The Annals of Applied Probability
1993, Vol. 3, No. 3, 682–695

PARALLEL AND TANDEM FLUID NETWORKS WITH DEPENDENT LÉVY INPUTS

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A group of stations in parallel is considered, where the input processes to the various stations are stochastically dependent nondecreasing Lévy processes and the release rates are deterministic linear flows. It is shown that the model of a tandem fluid network, with dependent or independent Lévy inputs to the stations, can be seen as a special case of this construction. In addition to other natural applications, the joint stochastic structure of the workload processes of the different classes in a preemptive resume priority M/G/1 queue can be viewed as a special case of the model considered. Using martingale and regenerative arguments, certain steady state characteristics are studied.

I. Introduction. Consider a number of stations in parallel that process raw material that may be accurately approximated as continuous matter. Some examples are in the food, chemical and petroleum industries, dams, rapid manufacturing of very small items and the transfer of (bits of) information over communication links. Regardless of the application, let us call the material that the stations process *fluid*. The inputs to the various stations are nondecreasing Lévy processes (subordinators); however, these processes may be stochastically dependent in a sense that will be made clear in the next section.

In this paper a general framework is first described, which is a multidimensional reflected Lévy process, having no negative jumps. An associated martingale is then defined and some consequences that serve as preliminary results are given. As a concrete example, it will be shown how to apply these ideas to study a tandem fluid network, as considered in Kella and Whitt (1992b), but having inputs to every station, rather than only to the first. Consequently, most of the results in Kella and Whitt (1992b) are generalized. In particular, the joint steady state distribution of the fluid levels of two nodes in tandem is found, when there are two independent (general) subordinators feeding into both nodes. In Kella and Whitt (1992b) this distribution was found only for the case where there is a compound Poisson process feeding into the first node. These results are then used to obtain the explicit covariance matrix of the limiting distribution of the inventory levels for the entire (*multi*dimensional) tandem network. Finally, Section 5 is a remark on applications to the M/G/1 queue with a preemptive priority discipline. It should be mentioned at the outset that in my opinion the most interesting

Received December 1991; revised January 1993.

AMS 1991 *subject classifications*. Primary 60J30; secondary 60K25, 60K30.

Key words and phrases. Lévy process, reflected Lévy process, multi-dimensional Lévy process, martingale, queues, M/G/1 queue, fluid models, queueing networks, storage networks, generalized Pollaczek–Khinchine formula.

and definite results in this paper are Theorems 3.1, 4.1 and 4.2 and Corollaries 4.1 and 4.2.

For earlier works on fluid and related storage models, see Gaver and Miller (1962), Miller (1963), Meyer, Rothkopf and Smith (1979, 1983), Newell (1982), Anick, Mitra and Sondhi (1982), Mitra (1988), Chen and Mandelbaum (1991), Chen and Yao (1992), Kella and Whitt (1992a, b) and further references in these sources. For background on Lévy processes see, for example, Bingham (1975), Breiman (1968), Fristedt (1974), Prabhu (1980), Jacod and Shiryaev (1987) and Protter (1990).

2. The model and preliminary results. Denote $\mathscr{R} = (-\infty, \infty)$ and $\mathscr{R}_+ = [0, \infty)$, so that $\mathscr{R}^n$ is the n-dimensional Euclidean space and $\mathscr{R}_+^n$ is the corresponding positive orthant. In this section n-dimensional vectors will always be column vectors and prime will denote transposition. Also denote by $a^+ \equiv a \vee 0 \equiv \max(a, 0)$ and by $a^- \equiv -a \wedge 0 \equiv -\min(a, 0)$. Endow $\mathscr{R}^n$ with its usual quadratic (L^2) norm, so that continuity and boundedness are always defined with respect to this metric. Throughout the paper LST abbreviates *Laplace—Stieltjes transform* and w.l.o.g. abbreviates *without loss of generality*.

Let us begin with an underlying standard (right-continuous and augmented) filtered probability space $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t | t \geq 0\})$. Throughout, *adapted*, *martingales* and *stopping times* will be defined with respect to this filtration. Call $Y = \{Y(t) | t \geq 0\}$, with $Y(t) = (Y_1(t), \dots, Y_n(t))'$ and $Y(0) = 0$, an n-dimensional Lévy process if it is adapted, continuous in probability, $Y(s+t) - Y(t)$ is independent of $\mathscr{F}_t$ and distributed like $Y(s)$ for every nonnegative s, t (stationary independent increments). There always exists a version with paths that are right continuous having left limits, which is therefore strong Markov. Throughout this paper, every Lévy process mentioned is assumed to be such.

It is known that for every $\alpha \in R^n$ and $t \geq 0$, $E \exp(i\alpha' Y(t)) = \exp(t\psi(\alpha))$, where, given some *centering* or *truncation* function $h: \mathscr{R}^n \to \mathscr{R}^n$ [i.e., bounded with $h(x) = x$ in a neighborhood of zero],

$$\psi(\alpha) = ic'\alpha - \frac{1}{2}\alpha'\Sigma\alpha + \int_{\mathscr{R}^n} \left(e^{i\alpha' x} - 1 - i\alpha' h(x)\right)\mu(dx), \tag{2.1}$$

where $c \in \mathscr{R}^n$, $\Sigma \in R^{n \times n}$ is a symmetric positive semindefinite matrix and the *Lévy measure* μ is a measure on $\mathscr{R}^n$ satisfying $\mu(\mathscr{R}^n \setminus B(0,1)) < \infty$, $\mu(\{0\}) = 0$ and $\int_{B(0,1)} x'x\mu(dx) < \infty$, with $B(0,1) = \{x | x'x \leq 1\}$. Equation (2.1) is a generalization of the one-dimensional Lévy–Khinchine formula and it is known that for every centering function, the triplet (c, Σ, μ) is unique [e.g., Jacod and Shiryaev (1987), page 86, Lemma 2.4]. $\psi(\cdot)$ is called the (Fourier) exponent of the Lévy process. It should be noted that a necessary and sufficient condition for $Y_1, \dots, Y_n$ to be independent processes is that Σ be diagonal and the support of μ be contained in the axes. In this case, it is easy to see that $\psi(\alpha) = \sum_{j=1}^n \psi_j(\alpha_j)$, where or every $1 \leq j \leq n$, ψ_j is the exponent of Y_j.

In this paper the focus will be only on the case where the process does not have negative jumps in any coordinate, that is, assume from now on that the Lévy measure is concentrated on $\mathscr{R}_+^n$. In this case it is known that $E\exp(-\alpha'Y(t)) = \exp(t\varphi(\alpha)) < \infty$ $(\alpha \in \mathscr{R}_+^n)$, where

$$\varphi(\alpha) = \psi(i\alpha) = -c'\alpha + \frac{1}{2}\alpha'\Sigma\alpha + \int_{\mathscr{R}_+^n} e^{-\alpha'x} - 1 + \alpha'h(x)\mu(dx). \tag{2.2}$$

From now on, *exponent* will mean the *Laplace–Stieltjes exponent* $\varphi(\cdot)$ and i is reserved as an index notation rather than $\sqrt{-1}$.

Because Σ is positive semidefinite and $g(\alpha) = e^{-\alpha'x} - 1 + \alpha'h(x)$ is convex for every fixed $x \in \mathscr{R}_+^n$, it is easy to see that $\varphi(\cdot)$ is a convex function on $\mathscr{R}_+^n$ with $\varphi(0) = 0$. It is known [e.g., Protter (1990), pages 25 and 31, Theorems 34 and 40, respectively] that without negative jumps $EY_j(1)^- < \infty$ for every $1 \le j \le n$, so that $EY_j(1)$ always exists, but may be infinite. If $EY_j(1) = -(\partial/\partial\alpha_j)\varphi(0) < 0$ for every $j = 1, \dots, n$ then it immediately follows by the convexity of $\varphi(\cdot)$ that for every $\alpha \in \mathscr{R}_+^n \setminus \{0\}$, $f_\alpha(\beta) = \varphi(\beta\alpha)$ is strictly increasing and continuous on $\mathscr{R}^+$, with $f_\alpha(0) = 0$. Therefore there exists an inverse $f_\alpha^{-1}(\cdot)$. Now for $x > 0$ let $T_\alpha(x) = \inf\{t \mid \alpha'Y(t) = -x\}$. Then because $\alpha'Y$ is a one-dimensional Lévy process with no negative jumps and exponent $f_\alpha(\cdot)$, it follows [e.g., Fristedt (1974), Bingham (1975) or Kella and Whitt (1992c)] that

$$Ee^{-\beta T_\alpha(x)} = e^{-f_\alpha^{-1}(\beta)x} \quad \text{and} \quad ET_\alpha(x) = x/f'_\alpha(0). \tag{2.3}$$

In particular, with e_j being the unit vector with 1 in the jth coordinate and 0 elsewhere, it will be convenient to denote

$$\varphi_j(\cdot) = f_{e_j}(\cdot) \quad \text{and} \quad T_j(x) = T_{e_j}(x). \tag{2.4}$$

Given some random vector $0 \le Z(0) \in \mathscr{F}_0$ (i.e., independent of Y), let

$$\begin{aligned} I_j(t) &= -\inf_{0\le s\le t} \left(Z_j(0) + Y_j(t)\right)^- \quad \text{and} \\ Z_j(t) &= Z_j(0) + Y_j(t) + I_j(t). \end{aligned} \tag{2.5}$$

Then Z_j is a reflected Lévy process with I_j being its local time at zero. Because Y_j has no negative jumps, I_j is continuous (clearly nondecreasing) with $I_j(0) = 0$. It is known that under the assumption that $EY_j(1) < 0$,

$$\lim_{t\to\infty} Ee^{-\alpha_j Z_j(t)} = \alpha_j \varphi_j'(0)/\varphi_j(\alpha_j) \tag{2.6}$$

[see Bingham (1975), Harrison (1977), Kella and Whitt (1991, 1992c), among others]. In this article the process $Z = (Z_1, \dots, Z_n)'$ will be the underlying general model. Let us begin with two general results that will then be used later on as a tool. Set $I = (I_1, \dots, I_n)'$.

LEMMA 2.1.

$$M(t) = \varphi(\alpha)\int_0^t e^{-\alpha' Z(s)}\,ds + e^{-\alpha' Z(0)} - e^{-\alpha' Z(t)} - \sum_{j=1}^n \alpha_j \int_0^t e^{-\alpha' Z(s)}\,dI_j(s) \tag{2.7}$$

is a martingale. If $\varphi_j'(0) > 0$ *for all* $1 \le j \le n$ [*hence* $T_j(x) < \infty$ *a.s. for every* $x \ge 0$], *then*

$$M(t) = \varphi(\alpha)\int_0^t e^{-\alpha' Z(s)}\,ds + e^{-\alpha' Z(0)} - e^{-\alpha' Z(t)} - \sum_{j=1}^n \alpha_j \int_0^{I_j(t)} e^{-\alpha' Z(T_j(x))}\,dx \tag{2.8}$$

is a martingale as well.

PROOF. Observe that $\alpha' Y$ is a Lévy process with no negative jumps and exponent $f_\alpha(\cdot)$. Also note that $\alpha' I$ is an adapted continuous process and it is known that $EI_j(t) < \infty$ for every $t \ge 0$ and $1 \le j \le n$. Applying Theorem 2 of Kella and Whitt (1992c) with the choice $\varphi = f_\alpha$, $\alpha = 1$, $Y^c = Y = \alpha'(Z(0) + I)$ and $Z = \alpha' Z$ (left and middle sides are in Kella and Whitt notation and right sides are in the notation used here) gives that (2.7) is a martingale. To obtain (2.8), make the change of variables $s = T_j(x)$ of the jth integral in the sum and observe that I_j is continuous and nondecreasing having the pseudo-inverse T_j [i.e., $I_j(T_j(x)) = x$ for every $x \ge 0$]. □

COROLLARY 2.1. *For every stopping time* T *such that* $P[T < \infty] = 1$,

$$\varphi(\alpha) E\int_0^T e^{-\alpha' Z(s)}\,ds + Ee^{-\alpha' Z(0)} - Ee^{-\alpha' Z(T)} = \sum_{j=1}^n \alpha_j E\int_0^T e^{-\alpha' Z(s)}\,dI_j(t) \tag{2.9}$$

(*meaning that either both sides are finite and equal or both are infinite*). *If, in addition,* $\varphi_j'(0) > 0$ *for all* $1 \le j \le n$, *then the right side of* (2.9) *can be replaced by* $\sum_{j=1}^n \alpha_j E\int_0^{I_j(T)} e^{-\alpha' Z(T_j(x))}\,dx$.

PROOF. Apply Doob's optional sampling theorem to the bounded stopping times $T \wedge t$ and then let $t \to \infty$, whereby limits and expectations can be interchanged from the bounded (for $e^{-\alpha' Z(T \wedge t)}$) and monotone convergence theorems (for the two integrals). □

COROLLARY 2.2. *If $\varphi_j'(0) > 0$ for all $1 \le j \le n$, then*

$$\lim_{t\to\infty}\left[\varphi(\alpha)\frac{1}{t}\int_0^t Ee^{-\alpha' Z(s)}\,ds - \sum_{j=1}^n \alpha_j\varphi_j'(0)E\frac{1}{I_j(t)}\int_0^{I_j(t)} e^{-\alpha' Z(T_j(x))}\,dx\right]=0 \tag{2.10}$$

and

$$\lim_{t\to\infty}\left[\varphi(\alpha)\frac{1}{t}\int_0^t Ee^{-\alpha' Z(s)}\,ds - \sum_{j=1}^n \alpha_j\varphi_j'(0)\frac{1}{EI_j(t)}E\int_0^{I_j(t)} e^{-\alpha' Z(T_j(x))}\,dx\right]=0. \tag{2.11}$$

PROOF. It can be shown that $t^{-1}I_j(t) \to \varphi_j'(0)$ in L^1 for every $1 \le j \le n$. Hence the results follow from the fact that $\int_0^{I_j(t)}e^{-\alpha' Z(T_j(x))}\,dx \le I_j(t)$, for all $t \ge 0$, and that $EM_t = 0$. □

COROLLARY 2.3. *Under the condition of Corollary* 2.2, *if as $t \to \infty$, $Z(t) \to Z^*$ and $Z(T_j(t)) \to Z^{j*}$ ($Z_j^{j*} \equiv 0$) for $1 \le j \le n$, all in distribution in the ergodic sense, then*

$$\begin{aligned}\varphi(\alpha)Ee^{-\alpha' Z^*} &= \sum_{j=1}^n \alpha_j\varphi_j'(0)Ee^{-\alpha' Z^{j*}}\\ &= \sum_{j=1}^n \varphi_j(\alpha_j)Ee^{-\alpha_j Z_j^*}Ee^{-\alpha' Z^{j*}}.\end{aligned} \tag{2.12}$$

PROOF. For the first equality it suffices to show that for each $1 \le j \le n$ in (2.11), one can replace $I_j(t)$ by $\varphi_j'(0)t$ (or equivalently, by t) without affecting the limit. To see that this is valid, observe that

$$\left|\int_0^{I_j(t)} e^{-\alpha' Z(T_j(x))}\,dx - \int_0^{\varphi^{*\prime}(0)t} e^{-\alpha' Z(T_j(x))}\,dx\right| \le \left|I_j(t) - \varphi'(0)t\right|, \tag{2.13}$$

so that, recalling that $t^{-1}I_j(t) \to \varphi_j'(0)$ in L^1, the replacement is justified. The second equality follows from (2.6). □

It would be useful if (2.12) (if valid) determined the distribution of Z^*. At this point I do not know of a proof, and it is possible that (2.12) has multiple solutions for $n \ge 2$. For $n = 1$, (2.12) gives the generalized Pollaczek–Khinchine formula, as expected.

Note that a special case of the foregoing model is when $\mu(\mathscr{R}_+^n) = 0$, in which case Y is an n-dimensional Brownian motion, so that Z is a special case of a *Brownian network* [see Harrison (1988)]. In this paper the focus will be only on the case $Y_j(t) = X_j(t) - r_j t$, where X_j is nondecreasing (subordinator) and $r_j > EX_j(1)$. X_j therefore models the input of fluid to station j, where subtracting $r_j t$ models our assumption that the station processes the fluid at a constant deterministic rate (whenever the fluid level is positive). The amount of fluid at station j at time t is then given by $Z_j(t)$. To be more precise, setting $X = (X_1, \dots, X_n)'$ and $r = (r_1, \dots, r_n)'$, assume that X is an n-dimensional subordinator, that is, a Lévy process, having the exponent

$$-\xi(\alpha) = -b'\alpha - \int_{\mathscr{R}_+^n} (1 - e^{-\alpha' x}) \mu(dx), \tag{2.14}$$

where for all $1 \le j \le n$, $b_j \ge 0$ and $\int_{\mathscr{R}_+^n} x_j \mu(dx) < \infty$. This implies that Y is an n-dimensional Lévy process with no negative jumps with exponent $\varphi(\alpha) = r'\alpha - \xi(\alpha)$.

3. A tandem fluid network with external inputs to every station. Kella and Whitt (1992b) considered a tandem fluid model with a (nondecreasing) Lévy input only to the first station, where the jth station feeds the $j+1$st at a constant rate r_j. The assumption there was that, w.l.o.g., $r_1 > \cdots > r_n$. Here generality is lost by assuming this condition. However, for the results of the parallel network to be applicable here, the assumption that $r_1 \ge \cdots \ge r_n$ has to be made, and it is. A generalization is considered where there are inputs to nodes other than the first and those inputs are subordinators. At first the subordinators are allowed to be dependent. However, for stronger results, independence will have to be assumed. Let J_j be the input process to node j and let

$$-\eta(\alpha) = -b'\alpha - \int_{\mathscr{R}_+^n} (1 - e^{-\alpha x}) \mu(dx) \tag{3.1}$$

be the exponent of J, with $\eta_j(\alpha_j) = \eta(0, \dots, 0, \alpha_j, 0, \dots, 0)$. For any $1 \le j \le n$, $\eta_j \equiv 0$ (e.g., $J_j \equiv 0$) is allowed. To simplify notation, it is assumed throughout that $Z(0) = 0$. Denote $\rho_j = \eta_j'(0)$ and impose the condition $\rho_1 + \cdots + \rho_n < r_n$. In particular, because $r_1 \ge \cdots \ge r_n$, this implies that $\rho_1 + \cdots + \rho_j < r_j$ for every $1 \le j \le n$. A critical observation is that as long as at least one of the stations $1, \dots, j$ is not empty, the fluid is flowing out of station j with rate r_j. This would not be true without the assumption that $r_1 \ge \cdots \ge r_n$, even if $\rho_1 + \cdots + \rho_j < r_j$ for every $1 \le j \le n$. It should, therefore, be clear that letting $Z_j(t)$ be the total amount of fluid contained in stations $1, \dots, j$,

$$\begin{aligned} Z_j(t) &= \sum_{i=1}^{j} J_i(t) - r_j t + I_j(t), \qquad \text{with} \\ I_j(t) &= -\inf_{0 \le s \le t} \left(\sum_{i=1}^{j} J_i(t) - r_j t \right). \end{aligned} \tag{3.2}$$

Thus, Z has the structure of a parallel network with $X_j = \sum_{i=1}^{j} J_i$, with exponent

$$(3.3)\qquad -\xi(\alpha) = -\eta\left(\sum_{i=1}^{n}\alpha_i, \sum_{i=2}^{n}\alpha_i, \ldots, \alpha_{n-1}+\alpha_n, \alpha_n\right).$$

Let $W = (W_1, \ldots, W_n)'$ be the fluid level processes at the corresponding stations. As is obvious from the description of the model,

$$(3.4)\qquad \begin{aligned} W_j(t) &= Z_j(t) - Z_{j-1}(t) \\ &= J_j(t) + (r_{j-1} - r_j)t - I_{j-1}(t) + I_j(t). \end{aligned}$$

LEMMA 3.1. *$Z(t) \to Z^*$ and $W(t) \to W^*$ in distribution for some proper Z^* and W^*.*

PROOF. It should be clear that $T_n(x)$ [see (2.4)] is a regeneration epoch for the processes Z and W, because at this instant the entire network is empty. It is well known that $\{T_n(x) | x > 0\}$ is a subordinator with drift x; thus, it can be chosen to have a nonarithmetic distribution (with a proper choice of x). Also, as in (2.3), $ET_n(x) = x/(r_n - \sum_{i=1}^{n}\rho_i) < \infty$. Because both Z and W are right continuous, the result follows [e.g., Asmussen (1987), Chapter V]. □

From (2.6), the (marginal) LST of Z_j^* is given by

$$(3.5)\qquad Ee^{-\alpha_j Z_j^*} = \alpha_j\left(r_j - \sum_{i=1}^{j}\rho_i\right)\Big/\left(\alpha_j r_j - \eta(\alpha_j, \ldots, \alpha_j, 0, \ldots, 0)\right),$$

where the first j coordinates of η have α_j and the rest 0. Thus, with

$$(3.6)\qquad S_{2j} = \sum_{1\le i,k\le j}\frac{\partial^2}{\partial\alpha_i\partial\alpha_k}\eta(0), \qquad S_{3j} = \sum_{1\le i,k,l\le j}\frac{\partial^3}{\partial\alpha_i\partial\alpha_k\partial\alpha_l}\eta(0)$$

(noting that $S_{2j} = \mathrm{Var}[\sum_{i=1}^{j} J_i(1)]$), we have, via some messy differentiation, that

$$(3.7)\qquad \begin{aligned} EZ_j^* &= \frac{S_{2j}}{2\left(r_j - \sum_{i=1}^{j}\rho_i\right)}, \\ \mathrm{Var}(Z_j^*) &= \frac{S_{3j}}{3\left(r_j - \sum_{i=1}^{j}\rho_i\right)} + \left(EZ_j^*\right)^2. \end{aligned}$$

In order to obtain the results that follow it will need to be assumed that $J_1, \ldots, J_n$ are independent subordinators; that is, that $\eta(\alpha) = \sum_{j=1}^{n}\eta_j(\alpha_j)$. From now on this assumption is in force. As throughout this article, denote $\varphi_j(\alpha_j) = r_j\alpha_j - \sum_{i=1}^{j}\eta_i(\alpha_j)$. Because $\rho_1 + \cdots + \rho_j < r_j$, φ_j has an inverse that is the negative of the exponent of the subordinator T_j [see (2.3) and (2.4)]. Before continuing let us state a well known result.

LEMMA 3.2. *Let B be a right continuous process with no negative jumps. $C(t) \equiv -\inf\{B(s)|0 \le s \le t\}^-$ is the unique nondecreasing, continuous process for which $C(0) = 0$, $A \equiv B + C \ge 0$ and $\int_0^\infty A(t)\, dC(t) = 0$.*

The simple proof, for the case where B is continuous, is found in Harrison [(1985), Chapter 2] and applies here without change. The result that is now stated and proved is what I view as the most interesting result of this section.

THEOREM 3.1. *For every $1 \le j \le n - 1$, $W^j = \{W_{j+1}(T_j(x)), \dots, W_n(T_j(x)) | x \ge 0\}$ are the fluid level processes of a tandem fluid network of $n - j$ stations, with dependent subordinator inputs with exponent*

$$- \eta^j(\alpha_{j+1}, \dots, \alpha_n) = -\varphi_j^{-1}\left[\sum_{i=j+1}^{n} [\eta_i(\alpha_i) + (r_{i+1} - r_i)\alpha_i]\right] \tag{3.8}$$

with input rates $\rho_k^j = (\rho_k + r_{k-1} - r_k)/(r_j - \sum_{i=1}^j \rho_i)$, for $j + 1 \le k \le n$, satisfying $\sum_{i=j+1}^n \rho_i^j = 1 - (r_n - \sum_{i=1}^n \rho_i)/(r_j - \sum_{i=1}^j \rho_i) < 1$, where the flow rates out of the stations are $r_{j+1}^j = \cdots = r_n^j = 1$.

PROOF. It suffices to show that the result holds for $j = 1$, because the argument for each $1 \le j \le n - 1$ is exactly the same. First recall that $I_1(T_1(x) = x$ and observe that $0 = Z_1(T_1(x)) = J_1(T_1(x)) - r_j T_1(x) + x$. Thus, noting that $r_1 - r_k = \sum_{i=1}^k (r_{i+1} - r_i)$,

$$\begin{aligned} Z_k(T_1(x)) = \sum_{i=2}^{k} [J_i(T_1(x)) + (r_{i-1} - r_i)T_1(x)] \\ - x + I_k(T_1(x)). \end{aligned} \tag{3.9}$$

For $2 \le j \le n$, setting $Z_j^1(x) = Z_j(T_1(x))$, $J_j^1(x) = J_j(T_1(x)) + (r_{j-1} - r_j)T_1(x)$ and $I_k^1(x) = I_j(T_1(x))$ gives that $Z_k^1(x) = \sum_{i=2}^k J_i^1(x) - x + I_x^1(x)$, for every $2 \le k \le n$ and $x \ge 0$. Hence, in order to establish the result, two things must be shown. The first is that $J^1 = (J_2^1, \dots, J_n^1)'$ is an $(n - 1)$-dimensional subordinator with respect to the (standard) filtration $\{\mathscr{F}_{T_1(x)} | x \ge 0\}$, with exponent given by (3.8). The second is that for every $x \ge 0$, $I_k^1(x) = -\inf_{0 \le y \le x}(\sum_{i=2}^k J_i^1(y) - y)$.

Because $T_1(x)$ is right continuous by definition, then so is J^1 (and $\mathscr{F}_{T_1(\cdot)}$). Now observe that, for x, $y \ge 0$,

$$\begin{aligned} & J_i^1(x + y) - J_i^1(x) \\ & \quad = J_i([T_1(x + y) - T_1(x)] + T_1(x)) \\ & \qquad - J_i(T_1(x)) + (r_{i-1} - r_i)[T_1(x + y) - T_1(x)], \end{aligned} \tag{3.10}$$

so that the stationary independent increment properties follow from the strong Markov property for J, the facts that $T_1(x) \le T_1(x + y)$ are stopping times that are independent of $(J_2, \dots, J_n)$ and the fact that T_1 itself is a subordinator. To establish (3.8), condition on $T_1(x)$ and then apply (2.3). ρ_i^j is obtained via differentiation from (3.8).

As for I_k^1, the main observation is that, from the construction, every point of increase of I_k is also a point of increase of I_1. In other words the (random) measure on $\mathscr{R}_+$ induced by I_k is absolutely continuous with respect to the one induced by I_1. Hence, thanks to the Radon–Nikodym theorem, there exists a nonnegative Borel measurable process U_k, which is I_1-integrable on compacts, for which $I_k(t) = \int_0^t U_k(s)\, dI_1(s)$. Performing the change of variables $s = T_1(y)$ as in the proof of Lemma 2.1 gives that $I_k^1(x) = \int_0^x U_k(T_1(y))\, dy$. Thus I_k^1 is absolutely continuous with respect to Lebesgue measure. In particular, it is continuous, nondecreasing with $I_k^1(0) = 0$. Finally observe that by the definition of I_k and by change of variables (with the left side $= 0$),

$$\begin{aligned}\int_0^\infty Z_k(t)\, dI_k(t) &= \int_0^\infty Z_k(t) U_k(t)\, dI_1(t)\\ &= \int_0^\infty Z_k^1(x) U_k(T_1(x))\, dx = \int_0^\infty Z_k^1(x)\, dI_k^1(x).\end{aligned} \tag{3.11}$$

Hence all of the conditions of Lemma 3.2 are met with $A = Z_k^1$, $B(x) = \sum_{i=2}^k J_i^1(x) - x$ and $C = I_k^1$, which completes the proof of Theorem 3.1. □

From Theorem 3.1 and Lemma 3.1, the following corollary is immediate.

COROLLARY 3.1. *The assumptions, hence the conclusions, of Corollary* 2.3 *are satisfied.*

In our case $Z_1^{j*} \equiv \cdots \equiv Z_j^{j*} \equiv 0$ and (2.12) becomes

$$Ee^{-\alpha' Z^*} = \frac{\sum_{j=1}^n \alpha_j \left(r_j - \sum_{i=1}^j \rho_i\right) E\exp\left(-\sum_{i=j+1}^n \alpha_i Z_i^{j*}\right)}{\sum_{j=1}^n \left[r_j \alpha_j - \eta_j\left(\sum_{i=j}^n \alpha_i\right)\right]} \tag{3.12}$$

Also it is easy to verify, using the same argument that yields Lemma 2.1 and the subsequent corollaries, that one may replace α_j by $\alpha_j - \alpha_{j+1}$ in (3.12) (even if $a_j < a_{j+1}$) to give [see also Kella and Whitt (1992b)]

$$Ee^{-\alpha' W^*} = \frac{\sum_{j=1}^n (\alpha_j - \alpha_{j+1})\left(r_j - \sum_{i=1}^j \rho_i\right) E\exp\left(-\sum_{i=j+1}^n \alpha_i W_i^{j*}\right)}{\sum_{j=1}^n \left[r_j(\alpha_j - \alpha_{j+1}) - \eta_j(\alpha_j)\right]} \tag{3.13}$$

with $W_i^{j*} = Z_i^{j*} - Z_{i-1}^{j*}$ and $\alpha_{n+1} = 0$.

Although it would be easy at first to hope that (3.13) can be used *inductively* to characterize the complete joint distribution of W^*, Theorem 3.1 explains why one should not be so optimistic. After one iteration one is left with a lower dimensional problem, but in which the inputs are *dependent*, thereby losing the structure. It might still be true that (3.13) has a unique solution, but if it is, a different approach seems to be needed.

4. The two-dimensional case and the correlation structure of W^*. In Kella and Whitt (1992b), the joint steady state distribution of the fluid levels of two stations in tandem with compound Poisson input to the first station only was found by applying Kella and Whitt (1992a). This approach does not directly apply when the input is not compound Poisson, and does not apply at all when there is an additional input process feeding into the second station, even if both inputs are compound Poisson. In this section, a two-fold generalization will therefore be achieved, because two inputs are allowed and both can be general subordinators.

It should be noted that if it is possible to find the joint distribution of (W_1, W_2), then this will immediately give the joint distribution of $(Z_i, Z_j - Z_i)$ for any $i < j$. This is true because one can replace the input to station 1 by $J_1 + \cdots + J_i$, the input to station 2 by $J_{i+1} + \cdots + J_j$, the output rate of station 1 by r_i and that of station 2 by r_j. Hence only the joint distribution of (W_1, W_2) will be studied and other results will be obtained via substitution. For the next theorem and what follows the following notation is used:

$$\varphi_1(\beta) = r_1\beta - \eta_1(\beta), \qquad \sigma_i^2 = \eta_i''(0),$$
$$\text{for } i = 1,2, \qquad \hat{J}_2^* = J_2^{1*}, \qquad \hat{W}_2^* = W_2^{1*},$$
$$\hat{\eta}_2(\beta) \equiv \eta_2^1(\beta) = \varphi_1^{-1}[\eta_2(\beta) + (r_1 - r_2)\beta],$$
$$\hat{\rho}_2 \equiv \rho_2^1 = 1 - (r_2 - \rho_1 - \rho_2)/(r_1 - \rho_1),$$
$$\hat{\sigma}_2^2 = \hat{\eta}_2''(0) = (\hat{\rho}_2^2\sigma_1^2 + \sigma_2^2)/(r_1 - \rho_1).$$

The caret ("hat") notation instead of the superscript 1 will be more convenient for the purpose of differentiation and power notation that will be needed in this section. The following is the main result of this section and, together with Theorem 4.2 (to come), is arguably the most important single contribution of this paper.

THEOREM 4.1. *The LST of the joint distribution of* (W_1^*, W_2^*) *is given by*

$$Ee^{-(\alpha_1 W_1^* + \alpha_2 W_2^*)} = \frac{(r_1 - \rho_1)[\alpha_1 - \hat{\eta}_2(\alpha_2)]}{\varphi_1(\alpha_1) - \varphi_1(\hat{\eta}_2(\alpha_2))}\,\frac{(1 - \hat{\rho}_2)\alpha_2}{\alpha_2 - \hat{\eta}_2(\alpha_2)}. \tag{4.1}$$

PROOF. Specializing (3.13) to the case $n = 2$ gives

$$Ee^{-(\alpha_1 W_1^* + \alpha_2 W_2^*)} = \frac{(\alpha_1 - \alpha_2)(r_1 - \rho_1)Ee^{-\alpha_2 \hat{W}_2^*} + \alpha_2(r_2 - \rho_1 - \rho_2)}{r_1(\alpha_1 - \alpha_2) + r_2\alpha_2 - \eta_1(\alpha_1) - \eta_2(\alpha_2)}. \tag{4.2}$$

Theorem 3.1 and (2.6) imply that $Ee^{-\alpha_2 \hat{W}_2^*} = (1 - \hat{\rho}_2)\alpha_2/(\alpha_2 - \hat{\eta}_2(\alpha_2))$. The rest is through straightforward manipulations. □

Note that if one sets $\alpha_1 = \alpha_2$, then (4.1) gives the LST of Z_2^*, and setting $\alpha_2 = 0$ gives the one for $W_1^* = Z_2^*$, both of which are consistent with (2.6) or

(3.5). Setting $\alpha_1 = 0$ gives the interesting formula

$$(4.3)\quad Ee^{-\alpha_2 W_2^*} = \frac{(r_1 - \rho_1)\hat{\eta}_2(\alpha_2)}{\varphi_1(\hat{\eta}_2(\alpha_2))} \frac{(1 - \hat{\rho}_2)\alpha_2}{\alpha_2 - \hat{\eta}_2(\alpha_2)} = Ee^{-\hat{\eta}_2(\alpha_2) W_1^*} Ee^{-\alpha_2 \hat{W}_2^*},$$

which gives the following decomposition result.

THEOREM 4.2. *Let U_1^* and $\hat{U}_2^*$ be independent random variables that are distributed like W_1^* and $\hat{W}_2^*$, respectively, and that are also independent of the process $\hat{J}_2$. Then*

$$(4.4)\qquad W_2^* =_d \hat{J}_2(U_1^*) + \hat{U}_2^*.$$

In particular,

$$(4.5)\qquad \begin{aligned} EW_2^* &= \hat{\rho}_2 EW_1^* + E\hat{W}_2^*, \\ \mathrm{Var}(W_2^*) &= \hat{\sigma}_2^2 EW_1^* + \hat{\rho}_2^2 \,\mathrm{Var}(W_1^*) + \mathrm{Var}\big(\hat{W}_2^*\big). \end{aligned}$$

Formulae for the means and variances of W_1^* and $\hat{W}_2^*$ are immediate with the aid of (3.7). In particular, it is easy to check that computing EW_2^* via (4.5) gives the same answer as from $EW_2^* = EZ_2^* - EZ_1^*$, which is

$$(4.6)\qquad EW_2^* = \big(\hat{\rho}_2 \sigma_1^2 + \sigma_2^2\big)/(2(r_2 - \rho_1 - \rho_2)).$$

Cumbersome, but unavoidable, differentiations give

$$(4.7)\qquad \hat{\eta}_2'''(0) = \big(\hat{\rho}_2^3 \eta_1'''(0) + \eta_2'''(0) + 3\hat{\rho}_2 \sigma_1^2 \hat{\sigma}_2^2\big)/(r_1 - \rho_1),$$

which yields, after some more tedious manipulations,

$$(4.8)\qquad \mathrm{Var}(W_2^*) = \frac{\hat{\rho}_2^2 \eta_1'''(0) + \eta_2'''(0)}{3(r_2 - \rho_1 - \rho_2)} + (EW_2^*)^2 + 2EW_1^* E\hat{W}_2^*$$

and because $\mathrm{Var}(Z_2^*) = \mathrm{Var}(W_1^*) + \mathrm{Var}(W_2^*) + 2\,\mathrm{Cov}(W_1^*, W_2^*)$, the following surprisingly simple formulae are obtained:

$$(4.9)\qquad \begin{aligned} \mathrm{Cov}(W_1^*, W_2^*) &= \frac{\hat{\rho}_2 E(W_1^*)^2}{2} \\ &= \frac{\hat{\rho}_2}{2}\left[\frac{\eta_1'''(0)}{3(r_1 - \rho_1)} + 2\left(\frac{\sigma_1^2}{2(r_1 - \rho_1)}\right)^2\right], \end{aligned}$$

so that

$$(4.10)\qquad \begin{aligned} \mathrm{Cov}(Z_1^*, Z_2^*) &= \left(1 + \frac{\hat{\rho}_2}{2}\right)\frac{\eta_1'''(0)}{3(r_1 - \rho_1)} \\ &\quad + (1 + \hat{\rho}_2)\left(\frac{\sigma_1^2}{2(r_1 - \rho_1)}\right)^2. \end{aligned}$$

In particular W_1^* and W_2^* are positively correlated. Moreover, the following result holds.

COROLLARY 4.1. *The correlation coefficient of* W_1^* *and* W_2^* *can take on only and all values in the interval* $(0, 1/\sqrt{3})$.

PROOF. In Kella and Whitt (1992b) it was shown that for the special case considered there this result is correct. Replacing σ_2^2 and (necessarily also) $\eta_2'''(0)$ by zero decreases $\mathrm{Var}(W_2^*)$, which is the only term in the correlation coefficient where these two values appear. Also, for this case the external input to the second station is the deterministic function $\rho_2 t$. Hence, it is easy to see that this model is equivalent to a model with no input to the second station and where the output rate from that station is $r_2 - \rho_2$. Hence, the arguments in Kella and Whitt (1992b) apply with no change for the more general situation considered here. □

Recalling the second paragraph of this section and letting

$$\hat{\rho}_{ij} = 1 - \left(r_j - \sum_{k=1}^{j} \rho_k\right) \Bigg/ \left(r_i - \sum_{k=1}^{i} \rho_k\right), \tag{4.11}$$

for $i \le j$, emphasizing that $\hat{\rho}_{ii} = 0$, the results are summarized as follows.

COROLLARY 4.2. *For every* $1 \le i, j \le n$,

$$\begin{aligned} \mathrm{Cov}(W_i^*, W_j^*) = {} & \mathrm{Cov}(Z_i^*, Z_j^*) - \mathrm{Cov}(Z_i^*, Z_{j-1}^*) \\ & - \mathrm{Cov}(Z_{i-1}^*, Z_j^*) + \mathrm{Cov}(Z_{i-1}^*, Z_{j-1}^*), \end{aligned} \tag{4.12}$$

where, for $1 \le i \le j \le n$,

$$\begin{aligned} \mathrm{Cov}(Z_i^*, Z_j^*) = {} & \left(1 + \frac{\hat{\rho}_{ij}}{2}\right) \frac{\sum_{k=1}^{i} \eta_k'''(0)}{3\left(r_i - \sum_{k=1}^{i} \rho_k\right)} \\ & + (1 + \hat{\rho}_{ij}) \left(\frac{\sum_{k=1}^{i} \sigma_k^2}{2\left(r_i - \sum_{k=1}^{i} \rho_k\right)}\right)^2. \end{aligned} \tag{4.13}$$

5. A remark on preemptive resume priorities in the M / G / 1 queue. For an M/G/1 queue with preemptive resume priority discipline [e.g., Conway, Maxwell and Miller (1967) and Jaiswal (1968)], let the priority classes in the model be denoted by $1, \ldots, n$, where if $i < j$, then a customer of class i has priority over one of class j. For this model it should be clear that as far as classes $1, \ldots, j$ are concerned, classes $j + 1, \ldots, n$ can be disregarded. The work inputs of the various priority classes are independent compound Poisson processes. Hence, the total workload process of classes $1, \dots, j$ is given by (3.2) with J_i being the input compound Poisson process of class i, and where $r_j = 1$ for all $1 \le j \le n$. The latter is true because the server is serving at unit rate, as is customary in that literature. Thus, the (joint) stochastic structure of the workload processes in an M/G/1 queue with preemptive resume priority discipline is a special case of our tandem fluid network with independent subordinator inputs. Hence, all of the results in Sections 3 and 4 hold for this case. I do not know of similar results in the literature of priority queues; therefore, indirectly, this article has contributed to this old field as well. The one-dimensional results give, via PASTA [see Wolff (1982)], the steady state distributions of the waiting times for the various classes.

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