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ORDERINGS FOR POSITIVE DEPENDENCE ON MULTIVARIATE EMPIRICAL DISTRIBUTIONS

BY MAGDY H. METRY AND ALLAN R. SAMPSON[1]

University of Northern Colorado and University of Pittsburgh

The study of orderings for positive dependence on bivariate empirical distributions can be viewed as the study of partial orderings on the set S_N of all permutations of the integers $1, \dots, N$. This paper extends earlier bivariate results to multivariate empirical distributions, with focus on the trivariate case. In terms of a newly defined notion of relative rearrangement, characterizations are given of the more positively upper orthant dependent ordering and related orderings. A new partial ordering describing concordance on $(S_N)^m$ is also introduced and connected with the positively upper orthant dependence ordering.

1. Introduction. An ordering for positive dependence allows the comparison of two multivariate distributions to determine if one distribution is more positively dependent in a certain sense than the other distribution, thereby partially ordering classes of multivariate distributions according to a degree of positive dependence. Orderings for positive dependence applied to bivariate distributions have provided approaches for stochastically comparing certain distributions of statistics [e.g., Tchen (1980); Schriever (1985), (1987a)], evaluating the meaningfulness of parameters in one-parameter families of fixed marginal distributions [e.g., Kimeldorf and Sampson (1987)], inducing orderings on permutations via orderings for positive dependence on bivariate empirical c.d.f.s [e.g., Block, Chhetry, Fang and Sampson (1990); Metry and Sampson (1993)], providing nonparametric statistical results [e.g., Yanagimoto and Okamoto (1969); Hollander, Proschan and Sethuraman (1977)] and studying the effects of random censoring on information [e.g., Hollander, Proschan and Sconing (1990)].

One widely studied ordering on bivariate distributions is the more concordant ordering [Tchen (1980)] that we denote by $\geq_c$. This ordering has been extended in several ways to the multivariate case. Let $\mathbf{X} = (X_1, \dots, X_m)$ and $\mathbf{Y} = (Y_1, \dots, Y_m)$ be two m-dimensional random vectors whose one-dimensional marginal distributions correspondingly agree; that is, $X_k \sim Y_k$,

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for all $k = 1, \dots, m$. The random vector $\mathbf{X}$ is said to be *more positively upper orthant dependent* (more PUOD) than $\mathbf{Y}$, denoted by $\mathbf{X} \geq_U \mathbf{Y}$, if $P(\bigcap_{i=1}^m (X_i > a_i)) \geq P(\bigcap_{i=1}^m (Y_i > a_i))$, for all $(a_1, \dots, a_m) \in \mathbf{R}^m$ [e.g., Block and Sampson (1983)]. If all $>$ are replaced by $\leq$ in the preceding definition, then $\mathbf{X}$ is said to be *more positively lower orthant dependent* (more PLOD) than $\mathbf{Y}$, denoted by $\mathbf{X} \geq_L \mathbf{Y}$. If both $\mathbf{X} \geq_U \mathbf{Y}$ and $\mathbf{X} \geq_L \mathbf{Y}$, then $\mathbf{X}$ is said to be *more positively orthant dependent* (more POD) than $\mathbf{Y}$, denoted by $\mathbf{X} \geq_O \mathbf{Y}$. In the bivariate case, the four orderings $\geq_U$, $\geq_L$, $\geq_O$ and $\geq_c$ are equivalent.

An approach, described by Scarsini (1984), Schriever [(1985), Section (4.1)] and Block, Chhetry, Fang and Sampson (1990), to extending any of these four orderings that require agreement of one-dimensional marginals to orderings that do not is as follows. For $\mathbf{X} = (X_1, \dots, X_m)$ and $\mathbf{Y} = (Y_1, \dots, Y_m)$ with marginals c.d.f.s $F_1, \dots, F_m$ and $G_1, \dots, G_m$, respectively, define $\mathbf{X} \to \mathbf{Y}$ if and only if $(F_1(X_1), \dots, F_m(X_m)) \to (G_1(Y_1), \dots, G_m(Y_m))$, where $\to$ denotes one of $\geq_c$, $\geq_U$, $\geq_L$ or $\geq_O$. In this way we can compare $\mathbf{X}$ and $\mathbf{Y}$ if $F_i(X_i) \sim G_i(Y_i)$, for all $i = 1, \dots, m$. Two cases that allow this comparison are when the one-dimensional marginals are continuous and when $\mathbf{X}$ and $\mathbf{Y}$ are random vectors corresponding to empirical distributions of two samples of the same size with no ties in each coordinate.

Let $\mathbf{x}^{(k)} = (x_1^{(k)}, \dots, x_m^{(k)})$, $k = 1, \dots, N$ $(m \geq 2)$, be a random sample of size N from an m-dimensional c.d.f. H with continuous one-dimensional marginals and let $\hat{H}$ be the corresponding empirical distribution. Rank order the $x_1^{(k)}$, $k = 1, \dots, N$, as $x_{1(1)} < \cdots < x_{1(N)}$ and let $i_k^{(l)}$ denote the rank among $x_{l+1}^{(1)}, \dots, x_{l+1}^{(N)}$ of the $x_{l+1}^{(j)}$ that correspond to $x_{1(k)}$, for all $k = 1, \dots, N$ and all $l = 1, \dots, m-1$. With probability 1, $\mathbf{i}^{(l)} = (i_1^{(l)}, \dots, i_N^{(l)}) \in S_N$, the set of all permutations of $1, \dots, N$. In this case $\mathbf{i}^{(l)}$ is the permutation that gives the order of the observations of the $(l+1)$st coordinate relative to the order of the corresponding observations of the first coordinate. The *m-dimensional rank order* is defined as $(\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(m-1)})$ and the empirical rank distribution, $\hat{H}^R$, is the c.d.f. of an m-dimensional random vector $(S^{(0)}, S^{(1)}, \dots, S^{(m-1)})$, whose probability mass function is given by $P(S^{(0)} = k, S^{(1)} = i_k^{(1)}, \dots, S^{(m-1)} = i_k^{(m-1)}) = N^{-1}$, for all $k = 1, \dots, N$. We call $(S^{(0)}, S^{(1)}, \dots, S^{(m-1)})$ a *random vector related* to $(\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(m-1)})$.

Using one of $\geq_c$, $\geq_U$, $\geq_L$ and $\geq_O$ for comparison of two empirical distributions $\hat{H}_1$ and $\hat{H}_2$ of the same dimension and same sample size with no ties in each coordinate is equivalent to comparing their corresponding empirical rank distributions $\hat{H}_1^R$ and $\hat{H}_2^R$ [see the argument for the bivariate case in Block, Chhetry, Fang and Sampson (1990)]. This leads us to introduce these four orderings on the class $(S_N)^{(m-1)} = S_N \times \cdots \times S_N((m-1)$ times), in the sense that for one of these four orderings $\to$, we define $(\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(m-1)}) \to (\mathbf{j}^{(1)}, \dots, \mathbf{j}^{(m-1)})$, whenever $(S^{(0)}, \dots, S^{(m-1)}) \to (V^{(0)}, \dots, V^{(m-1)})$, where $(S^{(0)}, \dots, S^{(m-1)})$ and $(V^{(0)}, \dots, V^{(m-1)})$ are two random vectors related to $(\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(m-1)})$ and $(\mathbf{j}^{(1)}, \dots, \mathbf{j}^{(m-1)})$, respectively.

The ordering $\geq_c$ on S_N has been considered by Schriever (1987a, b) and Block, Chhetry, Fang and Sampson (1990), but was originally discussed by

Cayley (1849). Although our framework deals with $(S_N)^{(m-1)}$, $m \geq 2$, we focus primarily on $(S_N)^2$, which corresponds to the trivariate case.

There are two main results in this paper. One result characterizes each of the orderings $\geq_U$, $\geq_L$ and $\geq_O$ on $(S_N)^2$ using a newly defined notion of relative rearrangement. The other main result is a new partial ordering $\geq_B$, on $(S_N)^2$, motivated by a geometrical aspect of the ordering $\geq_c$ on S_N. In Section 2, we introduce some concepts that are useful in establishing our results and briefly review some results for the ordering $\geq_c$ on S_N. Section 3 gives characterizations of the positive dependence orderings $\geq_U$, $\geq_L$ and $\geq_O$ on $(S_N)^2$. In Section 4, the ordering $\geq_B$ on $(S_N)^2$ is introduced by focusing on the multivariate geometry of moving masses from discordancy to concordancy. The discussion in Section 5 briefly considers several possible extensions of our results.

2. Notation, some preliminaries and review of the ordering $\geq_c$ on S_N. Throughout this paper, whenever we write vectors such as $\mathbf{i}, \mathbf{j}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ we mean that they are elements of S_N. The permutation $\mathbf{i}$ is denoted by $\mathbf{i} = (i_1, \dots, i_N)$. The product operation is the composition of permutations. The identity permutation is denoted by $\mathbf{e} = (1, \dots, N)$ and the inverse of a permutation $\mathbf{i}$ will be denoted by $\mathbf{i}^*$. For $\mathbf{i} \in S_N$, let $i_k^c = N + 1 - i_{N+1-k}$, $k = 1, \dots, N$. The permutation $\mathbf{i}^c = (i_1^c, i_2^c, \dots, i_N^c)$ is called [Savage (1957)] the complement of $\mathbf{i}$; for example, $(32514)^c = (25143)$. If $\mathbf{i} = \mathbf{i}^c$, then $\mathbf{i}$ is called self-complementary.

For $\mathbf{i} \in S_N$, and any positive integer $k \leq N$, the *increasing rearrangement* of $i_{N-k+1}, i_{N-k+2}, \dots, i_N$, which are the last k elements of $\mathbf{i}$, is denoted by $i_{1,k} < i_{2,k} < \cdots < i_{k,k}$. Yanagimoto and Okamoto (1969) use this concept to characterize the ordering $\geq_c$ on S_N.

THEOREM 2.1 [Yanagimoto and Okamoto (1969)]. *For $\mathbf{i},\mathbf{j} \in S_N$, $\mathbf{i} \geq_c \mathbf{j}$ if and only if $i_{l,k} \geq j_{l,k}$, for all $1 \leq l \leq k \leq N-1$.*

Metry and Sampson (1993) additionally show that

$$\mathbf{i} \geq_c \mathbf{j} \textit{ if and only if } \mathbf{i}^* \geq_c \mathbf{j}^* \textit{ if and only if } \mathbf{i}^c \geq_c \mathbf{j}^c. \tag{2.1}$$

DEFINITION 2.2. We say that $(\mathbf{i},\mathbf{j})$ is more PUOD than $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, denoted by $(\mathbf{i},\mathbf{j}) \geq_U (\boldsymbol{\alpha}, \boldsymbol{\beta})$, if $(R, S, T) \geq_U (U, V, W)$, where (R, S, T) and (U, V, W) are, respectively, random vectors related to $(\mathbf{i},\mathbf{j})$ and $(\boldsymbol{\alpha}, \boldsymbol{\beta})$. The more PLOD and more POD orderings on $(S_N)^2$ are defined similarly.

It is direct to show that the orderings $\geq_U$, $\geq_L$ and $\geq_O$ on $(S_N)^2$ are all partial orderings.

Let (R, S, T) be related to a given pair of permutations $(\mathbf{i},\mathbf{j}) \in (S_N)^2$. In order to express $P(R \geq r,\ S \geq s,\ T \geq t)$ in terms of $\mathbf{i}$ and $\mathbf{j}$, we define the connector function $\gamma_{\mathbf{i},\mathbf{j}}(\cdot, \cdot, \cdot)$ by

$$\gamma_{\mathbf{i},\mathbf{j}}(r, s, t) = \#\{(i_m, j_m) : r \leq m \leq N; i_m \geq s; j_m \geq t\}. \tag{2.2}$$

It is clear that

$$P(R \geq r, S \geq s, T \geq t) = N^{-1}\boldsymbol{\gamma}_{\mathbf{i},\mathbf{j}}(r, s, t). \tag{2.3}$$

We need to extend the notion of increasing rearrangements to deal with pairs of permutations. For a pair of permutations $(\mathbf{i},\mathbf{j}) \in (S_N)^2$ and for any positive integer $k \leq N$, define

$$j_m^{(k)} = \mathbf{j} \cdot \mathbf{i}^*(i_{m,k}), \quad \text{for all } m = 1,\dots,k. \tag{2.4}$$

The k-dimensional vector $(j_1^{(k)}, j_2^{(k)}, \dots, j_k^{(k)})$ is called by us the *relative rearrangement* of $(j_{N-k+1}, j_{N-k+2}, \dots, j_N)$ with respect to $(i_{N-k+1}, i_{N-k+2}, \dots, i_N)$. This notion is motivated in part by the fact that $j_m^{(k)} = j_\alpha \Leftrightarrow \mathbf{i}^*(i_{m,k}) = \alpha \Leftrightarrow i_{m,k} = i_\alpha$ and $\{(i_m, j_m)\colon\ m = N+1-k,\dots,N\} = \{(i_{m,k}, j_m^{(k)})\colon\ m = 1,\dots,k\}$. For any positive integer $r \leq k$, let $j_{1,r}^{(k)} < j_{2,r}^{(k)} < \cdots < j_{r,r}^{(k)}$ denote the increasing rearrangement of $j_{k-r+1}^{(k)}, j_{k-r+2}^{(k)}, \dots, j_k^{(k)}$. The following special cases are of interest:

1. For $k = N$, $j_m^{(N)} = \mathbf{j} \cdot \mathbf{i}^*(m)$, for all $1 \leq m \leq N$; that is, $(j_1^{(N)}, j_2^{(N)}, \dots, j_N^{(N)}) = \mathbf{j} \cdot \mathbf{i}^*$.
2. For $r = k$, $j_{m,k}^{(k)} = j_{m,k}$, for all $1 \leq m \leq k$.
3. For $\mathbf{i} = \mathbf{e}$, $j_m^{(k)} = j_{N-k+m}$, for all $1 \leq m \leq k$, and $j_{l,r}^{(k)} = j_{l,r}$, for all $1 \leq l \leq r \leq k$.

To illustrate this notation, let $\mathbf{i} = (16432587)$ and $\mathbf{j} = (14327658)$. Then for $k = 5$, $(i_{1,5}, i_{2,5}, i_{3,5}, i_{4,5}, i_{5,5}) = (2,3,5,7,8)$, $(j_{1,5}, j_{2,5}, j_{3,5}, j_{4,5}, j_{5,5}) = (2,5,6,7,8)$ and $(j_1^{(5)}, j_2^{(5)}, j_3^{(5)}, j_4^{(5)}, j_5^{(5)}) = (7,2,6,8,5)$. Furthermore, $j_{1,1}^{(5)} = 5$, $(j_{1,2}^{(5)}, j_{2,2}^{(5)}) = (5,8)$, $(j_{1,3}^{(5)}, j_{2,3}^{(5)}, j_{3,3}^{(5)}) = (5,6,8)$, $(j_{1,4}^{(5)}, j_{2,4}^{(5)}, j_{3,4}^{(5)}, j_{4,4}^{(5)}) = (2,5,6,8)$ and $(j_{1,5}^{(5)}, j_{2,5}^{(5)}, j_{3,5}^{(5)}, j_{4,5}^{(5)}, j_{5,5}^{(5)}) = (2,5,6,7,8) = (j_{1,5}, j_{2,5}, j_{3,5}, j_{4,5}, j_{5,5})$. For $k = 8$, $(j_1^{(8)}, j_2^{(8)}, j_3^{(8)}, j_4^{(8)}, j_5^{(8)}, j_6^{(8)}, j_7^{(8)}, j_8^{(8)}) = (17236485) = \mathbf{j} \cdot \mathbf{i}^*$.

By using the relative rearrangement notation, we have

$$\begin{aligned}\boldsymbol{\gamma}_{\mathbf{i},\mathbf{j}}(N+1-k, s, t) &= \#\{(i_m, j_m)\colon N+1-k \leq m \leq N;\, i_m \geq s;\, j_m \geq t\} \\ &= \#\big\{\big(i_{m,k}, j_m^{(k)}\big)\colon m = 1,\dots,k;\, i_{m,k} \geq s;\, j_m^{(k)} \geq t\big\}.\end{aligned}$$

This notation enables us to establish the following result that will be used in the proof of Theorem 3.3. For any positive integers l, r and k such that $1 \leq l \leq r \leq k \leq N$, we have

$$\boldsymbol{\gamma}_{\mathbf{i},\mathbf{j}}\big(N+1-k, i_{k+1-r,k}, j_{r+1-l,r}^{(k)}\big) = l. \tag{2.5}$$

This result follows from the equalities

$$\begin{aligned}&\boldsymbol{\gamma}_{\mathbf{i},\mathbf{j}}\big(N+1-k, i_{k+1-r,k}, j_{r+1-l,r}^{(k)}\big) \\ &\quad = \#\big\{\big(i_{m,k}, j_m^{(k)}\big)\colon 1 \leq m \leq k; \\ &\qquad\qquad i_{m,k} \geq i_{k+1-r,k};\, j_m^{(k)} \geq j_{r+1-l,r}^{(k)}\big\} \\ &\quad = \#\big\{j_m^{(k)}\colon k+1-r \leq m \leq k;\, j_m^{(k)} \geq j_{r+1-l,r}^{(k)}\big\} \\ &\quad = \#\big\{j_{m,r}^{(k)}\colon 1 \leq m \leq r;\, j_{m,r}^{(k)} \geq j_{r+1-l,r}^{(k)}\big\} = l.\end{aligned}$$

3. Properties of orderings for positive dependence on $(S_N)^2$. In this section, we extensively study properties of the three positive dependence partial orderings $\geq_U$, $\geq_L$ and $\geq_O$ on $(S_N)^2$.

The next lemma follows directly from the fact that (R, S, T) is related to $(\mathbf{i}, \mathbf{j})$ implies (R, S), (R, T) and (S, T) are, respectively, related to $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{j} \cdot \mathbf{i}^*$.

LEMMA 3.1. *A necessary condition for* $(\mathbf{i}, \mathbf{j}) \geq_U (\boldsymbol{\alpha}, \boldsymbol{\beta})$ *is that* $\mathbf{i} \geq_c \boldsymbol{\alpha}$, $\mathbf{j} \geq_c \boldsymbol{\beta}$ *and* $\mathbf{j} \cdot \mathbf{i}^* \geq_c \boldsymbol{\beta} \cdot \boldsymbol{\alpha}^*$.

Lemma 3.1 is also true if we replace $\geq_U$ by $\geq_L$ or $\geq_O$. The following example shows that the necessary condition of Lemma 3.1 is not sufficient.

EXAMPLE 3.2. Let $\mathbf{i} = (1342)$, $\mathbf{j} = (1324)$, $\boldsymbol{\alpha} = (4231)$ and $\boldsymbol{\beta} = (2431)$. Use Theorem 2.1 to verify that $\mathbf{i} \geq_c \boldsymbol{\alpha}$, $\mathbf{j} \geq_c \boldsymbol{\beta}$ and $\mathbf{j} \cdot \mathbf{i}^* \geq_c \boldsymbol{\beta} \cdot \boldsymbol{\alpha}^*$. To show that $(\mathbf{i}, \mathbf{j}) \geq_U (\boldsymbol{\alpha}, \boldsymbol{\beta})$ does not hold, let (R, S, T) and (U, V, W) be related to $(\mathbf{i}, \mathbf{j})$ and $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, respectively, and note that $P(R \geq 3,\ S \geq 3,\ T \geq 3) = 0 < \frac{1}{4} = P(U \geq 3, V \geq 3, W \geq 3)$.

By adding some conditions to those of Lemma 3.1, we obtain the following theorem, which gives a necessary and sufficient condition for $(\mathbf{i}, \mathbf{j}) \geq_U (\boldsymbol{\alpha}, \boldsymbol{\beta})$. This result can also be viewed as an extension of the result of Theorem 2.1 to the trivariate case.

THEOREM 3.3. *Let* $\mathbf{i}$, $\mathbf{j}$, $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta} \in S_N$. *For any* $\mathbf{t} \in S_N$ *and any positive integer* $k \leq N$, *let* $t_{1,k} < t_{2,k} < \cdots < t_{k,k}$ *denote the increasing rearrangement of* $t_{N-k+1}, t_{N-k+2}, \dots, t_N$. *Let* $j_m^{(k)} = \mathbf{j} \cdot \mathbf{i}^*(i_{m,k})$ *and* $\beta_m^{(k)} = \boldsymbol{\beta} \cdot \boldsymbol{\alpha}^*(\alpha_{m,k})$, *for all* $m = 1, \dots, k$, *be, respectively, the relative rearrangements corresponding to* $(\mathbf{i}, \mathbf{j})$ *and* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$. *For any positive integer* $r \leq k$, *let* $j_{1,r}^{(k)} < j_{2,r}^{(k)} < \cdots < j_{r,r}^{(k)}$ *and* $\beta_{1,r}^{(k)} < \beta_{2,r}^{(k)} < \cdots < \beta_{r,r}^{(k)}$ *be, respectively, the increasing rearrangement of* $j_{k-r+1}^{(k)}, j_{k-r+2}^{(k)}, \dots, j_k^{(k)}$ *and of* $\beta_{k-r+1}^{(k)}, \beta_{k-r+2}^{(k)}, \dots, \beta_k^{(k)}$. *A necessary and sufficient condition that* $(\mathbf{i}, \mathbf{j}) \geq_U (\boldsymbol{\alpha}, \boldsymbol{\beta})$ *is that the following two conditions are satisfied*:

(a) $\mathbf{i} \geq_c \boldsymbol{\alpha}$, $\mathbf{j} \geq_c \boldsymbol{\beta}$ *and* $\mathbf{j} \cdot \mathbf{i}^* \geq_c \boldsymbol{\beta} \cdot \boldsymbol{\alpha}^*$,

(b) $\beta_{r+1-l,r}^{(k)} \leq j_{m+1-l,m}^{(k)}$, *for all* $1 \leq l \leq r < k < N$, *where* $m = m(r,k) = \#\{i_{k+1-l,k}: l = 1, \dots, k;\ i_{k+1-l,k} \geq \alpha_{k+1-r,k}\}$.

PROOF. Let $\boldsymbol{\gamma}_{\mathbf{i},\mathbf{j}}(\cdot, \cdot, \cdot)$ and $\boldsymbol{\gamma}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\cdot, \cdot, \cdot)$ be the two functions given by (2.2). By using Definition 2.2 and (2.3) we have that $(\mathbf{i}, \mathbf{j}) \geq_U (\boldsymbol{\alpha}, \boldsymbol{\beta})$ if and only if $\boldsymbol{\gamma}_{\mathbf{i},\mathbf{j}}(r, s, t) \geq \boldsymbol{\gamma}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(r, s, t)$, for all $(r, s, t) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N}$, where $\mathbf{N} = \{1, \dots, N\}$, which holds if and only if

$$\boldsymbol{\gamma}_{\mathbf{i},\mathbf{j}}(N + 1 - k, s, t) \geq \boldsymbol{\gamma}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(N + 1 - k, s, t), \tag{3.1}$$

for all $(s, t) \in \mathbf{N} \times \mathbf{N}$ and all $k \in \mathbf{N}$.

The idea of the proof is to fix $k \in \mathbf{N}$ and to find a necessary and sufficient condition for (3.1) to be satisfied for all $(s,t) \in \mathbf{N} \times \mathbf{N}$. The collection of these necessary and sufficient conditions for all $k = 1, \dots, N$ gives a necessary and sufficient condition for $(\mathbf{i},\mathbf{j}) \geq_U (\boldsymbol{\alpha}, \boldsymbol{\beta})$.

For fixed $k \in \mathbf{N}$, we show that (3.1) is satisfied for all $(s,t) \in \mathbf{N} \times \mathbf{N}$ if and only if it is satisfied for all $(s,t) \in \{\alpha_{k+1-r,k}\colon r = 1,\dots,k\} \times \mathbf{N}$. Fix $s \in \mathbf{N}$; then there are three cases.

CASE 1. $s \leq \alpha_{1,k}$. In this case we have

$$\begin{aligned}\gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}(N+1-k,s,t) &= \gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}(N+1-k,\alpha_{1,k},t)\\ &\leq \gamma_{\mathbf{i},\mathbf{j}}(N+1-k,\alpha_{1,k},t)\\ &\leq \gamma_{\mathbf{i},\mathbf{j}}(N+1-k,s,t).\end{aligned}$$

CASE 2. $\alpha_{l,k} < s \leq \alpha_{l+1,k}$, for some integer $l = 1,\dots,k-1$. Then we have

$$\begin{aligned}\gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}(N+1-k,s,t) &= \gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}(N+1-k,\alpha_{l+1,k},t)\\ &\leq \gamma_{\mathbf{i},\mathbf{j}}(N+1-k,\alpha_{l+1,k},t)\\ &\leq \gamma_{\mathbf{i},\mathbf{j}}(N+1-k,s,t).\end{aligned}$$

CASE 3. $s > \alpha_{k,k}$. This case is obvious, where $\gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}(N+1-k,s,t) = 0$.

By using the same argument we can show that for every $r = 1,\dots,k$, (3.1) is satisfied for all $(s,t) \in \{\alpha_{k+1-r,k}\} \times \mathbf{N}$ if and only if it is satisfied for all $(s,t) \in \{\alpha_{k+1-r,k}\} \times \{\beta^{(k)}_{r+1-l,r}\colon l = 1,\dots,r\}$, which is equivalent to the following two conditions [by using (2.5) with $(\boldsymbol{\alpha},\boldsymbol{\beta})$ in place of $(\mathbf{i},\mathbf{j})$]:

$$m(r,k) \geq r \tag{3.2}$$

and

$$\beta^{(k)}_{r+1-l,r} \leq j^{(k)}_{m(r,k)+1-l,m(r,k)}, \quad \text{for all } l = 1,\dots,r, \tag{3.3}$$

where $m(r,k) = \#\{i_{k+1-l,k}\colon l = 1,\dots,k;\ i_{k+1-l,k} \geq \alpha_{k+1-r,k}\}$.

Thus, $(\mathbf{i},\mathbf{j}) \geq_U (\boldsymbol{\alpha},\boldsymbol{\beta})$ if and only if conditions (3.2) and (3.3) are satisfied for all $1 \leq r \leq k \leq N$. The result of the theorem follows immediately from the following facts:

(i) Condition (3.2) holding for all $1 \leq r \leq k \leq N$ is equivalent to $\mathbf{i} \geq_c \boldsymbol{\alpha}$, where $m(r,k) \geq r$ is equivalent to $i_{k+1-r,k} \geq \alpha_{k+1-r,k}$.

(ii) Condition (3.3) holding for all $1 \leq r \leq k = N$ is equivalent to $\mathbf{j} \cdot \mathbf{i}^* \geq_c \boldsymbol{\beta} \cdot \boldsymbol{\alpha}^*$, where $m(r,N) = r$, $(\beta_1^{(N)},\dots,\beta_N^{(N)}) = \boldsymbol{\beta} \cdot \boldsymbol{\alpha}^*$ and $(j_1^{(N)},\dots,j_N^{(N)}) = \mathbf{j} \cdot \mathbf{i}^*$.

(iii) Conditions (3.2) and (3.3) holding for all $1 \leq r = k < N$ are equivalent to $\mathbf{j} \geq_c \boldsymbol{\beta}$, where $m(k,k) = k$, $\beta^{(k)}_{k+1-l,k} = \beta_{k+1-l,k}$ and $j^{(k)}_{k+1-l,k} = j_{k+1-l,k}$, for all $1 \leq l \leq k$. $\square$

For the simple case where $\mathbf{i} = \boldsymbol{\alpha} = \mathbf{e}$, we readily have that $(\mathbf{e},\mathbf{j}) \geq_U (\mathbf{e},\boldsymbol{\beta})$ if and only if $\mathbf{j} \geq_c \boldsymbol{\beta}$. To see how $\mathbf{j} \geq_c \boldsymbol{\beta}$ implies condition b of Theorem 3.3, observe that $m(r,k) = r$, $j_{l,r}^{(k)} = j_{l,r}$ and $\beta_{l,r}^{(k)} = \beta_{l,r}$, for all $1 \leq l \leq r \leq k \leq N$.

An analogue of Theorem 3.3 for the orderings $\geq_L$ and $\geq_O$ follows as a corollary, using the next lemma.

LEMMA 3.4. $(\mathbf{i},\mathbf{j}) \geq_U (\boldsymbol{\alpha},\boldsymbol{\beta})$ *if and only if* $(\mathbf{i}^c,\mathbf{j}^c) \geq_L (\boldsymbol{\alpha}^c,\boldsymbol{\beta}^c)$.

PROOF. The result follows directly from Definition 2.2 and the result that (R,S,T) is related to $(\mathbf{i},\mathbf{j})$ if and only if $(N+1-R,\ N+1-S,\ N+1-T)$ is related to $(\mathbf{i}^c,\mathbf{j}^c)$. □

For general random vectors, it is known that the orderings more PUOD and more PLOD are different. However, as we now demonstrate, even on $(S_N)^2$ these two orderings differ.

EXAMPLE 3.5. Let $\mathbf{i} = (3124)$, $\mathbf{j} = (1324)$, $\boldsymbol{\alpha} = (4231)$ and $\boldsymbol{\beta} = (4213)$. We show $(\mathbf{i},\mathbf{j}) \geq_U (\boldsymbol{\alpha},\boldsymbol{\beta})$ and $(\mathbf{i},\mathbf{j}) \not\geq_L (\boldsymbol{\alpha},\boldsymbol{\beta})$. We use Theorem 3.3 to show that $(\mathbf{i},\mathbf{j}) \geq_U (\boldsymbol{\alpha},\boldsymbol{\beta})$. Condition (a) of Theorem 3.3 follows directly using Theorem 2.1. The following direct calculations demonstrate that condition b of Theorem 3.3 is satisfied: $m(1,2) = 1$ and $\beta_{1,1}^{(2)} = 1 < 4 = j_{1,1}^{(2)}$; $m(1,3) = 1$ and $\beta_{1,1}^{(3)} = 1 < 4 = j_{1,1}^{(3)}$; $m(2,3) = 2$ and $\beta_{1,2}^{(3)} = 1 < 2 = j_{1,2}^{(3)}$, $\beta_{2,2}^{(3)} = 2 < 4 = j_{2,2}^{(3)}$. From Lemma 3.4, showing that $(\mathbf{i},\mathbf{j}) \not\geq_L (\boldsymbol{\alpha},\boldsymbol{\beta})$ is equivalent to showing that $(\mathbf{i}^c,\mathbf{j}^c) \not\geq_U (\boldsymbol{\alpha}^c,\boldsymbol{\beta}^c)$, which is demonstrated in Example 3.2, where $\mathbf{i}^c = (1342)$, $\mathbf{j}^c = (1324)$, $\boldsymbol{\alpha}^c = (4231)$ and $\boldsymbol{\beta}^c = (2431)$.

4. Another partial ordering on $(S_N)^2$. In this section another partial ordering for positive dependence is defined on $(S_N)^2$. This new ordering is motivated from a geometric aspect of the ordering $\geq_c$ on S_N. Tchen (1980) noted that the ordering $\geq_c$ on S_N can be expressed as "moving masses" from discordancy to concordancy. For $\mathbf{i},\mathbf{j} \in S_N$, $\mathbf{i}$ is obtained from $\mathbf{j}$ by correcting an inversion, denoted by $\mathbf{i} \to_c \mathbf{j}$, or equivalently $\mathbf{j} \leftarrow_c \mathbf{i}$, if there exist two positive integers l and m, $l < m \leq N$, such that $i_l = j_m < j_l = i_m$ and $i_k = j_k$ for all $k \neq l, m$. Based on this notion of correcting an inversion, another characterization of the ordering $\geq_c$ on S_N is given by $\mathbf{i} \geq_c \mathbf{j}$ if $\mathbf{i} = \mathbf{j}$ or there exist elements $\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(p)} \in S_N$ satisfying $\mathbf{i} \to_c \mathbf{i}^{(1)} \to_c \cdots \to_c \mathbf{i}^{(p)} = \mathbf{j}$; that is, $\mathbf{i} \geq_c \mathbf{j}$ if $\mathbf{i} = \mathbf{j}$ or $\mathbf{i}$ is obtained from $\mathbf{j}$ in a number of steps each of which consists of correcting an inversion.

Focusing on the multivariate geometry of moving masses from discordancy to concordancy, we are led to a new positive dependence partial ordering, $\geq_B$, on $(S_N)^2$, that is based on a suitable notion of "correcting multivariate inversions."

DEFINITION 4.1. We say that $(\mathbf{i},\mathbf{j})$ is obtained from $(\boldsymbol{\alpha},\boldsymbol{\beta})$ by correcting an inversion, denoted by $(\mathbf{i},\mathbf{j}) \to_B (\boldsymbol{\alpha},\boldsymbol{\beta})$, or equivalently $(\boldsymbol{\alpha},\boldsymbol{\beta}) \leftarrow_B (\mathbf{i},\mathbf{j})$, if there exist two positive integers l and $m, l < m \leq N$, such that the following three conditions are satisfied:

$$\text{at least one of } (\alpha_m - \alpha_l) \text{ and } (\beta_m - \beta_l) \text{ is negative;} \tag{4.1}$$

$$i_l = \min\{\alpha_l, \alpha_m\}, \quad i_m = \max\{\alpha_l, \alpha_m\}, \quad i_k = \alpha_k, \quad \text{for all } k \neq l, m; \tag{4.2}$$

$$j_l = \min\{\beta_l, \beta_m\}, \quad j_m = \max\{\beta_l, \beta_m\}, \quad j_k = \beta_k, \quad \text{for all } k \neq l, m. \tag{4.3}$$

DEFINITION 4.2. We say that $(\mathbf{i},\mathbf{j})$ is *multivariate better ordered* than $(\boldsymbol{\alpha},\boldsymbol{\beta})$, denoted by $(\mathbf{i},\mathbf{j}) \geq_B (\boldsymbol{\alpha},\boldsymbol{\beta})$, if $(\mathbf{i},\mathbf{j}) = (\boldsymbol{\alpha},\boldsymbol{\beta})$ or there exist elements $(\mathbf{i}^{(1)},\mathbf{j}^{(1)}), (\mathbf{i}^{(2)},\mathbf{j}^{(2)}), \dots, (\mathbf{i}^{(p)},\mathbf{j}^{(p)})$ in $(S_N)^2$ satisfying $(\mathbf{i},\mathbf{j}) \to_B (\mathbf{i}^{(1)},\mathbf{j}^{(1)}) \to_B \cdots \to_B (\mathbf{i}^{(p)},\mathbf{j}^{(p)}) = (\boldsymbol{\alpha},\boldsymbol{\beta})$.

For example, $((213456), (124356)) \geq_B ((253146), (134265))$ because $((253146), (134265) \leftarrow_B ((253146), (134256)) \leftarrow_B ((243156), (134256)) \leftarrow_B ((213456), (124356))$, where the choices of l, m for each of the three steps with $\leftarrow_B$ are $l = 5$, $m = 6$; $l = 2$, $m = 5$; and $l = 2$, $m = 4$, respectively.

From Definition 4.2, one can show that the ordering $\geq_B$ on $(S_N)^2$ is reflexive and transitive. Antisymmetry will follow immediately from Theorem 4.5.

To study the implications among the orderings $\geq_B$ and $\geq_O$ on $(S_N)^2$, we need the following lemma.

LEMMA 4.3. $(\mathbf{i},\mathbf{j}) \to_B (\boldsymbol{\alpha},\boldsymbol{\beta})$ *if and only if* $(\mathbf{i}^c,\mathbf{j}^c) \to_B (\boldsymbol{\alpha}^c,\boldsymbol{\beta}^c)$.

PROOF. Let $(\mathbf{i},\mathbf{j}) \to_B (\boldsymbol{\alpha},\boldsymbol{\beta})$. By definition there exist two positive integers $l < m \leq N$, such that conditions (4.1), (4.2) and (4.3) are satisfied. Let $l^* = N + 1 - m$ and $m^* = N + 1 - l$, so that $l^* < m^* \leq N$, and the following three conditions are satisfied:

(i) At least one of $(\alpha^c_{m^*} - \alpha^c_{l^*})$ and $(\beta^c_{m^*} - \beta^c_{l^*})$ is negative.

(ii) $i^c_{l^*} = \min\{\alpha^c_{l^*}, \alpha^c_{m^*}\}$, $i^c_{m^*} = \max\{\alpha^c_{l^*}, \alpha^c_{m^*}\}$ and $i^c_k = \alpha^c_k$, for all $k \neq l^*, m^*$.

(iii) $j^c_{l^*} = \min\{\beta^c_{l^*}, \beta^c_{m^*}\}$, $j^c_{m^*} = \max\{\beta^c_{l^*}, \beta^c_{m^*}\}$ and $j^c_k = \beta^c_k$, for all $k \neq l^*, m^*$.

Hence, $(\mathbf{i}^c,\mathbf{j}^c) \to_B (\boldsymbol{\alpha}^c,\boldsymbol{\beta}^c)$. The converse is obvious. □

REMARK 4.4. $(\mathbf{i},\mathbf{j}) \geq_B (\boldsymbol{\alpha},\boldsymbol{\beta})$ if and only if $(\mathbf{i}^c,\mathbf{j}^c) \geq_B (\boldsymbol{\alpha}^c,\boldsymbol{\beta}^c)$.

This remark follows directly from Definition 4.2 and Lemma 4.3.

Because we can often verify more easily the ordering $\geq_B$ than the ordering $\geq_O$, the following theorem is useful.

THEOREM 4.5. *For* $(\mathbf{i},\mathbf{j})$ *and* $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in (S_N)^2$, $(\mathbf{i},\mathbf{j}) \geq_B (\boldsymbol{\alpha},\boldsymbol{\beta}) \Rightarrow (\mathbf{i},\mathbf{j}) \geq_O (\boldsymbol{\alpha},\boldsymbol{\beta})$.

PROOF. By transitivity of the ordering $\geq_O$ and Lemmas 3.4 and 4.3, it suffices to show $(\mathbf{i},\mathbf{j}) \rightarrow_B (\boldsymbol{\alpha},\boldsymbol{\beta}) \Rightarrow (\mathbf{i},\mathbf{j}) \geq_U (\boldsymbol{\alpha},\boldsymbol{\beta})$. This latter implication is equivalent to showing $(\mathbf{i},\mathbf{j}) \rightarrow_B (\boldsymbol{\alpha},\boldsymbol{\beta}) \Rightarrow \boldsymbol{\gamma}_{\mathbf{i},\mathbf{j}}(r,s,t) \geq \gamma_{\alpha,\beta}(r,s,t)$, for all $1 \leq r,s,t \leq N$, where $\boldsymbol{\gamma}_{\mathbf{i},\mathbf{j}}(\cdot,\cdot,\cdot)$ and $\boldsymbol{\gamma}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\cdot,\cdot,\cdot)$ are given by (2.2). It is direct to observe that $(\mathbf{i},\mathbf{j}) \rightarrow_B (\boldsymbol{\alpha},\boldsymbol{\beta})$ implies that, for all $1 \leq r,s,t \leq N$, either $\boldsymbol{\gamma}_{\mathbf{i},\mathbf{j}}(r,s,t) = \boldsymbol{\gamma}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(r,s,t)$ or $\boldsymbol{\gamma}_{\mathbf{i},\mathbf{j}}(r,s,t) = \boldsymbol{\gamma}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(r,s,t) + 1$. □

The following example shows that the implication of Theorem 4.5 is strict.

EXAMPLE 4.6. Let $\mathbf{i} = (1324)$, $\mathbf{j} = (2143)$, $\boldsymbol{\alpha} = (3142)$ and $\boldsymbol{\beta} = (3412)$. We show $(\mathbf{i},\mathbf{j}) \geq_O (\boldsymbol{\alpha},\boldsymbol{\beta})$ and $(\mathbf{i},\mathbf{j}) \not\geq_B (\boldsymbol{\alpha},\boldsymbol{\beta})$. Because $\mathbf{i}$, $\mathbf{j}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are all self-complementary permutations, to show $(\mathbf{i},\mathbf{j}) \geq_O (\boldsymbol{\alpha},\boldsymbol{\beta})$, by Lemma 3.4 we need only show that $(\mathbf{i},\mathbf{j}) \geq_U (\boldsymbol{\alpha},\boldsymbol{\beta})$. This can be proved using Theorem 3.3, where condition (b) of Theorem 3.3 is satisfied because of the calculations $m(1,2) = 1$ and $\beta^{(2)}_{1,1} = 1 < 3 = j^{(2)}_{1,1}$; $m(1,3) = 1$ and $\beta^{(3)}_{1,1} = 1 < 3 = j^{(3)}_{1,1}$; $m(2,3) = 3$ and $\beta^{(3)}_{1,2} = 1 < 3 = j^{(3)}_{2,3}$, $\beta^{(3)}_{2,2} = 2 < 4 = j^{(3)}_{3,3}$.

To show that $(\mathbf{i},\mathbf{j}) \not\geq_B (\boldsymbol{\alpha},\boldsymbol{\beta})$, begin by letting $\boldsymbol{\Gamma} = \{(\boldsymbol{\gamma},\boldsymbol{\delta})\colon (\boldsymbol{\gamma},\boldsymbol{\delta}) \rightarrow_B (\boldsymbol{\alpha},\boldsymbol{\beta})\}$. It can be shown computationally that $\boldsymbol{\Gamma} = \{((3124),(3412)),((2143),(2413)),((1342),(3412)),((3142),(3214)),((3142),(3142)),((3142),(1432))\}$. From Definition 4.2, observe that $(\mathbf{i},\mathbf{j}) \geq_B (\boldsymbol{\alpha},\boldsymbol{\beta})$, if and only if $(\mathbf{i},\mathbf{j}) \geq_B (\boldsymbol{\gamma},\boldsymbol{\delta})$, for some $(\boldsymbol{\gamma},\boldsymbol{\delta}) \in \boldsymbol{\Gamma}$. Thus, $(\mathbf{i},\mathbf{j}) \not\geq_B (\boldsymbol{\alpha},\boldsymbol{\beta})$, if and only if $(\mathbf{i},\mathbf{j}) \not\geq_B (\boldsymbol{\gamma},\boldsymbol{\delta})$, for all $(\boldsymbol{\gamma},\boldsymbol{\delta}) \in \boldsymbol{\Gamma}$. Hence, by Theorem 4.5, to show that $(\mathbf{i},\mathbf{j}) \not\geq_B (\boldsymbol{\alpha},\boldsymbol{\beta})$, it suffices to show $(\mathbf{i},\mathbf{j}) \not\geq_U (\boldsymbol{\gamma},\boldsymbol{\delta})$, for all $(\boldsymbol{\gamma},\boldsymbol{\delta}) \in \boldsymbol{\Gamma}$. The direct calculations in Table 4.1 show that $(\mathbf{i},\mathbf{j}) \not\geq_U (\boldsymbol{\gamma},\boldsymbol{\delta})$, for all $(\boldsymbol{\gamma},\boldsymbol{\delta}) \in \boldsymbol{\Gamma}$, where (R,S,T) and (U,V,W) are random vectors related to $(\mathbf{i},\mathbf{j})$ and $(\boldsymbol{\gamma},\boldsymbol{\delta})$, respectively.

In the following theorem, we summarize the implications among the four partial orderings $\geq_B$, $\geq_O$, $\geq_L$ and $\geq_U$ on $(S_N)^2$.

THEOREM 4.7. *For* $\geq_B$, $\geq_O$, $\geq_L$ *and* $\geq_U$ *on* $(S_N)^2$, *the following are true*:

(a) *The ordering* $\geq_B$ *implies* $\geq_O$, *which implies both* $\geq_L$ *and* $\geq_U$.

(b) *The ordering* $\geq_U$ *neither implies nor is implied by the ordering* $\geq_L$, *and all the implications in* (a) *are strict.*

TABLE 4.1
Calculations for Example 4.6

$(\boldsymbol{\gamma},\boldsymbol{\delta})$	(a,b,c)	$P(R \geq a, S \geq b, T \geq c)$	$P(U \geq a, V \geq b, W \geq c)$
((3124), (3412))	(1, 3, 2)	1/4	1/2
((2143), (2413))	(3, 3, 1)	1/4	1/2
((1342), (3412))	(2, 3, 4)	0	1/4
((3142), (3214))	(4, 2, 4)	0	1/4
((3142), (3142))	(3, 4, 4)	0	1/4
((3142), (1432))	(2, 1, 2)	1/2	3/4

PROOF. Part (a) follows from Definition 2.2 and Theorem 4.5. Part (b) follows from Examples 3.5 and 4.6 and Lemma 3.4. □

For $\geq_c$ on S_N, (2.1) gives the self-duality results with respect to complementation and inverses. The duality of $\geq_U$ and $\geq_L$, and self-duality of $\geq_O$ and $\geq_B$, with respect to complementation follow from Lemma 3.4 and Remark 4.4. The following example shows that there are no duality results with respect to inverses among the four orderings $\geq_U$, $\geq_L$, $\geq_O$ and $\geq_B$.

EXAMPLE 4.8. Let $\mathbf{i} = (4231)$, $\mathbf{j} = (2341)$, $\boldsymbol{\alpha} = (4321)$ and $\boldsymbol{\beta} = (2431)$, so that $\mathbf{i}^* = (4231)$, $\mathbf{j}^* = (4123)$, $\boldsymbol{\alpha}^* = (4321)$ and $\boldsymbol{\beta}^* = (4132)$. To show that $(\mathbf{i},\mathbf{j}) \geq_X (\boldsymbol{\alpha},\boldsymbol{\beta})$ and $(\mathbf{i}^*,\mathbf{j}^*) \not\geq_X (\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)$, where $\geq_X$ denotes one of the orderings $\geq_U$, $\geq_L$, $\geq_O$ or $\geq_B$, it suffices to show by Theorem 4.7 that $(\mathbf{i},\mathbf{j}) \geq_B (\boldsymbol{\alpha},\boldsymbol{\beta})$, $(\mathbf{i}^*,\mathbf{j}^*) \not\geq_U (\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)$ and $(\mathbf{i}^*,\mathbf{j}^*) \not\geq_L (\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)$. It is obvious that $(\mathbf{i},\mathbf{j}) \geq_B (\boldsymbol{\alpha},\boldsymbol{\beta})$. To show that $(\mathbf{i}^*,\mathbf{j}^*) \not\geq_U (\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)$ and $(\mathbf{i}^*,\mathbf{j}^*) \not\geq_L (\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)$, let (R,S,T) and (U,V,W) be two random vectors related to $(\mathbf{i}^*,\mathbf{j}^*)$ and $(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)$, respectively. Then note that $P(R \geq 3,\ S \geq 2,\ T \geq 3) = 0 < 1/4 = P(U \geq 3,\ V \geq 2,\ W \geq 3)$ and $P(R \leq 4,\ S \leq 1,\ T \leq 2) = 0 < 1/4 = P(U \leq 4,\ V \leq 1,\ W \leq 2)$.

5. Discussion. We note that the ordering $\geq_B$ on $(S_N)^2$ can be directly extended to $(S_N)^m$, $m \geq 2$. In fact, all of the results of Sections 3 and 4 (except Theorem 3.3) can also be extended in the obvious fashion to deal with partial orderings on $(S_N)^m$.

The ordering $\geq_c$ on S_N can also be extended to $\mathbf{R}^N$ and $(\mathbf{R}^N)^m$. Metry and Sampson (1993) considered a simple extension of the ordering $\geq_c$ from S_N to $\mathbf{R}^N$. For $\mathbf{x},\mathbf{y} \in \mathbf{R}^N$, $\mathbf{x}$ is said to be more concordant than $\mathbf{y}$ (($\mathbf{x} \geq_c \mathbf{y}$) if $(R,S) \geq_c (U,V)$, where (R,S) and (U,V) are, respectively, bivariate random variables related to $\mathbf{x}$ and $\mathbf{y}$. They show that for $\mathbf{x},\mathbf{y} \in \mathbf{R}^N$, $\mathbf{x} \geq_c \mathbf{y}$ if and only if one of the following two equivalent conditions is satisfied:

1. $\mathbf{y}$ is a permutation of $\mathbf{x}$ and $x_{l,k} \geq y_{l,k}$, for all $1 \leq l \leq k \leq N-1$.
2. $\mathbf{x} = \mathbf{y}$ or $\mathbf{x}$ can be obtained from $\mathbf{y}$ in a number of steps each of which consists of correcting an inversion.

The ordering $\geq_c$ on $\mathbf{R}^N$, used by Marshall and Olkin [(1979), page 160] to define arrangement increasing functions, can be directly extended to an ordering on $(\mathbf{R}^N)^m$. For $(\mathbf{x}_1,\dots,\mathbf{x}_m),(\mathbf{y}_1,\dots,\mathbf{y}_m) \in (\mathbf{R}^N)^m$, define $(\mathbf{x}_1,\dots,\mathbf{x}_m) \geq_c (\mathbf{y}_1,\dots,\mathbf{y}_m)$ if $\mathbf{x}_l \geq_c \mathbf{y}_l$, for all $l = 1,\dots,m$. This ordering is used implicitly by Boland and Proschan (1988) to define the multivariate arrangement increasing functions. We point out that the ordering $\geq_B$ can also be directly extended to $(\mathbf{R}^N)^m$ and is stronger than the preceding ordering $\geq_c$ on $(\mathbf{R}^N)^m$.

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DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF NORTHERN COLORADO
GREELEY, COLORADO 80639

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF PITTSBURGH
PITTSBURGH, PENNSYLVANIA 15260