|^2)^{-q}$, for $\frac{1}{2}d < q < 4$. We can then take $\beta(t) = t^{2q}$ and $\bar{f}(\omega) = |\omega|^{-2q}$. However, (2.4) does not uniquely define $\beta(\cdot)$, as we can just as well take $\beta(t) = c + t^{2q}$ for any constant c. As another example, consider $f(\omega) = \log(1 + |\omega|)/(a^2 + |\omega|^2)^q$ with $\frac{1}{2}d < q < 4$. Because $\log x$ is slowly varying, we can, for example, take $\beta(t) = t^{2q}/\log(2 + t)$ and $\bar{f}(\omega) = |\omega|^{-2q}$.

The boundary case covered by (2.6) includes an important special case in two dimensions. The spectral density $f(\omega) = (a^2 + |\omega|^2)^{-2}$, with corresponding covariance function $\pi a|x|K_1(a|x|)$, where K_1 is a modified Bessel function, was recommended by Whittle (1954) as a natural model for a process in two dimensions. If we take $\beta(t) = t^4$ and $\bar{f}(\omega) = |\omega|^{-4}$, then (2.6) holds with

$$(2\pi)^{d-4} \sum^* \bar{f}(J) = \frac{1}{4\pi^2} \sum^* |J|^{-4} \approx 0.153.$$

The value of the second term in (2.6) depends on a: for $a = 0.5$, 1 and 2, the corresponding values for the constant multiplying m^{-4} in this term are 0.0607, 0.0500 and 0.0332.

Quenouille (1949), Matérn (1986) and Iachan (1985) studied the asymptotic variance of systematic sampling when the distance between neighboring observations is a fixed distance Δ and the region of integration grows with the number of observations. Under some regularity conditions, the asymptotic variance is [Ripley (1981), page 24, gives the result in two dimensions]

$$\left(\frac{2\pi}{\Delta m}\right)^d \sum^* f\left(\frac{2\pi J}{\Delta}\right). \tag{2.7}$$

This result does not require that the observations be centered, in contrast to the results given here. Theorem 2 is written so as to highlight the similarity between it and (2.7). Indeed, taking $\Delta = 1/m$, (2.7) is asymptotically equivalent to the right-hand side of (2.5) divided by $\beta(m)$ when the conditions of Theorem 2 hold. However, for the fixed-domain setting, Theorem 2 is simpler than (2.7) in the sense that it gives the asymptotic variance as a simple function of m times a term not depending on m.

It is possible to extend Theorem 2 to $p > 4$ by changing the weights assigned to the observations near the edges. For example, the predictor denoted by $\hat{Z}_m(r; t_0, t_1, \dots, t_r)$ in Section 4 of Stein (1991) can be used for this purpose. Because the values of $t_1, \dots, t_r$ and r are irrelevant in the setting of this paper, I will write the predictor as $\hat{Z}_m(t_0)$. This predictor adjusts the weights assigned to observations within $(t_0 + 1)/m$ of a boundary of the unit cube, leaving a weight of m^{-d} for all observations farther than that from the boundary.

THEOREM 3. *Assume there exists a monotonic function $\beta(t)$ and $d < p < 2(t_0 - 1)$ such that $\beta(\cdot)$ is regularly varying with exponent p and a function $\bar{f}(\omega)$ such that for fixed $\nu, \omega \in \mathbb{R}^d$, (2.4) holds. Assuming $f(\omega)\beta(|\omega|)$ is bounded, (2.5) is valid if we use $\hat{Z}_m(t_0)$ to predict $\int_S Z(x)\,dx$.*

Moreover, $\hat{Z}_m(t_0)$ is asymptotically optimal for $p < 2(t_0 - 1)$ under the additional condition that $f(\cdot)$ is positive.

3. Numerical results. Table 1 reports some numerical results for centered systematic sampling in two dimensions with $f(\omega) = (1 + |\omega|^2)^{-q}$ for $q = 1.5$ and 3.0. For $q = 3.0$ and m large, by Theorem 1, the term $\int g_m(\omega; 0)\,d\omega$ should dominate the mean squared error and the right-hand side of (2.3) divided by m^4 should provide a good approximation to this term. We see that even for $m = 5$, (2.3) provides a reasonably accurate approximation of the mean squared error. For $q = 1.5$ and m large, by (2.5) of Theorem 2, $\int \Sigma^* g_m(\omega; J)\,d\omega$ should dominate the mean squared error and, taking $\beta(t) = t^{2q}$, we have that $(2\pi)^{2-2q} m^{-2q} \Sigma^* |J|^{-2q}$ should approximate this

TABLE 1

Exact and approximate mean squared errors using centered systematic sampling in two dimensions with m^2 observations and spectral density of the form $f(\omega) = (1 + |\omega|^2)^{-q}$. For $q = 3.0$, $\int g_m(\omega; 0)\, d\omega$ dominates the mean squared error; this integral is approximated using (2.3) of Theorem 1. For $q = 1.5$, $\int \Sigma^* g_m(\omega; J)\, d\omega$ dominates the mean squared error and is approximated using (2.5) of Theorem 2

q	m	Mean squared error	Approximation using (2.3)*	Approximation using (2.5)†	$\int \Sigma^* g_m(\omega; J)\, d\omega$
3.0	5	8.52×10^{-6}	8.17×10^{-6}	1.91×10^{-7}	2.10×10^{-7}
	10	5.16×10^{-7}	5.11×10^{-7}	2.99×10^{-9}	3.14×10^{-9}
	20	3.20×10^{-8}	3.19×10^{-8}	4.67×10^{-11}	4.79×10^{-11}
1.5	5	1.20×10^{-2}	‡	1.15×10^{-2}	1.16×10^{-2}
	10	1.47×10^{-3}	—	1.44×10^{-3}	1.44×10^{-3}
	20	1.82×10^{-4}	—	1.80×10^{-4}	1.80×10^{-4}

*$(1/576m^4)\int_{\mathbb{R}^2}\{\prod_{\alpha=1}^{2} \operatorname{sinc}^2(\omega_\alpha/2)\} f(\omega)|\omega|^4\, d\omega$. See Theorem 1.
†$(2\pi)^{2-2q} m^{-2q} \Sigma^* |J|^{-2q}$. See Theorem 2.
‡Theorem 1 does not apply for $q = 1.5$.

term. Again, even for $m = 5$ these approximations are quite good. The last two columns of Table 1 show that $(2\pi)^{2-2q} m^{-2q} \Sigma^* |J|^{-2q}$ provides a good approximation of $\int \Sigma^* g_m(\omega; J)\, d\omega$ for both values of q; however, it is only for $q = 1.5$ that this integral dominates the mean squared error. We see that even for moderate sample sizes, the asymptotic results of the previous section regarding which term on the right-hand side of (2.2) dominates the mean squared error are clearly applicable. The approximation using increasing-domain asymptotics given in (2.7) gives essentially the same results as the second to the last column in Table 1. Thus, the increasing-domain approximation does not capture the qualitative difference in results depending on whether $f(\omega)$ decays faster or slower than $|\omega|^{-4}$.

4. Proofs. To prove Theorem 1, first note that for fixed ω,

$$\lim_{m \to \infty} m^4 g_m(\omega; 0) = \frac{1}{576} \left\{ \prod_{\alpha=1}^{d} \operatorname{sinc}^2\left(\frac{\omega_\alpha}{2}\right) \right\} f(\omega) |\omega|^4 =_d G(\omega).$$

Furthermore, $m^4 g_m(\omega; 0)$ is dominated by some constant times $G(\omega)$ and $G(\omega)$ is integrable over $\mathbb{R}^d$. So, by dominated convergence, $m^4 \int g_m(\omega; 0)\, d\omega \to \int G(\omega)\, d\omega$. Theorem 1 follows by showing that $f(\omega) = o(|\omega|^{-4})$ implies $\int \Sigma^* g_m(\omega; J)\, d\omega = o(m^{-4})$.

To prove (2.5) of Theorem 2, it suffices to show

$$\lim_{m \to \infty} \beta(m) \int_{\mathbb{R}^d} \Sigma^* g_m(\omega; J)\, d\omega = (2\pi)^{d-p} \Sigma^* \bar{f}(J),$$

because in this case, $\int g_m(\omega; 0)\, d\omega$ will be asymptotically negligible. Using $f(\omega)\beta(|\omega|)$ bounded and $\beta(t)$ regularly varying with exponent $p > d$, for $J \neq 0$ there exists a constant C independent of J such that for all m

sufficiently large,

$$g_m(\omega; J) \le \frac{C}{\beta(2m\pi|J|)} \prod_{\alpha=1}^{d} \operatorname{sinc}^2\left(\frac{\omega_\alpha}{2}\right).$$

This bound along with the assumption on $\beta(\cdot)$ can be used to show that for any $\varepsilon > 0$ there exists R such that for all m sufficiently large,

$$\beta(m) \int_{\mathbb{R}^d} \sum_{|J|>R} g_m(\omega; J)\, d\omega < \varepsilon$$

and

$$\sum_{|J|>R} \bar{f}(J) < \varepsilon.$$

Because ε is arbitrary and there are only a finite number of terms with $|J| \le R$, to obtain (2.5) it suffices to show

$$\lim_{m\to\infty} \beta(m) \int_{\mathbb{R}^d} g_m(\omega; J)\, d\omega = (2\pi)^{d-p} \bar{f}(J). \tag{4.1}$$

Now $\beta(m)g_m(\omega; J)$ is dominated by some constant times $\prod_{\alpha=1}^{d} \operatorname{sinc}^2(\omega_\alpha/2)$, which is integrable over $\mathbb{R}^d$. Furthermore, for fixed ω, setting $t = 2\pi m$, $\omega = J$ and $\nu = \omega$ in (2.4),

$$\lim_{m\to\infty} \beta(m) g_m(\omega; J) = \frac{1}{(2\pi)^p} \left\{ \prod_{\alpha=1}^{d} \operatorname{sinc}^2\left(\frac{\omega_\alpha}{2}\right)\right\} \bar{f}(J),$$

so (4.1) and hence (2.5) follow by dominated convergence. (2.6) just combines the results of Theorem 1 and (2.5).

REFERENCES

CRESSIE, N. A. C. (1991). *Statistics for Spatial Data.* Wiley, New York.

FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications* **2**, 2nd ed. Wiley, New York.

HANNAN, E. J. (1962). Systematic sampling. *Biometrika* **49** 281–283.

IACHAN, R. (1985). Plane sampling. *Statist. Probab. Lett.* **3** 151–159.

JOURNEL, A. G. and HUIJBREGTS, CH. J. (1978). *Mining Geostatistics.* Academic, New York.

MATÉRN, B. (1986). *Spatial Variation, 2nd ed. Lecture Notes in Math.* **36**, Springer, Berlin. (First edition, 1960.)

MATHERON, G. (1965). *Les Variables Régionalisées et Leur Estimation.* Masson et Cie, Paris.

QUENOUILLE, M. H. (1949). Problems in plane sampling. *Ann. Math. Statist.* **20** 355–375.

RIPLEY, B. D. (1981). *Spatial Statistics.* Wiley, New York.

SCHOENFELDER, C. (1982). Random designs for estimating integrals of stochastic processes: asymptotics. Center for Stochastic Processes, Report 6, Dept. Statistics, Univ. North Carolina.

STEIN, M. L. (1991). Predicting integrals of random fields using observations on a lattice. Report 330, Dept. Statistics, Univ. Chicago.

TUBILLA, A. (1975). Error convergence rates for estimates of multidimensional integrals of random functions. Report 72, Dept. Statistics, Stanford Univ.

WHITTLE, P. (1954). On stationary processes in the plane. *Biometrika* **41** 434–449.

DEPARTMENT OF STATISTICS
UNIVERSITY OF CHICAGO
5734 UNIVERSITY AVENUE
CHICAGO, ILLINOIS 60637