The Annals of Applied Probability
1993, Vol. 3, No. 3, 840–862

A LIMIT THEOREM FOR FUNCTIONALS OF JUMPS OF BIRTH AND DEATH PROCESSES UNDER HEAVY TRAFFIC CONDITION

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For a class of birth and death processes under a heavy traffic condition, the asymptotic behavior of functionals of their jumps is investigated. It is shown that, under suitable normalization, the functionals converge in law to a process that is the sum of a Brownian motion and a constant times the local time of a reflecting Brownian motion at zero. Some applications to queueing processes are presented.

1. Introduction. We consider birth and death processes under the heavy traffic condition, which means that birth and death rates are nearly equal at the infinite level of the processes. Here, we study the asymptotic behavior of a certain type of functional of jumps of such birth and death processes; specifically, we consider a sequence of birth and death processes $\{Y_n(t)\}_{n\geq 1}$ on the nonnegative integers with birth rates $\lambda_n(\cdot)$ and death rates $\mu_n(\cdot)$ [$\mu_n(0)=0$], where $\lambda_n(x)$ and $\mu_n(x)$ are nearly equal when n and x are large (see Assumption 2). Our concern is to investigate the asymptotic behavior of the following type of functionals of jumps of $Y_n(t)$:

$$U_n(t) = \sum_{\substack{s \geq t \\ \Delta Y_n(s) \neq 0}} f(Y_n(s-), Y_n(s)), \qquad n \geq 1, \tag{1.1}$$

where $\Delta Y_n(s) = Y_n(s) - Y_n(s-)$. Such a problem often occurs in applications. For example, by taking $f(\cdot,\cdot)$ appropriately in (1.1), $U_n(t)$ represents, in the context of queueing theory, arrival or departure processes. The asymptotic analysis of such processes is important in connection with the investigation of the limiting behavior of the waiting time of a queue under heavy traffic (see Examples 5.2 and 5.3). Analyzing departure processes is also important because they can serve as arrival processes to other service stations (see Example 5.4). It will be shown that under the heavy traffic condition Assumption 2 and other suitable conditions for $\lambda_n(\cdot)$, $\mu_n(\cdot)$ and $f(\cdot,\cdot)$, the normalized processes $\tilde{U}_n(t) = (1/\sqrt{n})(U_n(nt) - nl_n t)$, $n \geq 1$, where the l_n's are suitably chosen constants, converge weakly to a process $U_0(t)$. The limit process $U_0(t)$ is characterized by $U_0(t) = \delta\xi(t) + B(t)$, where δ is a constant, $\xi(t)$ is the local time of a reflecting Brownian motion $X(t)$ at zero and $B(t)$ is a Brownian motion that is generally not independent of $X(t)$. To obtain this result, we follow the stochastic calculus approach used in Ikeda and Watan-

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Received September 1992; revised November 1992.

AMS 1991 *subject classifications*. Primary 60F17, 60K25; secondary 60H30.

Key words and phrases. Limit theorem, functionals of jumps, birth and death processes, heavy traffic condition, reflecting Brownian motion, local time.

abe ([2], Chapter 3, Section 4.4) and Papanicolaou, Stroock and Varadhan ([5], Section 3.5). More directly, we follow the approach in Yamada [10]. As we see later, this approach is particularly well suited for birth and death processes and it produces some useful results. As an example, the limiting behavior of some processes appearing in queueing models is discussed. However, to overcome some difficulties that lie in our approach, we have to impose some technical assumptions that restrict the scope of applications of our result. In choosing the class of birth and death processes for our study, the discussion in Serfozo [7], where the asymptotic behavior of the maximum of some class of birth and death processes under the heavy traffic condition is investigated, was helpful.

We denote by $D([0,\infty), R^d)$ the space of functions f: $[0,\infty) \to R^d$ that are right-continuous and admit left limits, and we endow this space with Skorohod's J_1 topology. Also "$\to_P$" and "$\to_{\mathscr{L}}$" denote convergence in probability and in law, respectively.

2. An outline of our approach with an example. In the next section we will state Assumptions 1–6 and our result. However, because some assumptions are rather technical in nature, in this section we will sketch briefly our approach with an example for the functional $U_n(t)$. This will explain the need for our assumptions. For that purpose, we suppose that the birth and death processes $\{Y_n(t)\}$, $n \ge 1$, represent a sequence of appropriate queueing models of a single station. Thus, $Y_n(t)$ is the number of customers in the nth queue at time t. Let $U_n(t)$ represent the number of busy cycles up to time t for the nth queue. $U_n(t)$ is represented by

$$U_n(t) = \sum_{s \le t} 1(Y_n(s-) = 0)1(\Delta Y_n(s) = 1).$$

Thus, in (1.1) $f(x, x+1) = 1(x = 0)$ and $f(x, x-1) = 0$. We want to investigate the asymptotic behavior of $U_n(t)$ as n tends to infinity under a heavy traffic condition, which means here that the arrival rate $\lambda_n(\cdot)$ and the service rate $\mu_n(\cdot)$ become equal at the infinite level of the queue when $n \to \infty$. In other words, if we set $\lambda_n = \lim_{x\to\infty} \lambda_n(x)$ and $\mu_n = \lim_{x\to\infty} \mu_n(x)$, then $\rho_n = \lambda_n/\mu_n \to 1$ and $n \to \infty$. Under this heavy traffic condition (see Assumption 2 for the exact condition) and some other technical conditions (Assumptions 1 and 3), it is known (Lemma 2) that the normalized processes $X_n(t) = (1/\sqrt{n})Y_n(nt)$, $n \ge 1$, converge in law to a reflecting Brownian motion $X(t)$ with a drift. We write $U_n(t)$ as

$$U_n(t) = \int_0^t 1(Y_n(s-) = 0)\, dN_n^1(s),$$

where $N_n^1(t) = \sum_{s\le t} 1(\Delta Y_n(s) = 1)$, the number of arrivals up to time t. Let $\tilde{N}_n^1(t) = N_n^1(t) - \int_0^t \lambda_n(Y_n(s))\, ds$. Then the normalized process $\tilde{U}_n(t)$ can be written as (take $l_n = 0$)

$$\tilde{U}_n(t) = \frac{1}{\sqrt{n}} \int_0^{nt} g_n(Y_n(s))\, ds + M_n(t) \qquad [= A_n(t) + M_n(t)],$$

where $g_n(x) = 1(x = 0)\lambda_n(x)$ and $M_n(t) = (1/\sqrt{n})\int_0^{nt} 1(Y_n(s-) = 0)\, d\tilde{N}_n^1(s)$. Thus, the investigation of the limit process of $\tilde{U}_n(t)$ is reduced to that of $(A_n(t), M_n(t))$. Using more or less standard functional central limit theorems for martingales [note that $M_n(t)$, $n \geq 1$, are martingales], we can show that the limit process of $M_n(t)$ is a Brownian motion $B(t)$. On the other hand, the investigation of the limit process of $A_n(t)$ is more technical. A heuristic argument for this is as follows. Write $A_n(t) = \int_0^t \sqrt{n}\, g_n(\sqrt{n} X_n(s))\, ds$. Suppose $X(t) > 0$ [recall that $X(t)$ is the limit process of $X_n(t)$]. Because $g_n(x)$ has compact support independent of n and $\sqrt{n} X_n(t) \to \infty$, $\sqrt{n}\, g_n(\sqrt{n} X_n(t)) \to 0$ as $n \to \infty$. This suggests that the limit process $A(t)$, if it exists, does not increase at time t. Thus, $A(t)$ is nonnegative and increasing and increases only when $X(t) = 0$; this indicates that $A(t)$ behaves like the local time of the process $X(t)$ at zero. Indeed, as is claimed in the Introduction, we can show that $A(t)$ is a constant times the local time of $X(t)$ at zero. Although the foregoing argument is not very precise, it may explain why we need the condition that $g_n(x)$ has compact support (not depending on n; see Assumption 6). Formally, we proceed as follows. For each $n \geq 1$, consider the equation

$$(2.1)\qquad \begin{gathered} (F_n(x+1) - F_n(x))\lambda_n(x) - (F_n(x) - F_n(x-1))\mu_n(x) = g_n(x), \\ F_n(0) = 0, \qquad x \in Z = \{0, 1, 2, \dots\}. \end{gathered}$$

[Note that if $\mathscr{L}_n$ represents the generator of the process $Y_n(t)$, the preceding equation can be written as $\mathscr{L}_n F_n(x) = g_n(x)$.] $F_n(x)$ is uniquely determined and $A_n(t)$ can be written as

$$A_n(t) = \frac{1}{\sqrt{n}} F_n\big(\sqrt{n} X_n(t)\big) - \frac{1}{\sqrt{n}} F_n\big(\sqrt{n} X_n(0)\big) + \hat{M}_n(t),$$

where $\hat{M}_n(t)$, $n \geq 1$, are martingales. [Note that, in general, $F_n(Y_n(t)) - \int_0^t \mathscr{L}_n F_n(Y_n(s))\, ds$ is a (local) martingale.] Because the asymptotic analysis of $\hat{M}_n(t)$, as martingales, is not difficult, the main problem in studying the limiting behavior of $A_n(t)$ is reduced to that of $(1/\sqrt{n})F_n(\sqrt{n} X_n(t))$. In view of the fact that $X_n(t) \to X(t) > 0$ for a.e. t, the problem is then to investigate the limiting behavior of $(1/\sqrt{n})F_n(\sqrt{n}\, x_n)$ when $x_n \to x > 0$. But, because $g_n(x)$ has compact support, the problem is finally reduced to determining the limiting behavior of $(1/\sqrt{n})(\alpha_n(1) + \cdots + \alpha_n(\sqrt{n}\, x_n))$ when $x_n \to x > 0$, where

$$(2.2)\qquad \alpha_n(x) = \frac{\mu_n(1)}{\lambda_n(1)} \cdots \frac{\mu_n(x)}{\lambda_n(x)}.$$

This analysis is, however, rather complicated under the presence of the index n. Thus we restrict our models to more specific ones so that the analysis becomes feasible. This is the reason why we impose Assumptions 4 and 5. For an implication of these assumptions, see Remark 2.

3. Basic result. We suppose that for each $n \geq 1$, the birth and death process $Y_n(t)$ is defined on a stochastic basis $(\Omega_n, \mathscr{F}_n, P_n; \mathscr{F}_t^n)$ satisfying the usual conditions (see Jacod and Shiryaev [4], page 2). We also make the following assumptions:

ASSUMPTION 1. $\lambda_n(x) > 0$ for all n and x, $\lambda_n = \lim_{x\to\infty} \lambda_n(x)$ and $\mu_n = \lim_{x\to\infty} \mu_n(x)$ exist. Moreover, $\sup_n \lambda_n(0) < \infty$ and

$$\sup_{n,x} |x(\lambda_n(x) - \lambda_n + \mu_n - \mu_n(x))| < \infty.$$

ASSUMPTION 2. $\mu_n \ge \lambda_n$, $n \ge 1$, $\lim_{n\to\infty} \sqrt{n}(\lambda_n - \mu_n) = c$, $-\infty < c \le 0$, and $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \mu_n = \lambda > 0$. Moreover, $\lambda_n(x) \ge \lambda_n$ and $\mu_n \ge \mu_n(x)$ for all $x \in Z = \{0, 1, 2, \dots\}$.

ASSUMPTION 3. $\lim_{x\to\infty, n\to\infty} x(\lambda_n(x) - \lambda_n + \mu_n - \mu_n(x)) = 0$.

ASSUMPTION 4. $\lambda_n(x)$ and $\mu_n(x)$ can be written as

$$\lambda_n(x) = \lambda_n \cdot \lambda_0(x) \cdot \tilde{\lambda}_n(x),$$
$$\mu_n(x) = \mu_n \cdot \mu_0(x) \cdot \tilde{\mu}_n(x)$$

and $\lim_{n\to\infty} \tilde{\lambda}_n(x)$ [$= \tilde{\lambda}(x)$] and $\lim_{n\to\infty} \tilde{\mu}_n(x)$ [$= \tilde{\mu}(x)$] exist.

ASSUMPTION 5.

(i) $$\lim_{x\to\infty} \prod_{k=1}^{x} \frac{\mu_0(k)}{\lambda_0(k)} = \alpha, \qquad 0 < \alpha < \infty,$$

(ii) $$\lim_{\substack{x\to\infty \\ n\to\infty}} \prod_{k=1}^{x} \frac{\tilde{\mu}_n(k)}{\tilde{\lambda}_n(k)} = \beta, \qquad 0 < \beta < \infty$$

and $\{\prod_{k=1}^{x}(\tilde{\mu}_n(k)/\tilde{\lambda}_n(k))\}$ are bounded in n and $x \in Z$.

ASSUMPTION 6. There exist constants l_n, $n \ge 1$, such that $g_n(x) := f(x, x+1)\lambda_n(x) + f(x, x-1)\mu_n(x) - l_n$, $x \in Z$, has a compact support not depending on n. Thus we assume that for a constant $x_0 \in Z$, $g_n(x) = 0$ for all $n \ge 1$ and $x \ge x_0$. Moreover, there exists a function $g(x)$ on Z such that $g_n(x) \to g(x)$ for all $x \in Z$ as $n \to \infty$.

REMARK 1. Suppose that the process $Y_n(t)$ represents the queue length of a single station queueing model. Then the condition $\lim_{n\to\infty}\sqrt{n}(\lambda_n - \mu_n) = c$ in Assumption 2 implies that when $n \to \infty$, the arrival and service rates become nearly equal when the queue length is very large. This condition is typically called the heavy traffic condition in queueing theory. When the arrival and service rates are independent of the queue length, this condition coincides with the usual one (Whitt [8]).

REMARK 2. By Assumptions 2 and 4, $\lim_{n\to\infty} \lambda_n(x) = \lambda \cdot \lambda_0(x) \cdot \tilde{\lambda}(x)$ [$:= \lambda(x)$] and $\lim_{n\to\infty} \mu_n(x)\mu_n(x) = \lambda \cdot \mu_0(x) \cdot \tilde{\mu}(x)$ [$:= \mu(x)$]. Then Assumption 5 implies that the birth and death process with birth rate $\lambda(x)$ and death rate $\mu(x)$ is null recurrent; that is, recurrent but not ergodic (see Asmussen [1], Chapter 3, Section 4, for the definition of "recurrent" and "ergodic"). Indeed, by Assumptions 4 and 5(ii), we have $\lim_{x\to\infty}\prod_{k=1}^{x}(\tilde{u}(k)/\tilde{\lambda}(k)) = \beta$. Thus

$$\lim_{x\to\infty}\prod_{k=1}^{x}\frac{\mu(x)}{\lambda(x)} = \alpha\beta > 0, \tag{3.1}$$

$$\sum_{x=1}^{\infty}\prod_{k=1}^{x}\frac{\mu(x)}{\lambda(x)} = \infty. \tag{3.2}$$

On the other hand, because $\lambda(x)$ is bounded [note that by Assumption 1, $\lambda_n(x)$ is bounded in n and x], (3.1) implies

$$\sum_{x=1}^{\infty}\frac{\lambda(0)\cdots\lambda(x-1)}{\mu(1)\cdots\mu(x)} = \lambda(0)\sum_{x=1}^{\infty}\frac{1}{\lambda(x)}\prod_{k=1}^{x}\frac{\lambda(k)}{\mu(k)} = \infty. \tag{3.3}$$

Thus, (3.2) and (3.3) imply our assertion ([1], Chapter 3, Section 2); that is, the process $Y_n(t)$ becomes null recurrent when $n\to\infty$.

Some of the foregoing assumptions were imposed for technical reasons and make our models restrictive. They are also not easy to check for arbitrarily given $\lambda_n(x)$ and $\mu_n(x)$. In particular, Assumption 5(ii) is generally difficult to verify. For example, if $\lambda_n(x)$ and $\mu_n(x)$ are given by $\lambda_n(x) = \lambda_n\cdot\lambda_0(x)$ and $\mu_n(x) = \mu_n\cdot\mu_0(x)$ [hence, $\tilde{\lambda}_n(x) = \tilde{\mu}_n(x) = 1$], then it only remains to check Assumption 5(i). On the other hand, if $\lambda_n(x)$ and $\mu_n(x)$ are given by $\lambda_n(x) = \lambda_n + \lambda_0(x)$ and $\mu_n(x) = \mu_n + \mu_0(x)$, it is not clear whether Assumption 5 holds or not. As another example where our assumptions make the model restrictive, let us consider the case $g_n(x) = \lambda_n(x) - \lambda_n$, which appears in Example 5.2 in Section 5, where we study the asymptotic behavior of arrival processes. In this case, Assumption 6 for $g_n(x)$ implies that the model under consideration is restricted to a class for which, for sufficiently large x, $\lambda_n(x)$ is a constant for each n. However, our assumptions are not too restrictive for the following reasons: First, Assumptions 1–5 are typically satisfied for models appearing in practical applications (see Example 3.1). Second, for an arbitrarily given birth and death process $Y(t)$ with birth rate $\lambda(x)$ and death rate $\mu(x)$, we can always choose a sequence of processes $Y_n(t)$, $n\geq 1$, with $\lambda_n(x)$ and $\mu_n(x)$ such that Assumptions 1–5 are satisfied and with $\lambda_n(x) = \mu(x)$ and $\mu_n(x) = \mu(x)$ for sufficiently large n as long as $\lambda(x)$ and $\mu(x)$ satisfy some appropriate conditions. Indeed, suppose that: (i) $\mu(\infty)\geq\lambda(\infty)$, where $\lambda(\infty) = \lim_{x\to\infty}\lambda(x)$ and $\mu(\infty) = \lim_{x\to\infty}\mu(x)\lim_{x\to\infty}\mu(x)$; (ii) $\lambda(x)\geq\lambda(\infty)$, $\mu(\infty)\geq\mu(x)$ and $x(\lambda(x)-\lambda(\infty)+\mu(\infty)-\mu(x))\to 0$ as $n\to\infty$; and (iii) $\lambda_0(x) := \lambda(x)/\lambda(\infty)$ and $\mu_0(x) := \mu(x)/\mu(\infty)$ satisfy Assumption 5(i). Then it is easy to see that we can take λ_n and μ_n so that $\lambda_n(x) := \lambda_n\cdot\lambda_0(x)$ and $\mu_n(x) := \mu_n\cdot\mu_0(x)$ satisfy Assumptions 1–5 and $\lambda_n(x) = \lambda(x)$ and $\mu_n(x) = \mu(x)$ for sufficiently large n if $\mu(\infty)$ and $\lambda(\infty)$ are nearly equal [Note that if $\lambda(\infty) = \mu(\infty)$, we can take $\lambda_n = \lambda(\infty)$ and $\mu_n = \mu(\infty)$.] Thus, for an arbitrary given function $f(\cdot,\cdot)$, $U(t) = \sum_{s \le t} f(Y(s-), Y(s))$ coincides with $U_n(t) = \sum_{s \le t} f(Y_n(s-), Y_n(s))$ for sufficiently large n, and the asymptotic behavior of $U(t)$ as $t \to \infty$ is approximated by that of $U_n(t)$ as $t \to \infty$ and $n \to \infty$.

EXAMPLE 3.1. For arbitrary fixed integers m_1 and m_2 ($m_1 > 0$, $m_2 > 0$), let

$$\lambda_n(x) = \sum_{k=1}^{m_1} \lambda_n^k \cdot \lambda^{(k)}(x) \cdot 1\big(x \in (K_{k-1}^n, K_k^n]\big),$$
$$\mu_n(x) = \sum_{k=1}^{m_2} \mu_n^k \cdot \mu^{(k)}(x) \cdot 1\big(x \in (L_{k-1}^n, L_k^n]\big),$$

where K_k^n and L_k^n are nonnegative integers such that $K_0^n = 0 < K_1^n < \cdots < K_{m_1}^n = \infty$ and $L_0^n = 0 < L_1^n < \cdots < L_{m_2}^n = \infty$. Let $\lambda^{(m_1)}(\infty) = \lim_{x \to \infty} \lambda^{(m_1)}(x)$ and $\mu^{(m_2)}(\infty) = \lim_{x \to \infty} \mu^{(m_2)}(x)$. Set

$$\lambda_n = \lambda_n^{m_1} \lambda^{(m_1)}(\infty), \qquad \mu_n = \mu_n^{m_2} \mu^{(m_2)}(\infty),$$
$$\lambda_0(x) = \lambda^{(m_1)}(x)/\lambda^{(m_1)}(\infty), \qquad \mu_0(x) = \mu^{(m_2)}(x)/\mu^{(m_2)}(\infty),$$
$$\tilde{\lambda}_n(x) = \sum_{k=1}^{m_1} (\lambda_n^k/\lambda_n^{m_1})(\lambda^{(k)}(x)/\lambda^{(m_1)}(x)) 1\big(x \in [K_{k-1}^n, K_k^n)\big),$$
$$\tilde{\mu}_n(x) = \sum_{k=1}^{m_2} (\mu_n^k/\mu_n^{m_2})(\mu^{(k)}(x)/\mu^{(m_2)}(x)) 1\big(x \in [L_{k-1}^n, L_k^n)\big).$$

We suppose that $\lim_{x \to \infty} x(\lambda_0(x) - \mu_0(x)) = 0$ and that Assumptions 2 and 5(i) hold. Then noting that $\tilde{\lambda}_n(x) = \tilde{\mu}_n(x) = 1$ for sufficiently large n and x, it is easy to see that all Assumptions 1—5 are satisfied. A typical example of $\mu_n(x)$ is given by

$$\mu_n(x) = \begin{cases} \hat{\mu}_n s_n, & x \ge s_n, \\ \hat{\mu}_n x, & x < s_n \end{cases} \tag{3.4}$$

and this represents the service rate for a multiserver queue.

Set

$$\begin{aligned} \tilde{F}_n(x) &= \frac{g_n(x-1)}{\lambda_n(x-1)} + \frac{\mu_n(x-1)}{\lambda_n(x-1)} \frac{g_n(x-2)}{\lambda_n(x-2)} \\ &\quad + \cdots + \frac{\mu_n(x-1)}{\lambda_n(x-1)} \frac{\mu_n(x-2)}{\lambda_n(x-2)} \cdots \frac{\mu_n(1)}{\lambda_n(1)} \frac{g_n(0)}{\lambda_n(0)}, \end{aligned} \tag{3.5}$$
$$\alpha_n(x) = \frac{\mu_n(1)}{\lambda_n(1)} \frac{\mu_n(2)}{\lambda_n(2)} \cdots \frac{\mu_n(x)}{\lambda_n(x)}$$

and let

$$(3.6) \qquad \gamma = \lim_{n\to\infty} \frac{\tilde{F}_n(x_0+1)}{\alpha_n(x_0)},$$

which exists by our assumptions. Then our basic result is contained in the following theorem.

THEOREM 1. *Let us define processes* $\{\tilde{U}_n(t)\}_{n\geq 1}$ *by*

$$\tilde{U}_n(t) = \frac{1}{\sqrt{n}}(U_n(nt) - l_n nt), \qquad n \geq 1.$$

Assume Assumptions 1–6 and suppose $Y_n(0) \to_{\mathscr{L}} Y(0)$, *where* $Y(0)$ *is a random variable. Then* $\tilde{U}_n(t) \to_{\mathscr{L}} U_0(t)$ *in* $D([0,\infty), R^1)$, *where*

$$U_0(t) = \delta\xi(t) + \sqrt{f_1(\infty)^2\lambda}\,B_1(t) + \sqrt{f_2(\infty)^2\lambda}\,B_2(t).$$

In the theorem, $\delta = \gamma\alpha\beta$, $B_1(t)$ and $B_2(t)$ are two independent standard Brownian motions and

$$\begin{aligned}\xi(t) &= -\inf_{s\leq t}\bigl(cs + \sqrt{\lambda}B_1(s) - \sqrt{\lambda}B_2(s)\bigr),\\ f_1(\infty) &= \lim_{x\to\infty} f(x, x+1),\\ f_2(\infty) &= \lim_{x\to\infty} f(x, x-1),\end{aligned}$$

where in the final two expressions we assume the limits exist.

REMARK 3. It is well known that $\xi(t)$ is characterized as a unique solution of the Skorohod equation (Ikeda and Watanabe [1], Chapter 3, Section 4.2):

$$X(t) = ct + \sqrt{\lambda}B_1(t) + \sqrt{\lambda}B_2(t) + \xi(t),$$

where $(X(t), \xi(t))$ are continuous processes such that $X(t) \geq 0$ for all $t \geq 0$ and $\xi(t)$ is increasing with $\xi(0) = 0$ and satisfies

$$\xi(t) = \int_0^t 1(X(s) = 0)\, d\xi(s).$$

Thus $\xi(t)$ is the local time of $X(t)$ at zero. Later it will be shown that $X(t)$ is the weak limit of $\{X_n(t)\}_{n\geq 1}$, where $X_n(t) = (1/\sqrt{n})Y_n(nt)$ [see Lemma 2 and (4.7)]. The proof of Theorem 1 uses this fact and the characterization of $\xi(t)$ mentioned previously. Taking into consideration the foregoing fact, it is sometimes more convenient to state Theorem 1 in the following form:

THEOREM 1′. *Under the same conditions in Theorem* 3.1, $(X_n(t), \tilde{U}_n(t)) \to_{\mathscr{L}} (X(t), U_0(t))$ *in* $D([0,\infty), R^2)$, *where* $X_n(t)$ *and* $X(t)$ *were given previously.*

REMARK 4. We can extend Theorem 1 in the following way. Set

$$V_n(t) = \sum_{s\leq t} h(Y_n(s-), Y_n(s)), \qquad n \geq 1.$$

We assume that Assumption 6 also holds for $h(\cdot,\cdot)$ with different l_n, x_0 and $g(x)$; that is, $\hat{l}_n$, $\hat{x}_0$ and $\hat{g}(x)$. Set

$$\tilde{V}_n(t) = \frac{1}{\sqrt{n}}\big(V_n(nt) - n\hat{l}_n t\big), \qquad n \ge 1.$$

Then under the same assumptions as in Theorem 1, we have $(\tilde{U}_n(t), \tilde{V}_n(t)) \to_{\mathscr{L}} (U_0(t), V_0(t))$ in $D([0,\infty), R^2)$, where $V_0(t)$ is defined as

$$\begin{aligned} V_0(t) &= \hat{\delta}\xi(t) + \sqrt{h_1(\infty)^2\lambda}\,B_1(t) + \sqrt{h_2(\infty)^2\lambda}\,B_2(t),\\ \hat{\delta} &= \hat{\gamma}\alpha\beta\colon \hat{\gamma} \text{ is defined in the same way as } \gamma \text{ using}\\ &\quad\; h(\cdot,\cdot) \text{ instead of } f(\cdot,\cdot),\\ h_1(\infty) &= \lim_{x\to\infty} h(x, x+1),\\ h_2(\infty) &= \lim_{x\to\infty} h(x, x-1). \end{aligned}$$

As before, we assume the existence of the limits $h_1(\infty)$ and $h_2(\infty)$. The proof of this fact can be done in the same way as the proof of Theorem 1.

4. Proving the theorem.

We begin with a lemma.

LEMMA 1. *For each $n \ge 1$, let $g_n(x)$ be any function defined on Z with compact support not depending on n. We suppose that $g_n(x) \to g(x)$ as $n \to \infty$ for each $x \in Z$. For the process $Y_n(t)$, we assume that $\lambda_n(x)$ and $\mu_n(x)$ are given by the form in Assumption 4 and that they satisfy the conditions in Assumptions 4 and 5. Moreover, we assume $\sqrt{n}(\lambda_n - \mu_n) = c$ and $\lim \lambda_n = \lim \mu_n = \lambda > 0$. Then*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} F_n\big(\big[\sqrt{n}\,x_n\big]\big) = \begin{cases} \delta(\lambda/c)(1 - \exp(-(c/\lambda)x)), & c \ne 0,\\ \delta x, & c = 0,\end{cases}$$

for any sequence $\{x_n\}$ such that $x_n \to x > 0$. ($[a]$ denotes the largest integer less than a.) In the preceding equation, $F_n(x)$, $x = 0, 1, \dots,$ is the unique solution of (2.1) *with $g_n(x)$ as before and $\delta = \gamma\alpha\beta$ [where γ is defined by* (3.6) *using the previous $g_n(x)$].*

PROOF. Set $\tilde{F}_n(x) = F_n(x) - F_n(x-1)$ for $x = 1, 2, \dots$. [$\tilde{F}_n(0)$ is arbitrarily defined.] Then

$$\tilde{F}_n(x+1)\lambda_n(x) - \tilde{F}_n(x)\mu_n(x) = g_n(x), \qquad x = 0, 1, \dots.$$

Because there exists x_0 such that $g_n(x) = 0$ for $x \ge x_0$,

$$\tilde{F}_n(x+1) = \alpha_n(x)\tilde{F}_n(x_0+1)/\alpha_n(x_0), \qquad x \ge x_0, \tag{4.1}$$

where $\alpha_n(x)$ was defined in (3.5). Hence

$$\frac{1}{\sqrt{n}} F_n\big(\big[\sqrt{n}\, x_n\big]\big) = \frac{1}{\sqrt{n}}\Big(\alpha_n(x_0) + \cdots + \alpha_n\big(\big[\sqrt{n}\, x_n\big] - 1\big)\Big) \times \frac{\tilde{F}_n(x_0+1)}{\alpha_n(x_0)} + \frac{F_n(x_0)}{\sqrt{n}}$$

if $[\sqrt{n}\, x_n] > x_0$. We can write $\alpha_n(x)$ as

$$\alpha_n(x) = \left(\frac{\mu_n}{\lambda_n}\right)^x \left(\prod_{k=1}^{x} \frac{\mu(k)}{\lambda(k)} - \alpha\right) \prod_{k=1}^{x} \frac{\tilde{\mu}_n(k)}{\tilde{\lambda}_n(k)} + \left(\frac{\mu_n}{\lambda_n}\right)^x \alpha\beta + \left(\frac{\mu_n}{\lambda_n}\right)^x \alpha \left(\prod_{k=1}^{x} \frac{\tilde{\mu}_n(k)}{\tilde{\lambda}_n(k)} - \beta\right) := \alpha_n^1(x) + \alpha_n^2(x) + \alpha_n^3(x).$$

Because $(\mu_n/\lambda_n)^{[\sqrt{n}\, x_n]}$ is bounded in n, by Assumption 5(ii), we have

$$\left|\frac{1}{\sqrt{n}}\Big(\alpha_n^1(x_0) + \cdots + \alpha_n^1\big(\big[\sqrt{n}\, x_n\big] - 1\big)\Big)\right| \le \frac{K}{\sqrt{n}}\Big(|\alpha(x_0)| + \cdots + \big|\alpha\big(\big[\sqrt{n}\, x_n\big] - 1\big)\big|\Big),$$

where $\alpha(x) = \prod_{k=1}^{x}(\mu(k)/\lambda(k)) - \alpha$. Then noting that $x_n \to x > 0$ and $\alpha([\sqrt{n}\, x_n] - 1) \to 0$,

$$\frac{1}{\sqrt{n}}\Big(\alpha_n^1(x_0) + \cdots + \alpha_n^1\big(\big[\sqrt{n}\, x_n\big] - 1\big)\Big) \to 0 \quad \text{as } n \to \infty.$$

We can also show that

$$\frac{1}{\sqrt{n}}\Big(\alpha_n^3(x_0) + \cdots + \alpha_n^3\big(\big[\sqrt{n}\, x_n\big] - 1\big)\Big) \to 0 \quad \text{as } n \to \infty. \tag{4.2}$$

Indeed, by Assumption 5(ii), for an arbitrary $\varepsilon > 0$ there exist $\bar{x}$ and $\bar{n}$ such that if $x \ge \bar{x}$ and $n \ge \bar{n}$, then

$$\left|\sum_{k=1}^{x} \frac{\tilde{\mu}_n(k)}{\tilde{\lambda}_n(k)} - \beta\right| \le \varepsilon.$$

Hence, the left-hand side of (4.2) is less than

$$\frac{1}{\sqrt{n}}\big(\alpha_n^3(x_0) + \cdots + \alpha_n^3(\bar{x})\big) + K\frac{1}{\sqrt{n}}\frac{1}{x_n}\varepsilon\big[\sqrt{n}\, x_n\big] \cdot x_n.$$

By Assumption 2 (i.e., $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \mu_n = \lambda$) and $x_n \to x > 0$ and Assumption 4, the preceding quantity tends to $K\varepsilon x$ as $n \to \infty$. Because ε was arbitrary, we get (4.2).

On the other hand, we have

$$\frac{1}{\sqrt{n}}\Big(\alpha_n^2(x_0) + \cdots + \alpha_n^2\big([\sqrt{n}\,x_n] - 1\big)\Big)$$

$$= \alpha\beta\frac{1}{\sqrt{n}}\left(\left(\frac{\mu_n}{\lambda_n}\right)^{x_0} + \cdots + \left(\frac{\mu_n}{\lambda_n}\right)^{[\sqrt{n}\,x_n]-1}\right)$$

$$= \alpha\beta\lambda_n\frac{1-(\mu_n/\lambda_n)^{[\sqrt{n}\,x_n]}}{\sqrt{n}\,(\lambda_n-\mu_n)} - \alpha\beta\lambda_n\frac{1-(\mu_n/\lambda_n)^{x_0}}{\sqrt{n}\,(\lambda_n-\mu_n)}$$

$$\to \alpha\beta\left(\frac{\lambda}{c}\right)\left(1-\exp\left(-\frac{c}{\lambda}x\right)\right) \quad \text{if } c \neq 0.$$

When $c = 0$, we have, for any $x \leq [\sqrt{n}\,x_n] - 1$,

$$1 \leq \left(\frac{\mu_n}{\lambda_n}\right)^{x} \leq \left(\frac{\mu_n}{\lambda_n}\right)^{[\sqrt{n}\,x_n]-1} \to 1 \quad \text{as } n \to \infty.$$

Hence

$$\frac{1}{\sqrt{n}}\Big(\alpha_n^2(x_0) + \cdots + \alpha_n^2\big([\sqrt{n}\,x_n] - 1\big)\Big) \to \alpha\beta x \quad \text{as } n \to \infty.$$

Combining these results, we reach the conclusion. □

The sequence of processes $\{X_n(t)\}_{n\geq 1}$ defined by

$$X_n(t) = \frac{1}{\sqrt{n}}Y_n(nt), \qquad n \geq 1,$$

was introduced in Section 2.

LEMMA 2. *Under Assumptions* 1–3, $X_n(t) \to_{\mathscr{L}} X(t)$ *in* $D([0,\infty), R^1)$, *where* $X(t)$ *is the unique nonnegative process satisfying the Skorohod equation*

$$X(t) = ct + \sqrt{2\lambda}\,B(t) + \xi(t),$$

with $B(t)$ *being a standard Brownian motion and* $\xi(t)$ *being a continuous increasing process such that* $\xi(0) = 0$ *and*

$$\xi(t) = \int_0^t 1(X(s) = 0)\,d\xi(s).$$

PROOF. In Yamada [11], for a sequence of birth and death processes which is slightly different from the one considered here, a similar result is given under the same conditions as ours. The argument used there is applicable here in almost the same way. Hence we omit the detailed discussion and present the facts that will be used later and some discussion not found in reference 11.

Let us define processes $N_n^1(t)$, $N_n^2(t)$, $\tilde{N}_n^1(t)$ and $\tilde{N}_n^2(t)$ by

$$N_n^1(t) = \sum_{s \le t} 1(\Delta Y_n(s) = 1),$$

$$N_n^2(t) = \sum_{s \le t} 1(\Delta Y_n(s) = -1),$$

$$\tilde{N}_n^1(t) = N_n^1(t) - \int_0^t \lambda_n(Y_n(s))\,ds,$$

$$\tilde{N}_n^2(t) = N_n^2(t) - \int_0^t \mu_n(Y_n(s))\,ds.$$

Then we have

$$Y_n(t) = Y_n(0) + N_n^1(t) - N_n^2(t)$$

and hence $X_n(t)$ can be written as

$$X_n(t) = X_n(0) + D_n(t) + \xi_n(t) + M_n^1(t) - M_n^2(t),$$

where

$$D_n(t) = \sqrt{n}\,(\lambda_n - \mu_n)t,$$

$$\xi_n(t) = \frac{1}{\sqrt{n}} \int_0^{nt} (\lambda_n(Y_n(s)) - \lambda_n - \mu_n(Y_n(s)) + \mu_n)\,ds,$$

$$M_n^1(t) = \frac{1}{\sqrt{n}} \tilde{N}_n^1(nt),$$

$$M_n^2(t) = \frac{1}{\sqrt{n}} \tilde{N}_n^2(nt).$$

We can show using Assumptions 1 and 2 that each family of processes $\{X_n\}$, $\{D_n\}$, $\{\xi_n\}$, $\{M_n^1\}$ and $\{M_n^2\}$ is C-tight in $D([0, \infty), R^1)$. Hence $\{X_n, D_n, \xi_n, M_n^1, M_n^2\}_{n \ge 1}$ is tight in $D([0, \infty), R^5)$ (Jacod and Shiryaev [4], Chapter 6, Corollary 3.3) and, if we let $\{X, D, \xi, M_1, M_2\}$ be any weak limit of $\{X_n, D_n, \xi_n, M_n^1, M_n^2\}_{n \ge 1}$ in $D([0, \infty), R^5)$, we have

$$X(t) = ct + \xi(t) + M_1(t) - M_2(t).$$

To identify the process $M_1(t) - M_2(t)$ that is a continuous martingale, we note that $\mathscr{L}\{t;\ X(t) = 0\} = 0$ with probability 1 ($\mathscr{L}$ denotes Lebesgue measure). Indeed, because $X(t)$ has, as a semimartingale, a local time, the density formula for local times (Jacod [3], page 188) implies

$$\int_0^t 1_{\{0\}}(X(s))\,d\langle M_1 - M_2\rangle(s) = 0.$$

Because $M_1(t)$ and $M_2(t)$ are orthogonal [note that $M_n^1(t)$ and $M_n^2(t)$ are orthogonal], this further implies

$$0 = \int_0^t 1_{\{0\}}(X(s))\,d\langle M_1\rangle(s). \tag{4.3}$$

On the other hand, because $\lambda_n(x)$ is bounded by Assumption 1, we have $|\langle M_n^1\rangle(t) - \langle M_n^1\rangle(s)| \le K|t - s|$ and, hence, by letting n tend to infinity, we can easily show that $\langle M_1\rangle(t)$ is absolutely continuous. That is, there exists a measurable process $\alpha(t)$ such that $\langle M_1\rangle(t) = \int_0^t \alpha(s)\,ds$. Moreover, because $\lambda_n(x)$ is bounded below [note that $\lambda_n(x) \ge \lambda_n$], we can choose the process $\alpha(t)$ such that $\alpha(t) \ge \alpha > 0$ for a constant α. Thus from (4.3),

$$0 = \int_0^t 1_{\{0\}}(X(s))\,\alpha(s)\,ds \ge \alpha\int_0^t 1_{\{0\}}(X(s))\,ds,$$

which implies $\int_0^t 1_{\{0\}}(X(s))\,ds = 0$; that is, $\mathscr{L}\{t;\ X(t) = 0\} = 0$. Next, using this fact and noting that $\lambda_n(x) \to \lambda$ and $\mu_n(x) \to \lambda$ as $n \to \infty$ and $x \to \infty$ (see Assumptions 2 and 3) and

$$\langle M_n^1 - M_n^2\rangle(t) = \int_0^t \left[\lambda_n\left(\sqrt{n}\,X_n(s)\right) + \mu_n\left(\sqrt{n}\,X_n(s)\right)\right] ds,$$

we have that $\langle M_n^1 - M_n^2\rangle(t) \to_P 2\lambda t$ for each t. Moreover, $|\Delta(M_n^1 - M_n^2)(t)| \le 2/\sqrt{n}$. Hence, by a standard functional central limit theorem for martingales (e.g., Rebolledo [6]), there exists a Brownian motion $B(t)$ such that $M_1(t) - M_2(t) = \sqrt{2\lambda}\,B(t)$.

We can also show, by using Assumption 3, that the process $\xi(t)$ increases only when $X(t) = 0$ as asserted in the lemma. The argument for this is as follows. We will show that $\xi(t)$ does not increase at t when $X(t) > 0$. For this we may assume that because $X(t)$ and $\xi(t)$ are continuous, $X_n(t) \to X(t)$ and $\xi_n(t) \to \xi(t)$ uniformly on each compact t-set with probability 1. Then we can take t_1 and t_2 and $\alpha > 0$ such that $t_1 < t < t_2$, $X(s) > 0$ for $s \in [t_1, t_2]$ and $X_n(s) \ge \alpha$ for $s \in [t_1, t_2]$ and for sufficiently large n. Then

$$\begin{aligned}
0 &\le \xi_n(t_2) - \xi_n(t_1) \\
&= \int_{t_1}^{t_2} \frac{1}{X_n(s)} \sqrt{n}\,X_n(s)\left(\lambda_n\left(\sqrt{n}\,X_n(s)\right) - \lambda_n - \mu_n\left(\sqrt{n}\,X_n(s)\right) + \mu_n\right) ds \\
&\le \frac{1}{\alpha}\int_{t_1}^{t_2} \sqrt{n}\,X_n(s)\left(\lambda_n\left(\sqrt{n}\,X_n(s)\right) - \lambda_n - \mu_n\left(\sqrt{n}\,X_n(s)\right) + \mu_n\right) ds \to 0,
\end{aligned}$$

because $X_n(s) \to X(s) > 0$ for $s \in [t_1, t_2]$. Thus $\xi(t_2) = \xi(t_1)$ and $\xi(\cdot)$ does not increase at t. □

REMARK 5. An intuitive argument for obtaining Lemma 2 goes as follows and explains why we impose Assumption 3: $X_n(t)$ can be written as

$$X_n(t) = X_n(0) + D_n(t) + \xi_n^1(t) + M_n^1(t) - M_n^2(t) + \xi_n^2(t),$$

where

$$\xi_n^1(t) = \int_0^t 1(X_n(s) > 0)\,d\xi_n(s),$$

$$\xi_n^2(t) = \int_0^t 1_{\{0\}}(X_n(s))\,d\xi_n(s).$$

$\xi_n^1(t)$ can be written as

$$\xi_n^1(t) = \int_0^t 1(X_n(s) > 0)(1/X_n(s))k_n(X_n(s))\,ds,$$

where

$$k_n(X_n(s)) = \sqrt{n}X_n(s)\big(\lambda_n(\sqrt{n}X_n(s)) - \lambda_n + \mu_n - \mu_n(\sqrt{n}X_n(s))\big).$$

At s where $X(s) > 0$, in view of Assumption 3, $k_n(X_n(s)) \to 0$ when $n \to \infty$. Thus if we admit the fact $\mathscr{L}\{t;\ X(t) = 0\} = 0$, then the weak limit of $\xi_n^1(t)$ will be a null process. On the other hand, because $\xi_n^2(t)$ increases only when $X_n(t) = 0$, the weak limit of $\xi_n^2(t)$ also increases only when $X(t) = 0$. Hence, in view of the discussion of the weak limit of $M_n^1(t) - M_n^2(t)$ in the proof of Lemma 2, the limit process $X(t)$ will be the process given in Lemma 2.

PROOF OF THEOREM 1. We can write $\tilde{U}_n(t)$ as

$$\tilde{U}_n(t) = A_n(t) + m_n^1(t) + m_n^2(t),$$

where

$$A_n(t) = \frac{1}{\sqrt{n}}\int_0^{nt} g_n(Y_n(s))\,ds,$$

$$m_n^1(t) = \frac{1}{\sqrt{n}}\int_0^{nt} f_1(Y_n(s-))\,d\tilde{N}_n^1(s),$$

$$m_n^2(t) = \frac{1}{\sqrt{n}}\int_0^{nt} f_2(Y_n(s-))\,d\tilde{N}_n^2(s),$$

$$f_1(x) = f(x, x+1), \qquad f_2(x) = f(x, x-1)$$

and $g_n(x)$ is defined in Assumption 6.

We will prove our result in several steps.

STEP 1. Let $\mathscr{A}_n(t) = (X_n(t), D_n(t), \xi_n(t), M_n^1(t), M_n^2(t), A_n(t), m_n^1(t), m_n^2(t))$. In this step, we will show that $\{\mathscr{A}_n(t)\}_{n\ge1}$ is tight in $D([0,\infty), R^8)$. For that purpose, it suffices to show that each family of processes of $\mathscr{A}_n(t)$ is C-tight in $D([0,\infty), R^1)$. The tightness of $\{X_n\}_{n\ge1}$, $\{D_n\}_{n\ge1}$, $\{\xi_n\}_{n\ge1}$, $\{M_n^1\}_{n\ge1}$ and $\{M_n^2\}_{n\ge1}$ is shown in Lemma 2. The C-tightness of $\{m_n^1\}_{n\ge1}$ and $\{m_n^2\}_{n\ge1}$ can be shown easily as in the case of $\{M_n^1\}$ and $\{M_n^2\}$. Thus it only remains to see that $\{A_n\}_{n\ge1}$ is tight in $D([0,\infty), R^1)$. For functions $g_n(x)$, $n \ge 1$, given in Assumption 6, let $F_n(x)$ be the unique solution of (2.1). Then, because

$$\begin{aligned}F_n(Y_n(t)) &= F_n(Y_n(0)) + \sum_{s\le t}\{F_n(Y_n(s)) - F_n(Y_n(s-))\}\\ &= F_n(Y_n(0)) + \int_0^t g_n(Y_n(s))\,ds + \int_0^t \tilde{F}_n(Y_n(s-)+1)\,d\tilde{N}_n^1(s)\\ &\quad - \int_0^t \tilde{F}_n(Y_n(s-))\,d\tilde{N}_n^2(s),\end{aligned}$$

we have

$$A_n(t) = A_n^1(t) - A_n^2(t), \tag{4.4}$$

where

$$A_n^1(t) = \frac{1}{\sqrt{n}} F_n\big(\sqrt{n}\, X_n(t)\big) - \frac{1}{\sqrt{n}} F_n\big(\sqrt{n}\, X_n(0)\big),$$

$$A_n^2(t) = \frac{1}{\sqrt{n}} \int_0^{nt} \tilde{F}_n(Y_n(s-) + 1)\, d\tilde{N}_n^1(s) - \frac{1}{\sqrt{n}} \int_0^{nt} \tilde{F}_n(Y_n(s-))\, d\tilde{N}_n^2(s).$$

We will show that $\{A_n^1\}_{n \ge 1}$ is C-tight and $\{A_n^2\}_{n \ge 1}$ is tight in $D([0,\infty), R^1)$, respectively. Note that this implies the tightness of $\{A_n\}_{n \ge 1}$ (Jacod and Shiryaev [4], Chapter 6, Corollary 3.3). As for $\{A_n^1\}_{n \ge 1}$, we have

$$|A_n^1(t) - A_n^1(s)| \le \frac{1}{\sqrt{n}} \sum_{k=k_1}^{k_2} |\tilde{F}_n(k)|,$$

where $k_1 = \min(\sqrt{n}\, X_n(s), \sqrt{n}\, X_n(t))$ and $k_2 = \max(\sqrt{n}\, X_n(s), \sqrt{n}\, X_n(t))$. Because $|\tilde{F}_n(k)| \le K(\mu_n/\lambda_n)^k$, where K is a constant not depending on n and k, we have

$$|A_n^1(t) - A_n^1(s)| \le \frac{K}{\sqrt{n}} \sum_{k=k_1}^{k_2} B^{k/\sqrt{n}},$$

where B is a constant that bounds $\{(\mu_n/\lambda_n)^{\sqrt{n}}\}_{n \ge 1}$. Thus we have

$$|A_n^1(t) - A_n^1(s)| \le KB^{\max(X_n(s),\, X_n(t))} |X_n(t) - X_n(s)|. \tag{4.5}$$

For any $\alpha \in D([0,\infty), R^1)$, let

$$w(\alpha; I) = \sup_{s,t \in I} |\alpha(s) - \alpha(t)|, \quad I \text{ an interval of } [0,\infty),$$

$$w_T(\alpha, \theta) = \sup\{w(\alpha; [t, t+\theta]); 0 \le t \le t+\theta \le T\}, \qquad \theta > 0.$$

Then, for any $\eta > 0$ and $N > 0$,

$$\begin{aligned} &P\big(w_T(A_n^1, \theta) > \eta\big) \\ &\quad \le P\Big(\sup_{0 \le u \le T} |X_n(u)| \ge N\Big) \\ &\qquad + P\Big(w_T\big(A_n^1, \theta\big) > \eta,\ \sup_{0 \le u \le T} |X_n(u)| < N\Big) \\ &\quad \le P\Big(\sup_{0 \le u \le T} |X_n(u)| \ge N\Big) + P\Big(w_T(X_n, \theta) > \frac{\eta}{K(N)}\Big), \end{aligned} \tag{4.6}$$

where $K(N)$ is a constant not depending on n and the last inequality is due to the fact that in view of (4.5),

$$|A_n^1(t) - A_n^1(s)| \le K(N) |X_n(t) - X_n(s)|$$

on the event $\{\sup_{0 \le u \le T} |X_n(u)| < N\}$. Because $\{X_n\}_{n \ge 1}$ is tight, the inequality

(4.6) implies

$$\lim_{n\to\infty} P\big(w_T(A_n^1, \theta) > \eta\big) = 0.$$

Similarly, using (4.5), we can show

$$\lim_{N\to\infty} \limsup_{n\to\infty} P\Big(\sup_{0\le t\le T} |A_n^1(t)| > N\Big) = 0.$$

Hence $\{A_n^1\}_{n\ge 1}$ is C-tight in $D([0,\infty), R^1)$ (Jacod and Shiryaev [4], Chapter 6, Proposition 3.26).

Next we will show the tightness of $\{A_n^2\}_{n\ge 1}$. To this end, it suffices to see that $\{\langle A_n^2\rangle(t),\ t\ge 0\}_{n\ge 1}$ is C-tight, because $A_n^2(t)$ is a locally square-integrable martingale (Jacod and Shiryaev [4], Chapter 6, Theorem 4.13). Because $\tilde{N}_n^1(t)$ and $\tilde{N}_n^2(t)$ are orthogonal, we have

$$\langle A_n^2\rangle(t) = \int_0^t \Big\{\tilde{F}_n\big(\sqrt{n}\,X_n(s)+1\big)^2 \lambda_n\big(\sqrt{n}\,X_n(s)\big) + \tilde{F}_n\big(\sqrt{n}\,X_n(s)\big)^2 \mu_n\big(\sqrt{n}\,X_n(s)\big)\Big\}\,ds.$$

Because $|\tilde{F}_n(k)| \le K(\mu_n/\lambda_n)^k$,

$$\begin{aligned}\tilde{F}_n\big(\sqrt{n}\,X_n(s)+1\big)^2 &\le K^2(\mu_n/\lambda_n)^{2\sqrt{n}X_n(s)+2}\\ &= K^2\Big[(\mu_n/\lambda_n)^{\sqrt{n}}\Big]^{2X_n(s)+2/\sqrt{n}}\\ &\le K^2 B^{2X_n(s)+2/\sqrt{n}}.\end{aligned}$$

Thus, owing to the same argument as before, on the event $\{\sup_{0\le u\le T}|X_n(u)| < N\}$, we have

$$\langle A_n^2\rangle(t) - \langle A_n^2\rangle(s) \le K(N)(t-s)$$

for $s < t \le T$. Then, as in the case of $\{A_n^1\}_{n\ge 1}$, we have the tightness of $\{A_n^2\}_{n\ge 1}$. Now we have shown the tightness of $\{A_n\}_{n\ge 1}$.

Here we note that any weak limit $A(t)$ of $\{A_n(t)\}_{n\ge 1}$ is a continuous process of bounded variation. Indeed, $A_n(t)$ can be written as

$$\begin{aligned}A_n(t) &= \frac{1}{\sqrt{n}}\int_0^{nt} g_n^+(Y_n(s))\,ds - \frac{1}{\sqrt{n}}\int_0^{nt} g_n^-(Y_n(s))\,ds\\ &:= A_n^+(t) - A_n^-(t),\end{aligned}$$

where $g_n^\pm(Y_n(s))$ are the positive and negative parts of $g_n(Y_n(s))$, respectively. Then, because $g_n^\pm(x)$ have compact support not depending on n, we can show tightness of $\{A_n^+\}_{n\ge 1}$ and $\{A_n^-\}_{n\ge 1}$ in exactly the same way as for $\{A_n\}_{n\ge 1}$. Because $A_n^+(t)$ and $A_n^-(t)$ are both increasing and continuous, their weak limits are also increasing and continuous. Hence $A(t)$ is the difference of the two increasing and continuous weak limits.

STEP 2. Let $\mathscr{A}(t) = (X(t), D(t), \xi(t), M_1(t), M_2(t), A(t), m_1(t), m_2(t))$ be any weak limit of $\{\mathscr{A}_n\}_{n\ge 1}$ in $D([0,\infty), R^8)$. In this step, we will identify the weak limit $\mathscr{A}(t)$. To this end we may assume that with probability 1, $\mathscr{A}_n(t) \to \mathscr{A}(t)$ uniformly on any compact t-set. We will show that the following two propositions hold:

PROPOSITION 1. $A(t) = \gamma\alpha\beta\xi(t)$.

PROPOSITION 2. *There exist two independent standard Brownian motions $B_1(t)$ and $B_2(t)$ such that*

$$M_1(t) = \sqrt{\lambda}\, B_1(t), \qquad M_2(t) = \sqrt{\lambda}\, B_2(t),$$
$$m_1(t) = \sqrt{f_1(\infty)^2 \lambda}\, B_1(t), \qquad m_2(t) = \sqrt{f_2(\infty)^2 \lambda}\, B_2(t).$$

Once propositions 1 and 2 have been established, the conclusion of Theorem 1 follows immediately. Indeed, we have $\tilde{U}_n(t) \to_{\mathscr{L}} U_0(t) := A(t) + m_1(t) + m_2(t)$ in $D([0,\infty), R^1)$. On the other hand, $X(t)$ satisfies

$$X(t) = ct + \sqrt{\lambda}\, B_1(t) - \sqrt{\lambda}\, B_2(t) + \xi(t), \tag{4.7}$$

where $\xi(t)$ is an increasing process with $\xi(0) = 0$ and satisfies

$$\xi(t) = \int_0^t 1(X(s) = 0)\, d\xi(s).$$

Hence $\xi(t)$ can be explicitly written as

$$\xi(t) = -\inf_{s \le t}\left(cs + \sqrt{\lambda}\, B_1(s) - \sqrt{\lambda}\, B_2(s)\right)$$

(Ikeda and Watanabe [2] Chapter 3, Lemma 4.2). Thus by Propositions 1 and 2,

$$U_0(t) = \gamma\alpha\beta\xi(t) + \sqrt{f_1(\infty)^2 \lambda}\, B_1(t) + \sqrt{f_2(\infty)^2 \lambda}\, B_2(t).$$

Now we will prove Propositions 1 and 2.

PROOF OF PROPOSITION 1. Let $A_2(t)$ be any weak limit of $\{A_n^2\}_{n \ge 1}$. Then we can assume that, with probability 1, $A_n^2(t) \to A_2(t)$ uniformly on any compact t-set, and similarly for $\mathscr{A}_n(t)$. Then we can show that there exists a standard Brownian motion $B(t)$ such that

$$A_2(t) = \sqrt{2\lambda} \int_0^t \alpha\beta\gamma \exp\left(-c\frac{X(s)}{\lambda}\right) dB(s).$$

This result can be obtained by using (3.6) and (4.1), Assumptions 2, 3 and 5 and (as in Lemma 2) a functional central limit theorem of Rebolledo [6]. Now letting n go to infinity in (4.4), we have, in view of Lemma 1,

$$A(t) = \begin{cases} \delta\left(\dfrac{\lambda}{c}\right)\left(1 - \exp\left(-\dfrac{c}{\lambda}X(t)\right)\right) - A_2(t), & c \ne 0, \\ \delta X(t) - A_2(t), & c = 0 \end{cases} \tag{4.8}$$

for t such that $X(t) > 0$. However, because $A(t)$ and $X(t)$ are continuous, (4.8) holds for all $t \geq 0$. On the other hand, because $X(t)$ satisfies (4.7), by Itô's formula, we have, for the case $c \neq 0$,

$$\exp\left(-\frac{c}{\lambda}X(t)\right) = 1 - \frac{c}{\lambda}\int_0^t \exp\left(-\frac{c}{\lambda}X(s)\right) d\xi(s) + m_0(t)$$
$$= 1 - \frac{c}{\lambda}\xi(t) + m_0(t),$$

where $m_0(t)$ is a continuous martingale. Then from (4.8),

$$A(t) - \delta\xi(t) = -\delta(\lambda/c)m_0(t) - A_2(t).$$

Because the process on the left-hand side is a continuous process of bounded variation and the process on the right-hand side is a continuous martingale, we have $A(t) = \delta\xi(t)$. Thus the proof is complete. □

PROOF OF PROPOSITION 2. By a standard argument, we can easily show that the weak limits $M_1(t)$, $M_2(t)$, $m_1(t)$ and $m_2(t)$ are continuous square-integrable martingales such that

$$\langle M_1\rangle(t) = \langle M_2\rangle(t) = \lambda t, \qquad \langle M_i, M_j\rangle(t) = 0, \qquad i \neq j,$$
$$\langle m_1\rangle(t) = f_1(\infty)^2 \lambda t, \qquad \langle m_2\rangle(t) = f_2(\infty)^2 \lambda t,$$
$$\langle m_1, m_2\rangle(t) = 0, \qquad \langle M_1, m_1\rangle(t) = f_1(\infty)\lambda t,$$
$$\langle M_2, m_2\rangle(t) = f_2(\infty)\lambda t, \qquad \langle M_i, m_j\rangle(t) = 0, \qquad i \neq j.$$

Then by a representation theorem for martingales (Ikeda and Watanabe [2], Chapter 2, Theorem 7.1), we reach the conclusion. □

5. Examples. In the examples that follow, the birth and death processes $Y_n(t)$, $n \geq 1$, are considered to represent a sequence of appropriate queueing models of a single station. Thus $Y_n(t)$ is the number of customers in the nth queue at time t.

EXAMPLE 5.1 (Number of busy cycles). Let $U_n(t)$ be the number of busy cycles for the nth queue (this was considered in Section 2). As we saw there, in Assumption 6 we take $l_n = 0$. Then $g_n(x) = 1(x = 0)\lambda_n(x)$ and we can take $x_0 = 1$ in Assumption 6. Thus $\gamma = 1$ in (3.6) and for processes $Y_n(t)$, $n \geq 1$, satisfying Assumptions 1–5, we have $\tilde{U}_n(t) \to \alpha\beta\xi(t)$. To calculate α and β explicitly, let us consider a simple model where $\lambda_n(x) = \lambda_n$ and $\mu_n(x)$ is given by (3.4). We let $\mu_n = \hat{\mu}_n s_n$, $\lambda_0(x) = \mu_0(x) = 1$, $\tilde{\lambda}_n(x) = 1$ and

$$\tilde{\mu}_n(x) = (x/s_n)1(x < s_n) + 1(s \geq s_n).$$

Then $\alpha = 1$ and, assuming $s_n \to s$,

$$\begin{aligned}\beta &= \lim_{\substack{x\to\infty\\ n\to\infty}} \prod_{k=1}^{x} \frac{\tilde{\mu}_n(k)}{\tilde{\lambda}_n(k)}\\ &= \lim_{\substack{x\to\infty\\ n\to\infty}} \prod_{k=1}^{x} \left[\left(\frac{k}{s_n}\right)1(k < s_n) + 1(k \ge s_n)\right]\\ &= \frac{s!}{s^s}.\end{aligned}$$

Hence

$$\tilde{U}_n(t) \to_{\mathscr{L}} -s!/s^s \inf_{u \le t}\left(cu + \sqrt{\lambda}B_1(u) - \sqrt{\lambda}B_2(u)\right),$$

where $\lambda = \lim_{n\to\infty} \lambda_n$.

EXAMPLE 5.2 (Arrival and departure processes of customers). Let us define $U_n(t)$ and $V_n(t)$ by

$$U_n(t) = \sum_{s \le t} 1(\Delta Y_n(s) = 1) \qquad [= N_n^1(t)],$$

$$V_n(t) = \sum_{s \le t} 1(\Delta Y_n(s) = -1) \qquad [= N_n^2(t)].$$

Thus $U_n(t)$ and $V_n(t)$ represent the arrival and departure processes of customers for the nth queueing model. We assume the following condition $(*)$:

$(*)$ Either $\lambda_n(x) - \lambda_n$ or $\mu_n(x) - \mu_n$ has compact support not depending on n.

Suppose, for example, the former case holds, and consider the process $U_n(t)$. Then we have $f(x, x+1) = 1$ and $f(x, x-1) = 0$ in (1.1) and take $l_n = \lambda_n$ in Assumption 6. Then by Theorem 1, if we set $\tilde{U}_n(t) = (1/\sqrt{n})(U_n(nt) - \lambda_n nt)$,

$$\tilde{U}_n(t) \to_{\mathscr{L}} \delta\xi(t) + \sqrt{\lambda}B_1(t), \qquad \delta = \gamma\alpha\beta.$$

Next we define $\tilde{V}_n(t)$ by

$$\tilde{V}_n(t) = (1/\sqrt{n})(V_n(nt) - \mu_n nt).$$

Then, because $Y_n(t) = Y_n(0) + N_n^1(t) - N_n^2(t)$, we have

$$\tilde{V}_n(t) = \frac{1}{\sqrt{n}}Y_n(0) + \tilde{U}_n(t) + \sqrt{n}(\lambda_n - \mu_n)t - X_n(t).$$

Hence, by Theorem 1′,

$$\begin{aligned}\left(\tilde{U}_n(t), \tilde{V}_n(t)\right) &\to_{\mathscr{L}} \left(\sigma\xi(t) + \sqrt{\lambda}B_1(t), \delta\xi(t) + \sqrt{\lambda}B_1(t) + ct - X(t)\right)\\ &= \left(\delta\xi(t) + \sqrt{\lambda}B_1(t), (\delta - 1)\xi(t) + \sqrt{\lambda}B_2(t)\right)\end{aligned}$$

in $D([0,\infty), R^2)$. Note that γ is calculated by (3.6) using $g_n(x) = \lambda_n(x) - \lambda_n$. Similarly, if the latter case holds in $(*)$, we have

$$\left(\tilde{U}_n(t), \tilde{V}_n(t)\right) \to_{\mathscr{L}} \left((1+\delta')\xi(t) + \sqrt{\lambda}B_1(t), \delta'\xi(t) + \sqrt{\lambda}B_2(t)\right),$$

Here $\delta' = \gamma\alpha\beta$ and γ is calculated by (3.6) using $g_n(x) = \mu_n(x) - \mu_n$.

We note that $\delta\xi(t) \geq 0$ and $(\delta - 1)\xi(t) \leq 0$ for all t in the former case in $(*)$, and a similar remark applies to the latter case. This is because $\lambda_n(x) \geq \lambda_n$ and $\mu_n(x) \leq \mu_n$ for all $x \in Z$ (see Assumption 2) and

$$\frac{1}{\sqrt{n}} \int_0^{nt} \left(\lambda_n(Y_n(s)) - \lambda_n\right) ds \to_{\mathscr{L}} \delta\xi(t), \tag{5.1}$$

$$\frac{1}{\sqrt{n}} \int_0^{nt} \left(\mu_n(Y_n(s)) - \mu_n\right) ds \to_{\mathscr{L}} (\delta - 1)\xi(t) \tag{5.2}$$

in $D([0,\infty), R^1)$. The convergence (5.1) is our result itself (see Proposition 1). The result (5.2) comes from the following argument [note that (5.2) is not our result: we are not assuming that $\mu_n(x) - \mu_n$ has compact support not depending on n]. Let $B_n(t)$ be the process on the left-hand side in (5.2). Then, because $B_n(t) = \tilde{V}_n(t) - M_n^2(t)$, $\{B_n\}_{n\geq 1}$ is tight. Let $B(t)$ be any weak limit of $\{B_n\}_{n\geq 1}$. Then $\tilde{V}_n(t) \to_{\mathscr{L}} B(t) + M_2(t)$, where $M_2(t)$ is the weak limit of $\{M_n^2\}_{n\geq 1}$, which is a continuous martingale. Thus $B(t) + M_2(t) = (\delta - 1)\xi(t) + \sqrt{\lambda}B_2(t)$ and $B(t) = (\delta - 1)\xi(t)$. Because $B(t) \leq 0$ for all t, $(\delta - 1)\xi(t) \leq 0$ for all t.

Consider a simple case: $\lambda_n(x) = \lambda_n$. Then $\delta = 0$ because $g_n(x) = 0$. Thus

$$\left(\tilde{U}_n(t), \tilde{V}_n(t)\right) \to_{\mathscr{L}} \left(\sqrt{\lambda}B_1(t), -\xi(t) + \sqrt{\lambda}B_2(t)\right)$$

in $D([0,\infty), R^2)$.

EXAMPLE 5.3 (Sojourn times of customers). In what follows, for simplicity of our discussion we assume $Y_n(0) = 0$ for all $n \geq 1$. Let $(N_n^1)^{-1}(t)$ and $(N_n^2)^{-1}(t)$ be the inverse processes of $N_n^1(t)$ and $N_n^2(t)$, respectively; that is,

$$\left(N_n^1\right)^{-1}(t) = \inf\{s; N_n^1(s) > t\}$$

and

$$\left(N_n^2\right)^{-1}(t) = \inf\{s; N_n^2(s) > t\}.$$

Then, for an arbitrary positive integer k,

$$W_n(k) = \left(N_n^2\right)^{-1}(k) - \left(N_n^1\right)^{-1}(k), \qquad n \geq 1,$$

is considered to be the sojourn time for the kth customer in the nth queueing model. Our interest is in finding the limit process of the normalized processes $\tilde{W}_n(t) = 1/\sqrt{n}\, W_n([nt])$, $n \geq 1$. We assume that the condition $(*)$ of Example 5.2 is also satisfied here.

By the result of Example 5.2, we have

$$(5.3)\qquad \left(\frac{1}{\sqrt{n}}\left(N_n^1(nt)-\lambda_n nt\right),\frac{1}{\sqrt{n}}\left(N_n^2(nt)-\lambda_n nt\right)\right) \to_{\mathscr{L}} (X_1(t),X_2(t))$$

in $D([0,\infty),R^2)$, where, for example, in the former case in $(*)$,

$$(5.4)\qquad \begin{aligned} X_1(t) &= \delta\xi(t)+\sqrt{\lambda}B_1(t),\\ X_2(t) &= (\delta-1)\xi(t)+\sqrt{\lambda}B_2(t)-ct. \end{aligned}$$

Because the inverse processes of $(1/n\lambda_n)N_n^1(n\,\cdot)$ and $(1/n\lambda_n)N_n^2(n\,\cdot)$ are $(1/n)(N_n^1)^{-1}(\lambda_n n\,\cdot)$ and $(1/n)(N_n^2)^{-1}(\lambda_n n\,\cdot)$, respectively, (5.3) implies (using Whitt [9]) that

$$\left(\lambda_n\sqrt{n}\left(\frac{1}{n}(N_n^1)^{-1}(\lambda_n nt)-t\right),\lambda_n\sqrt{n}\left(\frac{1}{n}(N_n^2)^{-1}(\lambda_n nt)-t\right)\right) \to_{\mathscr{L}} (-X_1(t),-X_2(t)) \quad \text{in } D([0,\infty),R^2).$$

This convergence leads to

$$\left(\frac{1}{\sqrt{n}}\left((N_n^1)^{-1}(nt)-\frac{1}{\lambda_n}nt\right),\frac{1}{\sqrt{n}}\left((N_n^2)^{-1}(nt)-\frac{1}{\lambda_n}nt\right)\right) \to_{\mathscr{L}} \left(-\frac{1}{\lambda}X_1\left(\frac{t}{\lambda}\right),-\frac{1}{\lambda}X_2\left(\frac{t}{\lambda}\right)\right) \quad \text{in } D([0,\infty),R^2).$$

Thus we have

$$\tilde{W}_n(t)=\frac{1}{\sqrt{n}}\left((N_n^2)^{-1}(nt)-(N_n^1)^{-1}(nt)\right) \to_{\mathscr{L}} -\frac{1}{\lambda}\left(X_2\left(\frac{t}{\lambda}\right)-X_1\left(\frac{t}{\lambda}\right)\right)=\frac{1}{\lambda}X\left(\frac{t}{\lambda}\right)$$

in $D([0,\infty),R^1)$. Note that the limit process $(1/\lambda)X(t/\lambda)$ is a reflecting Brownian motion. Furthermore, note that in obtaining the foregoing convergence, we have not used the fact that $X_1(t)$ and $X_2(t)$ are expressed as in (5.4), but used only the fact that (5.3) holds and $X(t)=X_1(t)-X_2(t)$.

EXAMPLE 5.4 (Queue lengths of tandem models under heavy traffic condition). Let us consider a sequence of tandem queues that consists of two stages. (We consider the case of two stages only for simplicity.)

For the nth model, let $Y_n^k(t)$, $k=1,2$, be the numbers of customers at stage k, $N_n^0(t)$ the number of customers arriving at stage 1 until time t and $N_n^k(t)$, $k=1,2$, the number of customers departing from stage k until time t.

Then we have

$$Y_n^1(t) = Y_n^1(0) + N_n^0(t) - N_n^1(t), \tag{5.5}$$

$$Y_n^2(t) = Y_n^2(0) + N_n^1(t) - N_n^2(t). \tag{5.6}$$

We assume that $N_n^k(t)$, $k = 0, 1, 2$, have no common discontinuities and have intensities $\lambda_n^0(Y_n^1(t))$, $\lambda_n^1(Y_n^1(t))$ and $\lambda_n^2(Y_n^2(t))$, respectively. $Y_n^1(t)$ is a birth and death process treated in Section 3. We make the following assumptions:

ASSUMPTION 7. For $\lambda_n^0(\cdot)$ and $\lambda_n^1(\cdot)$, in addition to Assumptions 1–5, where we set $\lambda_n(x) = \lambda_n^0(x)$ and $\mu_n(x) = \lambda_n^1(x)$, we also assume $(*)$ in Example 5.2.

ASSUMPTION 8. $\lim_{x\to\infty} \lambda_n^2(x) = \lambda_n^2(\infty) < \infty$, $\lambda_n^2(\infty) \geq \lambda_n^2(x)$ for all x and $\lim_{n\to\infty}\sqrt{n}\,(\lambda_n^1(\infty) - \lambda_n^2(\infty)) = d_2$. Moreover, $\lim_{n\to\infty} \lambda_n^1(\infty) = \lim_{n\to\infty} \lambda_n^2(\infty) = \lambda \geq 0$.

ASSUMPTION 9. $\lim_{n\to\infty,\, x\to\infty} x(\lambda_n^2(\infty) - \lambda_n^2(x)) = a_2$ and $\sup_{n,x} x(\lambda_n^2(\infty) - \lambda_n^2(x)) < \infty$.

Write $d_1 = \lim_{n\to\infty}\sqrt{n}\,(\lambda_n^0(\infty) - \lambda_n^1(\infty))$ and $\lambda = \lim_{n\to\infty} \lambda_n^0(\infty)$ in Assumption 1 for the processes $\{Y_n^1(t)\}_{n\geq 1}$. Then using the result of Example 5.2, we have the following proposition.

PROPOSITION 3. *Assume Assumption* 7 *for* $\{Y_n^1(t)\}_{n\geq 1}$ *and Assumptions* 8 *and* 9 *for* $\{Y_n^2(t)\}_{n\geq 1}$. *Moreover assume* $(Y_n^1(0), Y_n^2(0)) \to_{\mathscr{L}} (Y_1(0), Y_2(0))$, *a random vector. Let* $(X_n^1(t), X_n^2(t))_{n\geq 1}$ *be defined by*

$$X_n^1(t) = \frac{1}{\sqrt{n}} Y_n^1(nt) \quad \text{and} \quad X_n^2(t) = \frac{1}{\sqrt{n}} Y_n^2(nt).$$

then $(X_n^1(t), X_n^2(t)) \to_{\mathscr{L}} (X_1(t), X_2(t))$ *in* $D([0,\infty), R^2)$, *where* $(X_1(t), X_2(t))$ *is characterized as the unique solution of the following Skorohod equation*:

$$X_1(t) = d_1 t + \sqrt{\lambda}\, B_1(t) + \sqrt{\lambda}\, B_2(t) + \xi_1(t), \tag{5.7}$$

$$X_2(t) = d_2 t + \sqrt{\lambda}\, B_2(t) - \sqrt{\lambda}\, B_3(t) + (\delta - 1)\xi_1(t) + \xi_2(t), \tag{5.8}$$

where $(X_1(t), X_2(t))$ *and* $(\xi_1(t), \xi_2(t))$ *are nonnegative continuous processes such that* $\xi_1(0) = \xi_2(0) = 0$, $\xi_1(t)$ *and* $\xi_2(t)$ *are nondecreasing*,

$$\xi_1(t) = \int_0^t 1(X_1(s) = 0)\, d\xi_1(s),$$

$$\int_0^t X_2(s)\, d\xi_2(s) = a_2 t$$

and $B_1(t)$, $B_2(t)$ *and* $B_3(t)$ *are independent standard Brownian motions.* δ *in* (5.8) *is defined in Theorem* 1 *for the process* $Y_n^1(t)$.

A similar result was obtained in Yamada [11]. There the intensity $\lambda_n^0(\cdot)$ was assumed, for technical reasons, not to depend on the queue length $Y_n^1(t)$;

that is, $\lambda_n^0(\cdot) = \lambda_n^0$. [But the assumption on $\lambda_n^1(\cdot)$ in [11] is more general than the one made here.] As we see in the following text, we can remove this restriction by using the result of Example 5.2 (this is the purpose of this example). We must prove (5.7) and (5.8). To this end, write $X_n^2(t)$ as

$$X_n^2(t) = X_n^2(0) + \tilde{V}_n(t) + D_n^2(t) + M_n^2(t) + \xi_n^2(t),$$

where

$$\tilde{V}_n(t) = \frac{1}{\sqrt{n}}\left(N_n^1(nt) - \lambda_n^1(\infty)nt\right),$$

$$D_n^2(t) = \sqrt{n}\left(\lambda_n^1(\infty) - \lambda_n^2(\infty)\right)t,$$

$$M_n^2(t) = \frac{1}{\sqrt{n}}\tilde{N}_n^2(nt),$$

$$\xi_n^2(t) = \frac{1}{\sqrt{n}}\int_0^{nt}\left(\lambda_n^2(\infty) - \lambda_n^2\left(Y_n^2(s)\right)\right)ds,$$

$$\lambda_n^i(\infty) = \lim_{x\to\infty}\lambda_n^i(x) \quad \text{for } i = 0,1,2.$$

We know that $\{D_n^2\}_{n\geq 1}$ and $\{M_n^2\}_{n\geq 1}$ are C-tight. $\{\tilde{V}_n\}_{n\geq 1}$ is also C-tight by Example 5.2. Then using the discussion in [11], we can show that $\{X_n^2\}_{n\geq 1}$ and $\{\xi_n^2\}_{n\geq 1}$ are also C-tight in $R([0,\infty), R^1)$. Thus, setting $\tilde{U}_n(t) = (1/\sqrt{n})(N_n^0(nt) - \lambda_n^0(\infty)nt)$, $\mathscr{B}_n(t) = \{X_n^1(t), \tilde{U}_n(t), \tilde{V}_n(t), X_n^2(t), D_n^2(t), M_n^2(t), \xi_n^2(t)\}_{n\geq 1}$ is C-tight in $D([0,\infty), R^7)$. Let $\{X_1(t), \tilde{U}(t), \tilde{V}(t), X_2(t), D_2(t), M_2(t), \xi_2(t)\}$ be any weak limit of $\{\mathscr{B}_n\}_{n\geq 1}$. Then by using the result of Example 5.2 and by the discussions in the proof of Theorem 1 and in [11], it is not hard to see that (5.7) and (5.8) hold.

In conclusion, the point of this example is that as long as we are concerned with the limit process of $X_n^1(t)$ only, there is no need to use Theorem 1 (or the result of Example 5.2); all we need is Lemma 2. However, if we consider the joint convergence of $X_n^1(t)$ and $X_n^2(t)$, we need the help of our Theorem 1.

REMARK 6. The condition that $\lim \sqrt{n}(\lambda_n^0(\infty) - \lambda_n^1(\infty)) = d_1$, and $\lim_{n\to\infty}\sqrt{n}(\lambda_n^1(\infty) - \lambda_n^2(\infty)) = d_2$, is called the heavy traffic condition as in Remark 1. Concerning the condition that $x(\lambda_n^2(\infty) - \lambda_n^2(x)) \to a_2$ as $n\to\infty$ and $x\to\infty$, we can make the same comment as in Remark 5, but we omit the detailed discussion.

Acknowledgments. The authors are grateful to the referees for their various comments. A careful reading by one of the referees is particularly appreciated. We are also grateful to an Associate Editor for his comment on the improvement of the paper.

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