The Annals of Applied Probability
1993, Vol. 3, No. 4, 1189–1218

PRODUCTS OF 2 × 2 RANDOM MATRICES

BY DAVID MANNION

Royal Holloway College, University of London

The notion of the *shape* of a triangle can be used to define the shape of a 2×2 real matrix; we find that the shape of a matrix retains just the right amount of information required for determining the main features of the asymptotic behaviour, as $n \to \infty$, of $\mathbf{G}_n\mathbf{G}_{n-1} \cdots \mathbf{G}_1$, where the $\mathbf{G}_i$ are i.i.d. copies of a 2×2 random matrix $\mathbf{G}$. An alternative formula to the Furstenberg formula is proposed for the upper Lyapounov exponent of the probability distribution of $\mathbf{G}$. We find that in some cases, using our formula, the Lyapounov exponent is more susceptible to explicit calculation.

1. Introduction. Let $\mathscr{G} \equiv \mathrm{Gl}(2, \boldsymbol{R})$ be the set of 2×2 real, invertible matrices, let μ be the probability measure of a random matrix $\mathbf{G} \in \mathscr{G}$ and let

$$\mathbf{H}_n \equiv \mathbf{G}_n\mathbf{G}_{n-1} \cdots \mathbf{G}_1,$$

where the $\mathbf{G}_i$ are i.i.d. copies of $\mathbf{G}$.

Let $\wp$ be the set of *directions* in $\boldsymbol{R}^2$, where $\mathbf{z}_1$ and $\mathbf{z}_2 \in \boldsymbol{R}^2$ are said to have the same direction if there exists real λ such that $\mathbf{z}_1 = \lambda \mathbf{z}_2$. We say that the probability measure ν on $\wp$ is *μ-invariant* if for every bounded Borel function f on $\wp$

$$\int f(u)\,\nu(du) = \iint f(\overline{\mathbf{gz}})\,\mu(d\mathbf{g})\,\nu(du),$$

where $\mathbf{z} \in \boldsymbol{R}^2$ is a vector with direction u and $\overline{\mathbf{gz}}$ is the direction of $\mathbf{gz}$.

Furstenberg's theorem states that if $\det(\mathbf{G}) = \pm 1$ a.s. and conditions (2.3) and (2.4) of Section 2 hold, then there exists $\gamma > 0$ such that for each $\mathbf{z} \neq \mathbf{0}$, $\mathbf{z} \in \boldsymbol{R}^2$,

$$\frac{1}{n}\log(|\mathbf{H}_n\mathbf{z}|) \to \gamma \quad \text{a.s. as } n \to \infty. \tag{1.1}$$

Moreover there is a unique μ-invariant probability measure ν on $\wp$, and ν is continuous. γ is given by the formula

$$\gamma = \iint \log\left(\frac{|\mathbf{gz}|}{|\mathbf{z}|}\right)\mu(d\mathbf{g})\,\nu(du), \tag{1.2}$$

where $\mathbf{z}$ is a vector with direction u. γ is called the *upper Lyapounov exponent* of μ.

Received April 1992; revised March 1993.

AMS 1991 *subject classifications*. 60D05, 60J15

Key words and phrases. Contracting subsets of $\mathrm{Gl}(n, \boldsymbol{R})$, Lyapounov exponent, products of random matrices, shape.

Note that what we have called *Furstenberg's theorem* is really an amalgamation of several results that appear in Furstenberg [5] (Theorems 8.5 and 8.6). See also Bougerol and Lacroix [2] (Chapter 2, Theorems 4.1 and 4.4, and Chapter 3, Theorem 6.3).

Lyapounov exponents are difficult to calculate: Many of the examples that appear in the literature have, in some sense, been designed for simple calculation (see [2]). There are also some calculations of Lyapounov exponents to be found in Key [12], though he makes no explicit use of (1.2). In the calculation of a Lyapounov exponent, finding the μ-invariant probability measure ν is the difficult part. We shall derive an alternative formula for γ [formulae (1.4) and (1.5)] that, like (1.2), is also the integral of a function with respect to an invariant probability measure ψ. In some cases ψ is easier to find than ν.

Let a, b, c, d be real-valued random variables such that $ab \neq bc$ with probability 1. Let $\mathbf{G}$ be the 2×2 real matrix:

$$\mathbf{G} \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then $\mathbf{G} \in \mathscr{G}$ with probability 1. Let t be a constant and let t_1 be the random variable

$$t_1 \equiv \frac{at + b}{ct + d}$$

and let $\boldsymbol{Q}$ be the probability distribution of t_1:

$$\boldsymbol{Q}(t, B) \equiv \Pr[t_1 \in B],$$

where B is a Borel subset of $\boldsymbol{R}$. If condition (2.5) holds, there exists a unique probability distribution ψ on $\boldsymbol{R}$ such that

$$\psi(B) = \int \psi(dt) \boldsymbol{Q}(t, B). \tag{1.3}$$

For real numbers $u_1, u_2, \dots, u_p$, we define their *range*:

$$\operatorname{range}(u_1, u_2, \dots, u_p) \equiv \max(u_1, u_2, \dots, u_p) - \min(u_1, u_2, \dots, u_p).$$

For constant t, let

$$h(t) \equiv \log(\operatorname{range}(0, t, 1)),$$
$$H(t) \equiv \boldsymbol{E}[\log(\operatorname{range}(0, at + b, ct + d))].$$

We shall derive the following alternative formulae for γ:

$$\gamma = \int H(t) \psi(dt) - \int h(t) \psi(dt) \tag{1.4}$$

$$= \int \boldsymbol{E}[\log(|ct + d|)] \psi(dt). \tag{1.5}$$

Important in our derivation of (1.4) and (1.5) is the notion of the *shape* of a triangle where, according to the definition of D. G. Kendall, [8], [9], [11]:

> The *shape* of a geometrical object is what is left after attributes of its size, orientation and location have been filtered out.

To this definition we add the extra requirement that an object and its *reflection* in a hyperplane shall have the same shape.

The notion of the shape of a triangle (we shall give its precise definition in Section 2) can be used to define the shape of a 2×2 real matrix. Let

$$\mathbf{Z} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}.$$

The shape of $\mathbf{Z}$ is defined to be the shape of the triangle Γ whose vertices are the origin and the points with coordinates (x_1, y_1) and (x_2, y_2). The shape of Γ is determined by a pair of real coordinates (s, w). Thus, because for our purposes it is the shape of Z that matters, the *four* dimensions (x_1, y_1, x_2, y_2) of $\mathbf{Z}$ can be reduced to the *two* dimensions of its shape (s, w). Notice that only *invertible* matrices give triangles whose vertices are not collinear.

We know from Theorem 4.6 of [7] that if (2.5) holds, then the directions of the vectors formed by the rows of $\mathbf{H}_n$ converge in probability as $n \to \infty$ to a common limit. See also [2] (Chapter 3, Theorem 4.3). In shape terms, this translates as follows. Let (s_n, w_n) be the shape of $\mathbf{Z}_n \equiv \mathbf{H}_n \mathbf{Z}_0$, where $\mathbf{Z}_0$ is an arbitrary real, invertible, non-random matrix. Then there exists a random variable τ^*, $0 \le \tau^* \le 1$, such that

$$w_n \to 0 \text{ a.s.} \quad \text{and} \quad s_n \to \tau^* \text{ in probability as } n \to \infty.$$

Thus, the *four* dimensions of $\mathbf{Z}_n$ are replaced by the *two* dimensions of (s_n, w_n); in the long run, these are replaced by the *one* dimension of τ^*.

There are difficulties in a direct study of the shape process (s_n, w_n) since this is not a Markov process, (in Section 9.2 we define a shape process that is Markov and has the same asymptotic behaviour as (s_n, w_n)). Our preferred approach is by way of a sequence $\tilde{\mathbf{Z}}_n$, normalised versions of the $\mathbf{Z}_n$, which is Markov and which is very close in information content to the sequence (s_n, w_n).

We also discover that γ measures the rate at which $w_n \to 0$:

$$\frac{1}{n}\log(w_n) \to -2\gamma \quad \text{a.s. as } n \to \infty$$

and we shall see in Section 9 that there is a simple relation between the invariant probability measure ψ of (1.3) and the probability distribution of τ^*.

In Sections 6–8 we demonstrate our approach in three examples, for each of which we calculate both ψ and γ. In Section 6, the two rows of $\mathbf{G}$ are i.i.d. from a bivariate normal distribution, $N(\mathbf{0}, \Sigma)$. We discover that γ is a simple function of $\mathrm{tr}(\Sigma)$ and $\det(\Sigma)$. Cohen and Newman [4] have examined the case where the entries of $\mathbf{G}$, a matrix of arbitrary dimension, are i.i.d. from a standard symmetric stable distribution. In Section 7, $\mathbf{G}$ has a form that arises in the theory of Brocot sequences:

$$\mathbf{G} = \begin{bmatrix} 0 & 1 \\ 1 & X \end{bmatrix},$$

where X is a nonnegative random variable with a continuous distribution. Such $\mathbf{G}$ have been studied by Chassaing, Letac and Mora [3], and we extend some of their results. In Section 8, we describe a *triangles within triangles* problem, a detailed discussion of which appears in Mannion [13]–[15]. Of the three examples, the calculations involved here proved to be the most difficult.

We are currently looking at the possibility of using the shape approach to extend our results to higher dimensions. Formally, the setup is entirely similar to the two-dimensional case.

Let $\mathbf{G}$ be a random $p \times p$ real invertible matrix, let μ be the probability measure of $\mathbf{G}$ and let $\mathbf{G}_1, \mathbf{G}_2, \dots, \mathbf{G}_n, \dots$ be i.i.d. copies of $\mathbf{G}$. For real constants t_j, let $\mathbf{t}_0 \equiv (t_1, t_2, \dots, t_{p-1})$. Let $\mathbf{G}_1 \equiv [g_{ij}]$ and let

$$u_i \equiv g_{i1}t_1 + g_{i2}t_2 + \cdots + g_{ip-1}t_{p-1} + g_{ip}, \qquad i = 1, 2, \dots, p.$$

Define

$$v_i \equiv u_i / u_p, \qquad i = 1, 2, \dots, p-1,$$

and let $\mathbf{t}_1 \equiv (v_1, v_2, \dots, v_{p-1})$. We iterate this prescription in the obvious way to generate the sequence $\mathbf{t}_0, \mathbf{t}_1, \dots, \mathbf{t}_n, \dots$. Thus, to generate $\mathbf{t}_n$, use the preceding prescription replacing $\mathbf{t}_0$ by $\mathbf{t}_{n-1}$ and $\mathbf{G}_1$ by $\mathbf{G}_n$, $n = 2, 3, \dots$.

If μ is contracting, then there exists a $(p-1)$-dimensional random variable τ such that $\mathbf{t}_n \to \tau$ in probability as $n \to \infty$. Let ψ be the probability measure of τ and for a constant $\mathbf{t}$, let

$$h(\mathbf{t}) \equiv \log\big(\mathrm{range}(0, t_1, t_2, \dots, t_{p-1}, 1)\big),$$
$$H(\mathbf{t}) \equiv \boldsymbol{E}\big[\log\big(\mathrm{range}(0, u_1, u_2, \dots, u_p)\big)\big].$$

We hope to prove the higher dimensional analogue of (1.4):

$$\gamma = \int H(\mathbf{t})\psi(d\mathbf{t}) - \int h(\mathbf{t})\psi(d\mathbf{t})$$
$$= \int \boldsymbol{E}\big[\log(|u_p|)\big]\,\psi(d\mathbf{t}).$$

It should be noted that in two dimensions it is a weak requirement that μ should be contracting; in higher dimensions it is a stronger requirement.

2. Triangle shapes and 2 × 2 matrices. Points in the Euclidean plane $\boldsymbol{R}^2$ are identified by Cartesian coordinates (x, y), relative to a pair of (x, y) axes. To define the shape of a triangle Δ whose vertices are noncollinear we need first of all to label its vertices. If there is a unique longest side of Δ, we label its end points A_0 and A_2 and label the third vertex of the triangle A_1

where the labelling is such that A_1 is to the *left* of the directed line from A_0 to A_2. If there is more than one "longest" side, we choose one of them, label its end points as A_0 and A_2 and proceed as before. Whatever choice is made will lead to the same *shape*.

Let the coordinates of M, the midpoint of A_0A_2, be (α, β) and let θ be the angle the directed line A_0A_2 makes with the x-axis. Thus $\overline{A_0A_2} = |A_0A_2|(\cos(\theta), \sin(\theta))'$. New coordinate axes are set up by first translating the (x, y) axes so that the origin moves to M and then rotating the axes so obtained about M, through the angle θ in the positive sense, to get the final pair of axes—the (x', y') axes, say. By this construction the x'-axis lies along A_0A_2. A point with coordinates (a, b) relative to the original axes now has coordinates (a', b') relative to the (x', y') axes, where

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{pmatrix} a - \alpha \\ b - \beta \end{pmatrix}.$$

Let (α_1, β_1) be the coordinates of A_1 relative to the (x, y) axes and let (S, W) be the coordinates of A_1 relative to the (x', y') axes (see Figure 1). Then

$$\begin{pmatrix} S \\ W \end{pmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{pmatrix} \alpha_1 - \alpha \\ \beta_1 - \beta \end{pmatrix}.$$

Let L be the length of A_0A_2. We define the shape $\sigma(\Delta)$ of Δ by

$$\sigma(\Delta) \equiv (s, w),$$

where

$$s \equiv 2|S|/L, \qquad w \equiv 2W/L.$$

s and w are called the *shape coordinates* of Δ.

This definition differs from that of Kendall [9]. He has $s \equiv 2S/L$, so that according to his definition, a triangle and its reflection in a line have different shapes. The strict Kendall definition is not best suited to this study: it retains

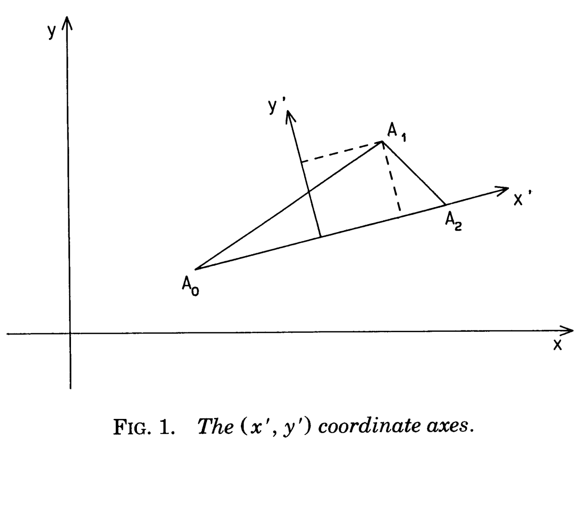

FIG. 1. *The (x', y') coordinate axes.*

information about the shape of a triangle that is not needed for tracking the evolution of the *size* of a sequence of products of i.i.d. random matrices. It would also make our argument very much more complicated. Under our definition, there is a small additional bonus in that there is no uncertainty in the assignment of a shape to an isosceles triangle for which the two equal sides are also the longest.

We see that (s, w) satisfies the inequalities

$$s \geq 0, \qquad w \geq 0, \qquad (1+s)^2 + w^2 \leq 4.$$

Accordingly we define S, the *shape space* of triangles, as (see Figure 2)

$$S \equiv \{(s, w): s \geq 0; w \geq 0; (1+s)^2 + w^2 \leq 4\}. \tag{2.1}$$

Let $\mathbf{Z}$ be an arbitrary 2×2 real, invertible matrix

$$\mathbf{Z} \equiv \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

and let $\Gamma = \Gamma(\mathbf{Z})$ be the triangle with vertices $(0, 0)$, (x_1, y_1), (x_2, y_2). We will say that Γ *is the triangle of* $\mathbf{Z}$ and we define the shape of $\mathbf{Z}$ by

$$\text{shape}(\mathbf{Z}) \equiv \sigma(\Gamma).$$

An observation that will be useful in the proof of Theorem 2.1 is the following: If L is the length of the longest side of Γ and $\sigma(\Gamma) = (s, w)$, then

$$|\det(\mathbf{Z})| = 2 \times \text{area of } \Gamma(\mathbf{Z}) = LW = \tfrac{1}{2}L^2 w. \tag{2.2}$$

We define $\mathscr{G}$ to be the set of 2×2 real invertible matrices. Let $\mathbf{G} \in \mathscr{G}$ be a random matrix and let μ be the probability measure of $\mathbf{G}$. Let $\mathbf{H}_n$ be the product of i.i.d. copies of $\mathbf{G}$:

$$\mathbf{H}_n \equiv \mathbf{G}_n \mathbf{G}_{n-1} \cdots \mathbf{G}_1.$$

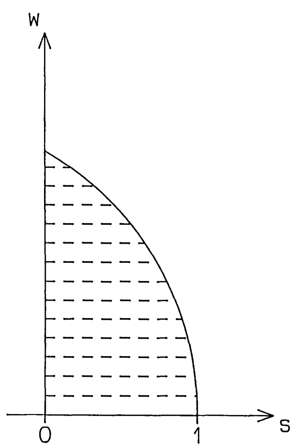

FIG. 2. *The shape space.*

We define $\mathbf{Z}_n \equiv \mathbf{H}_n \mathbf{Z}_0$, where $\mathbf{Z}_0$ is an arbitrary (nonrandom) matrix in $\mathscr{G}$, and let Γ_n denote the triangle of $\mathbf{Z}_n$. We let σ_n denote the shape and (s_n, w_n) the shape coordinates of Γ_n:

$$\sigma_n \equiv \sigma(\Gamma_n) = (s_n, w_n).$$

We define $\mathscr{G}_\mu$ to be the smallest closed subgroup of $\mathscr{G}$ that contains the support of μ and we define $\tilde{\mathscr{G}}_\mu$ to be the smallest closed semigroup of $\mathscr{G}$ that contains the support of μ.

A subset $\mathscr{S}$ of $\mathscr{G}$ is defined as *strongly irreducible* if, for any $\mathbf{g} \in \mathscr{S}$, there do *not* exist proper linear subspaces of $\boldsymbol{R}^2$, $V_1, V_2, \dots, V_k$ such that

$$\mathbf{g}\Big(\bigcup_{1 \le j \le k} V_j\Big) = \bigcup_{1 \le j \le k} V_j.$$

A subset $\mathscr{S}$ of $\mathscr{G}$ is said to be *contracting* if there exists a sequence $\{\mathbf{M}_n, n = 1, 2, \dots\}$ in $\mathscr{S}$ such that $\mathbf{M}_n / \|\mathbf{M}_n\|$ converges to a matrix of rank 1, where $\|\mathbf{g}\|$ is the usual norm of $\mathbf{g}$:

$$\|\mathbf{g}\| \equiv \sup_{|\mathbf{z}| \le 1} |\mathbf{g}\mathbf{z}|, \qquad \mathbf{z} = (x, y)' \in \boldsymbol{R}^2$$

and $|\mathbf{z}| \equiv \sqrt{(x^2 + y^2)}$.

We state four conditions that we may require of a particular μ:

$$\int |\log(\|\mathbf{g}\|)|\, \mu(d\mathbf{g}) < \infty; \tag{2.3}$$

(2.4) $\mathscr{G}_\mu$ is strongly irreducible and is not a compact subset of $\mathscr{G}$;

(2.5) $\tilde{\mathscr{G}}_\mu$ is strongly irreducible and contracting;

(2.6) for some $\alpha > 0$, $\int \|\mathbf{g}\|^\alpha \mu(d\mathbf{g}) < \infty$ and $\int \|\mathbf{g}^{-1}\|^\alpha \mu(d\mathbf{g}) < \infty$.

The *upper Lyapounov exponent* of μ is γ defined by

$$\gamma \equiv \lim_{n \to \infty} \frac{1}{n} \boldsymbol{E}\big[\log(\|\mathbf{H}_n\|)\big]. \tag{2.7}$$

The sequence $\{\boldsymbol{E}[\log(\|\mathbf{H}_n\|)]; n = 1, 2, \dots\}$ is subadditive. Thus, if condition (2.3) holds, the limit (2.7) exists. Furstenberg's theorem states that if det $(\mathbf{G}) = \pm 1$ a.s. and conditions (2.3) and (2.4) hold, then there exists $\gamma > 0$ such that for each $z \neq 0$,

$$\frac{1}{n} \log(|\mathbf{H}_n z|) \to \gamma \quad \text{a.s. as } n \to \infty. \tag{2.8}$$

The condition $\det(\mathbf{G}) = \pm 1$ is not very restrictive. If $\mathbf{G}$ is invertible, we may consider $\breve{\mathbf{G}} \equiv \mathbf{G}' / \sqrt{|\det(\mathbf{G})|}$, where $\mathbf{G}'$ is the transpose of $\mathbf{G}$. Let $\breve{\mu}$ denote the probability measure of $\breve{\mathbf{G}}$. Note that for Furstenberg's theorem, it is sufficient that in (2.3) the range of the integral is restricted to $\{\mathbf{g}: \|\mathbf{g}\| > 1\}$.

THEOREM 2.1. *If $\check{\mu}$ satisfies* (2.3) *and* (2.4) *and if $\check{\gamma}$ is the upper Lyapounov exponent of $\check{\mu}$, then $\check{\gamma} > 0$ and, provided $w_0 > 0$,*

$$\frac{1}{n}\log(w_n) \to -2\check{\gamma} \quad a.s. \text{ as } n \to \infty.$$

PROOF. Let L_n be the length of the longest side of Γ_n, the triangle of $\mathbf{Z}_n = \mathbf{H}_n\mathbf{Z}_0$. Let $\mathbf{v}_1 \equiv (1,0)'$ and $\mathbf{v}_2 \equiv (0,1)'$. Then the lengths of the sides of Γ_n are

$$|\mathbf{Z}_0'\mathbf{H}_n'\mathbf{v}_1|, \qquad |\mathbf{Z}_0'\mathbf{H}_n'\mathbf{v}_2|, \qquad |\mathbf{Z}_0'\mathbf{H}_n'(\mathbf{v}_2 - \mathbf{v}_1)|$$

and

$$L_n = \max\{|\mathbf{Z}_0'\mathbf{H}_n'\mathbf{v}_1|, |\mathbf{Z}_0'\mathbf{H}_n'\mathbf{v}_2|, |\mathbf{Z}_0'\mathbf{H}_n'(\mathbf{v}_2 - \mathbf{v}_1)|\}.$$

Let

$$K_n \equiv L_n/\sqrt{|\det(\mathbf{H}_n)|}, \qquad \mathbf{J}_n \equiv \mathbf{H}_n'/\sqrt{|\det(\mathbf{H}_n)|}.$$

Then

$$K_n = \max\{|\mathbf{Z}_0'\mathbf{J}_n\mathbf{v}_1|, |\mathbf{Z}_0'\mathbf{J}_n\mathbf{v}_2|, |\mathbf{Z}_0'\mathbf{J}_n(\mathbf{v}_2 - \mathbf{v}_1)|\}.$$

Recalling (2.2), we see that

$$w_n = 2(L_n)^{-2}|\det(\mathbf{Z}_n)|$$

and so

$$w_n = 2(K_n)^{-2}|\det(\mathbf{Z}_0)|.$$

With $\mathbf{v}$ any one of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2$, let $\mathbf{u}_n \equiv \mathbf{J}_n\mathbf{v}/|\mathbf{J}_n\mathbf{v}|$. Then

$$\log(|\mathbf{Z}_0'\mathbf{J}_n\mathbf{v}|) = \log(|\mathbf{Z}_0'\mathbf{u}_n|) + \log(|\mathbf{J}_n\mathbf{v}|).$$

By Furstenberg's theorem and, in particular, (2.8),

$$\frac{1}{n}\log(|\mathbf{J}_n\mathbf{v}|) \to \check{\gamma} \quad \text{a.s. as } n \to \infty.$$

Because $w_0 > 0$, it follows that $\det(\mathbf{Z}_0) \neq 0$ and $\inf_{|\mathbf{u}|=1}|\mathbf{Z}_0'\mathbf{u}| > 0$. Thus

$$\frac{1}{n}\log(|\mathbf{Z}_0'\mathbf{u}_n|) \to 0 \quad \text{a.s. as } n \to \infty$$

and hence

$$\frac{1}{n}\log(|\mathbf{Z}_0'\mathbf{J}_n\mathbf{v}|) \to \check{\gamma} \quad \text{a.s. as } n \to \infty.$$

Thus

$$\frac{1}{n}\log(K_n) \to \check{\gamma} \quad \text{a.s. as } n \to \infty$$

and

$$\frac{1}{n}\log(w_n) \to -2\check{\gamma} \quad \text{a.s. as } n \to \infty,$$

as required. □

COROLLARY 2.1.

$$w_n \to 0 \quad a.s. \text{ as } n \to \infty.$$

PROOF. Because $\check{\gamma} > 0$, the corollary is an immediate consequence of Theorem 2.1. □

COROLLARY 2.2. *If condition* (2.3) *holds and* $\boldsymbol{E}[|\log(|\det(\mathbf{G})|)|]$ *is finite, then* γ *and* $\check{\gamma}$ *are related by*

$$\check{\gamma} = \gamma - \tfrac{1}{2}\boldsymbol{E}[\log(|\det(\mathbf{G})|)]. \tag{2.9}$$

PROOF.

$$\begin{aligned}\frac{1}{n}\log(\|\mathbf{J}_n\|) &= \frac{1}{n}\log(\|\mathbf{H}_n\|) - \frac{1}{2n}\log(|\det(\mathbf{H}_n)|)\\ &= \frac{1}{n}\log(\|\mathbf{H}_n\|) - \frac{1}{2n}\log(|\det(\mathbf{G}_n\mathbf{G}_{n-1}\cdots\mathbf{G}_1)|)\\ &= \frac{1}{n}\log(\|\mathbf{H}_n\|) - \frac{1}{2n}\sum_{i=1}^{n}\log(|\det(\mathbf{G}_i)|).\end{aligned}$$

Thus

$$\frac{1}{n}\boldsymbol{E}[\log(\|\mathbf{J}_n\|)] = \frac{1}{n}\boldsymbol{E}[\log(\|\mathbf{H}_n\|)] - \frac{1}{2}\boldsymbol{E}[\log(|\det(\mathbf{G})|)]. \tag{2.10}$$

(2.9) follows directly from (2.7) and (2.10). □

3. The sequence of normalized matrices $\tilde{\mathbf{Z}}_n$. Recall that $\mathbf{Z}_n \equiv \mathbf{G}_n\mathbf{G}_{n-1}\cdots\mathbf{G}_1\mathbf{Z}_0$, where Z_0 is nonrandom. Because the random matrices $\mathbf{G}_n$ are mutually independent, $\{\mathbf{Z}_0, \mathbf{Z}_1, \dots, \mathbf{Z}_n, \dots\}$ is a Markov process with starting point $\mathbf{Z}_0$. We shall see that all that is required for the calculation of the upper Lyapounov exponent of μ is a knowledge of the asymptotic behaviour of the shape process $\sigma_n = \text{shape}(\mathbf{Z}_n)$. Without some appropriate additional construction, it is not the case that the sequence of shapes σ_n themselves form a Markov process. We have seen how to *assign a shape to a matrix*. To construct a Markov process of shapes we need to be able to *assign a matrix to a shape*. This is not such an entirely straightforward thing to do because, for a given shape σ, there are many choices of $\mathbf{Z}$ such that $\text{shape}(\mathbf{Z}) = \sigma$.

In Section 9 we show that it is possible to construct a shape process $\{\tilde{\sigma}_n, n = 0, 1, \dots\}$ that has the Markov property and is such that the asymptotic behaviour of $\mathscr{L}(\tilde{\sigma}_n)$ is the same as that of $\mathscr{L}(\text{shape}(Z_n))$, $n \to \infty$. Under the conditions of Theorem 9.2, much more is true. Namely,

$$\mathscr{L}(\tilde{\sigma}_0, \tilde{\sigma}_1, \dots, \tilde{\sigma}_n) = \mathscr{L}(\text{shape}(\mathbf{Z}_0), \text{shape}(\mathbf{Z}_1), \dots, \text{shape}(\mathbf{Z}_n)),$$

$n = 1, 2, \dots,$ where $\mathscr{L}$ stands for "the probability measure of."

However, we can make progress without resorting to this construction. We define a sequence of random matrices $\tilde{\mathbf{Z}}_n$, where each $\tilde{\mathbf{Z}}_n$ is a simple transformation of $\mathbf{Z}_n$, where the sequence $\{\tilde{\mathbf{Z}}_n,\ n = 0, 1, \dots\}$ is a Markov process and where the $\tilde{\mathbf{Z}}_n$ retain just the right amount of information we require: $\tilde{\mathbf{Z}}_n$ contains less information than $\mathbf{Z}_n$, and only a little more than σ_n.

Let $\mathbf{M}$ be the 2×2 real invertible matrix

$$\mathbf{M} \equiv \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

and let $x_2 = r\cos(\theta)$ and $y_2 = r\sin(\theta)$. Because $\mathbf{M}$ is invertible, $r > 0$.

We define the orthogonal matrix $\mathbf{U}$, a rotation about the origin through an angle θ in the clockwise sense:

$$\mathbf{U} \equiv \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

We denote by $\tilde{\mathbf{M}}$ the *normalized* version of $\mathbf{M}$ given by

$$\vartheta : \mathbf{M} \to \tilde{\mathbf{M}} \equiv r^{-1}\mathbf{M}\mathbf{U}'.$$

We define x and y by

$$x \equiv (x_1 x_2 + y_1 y_2) r^{-2}, \qquad y \equiv (-x_1 y_2 + y_1 x_2) r^{-2}. \tag{3.1}$$

Then

$$\tilde{\mathbf{M}} = \begin{bmatrix} x & y \\ 1 & 0 \end{bmatrix}.$$

Note that the transformation $\mathbf{M} \to \tilde{\mathbf{M}}$ involves only a rotation and a scaling, so

$$\operatorname{shape}(\tilde{\mathbf{M}}) = \operatorname{shape}(\mathbf{M}).$$

LEMMA 3.1. *For a given invertible matrix* $\mathbf{M}$, *if, for* $r_i > 0$ *and orthogonal matrices,* $\mathbf{U}_i$,

$$r_i^{-1}\mathbf{M}\mathbf{U}_i' = \begin{bmatrix} x_i & y_i \\ 1 & 0 \end{bmatrix},\ i = 1,2,$$

then $r_1 = r_2$ *and* $\mathbf{U}_1 = \mathbf{U}_2$.

PROOF. The lemma follows easily from the orthogonality of the $\mathbf{U}_i$ and

$$\mathbf{M} = r_1\begin{bmatrix} x_1 & y_1 \\ 1 & 0 \end{bmatrix}\mathbf{U}_1 = r_2\begin{bmatrix} x_2 & y_2 \\ 1 & 0 \end{bmatrix}\mathbf{U}_2. \qquad \square$$

COROLLARY 3.1.

$$\tilde{\mathbf{Z}}_n = \vartheta\left(\mathbf{G}_n \tilde{\mathbf{Z}}_{n-1}\right),\ n = 1, 2, \dots. \tag{3.2}$$

PROOF. Let

$$\tilde{\mathbf{Z}}_n = r_n^{-1}\mathbf{Z}_n\mathbf{U}_n = r_n^{-1}\mathbf{G}_n\mathbf{Z}_{n-1}\mathbf{U}_n; \quad \tilde{\mathbf{Z}}_{n-1} = r_{n-1}^{-1}\mathbf{Z}_{n-1}\mathbf{U}_{n-1}.$$

Then (3.2) follows from Lemma 3.1 and the observation

$$\tilde{\mathbf{Z}}_n = r_{n-1}r_n^{-1}\mathbf{G}_n\tilde{\mathbf{Z}}_{n-1}\mathbf{U}_{n-1}\mathbf{U}_n'. \qquad \square$$

[A relation similar to (3.2) appears in Goodall [6] in his cut/grow construction.]

We define x_n and y_n by

$$\tilde{\mathbf{Z}}_n \equiv \begin{bmatrix} x_n & y_n \\ 1 & 0 \end{bmatrix}. \tag{3.3}$$

THEOREM 3.1. *The sequence of* 2×2 *random matrices* $\{\tilde{\mathbf{Z}}_n,\ n = 0, 1, \dots\}$ *is a Markov process.*

PROOF. It is enough to note from (3.2) that $\tilde{\mathbf{Z}}_n = \vartheta(\mathbf{G}_n\mathbf{Z}_{n-1}) = \vartheta(\mathbf{G}_n\tilde{\mathbf{Z}}_{n-1})$, $n = 1, 2, \dots$. $\square$

THEOREM 3.2. *If* (2.5) *holds, then there exists a random variable* τ *such that*

$$(x_n, y_n) \to (\tau, 0) \quad \textit{in probability as } n \to \infty.$$

If, in addition, (2.6) *holds, then*

$$(x_n, y_n) \to (\tau, 0) \quad \textit{a.s. as } n \to \infty.$$

PROOF. Let

$$\mathbf{G}_n\mathbf{G}_{n-1} \cdots \mathbf{G}_1 \equiv \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}.$$

From (3.1), we see that x_n and y_n are given by the expressions

$$x_n = \{(A_n x_0 + B_n)(C_n x_0 + D_n) + A_n C_n y_0^2\}R_n^{-2},$$
$$y_n = \{(A_n D_n - B_n C_n) y_0\}R_n^{-2},$$

where $R_n^2 \equiv (C_n x_0 + D_n)^2 + (C_n y_0)^2$.

We define

$$t_{n1} \equiv A_n/C_n, \quad t_{n2} \equiv B_n/D_n, \qquad t_{n3} \equiv B_n/A_n, \qquad t_{n4} \equiv D_n/C_n,$$
$$R_{n0}^2 \equiv (x_0 + t_{n4})^2 + (y_0)^2.$$

Then

$$x_n = t_{n1}\{(x_0 + t_{n3})(x_0 + t_{n4}) + y_0^2\}R_{n0}^{-2}, \tag{3.4}$$

$$y_n = (t_{n1} - t_{n2})t_{n4} y_0 R_{n0}^{-2}. \tag{3.5}$$

Let μ' be the probability measure of $\mathbf{G}'$ and let (3.6) be the condition

(3.6) $$\tilde{\mathscr{G}}_{\mu'} \text{ is strongly irreducible and contracting.}$$

It is easy to show that if (2.5) holds, then so does (3.6).

From [7] (Theorem 4.6) and [2] (Chapter 3, Theorem 4.3) we know, using y/x for the *direction* of a vector $(x, y)'$, that if condition (2.5) holds, there exists a random variable $\tilde{\tau}$ with a continuous distribution (and hence $\tilde{\tau}$ is finite, with probability 1) such that

(3.7) $$t_{n3} \to \tilde{\tau}, \qquad t_{n4} \to \tilde{\tau} \quad \text{in probability as } n \to \infty.$$

Because (2.5) implies (3.6) we can further assert that there exists a random variable τ with a continuous distribution (and hence τ is finite, with probability 1) such that

(3.8) $$t_{n1} \to \tau, \qquad t_{n2} \to \tau \quad \text{in probability as } n \to \infty.$$

It follows from (3.4)–(3.8) that

(3.9) $$(x_n, y_n) \to (\tau, 0) \quad \text{in probability as } n \to \infty.$$

If, in addition to (2.5), condition (2.6) also holds, then *in probability* in (3.7), (3.8) and (3.9) may be replaced by *a.s.* See [7] (Theorem 5.18) and [2] (Chapter 4, Theorem 3.1). □

4. The sequence of random matrices T_n. Because $y_n \to 0$ a.s. as $n \to \infty$, we should expect that what determines the asymptotic behaviour of the size of $\mathbf{Z}_n$—and hence γ—is the asymptotic behaviour of x_n.

We define a sequence of random matrices $\mathbf{T}_n$ as follows. For an arbitrary real constant t_0, let

$$\mathbf{T}_0 \equiv \begin{bmatrix} t_0 & 0 \\ 1 & 0 \end{bmatrix}$$

and

$$\mathbf{T}_n \equiv \vartheta(\mathbf{G}_n \mathbf{T}_{n-1}), \qquad n = 1, 2, \dots .$$

With

$$\mathbf{G}_n \equiv \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix},$$

we define

(4.1) $$t_n \equiv (a_n t_{n-1} + b_n)/(c_n t_{n-1} + d_n), \qquad n = 1, 2, \dots .$$

THEOREM 4.1.

$$\mathbf{T}_n = \begin{bmatrix} t_n & 0 \\ 1 & 0 \end{bmatrix}.$$

If condition (2.5) *holds, then*

$$t_n \to \tau \quad \textit{in probability as } n \to \infty.$$

If, in addition to (2.5), (2.6) *also holds, then*

$$t_n \to \tau \quad \textit{a.s. as } n \to \infty.$$

PROOF. The first part follows directly from the definition $\mathbf{T}_n \equiv \vartheta(\mathbf{G}_n\mathbf{T}_{n-1})$. Notice that, with $x_0 = t_0$, and $y_0 = 0$, then $t_n = x_n$ and, from (3.3),

$$t_n = t_{n1}(t_0 + t_{n3})(t_0 + t_{n4})^{-1}.$$

The probability distribution of $\tilde{\tau}$ is continuous, hence

$$\Pr\big[|t_{n3} - t_{n4}| > \delta|t_0 + t_{n4}|\big] \to 0 \quad \text{as } n \to \infty \qquad \text{for } \delta > 0;$$

thus

$$(t_0 + t_{n3})(t_0 + t_{n4})^{-1} \to 1 \quad \text{in probability as } n \to \infty.$$

From (3.8), we now have

$$t_n \to \tau \quad \text{in probability as } n \to \infty.$$

If (2.6) holds and if also $t_0 + \tilde{\tau} \neq 0$, then

$$t_n \to \tau \quad \text{a.s. as } n \to \infty. \qquad \square$$

Let $\psi(dt)$ be the probability measure of τ and let $\mathbf{z}_n \equiv (x_n, y_n)$. $\mathbf{z}_0, \mathbf{z}_1, \dots, \mathbf{z}_n, \dots$ is a Markov process with starting point $\mathbf{z}_0$. Let $\boldsymbol{P}_n(\mathbf{z}_0, B)$ denote the probability measure of $\mathbf{z}_n$ and let $\overline{\boldsymbol{P}}_n(\mathbf{z}_0, B)$ be the average

$$\overline{\boldsymbol{P}}_n(\mathbf{z}_0, B) \equiv \frac{1}{n}\sum_{m=0}^{n-1} \boldsymbol{P}_m(\mathbf{z}_0, B).$$

THEOREM 4.2. *Let $\breve{q}$ be a bounded, measurable, real-valued function on $\boldsymbol{R}^2$ and let $q(t) \equiv \breve{q}((t,0))$ be a continuous function on $\boldsymbol{R}$. Then, if* (2.5) *holds,*

$$\lim_{n\to\infty} \int \breve{q}(\mathbf{z})\overline{\boldsymbol{P}}_n(\mathbf{z}_0, d\mathbf{z}) = \int q(t)\psi(dt).$$

PROOF. Let $\breve{B}$ be an arbitrary Borel subset of the plane, and let

$$B \equiv \breve{B} \cap \{\mathbf{z}: \mathbf{z} = (x, 0),\ -\infty < x < \infty\}.$$

We define the probability measure $\breve{\psi}$ on $\boldsymbol{R}^2$ by

$$\breve{\psi}(\breve{B}) \equiv \psi(B).$$

Because $\mathbf{z}_n \to (\tau, 0)$ in probability as $n \to \infty$, the probability distribution $\overline{\boldsymbol{P}}_n(\mathbf{z}_0, \cdot)$ converges weakly to the measure $\breve{\psi}(\cdot)$. Thus

$$\lim_{n\to\infty} \int \breve{q}(\mathbf{z})\overline{\boldsymbol{P}}_n(\mathbf{z}_0, d\mathbf{z}) = \int q(t)\psi(dt).$$

That it is only q that is required to be continuous is a consequence of a weak convergence theorem, for which see Billingsley [1] (Theorem 25.7) or Pollard [16] (Chapter 4, Section 2, Continuous Mapping Theorem). $\square$

Let

$$\mathbf{G} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and let $t_1 \equiv (at + b)/(ct + d)$, where t is a real constant. Let

$$\boldsymbol{Q}(t, B) \equiv \Pr[t_1 \in B],$$

where B is a Borel subset of the real line.

THEOREM 4.3. *If* (2.5) *holds, then* ψ, *the probability measure of* τ, *is the unique* (*up to scaling*) *solution of the integral equation*

$$\psi(B) = \int \psi(dt) \boldsymbol{Q}(t, B). \tag{4.2}$$

PROOF. That ψ satisfies (4.2) follows from (4.1) and Theorem 4.1. That ψ is unique follows from [7] (Theorem 4.6). □

5. The Lyapounov exponent. In Section 1 we defined, for constant t,

$$h(t) \equiv \log(\text{range}(0, t, 1)),$$
$$H(t) \equiv \boldsymbol{E}[\log(\text{range}(0, at + b, ct + d))].$$

THEOREM 5.1. *If* μ, *the probability measure of the* 2×2 *real invertible random matrix* $\mathbf{G}$, *satisfies* (2.3) *and* (2.4) *and if* $\boldsymbol{E}[|\log(|\det(\mathbf{G})|)|]$ *is finite, then the upper Lyapounov exponent of* μ *is* γ *given by*

$$\gamma = \int H(t) \psi(dt) - \int h(t) \psi(dt)$$
$$= \int \boldsymbol{E}[\log(|ct + d|)] \psi(dt).$$

PROOF. We suppose that $w_0 > 0$. Because $\det(\mathbf{G}) \neq 0$ a.s., $w_n > 0$ a.s. for each $n = 1, 2, \dots$ and so we may define $Q_n \equiv w_n / w_{n-1}$. Thus $Q_n > 0$ a.s. for each $n = 1, 2, \dots$. Clearly

$$w_n = Q_n Q_{n-1} \cdots Q_1 w_0,$$

so

$$\frac{1}{n} \log(w_n) = \frac{1}{n} \sum_{m=1}^{n} \log(Q_m) + \frac{1}{n} \log(w_0). \tag{5.1}$$

Let $\mathscr{F}_n \equiv \sigma(\mathbf{Z}_0, \mathbf{Z}_1, \dots, \mathbf{Z}_n)$ be the sigma-field of events generated by $\mathbf{Z}_0$, $\mathbf{Z}_1, \dots, \mathbf{Z}_n$. The sequence of random matrices $\{\mathbf{Z}_n,\ n = 0, 1, \dots\}$ is a Markov process, so

$$\boldsymbol{E}[\log(Q_n) | \mathscr{F}_{n-1}] = \boldsymbol{E}[\log(Q_n) | \mathbf{Z}_{n-1}].$$

Let

$$\Lambda(\mathbf{Z}_{n-1}) \equiv \boldsymbol{E}[\log(Q_n) | \mathbf{Z}_{n-1}].$$

Then, from (5.1) we have

$$\frac{1}{n}\boldsymbol{E}[\log(w_n)|\mathbf{Z}_0] = \frac{1}{n}\sum_{m=0}^{n-1}\boldsymbol{E}[\Lambda(\mathbf{Z}_m)|\mathbf{Z}_0] + \frac{1}{n}\log(w_0). \tag{5.2}$$

From (2.10), we know that $w_n = 2(L_n)^{-2}|\det(\mathbf{Z}_n)|$ and hence

$$\begin{aligned} Q_n = w_n/w_{n-1} &= (L_n/L_{n-1})^{-2}|\det(\mathbf{Z}_n)/\det(\mathbf{Z}_{n-1})| \\ &= (L_n/L_{n-1})^{-2}|\det(\mathbf{G}_n)|. \end{aligned}$$

Thus

$$\log(Q_n) = -2\log(L_n/L_{n-1}) + \log(|\det(\mathbf{G}_n)|).$$

It follows that

$$\Lambda(\mathbf{Z}_0) = -2\boldsymbol{E}[\log(L_1/L_0)|\mathbf{Z}_0] + \boldsymbol{E}[\log(|\det(\mathbf{G})|)]. \tag{5.3}$$

Let $\tilde{L}_0$ be the length of the longest side of $\tilde{\Gamma}_0$, the triangle of $\tilde{\mathbf{Z}}_0$, and let $\hat{L}_1$ be the length of the longest side of $\hat{\Gamma}_1$, the triangle of $\mathbf{G}_1\tilde{\mathbf{Z}}_0$. Then

$$L_1/L_0 = \hat{L}_1/\tilde{L}_0.$$

We define $\lambda(\mathbf{z})$ by

$$\lambda(\mathbf{z}) \equiv \boldsymbol{E}\Big[\log\big(\hat{L}_1/\tilde{L}_0\big)|\mathbf{z}_0 = \mathbf{z}\Big]. \tag{5.4}$$

From (5.2), (5.3) and (5.4) we have

$$\begin{aligned} \frac{1}{n}\boldsymbol{E}[\log(w_n)|\mathbf{Z}_0] &= -\frac{2}{n}\sum_{m=0}^{n-1}\boldsymbol{E}[\lambda(\mathbf{z}_m)|\mathbf{z}_0] \\ &\quad + \boldsymbol{E}[\log(|\det(\mathbf{G})|)] + \frac{1}{n}\log(w_0). \end{aligned} \tag{5.5}$$

Let γ be the Lyapounov exponent of μ and let $\breve{\gamma}$ be the Lyapounov exponent of $\breve{\mu}$. Then, from Theorems 2.1 and 2.2,

$$\begin{aligned} \gamma &= \breve{\gamma} + \frac{1}{2}\boldsymbol{E}[\log(|\det(\mathbf{G})|)] \\ &= \lim_{n\to\infty} -\frac{1}{2n}\log(w_n) + \frac{1}{2}\boldsymbol{E}[\log(|\det(\mathbf{G})|)]. \end{aligned}$$

Combining this with (5.5) we get

$$\begin{aligned} \gamma &= \lim_{n\to\infty}\frac{1}{n}\sum_{m=0}^{n-1}\boldsymbol{E}[\lambda(\mathbf{z}_m)|\mathbf{z}_0] \\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{m=0}^{n-1}\int\lambda(\mathbf{z})\boldsymbol{P}_m(\mathbf{z}_0, d\mathbf{z}) \\ &= \lim_{n\to\infty}\int\lambda(\mathbf{z})\overline{\boldsymbol{P}}_n(\mathbf{z}_0, d\mathbf{z}). \end{aligned}$$

(2.3) guarantees that $\lambda(\mathbf{z})$ is bounded, so we may apply Theorem 4.2 to get

$$\gamma = \int \zeta(t)\psi(dt),$$

where $\zeta(t) \equiv \lambda((t,0))$. When we put $\mathbf{z}_0 = (t,0)$, we get

$$\hat{L}_1 = \text{range}(0, at+b, ct+d), \qquad \tilde{L}_0 = \text{range}(0,t,1),$$

from which it follows that

$$\zeta(t) = H(t) - h(t).$$

Thus

$$\gamma = \int H(t)\psi(dt) - \int h(t)\psi(dt).$$

Now

$$\begin{aligned}\int H(t)\psi(dt) &= \int \boldsymbol{E}[\log(\text{range}(0, at+b, ct+d))]\psi(dt)\\ &= \int \boldsymbol{E}[\log(\text{range}(0,(at+b)/|ct+d|,\\ &\qquad\qquad (ct+d)/|ct+d|))]\psi(dt)\\ &\quad + \int \boldsymbol{E}[\log(|ct+d|)]\psi(dt)\end{aligned}$$

and

$$\begin{aligned}&\int \boldsymbol{E}[\log(\text{range}(0,(at+b)/|ct+d|,(ct+d)/|ct+d|))]\psi(dt)\\ &= \int \boldsymbol{E}_{\{ct+d\geq 0\}}[\log(\text{range}(0,(at+b)/(ct+d),1))]\psi(dt)\\ &\quad + \int \boldsymbol{E}_{\{ct+d<0\}}[\log(\text{range}(0,-(at+b)/(ct+d),-1))]\psi(dt)\\ &= \int \boldsymbol{E}_{\{ct+d\geq 0\}}[\log(\text{range}(0,(at+b)/(ct+d),1))]\psi(dt)\\ &\quad + \int \boldsymbol{E}_{\{ct+d<0\}}[\log(\text{range}(0,(at+b)/(ct+d),1))]\psi(dt)\\ &\qquad\qquad [\text{because range}(0,-x,-1) = \text{range}(0,x,1), \text{ for all } x]\\ &= \int \boldsymbol{E}[\log(\text{range}(0,(at+b)/(ct+d),1))]\psi(dt)\\ &= \int \psi(dt)\int \boldsymbol{Q}(t,ds)\log(\text{range}(0,s,1))\\ &= \int \log(\text{range}(0,t,1))\psi(dt) = \int h(t)\psi(dt).\end{aligned}$$

Thus

$$\gamma = \int H(t)\psi(dt) - \int h(t)\psi(dt) = \int \boldsymbol{E}[\log(|ct+d|)]\psi(dt),$$

as required. □

6. Example 1. Perhaps the hardest part in the task of determining a Lyapounov exponent is that of calculating the invariant distribution $\psi(dt)$, a solution of (4.2). However, it can happen that $\boldsymbol{Q}(t, B)$ is independent of t, in which case

$$\psi(B) = \boldsymbol{Q}(t, B), \quad \text{for all } t.$$

Our first example is an instance of this.

THEOREM 6.1. *With*

$$\mathbf{G} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we suppose that $(a, b)'$ and $(c, d)'$ are i.i.d. $N(0, \Sigma)$ random variables. Then the invariant probability measure that satisfies (4.2) *is $\psi(dt) = g(t)\,dt$, where*

$$g(t) = \frac{1}{\pi(1+t^2)}, \qquad -\infty < t < \infty.$$

The upper Lyapounov exponent of $\mathbf{G}$ is γ given by

$$\gamma = -\tfrac{1}{2}\hat{\gamma} + \tfrac{1}{2}\log\left(\tfrac{1}{2}\operatorname{tr}(\Sigma) + \sqrt{\det(\Sigma)}\right),$$

where $\hat{\gamma}$ is Euler's constant:

$$\hat{\gamma} = -\int_0^{+\infty} \log(x) e^{-x}\,dx = 0.577215669\ldots.$$

PROOF. It is easy to check that μ, the probability measure of $\mathbf{G}$, satisfies conditions (2.3)–(2.5), required for Theorems 4.3 and 5.1.

For constant t, let $\sigma^2(t) \equiv \operatorname{var}[at + b] = \operatorname{var}[ct + d] = (t, 1)\Sigma(t, 1)'$. Let $u \equiv (at + b)/\sigma(t)$ and $v \equiv (ct + d)/\sigma(t)$. Then u, v are i.i.d. $N(0, 1)$ random variables. Thus $t_1 \equiv (at + b)/(ct + d) = u/v$ is the quotient of i.i.d. $N(0, 1)$ random variables, and hence has the standard Cauchy distribution. Thus $\boldsymbol{Q}(t, B)$, the probability measure of t_1, is given by $\boldsymbol{Q}(t, ds) = \psi(ds) = g(s)\,ds$, where

$$g(s) = \frac{1}{\pi(1+s^2)}, \qquad -\infty < s < \infty.$$

Recall (1.5), the formula for γ:

$$\begin{aligned}
\gamma &= \int E[\log(|ct + d|)]\,\psi(dt) \\
&= \int E[\log(|ct + d|/\sigma(t))]\,\psi(dt) + \int \log(\sigma(t))\,\psi(dt) \\
&= E[\log(|v|)] + \int_{-\infty}^{+\infty} \log(\sigma(t)) g(t)\,dt,
\end{aligned}$$

where

(i) $\qquad E[\log(|v|)] = -\tfrac{1}{2}\log(2) - \tfrac{1}{2}\hat{\gamma} = -0.635181422\ldots$

and

$$\text{(ii)} \quad \int_{-\infty}^{+\infty} \log(\sigma(t))g(t)\,dt = \tfrac{1}{2}\log(2) + \tfrac{1}{2}\log\left(\tfrac{1}{2}\operatorname{tr}(\Sigma) + \sqrt{\det(\Sigma)}\right).$$

Thus

$$\gamma = -\tfrac{1}{2}\hat{\gamma} + \tfrac{1}{2}\log\left(\tfrac{1}{2}\operatorname{tr}(\Sigma) + \sqrt{\det(\Sigma)}\right),$$

as required.

(i) follows from the observation that, because v is standard normal, we can write $|v| = \sqrt{2Z}\,|\cos(\Theta)|$, where Z and Θ are independent random variables, Z being negative-exponentially distributed with mean 1 and Θ being uniformly distributed on $[0, 2\pi]$. Thus

$$\begin{aligned}
\boldsymbol{E}[\log(|v|)] &= \boldsymbol{E}\left[\log(\sqrt{2Z}\,|\cos(\Theta)|)\right]\\
&= \frac{1}{2}\log(2) + \frac{1}{2}\boldsymbol{E}[\log(Z)] + \boldsymbol{E}\left[\log(|\cos(\Theta)|)\right]\\
&= \frac{1}{2}\log(2) + \frac{1}{2}\int_0^{+\infty}\log(z)e^{-z}\,dz + \frac{1}{2\pi}\int_0^{2\pi}\log(|\cos(\theta)|)\,d\theta\\
&= \frac{1}{2}\log(2) - \frac{1}{2}\hat{\gamma} + \frac{2}{\pi}\int_0^{\pi/2}\log(\cos(\theta))\,d\theta\\
&= \frac{1}{2}\log(2) - \frac{1}{2}\hat{\gamma} + \frac{2}{\pi}\left(-\frac{\pi}{2}\log(2)\right) = -\frac{1}{2}\log(2) - \frac{1}{2}\hat{\gamma}.
\end{aligned}$$

Let η_1, η_2 be the eigenvalues of Σ, $\eta_1 \geq \eta_2 > 0$. Then (ii) follows from

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\log(\sigma(t))g(t)\,dt\\
&\quad= \int_{-\infty}^{+\infty}\log((t,1)\Sigma(t,1)')\frac{1}{2\pi(1+t^2)}\,dt\\
&\quad= \int_{-\infty}^{+\infty}\log(\eta_1 t^2 + \eta_2)\frac{1}{2\pi(1+t^2)}\,dt\\
&\quad= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\log(\eta_1 t^2 + \eta_2)\,d\theta\\
&\quad= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\log(\eta_1\sin^2(\theta) + \eta_2\cos^2(\theta))\,d\theta\\
&\qquad\quad - \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\log(\cos(\theta))\,d\theta\\
&\quad= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\log(\eta_1\sin^2(\theta) + \eta_2\cos^2(\theta))\,d\theta + \log(2)
\end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \log\left(\frac{1}{2}(\eta_1 + \eta_2) + \frac{1}{2}(\eta_2 - \eta_1)\cos(2\theta)\right) d\theta + \log(2)$$

$$= \frac{1}{2}\log\left(\frac{1}{2}(\eta_1 + \eta_2) + \sqrt{\eta_1\eta_2}\right) + \log(2)$$

$$= \frac{1}{2}\log\left(\frac{1}{2}\operatorname{tr}(\Sigma) + \sqrt{\det(\Sigma)}\right) + \log(2). \qquad \square$$

7. Example 2. In our second example we look at the form that **G** takes in the study of Brocot sequences by Chassaing, Letac and Mora [3] (see also [2]):

$$\mathbf{G} = \begin{bmatrix} 0 & 1 \\ 1 & T \end{bmatrix},$$

where T is a nonnegative random variable. The authors calculate ψ and γ in the case when T has the negative exponential distribution. We shall show that for fairly general T, there is a simple and interesting equation for ψ and a simple formula for γ. We calculate ψ and γ explicitly in the case when T has a gamma distribution.

We suppose that T has a probability density f. For constant s, let $t_1 \equiv 1/(s + X)$ and let $g(s,t)$ be the probability density of t_1. Then

$$g(s,t) = \frac{1}{t^2} f\left(\frac{1}{t} - s\right), \qquad 0 < t < \frac{1}{s} \ (= 0 \text{ otherwise}).$$

The invariant probability measure ψ of (4.2) has a density g, given by

$$g(t) = \int_0^\infty g(s) g(s,t)\, ds = \int_0^{t^{-1}} g(s) \frac{1}{t^2} f\left(\frac{1}{t} - s\right) ds, \qquad t > 0.$$

Thus

$$\frac{1}{t^2} g\left(\frac{1}{t}\right) = \int_0^t g(s) f(t-s)\, ds, \qquad t > 0. \tag{7.1}$$

We show that the Lyapounov exponent γ is given by

$$\gamma = -\int_0^{+\infty} \log(t) g(t)\, dt. \tag{7.2}$$

We have from (1.5),

$$\gamma = \int_0^{+\infty} E[\log(t + T)] g(t)\, dt$$

$$= \int_0^{+\infty}\int_0^{+\infty} \log(t+s) f(s) g(t)\, ds\, dt = \int_0^{+\infty} \log(t)\, dt \int_0^t f(s) g(t-s)\, ds$$

$$= \int_0^{+\infty} \log(t) \frac{1}{t^2} g\left(\frac{1}{t}\right) dt = -\int_0^{+\infty} \log(t) g(t)\, dt.$$

THEOREM 7.1. *If T has the gamma distribution, $\mathscr{G}(\alpha, \beta)$, with probability density f,*

$$f(t) = \alpha \exp(-\alpha t)(\alpha t)^{\beta - 1}/\Gamma(\beta), \qquad t > 0,$$

then

$$(7.3)\qquad g(t) = \frac{1}{t^{\beta+1}}\exp\left(-\alpha\left(t + \frac{1}{t}\right)\right)\Big/ 2K_\beta(2\alpha), \qquad t > 0,$$

where $K_\beta(2\alpha)$ *is the modified Bessel function defined by*

$$K_\beta(2\alpha) \equiv \int_0^{+\infty} \cosh(\beta t)\exp(-2\alpha\cosh(t))\,dt.$$

The upper Lyapounov exponent $\gamma = \gamma_{\alpha,\beta}$ *is given by*

$$(7.4)\qquad \gamma_{\alpha,\beta} = \left[K_\beta(2\alpha)\right]^{-1}\int_0^{+\infty} t\sinh(\beta t)\exp(-2\alpha\cosh(t))\,dt.$$

PROOF. It is easy to check that g, given by (7.3), is a solution of (7.1). Moreover it is the only solution because, under condition (2.5) (satisfied here), there is only one invariant probability measure that satisfies (4.2). From (7.2),

$$\begin{aligned}\gamma_{\alpha,\beta} &= -\left[2K_\beta(2\alpha)\right]^{-1}\int_0^{+\infty}\log(t)\frac{1}{t^{\beta+1}}\exp\left(-\alpha\left(t+\frac{1}{t}\right)\right)dt\\ &= -\left[2K_\beta(2\alpha)\right]^{-1}\int_0^{+\infty} t\exp(-\beta t)\exp(-2\alpha\cosh(t))\,dt\\ &= \left[K_\beta(2\alpha)\right]^{-1}\int_0^{+\infty} t\sinh(\beta t)\exp(-2\alpha\cosh(t))\,dt,\end{aligned}$$

as required. □

Notice that when $\beta = 1$, after an integration by parts, we get

$$\gamma_{\alpha,1} = K_0(2\alpha)\left[2\alpha K_1(2\alpha)\right]^{-1}.$$

There is a mistake in the calculations of Chassaing, Letac and Mora [3], who looked at this case. For their calculation of γ, they have $K_0(2\alpha)$ $[\alpha K_1(2\alpha)]^{-1}$. The mistake is repeated in Bougerol and Lacroix [2].

When β is a positive integer we may express γ in terms of Bessel functions. Let $\beta = n$, a positive integer, $n \geq 2$.

COROLLARY 7.1.

$$\gamma_{\alpha,n} = -\left[K_n(2\alpha)\right]^{-1}\sum_{r=0}^{n} c_{r,n}K_r(2\alpha),$$

where the coefficients $c_r = c_{r,n}$, $r = 0, 1, \dots, n$, *are given by*

$$\alpha c_2 + c_1 - 2\alpha c_0 = 0, \qquad c_n = 0, \qquad c_{n-1} = -\alpha^{-1},$$
$$\alpha c_{r+1} + r c_r - \alpha c_{r-1} = 0, \qquad 2 \leq r \leq n-1.$$

In particular,

$$\gamma_{\alpha,2} = \left[K_2(2\alpha)\right]^{-1}\left[\tfrac{1}{2}\alpha^{-2}K_0(2\alpha) + \alpha^{-1}K_1(2\alpha)\right],$$

$$\gamma_{\alpha,3} = \left[K_3(2\alpha)\right]^{-1}\left[\left(\tfrac{1}{2}\alpha^{-1} + \alpha^{-3}\right)K_0(2\alpha) + 2\alpha^{-2}K_1(2\alpha) + \alpha^{-1}K_2(2\alpha)\right],$$

$$\gamma_{\alpha,4} = \left[K_4(2\alpha)\right]^{-1}\left[(2\alpha^{-2} + 3\alpha^{-4})K_0(2\alpha) + (\alpha^{-1} + 6\alpha^{-3})K_1(2\alpha) + 3\alpha^{-2}K_2(2\alpha) + \alpha^{-1}K_3(2\alpha)\right].$$

PROOF. Let

$$I(t) \equiv \int_0^t \sinh(\beta s)\exp(-2\alpha\cosh(s))\,ds.$$

Then

$$\gamma_{\alpha,\beta} = -\left[K_\beta(2\alpha)\right]^{-1}\int_0^{+\infty} I(t)\,dt.$$

The remaining part of the proof follows on noting that, for the given c_r,

$$I(t) = \sum_{r=0}^{n} c_r \cosh(rt)\exp(-2\alpha\cosh(t)). \qquad \square$$

We notice from the formula

$$\gamma = -\int_0^{+\infty}\log(t)g(t)\,dt$$

that if $g(t) = e^{-t}$, $t \geq 0$, then the Lyapounov exponent is Euler's constant. The question is then: What is the corresponding $f(t)$? (7.1) provides the answer:

$$f(t) = e^{-t}\frac{d}{dt}\left(t^{-2}e^{t-1/t}\right) = (1-t)^2 t^{-4} t e^{-1/t}, \qquad t > 0.$$

Similarly, if $g(t) = \alpha e^{-\alpha t}(\alpha t)^{n-1}/(n-1)!$, $\alpha > 0$, $t > 0$, then

$$f(t) = e^{-\alpha t}\frac{d^n}{dt^n}\left(e^{\alpha(t-1/t)}t^{-n-1}\right)/(n-1)!, \qquad t > 0.$$

8. Example 3. Our last example, the topic of a series of articles by Mannion [13]–[15], arises from a problem proposed to us by David Kendall when he and his colleague Hui-Lin Le were investigating the shape distribution of a triangle whose vertices are three points chosen at random in the interior of a convex polygon (Kendall and Le [10]).

Three points are chosen at random in the interior of a *parent* triangle Δ_0, these points to be the vertices of the *daughter* triangle Δ_1. Δ_1 now serves as the new parent triangle for a new daughter triangle Δ_2, created by choosing three points at random in the interior of Δ_1. Repeating this prescription, with Δ_n being the daughter of parent Δ_{n-1}, gives rise to a chain of nested triangles $\{\Delta_n, n = 0, 1, \ldots\}$.

We suppose that the vertices of Δ_0 are noncollinear. Let them be labelled P_{00}, P_{01}, P_{02} and let their coordinates be (x_{0j}, y_{0j}), $j = 0, 1, 2$. Let $\mathscr{H}$ be the triangle in $\boldsymbol{R}^3$,

$$\mathscr{H} \equiv \left\{\mathbf{u}\colon u_0 + u_1 + u_2 = 1;\ u_j \geq 0,\ j = 0, 1, 2\right\}$$

and let $\eta(d\mathbf{u})$ be a probability measure on $\mathscr{H}$ such that each of the probability laws $\mathscr{L}(u_i, u_j) = \mathscr{L}(u_1, u_2)$, $i \neq j$. Now let $\mathbf{u}_i = (u_{i0}, u_{i1}, u_{i2})$, $i = 0, 1, 2$, be three independently η-chosen points in $\mathscr{H}$.

The vertices P_{10}, P_{11}, P_{12} of Δ_1 are the points with coordinates (x_{1j}, y_{1j}), $j = 0, 1, 2$, respectively, where

$$x_{1j} = u_{i0}x_{00} + u_{i1}x_{01} + u_{i2}x_{02}, \qquad i = 0, 1, 2,$$
$$y_{1j} = u_{i0}y_{00} + u_{i1}y_{01} + u_{i2}y_{02}, \qquad i = 0, 1, 2.$$

Let

$$\mathbf{Z}_0 \equiv \begin{bmatrix} x_{01} - x_{00} & y_{01} - y_{00} \\ x_{02} - x_{00} & y_{02} - y_{00} \end{bmatrix},$$

$$\mathbf{Z}_1 \equiv \begin{bmatrix} x_{11} - x_{10} & y_{11} - y_{10} \\ x_{12} - x_{10} & y_{12} - y_{10} \end{bmatrix},$$

$$\mathbf{G} \equiv \begin{bmatrix} u_{11} - u_{01} & u_{12} - u_{02} \\ u_{21} - u_{01} & u_{22} - u_{02} \end{bmatrix}.$$

Then

$$\mathbf{Z}_1 = \mathbf{G}\mathbf{Z}_0.$$

Let $\mathbf{G}_1, \mathbf{G}_2, \dots, \mathbf{G}_n$ be i.i.d. copies of $\mathbf{G}$, let $\mathbf{Z}_n \equiv \mathbf{G}_n\mathbf{G}_{n-1} \cdots \mathbf{G}_1\mathbf{Z}_0$ and let Γ_n be the triangle of Z_n, $n = 0, 1, \dots$. Then for each n, Γ_n is a transformation of Δ_n obtained by translating one of the vertices of Δ_n to the origin. Typically, depending on $\eta(d\mathbf{u})$, the areas of the Δ_n will converge rapidly to zero, as $n \to \infty$. But what of the limiting probability distribution of the *shapes* of the Δ_n? We discover what this is in the case when $\eta(d\mathbf{u})$ is the *uniform* distribution.

Let $\mathbf{c}_i \equiv (u_{1i} - u_{0i}, u_{2i} - u_{0i})'$, $i = 1, 2$, and let $\mathbf{c}_0 \equiv -\mathbf{c}_1 - \mathbf{c}_2$. Then $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2$ are pairwise exchangeable. Accordingly, from Theorem 9.1, the shapes $\sigma(\Gamma_n)$, and hence also the shapes $\sigma(\Delta_n)$, form a Markov process.

For a constant t, let

$$v_i \equiv u_{i1}t + u_{i2}, \qquad i = 0, 1, 2.$$

Then

$$t_1 \equiv (g_{11}s + g_{12})/(g_{21}s + g_{22}) = (v_1 - v_0)/(v_2 - v_0),$$

where $g_{ij} \equiv u_{ij} - u_{0j}$, $i, j = 1, 2$.

We suppose that the v_i have a probability density $p(s, v)$. Then the probability measure $\boldsymbol{Q}(s, B)$ of t_1 has a density $g(s, t)$: $g(s, t)\,dt = \boldsymbol{Q}(s, dt)$. If $0 \le s \le 1$, then $0 \le t_1 \le 1$ and

$$g(s, t) = \int_{0 \le x \le y \le 1} 6(y - x)p(s, x)p(s, x + t(y - x))p(s, y)\,dx\,dy,$$
$$0 \le s, t \le 1.$$

Moreover, there is an invariant density $g(t)$ for $g(s,t)$ and

$$g(t) = \int_0^1 g(s)g(s,t)\,ds.$$

Because of the exchangeability of the $\mathbf{c}_i$, it follows that $p(s,v) = p(1-s, 1-v)$, and hence that $g(1-s, 1-t) = g(s,t)$ and that $g(1-t) = g(t)$.

In the case when $\eta(d\mathbf{u})$ is the *uniform* distribution on $\mathscr{H}$ we find

$$p(s,v) = \min\left(\frac{2v}{s}, \frac{2(1-v)}{1-s}\right), \qquad 0 \le s, v \le 1.$$

Let

$$\alpha_1(s) \equiv 1 + s^2, \qquad \beta_1(t) \equiv \frac{2}{3}(2-t),$$

$$\alpha_2(s) \equiv s^3(3-s)(1-s)^{-2}, \qquad \beta_2(t) \equiv -\frac{2}{15}(t^{-2} - 3 + 2t),$$

$$\alpha_3(s) \equiv s^4(1-s)^{-2}, \qquad \beta_3(t) \equiv \frac{2}{15}(t^{-3} - 4 + 3t).$$

The transition probability density $g(s,t)$ is given by

$$\begin{aligned} g(s,t) &= \alpha_1(s)\beta_1(t) + \alpha_2(s)\beta_2(t) + \alpha_3(s)\beta_3(t), \quad 0 \le s \le t \le 1,\\ g(s,t) &= \alpha_1(1-s)\beta_1(1-t) + \alpha_2(1-s)\beta_2(1-t)\\ &\quad + \alpha_3(1-s)\beta_3(1-t), \qquad 0 \le t \le s \le 1. \end{aligned}$$

The hard part of our calculations was to find the invariant density $g(t)$:

$$g(t) = \frac{3}{\pi^2}(1 + \omega(t) + \omega(1-t)), \qquad 0 \le t \le 1,$$

where

$$\omega(t) \equiv \frac{1}{t^2} + \frac{1}{2t^3}(2 + 3t + 2t^2)(1-t)^2 \log(1-t), \qquad 0 < t \le 1,$$

$$\omega(0) \equiv \frac{11}{12}.$$

We also find

$$\begin{aligned} H(t) &= \int_{0 \le x \le y \le 1} \log(z-x) 6p(t,x)p(t,y)p(t,z)\,dx\,dy\,dz\\ &= \frac{1}{15}\left[\frac{3t^3\log(t)}{(1-t)^2} + \frac{3(1-t)^3\log(1-t)}{t^2} + \frac{3}{t(1-t)} - 5t(1-t)\right]\\ &\quad - \frac{107}{60}. \end{aligned}$$

Notice that

$$\int_0^1 h(t)g(t)\,dt = 0,$$

so that, from (1.4), we get

$$\gamma = \int_0^1 H(t)g(t)\,dt = -\frac{7}{6}.$$

$|\det(\mathbf{G})|$ is the area of a triangle whose vertices are chosen uniformly at random in the interior of another triangle of unit area. We discover that

$$\boldsymbol{E}\big[\log(|\det(\mathbf{G})|)\big] = \frac{2}{45}\pi^2 - \frac{7}{2},$$

from which we get $\breve{\gamma}$, the upper Lyapounov exponent of $\breve{\mu}$, the probability measure of $\breve{\mathbf{G}} = \mathbf{G}'/\sqrt{|\det(\mathbf{G})|}$:

$$\breve{\gamma} = \frac{7}{12} - \frac{1}{45}\pi^2.$$

9. The shape process. The relation between $\tilde{\mathbf{Z}}_n$ and $\sigma_n = \text{shape}(\mathbf{Z}_n)$ is particularly simple. To see this we define the partition of the plane $\{\mathbf{S}^{(j)}, j = 1, 2, \dots, 12\}$:

$$\boldsymbol{S}^{(1)} \equiv \{(x, y): x \geq 0; y \geq 0; (1+x)^2 + y^2 \leq 4\},$$

$$\boldsymbol{S}^{(2)} \equiv \{(x, y): x \geq 0; y < 0; (1+x)^2 + y^2 \leq 4\},$$

$$\boldsymbol{S}^{(3)} \equiv \{(x, y): x < 0; y \geq 0; (1-x)^2 + y^2 \leq 4\},$$

$$\boldsymbol{S}^{(4)} \equiv \{(x, y): x < 0; y < 0; (1-x)^2 + y^2 \leq 4\},$$

$$\boldsymbol{S}^{(5)} \equiv \{(x, y): x \geq 0; y \geq 0; 4 < (1+x)^2 + y^2 \leq 16\},$$

$$\boldsymbol{S}^{(6)} \equiv \{(x, y): x \geq 0; y < 0; 4 < (1+x)^2 + y^2 \leq 16\},$$

$$\boldsymbol{S}^{(7)} \equiv \{(x, y): x < 0; y \geq 0; 4 < (1-x)^2 + y^2 \leq 16\},$$

$$\boldsymbol{S}^{(8)} \equiv \{(x, y): x < 0; y < 0; 4 < (1-x)^2 + y^2 \leq 16\},$$

$$\boldsymbol{S}^{(9)} \equiv \{(x, y): x \geq 0; y \geq 0; 16 < (1+x)^2 + y^2\},$$

$$\boldsymbol{S}^{(10)} \equiv \{(x, y): x \geq 0; y < 0; 16 < (1+x)^2 + y^2\},$$

$$\boldsymbol{S}^{(11)} \equiv \{(x, y): x < 0; y \geq 0; 16 < (1-x)^2 + y^2\},$$

$$\boldsymbol{S}^{(12)} \equiv \{(x, y): x < 0; y < 0; 16 < (1-x)^2 + y^2\}.$$

Recall $\boldsymbol{S}$, the shape space, defined in (2.1). For a given $\sigma = (s, w) \in \boldsymbol{S}$, let Δ be a triangle with shape σ, and let A_0, A_1, A_2 be its vertices, where A_0 has coordinates $(0, 0)$ and A_2 has coordinates $(1, 0)$. There are just 12 points in $\boldsymbol{R}^2$ for the position of the third vertex A_1 so that Δ has the given shape σ.

Let $(x^{(j)}(\sigma), y^{(j)}(\sigma))$, $j = 1, 2, \dots, 12$ be their coordinates. With $x^{(j)} \equiv x^{(j)}(\sigma)$, $y^{(j)} \equiv y^{(j)}(\sigma)$, we see that

$$
\begin{aligned}
(x^{(1)}, y^{(1)}) &\equiv \left(\frac{1}{2}(1+s), \frac{1}{2}w\right),\\
(x^{(2)}, y^{(2)}) &\equiv \left(\frac{1}{2}(1+s), -\frac{1}{2}w\right),\\
(x^{(3)}, y^{(3)}) &\equiv \left(\frac{1}{2}(1-s), \frac{1}{2}w\right),\\
(x^{(4)}, y^{(4)}) &\equiv \left(\frac{1}{2}(1-s), -\frac{1}{2}w\right),\\
(x^{(5)}, y^{(5)}) &\equiv \left(\frac{2(1+s)}{(1+s)^2+w^2}, \frac{2w}{(1+s)^2+w^2}\right),\\
(x^{(6)}, y^{(6)}) &\equiv \left(\frac{2(1+s)}{(1+s)^2+w^2}, \frac{-2w}{(1+s)^2+w^2}\right),\\
(x^{(7)}, y^{(7)}) &\equiv \left(1-\frac{2(1+s)}{(1+s)^2+w^2}, \frac{2w}{(1+s)^2+w^2}\right),\\
(x^{(8)}, y^{(8)}) &\equiv \left(1-\frac{2(1+s)}{(1+s)^2+w^2}, \frac{-2w}{(1+s)^2+w^2}\right),\\
(x^{(9)}, y^{(9)}) &\equiv \left(\frac{2(1-s)}{(1-s)^2+w^2}, \frac{2w}{(1-s)^2+w^2}\right),\\
(x^{(10)}, y^{(10)}) &\equiv \left(\frac{2(1-s)}{(1-s)^2+w^2}, \frac{-2w}{(1-s)^2+w^2}\right),\\
(x^{(11)}, y^{(11)}) &\equiv \left(1-\frac{2(1-s)}{(1-s)^2+w^2}, \frac{2w}{(1-s)^2+w^2}\right),\\
(x^{(12)}, y^{(12)}) &\equiv \left(1-\frac{2(1-s)}{(1-s)^2+w^2}, \frac{-2w}{(1-s)^2+w^2}\right).
\end{aligned}
$$

To check the following lemmas is straightforward.

LEMMA 9.1.

$$(2x^{(j)} - 1, 2y^{(j)}) \in \boldsymbol{S}^{(j)}, \qquad j = 1, 2, \dots, 12.$$

In Figure 3 we see the regions $\boldsymbol{S}^{(j)}$; for $\sigma = (s, w) = (\frac{1}{3}, \frac{2}{3})$, the twelve points $(2x^{(j)} - 1, 2y^{(j)})$, are also included in the figure.

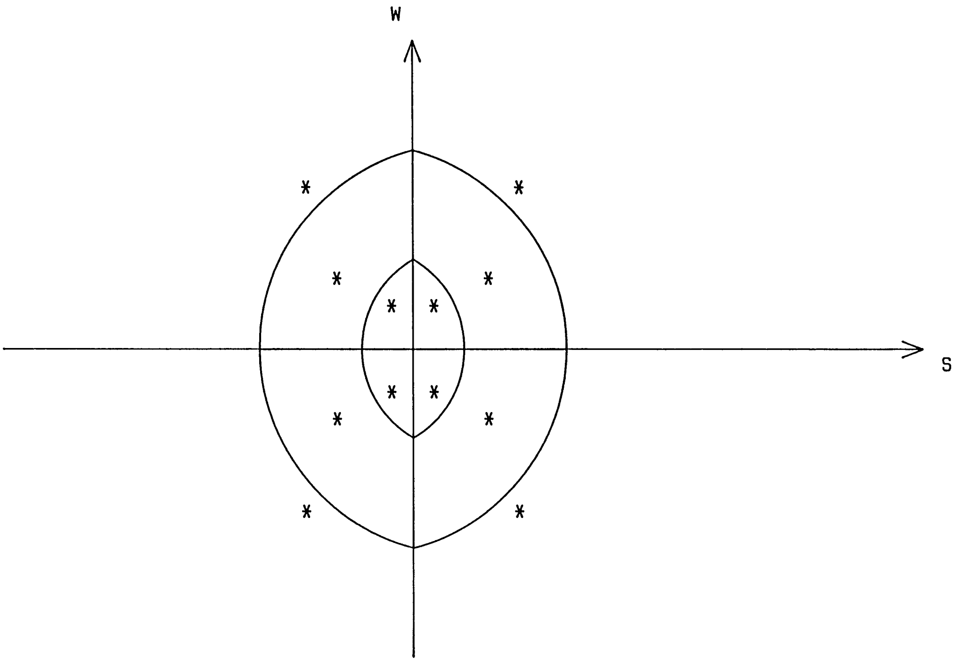

FIG. 3. *The* 12 *points corresponding to the shape* $(\frac{1}{3}, \frac{2}{3})$.

LEMMA 9.2. *If*

$$\tilde{\mathbf{M}} = \begin{bmatrix} x & y \\ 1 & 0 \end{bmatrix},$$

$\sigma = \text{shape}(\tilde{\mathbf{M}})$ *and* $(2x - 1, 2y) \in \mathbf{S}^{(j)}$, *then*

$$(x, y) = (x^{(j)}(\sigma), y^{(j)}(\sigma)).$$

Thus, if

$$\tilde{\mathbf{Z}}_n = \begin{bmatrix} x_n & y_n \\ 1 & 0 \end{bmatrix},$$

$\sigma_n = \text{shape}(\tilde{\mathbf{Z}}_n)$ and $(2x_n - 1, 2y_n) \in \mathbf{S}^{(j)}$, then

$$(x_n, y_n) = (x^{(j)}(\sigma_n), y^{(j)}(\sigma_n)). \tag{9.1}$$

We define a random variable ξ_n as $\xi_n = j$ if $(2x_n - 1, 2y_n) \in \mathbf{S}^{(j)}$.

THEOREM 9.1. *There is a one-to-one correspondence between* $\tilde{\mathbf{Z}}_n$ *and* (σ_n, ξ_n).

PROOF. σ_n and ξ_n determine $\tilde{\mathbf{Z}}_n$ through (9.1). Conversely, $\sigma_n = \text{shape}(\tilde{\mathbf{Z}}_n)$ and $\xi_n = j$ if $(2x_n - 1, 2y_n) \in \mathbf{S}^{(j)}$. □

9.1. *The "exchangeable" case.* There are some cases where a knowledge of $\sigma_{n-1} = \text{shape}(\mathbf{Z}_{n-1})$ alone is sufficient to calculate the probability measure of $\sigma_n = \text{shape}(\mathbf{Z}_n)$. Let

$$\mathbf{G} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and let $\mathbf{c}_1 \equiv (a, c)'$, $\mathbf{c}_2 \equiv (b, d)'$ and $\mathbf{c}_0 \equiv -\mathbf{c}_1 - \mathbf{c}_2$. We say $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2$ are *pairwise exchangeable* if

$$\mathscr{L}(\mathbf{c}_i, \mathbf{c}_j) = \mathscr{L}(\mathbf{c}_1, \mathbf{c}_2), \qquad i, j = 0, 1, 2,\ i \neq j, \tag{9.2}$$

where we recall that $\mathscr{L}$ denotes "the probability measure of."

THEOREM 9.2. *If $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2$ are pairwise exchangeable and if*

$$\text{shape}(\mathbf{M}_1) = \text{shape}(\mathbf{M}_2),$$

then

$$\mathscr{L}(\text{shape}(\mathbf{GM}_1)) = \mathscr{L}(\text{shape}(\mathbf{GM}_2)). \tag{9.3}$$

PROOF. It is enough to fix a shape $\sigma = (s, w)$ and to demonstrate the theorem for $\mathbf{M}_1$ and $\mathbf{M}_2$ given by

$$\mathbf{M}_1 = \begin{bmatrix} x^{(1)} & y^{(1)} \\ 1 & 0 \end{bmatrix},$$

$$\mathbf{M}_2 = \begin{bmatrix} x^{(j)} & y^{(j)} \\ 1 & 0 \end{bmatrix}, \quad \text{for some } j = 2, 3, \dots, 12.$$

We prove (9.3) for $j = 7$; the other cases are entirely similar. Let $x \equiv \frac{1}{2}(1 + s)$, $y \equiv \frac{1}{2}w$ and $r^2 \equiv x^2 + y^2$. Then

$$\mathbf{GM}_1 = \begin{bmatrix} ax + b & ay \\ cx + d & cy \end{bmatrix},$$

$$\mathbf{GM}_2 = \begin{bmatrix} a(1 - xr^{-2}) + b & ayr^{-2} \\ c(1 - xr^{-2}) + d & cyr^{-2} \end{bmatrix}$$

and, with

$$\mathbf{U} \equiv \begin{bmatrix} -xr^{-1} & yr^{-1} \\ -yr^{-1} & xr^{-1} \end{bmatrix},$$

we see that

$$r\mathbf{GM}_2\mathbf{U}' = \begin{bmatrix} -(a + b)x + a & -(a + b)y \\ -(c + d)x + c & -(c + d)y \end{bmatrix}.$$

The transformation $\mathbf{GM}_2 \to r\mathbf{GM}_2\mathbf{U}'$ involves only a scaling and a rotation, so

$$\text{shape}(r\mathbf{GM}_2\mathbf{U}') = \text{shape}(\mathbf{GM}_2).$$

Under the exchangeability condition,

$$\mathscr{L}(r\mathbf{GM}_2\mathbf{U}') = \mathscr{L}(\mathbf{GM}_1).$$

Accordingly,

$$\mathscr{L}(\text{shape}(\mathbf{GM}_2)) = \mathscr{L}(\text{shape}(\mathbf{GM}_1)),$$

as required. □

If the pairwise exchangeability condition (9.2) holds, the process of shapes $\{\sigma_n,\ n = 0, 1, \dots\}$ has the Markov property, as the following construction of a single transition shows. If $\sigma_{n-1} = (s, w) \in \mathbf{S}$, then

$$\sigma_n = \text{shape}(\mathbf{G}_n\mathbf{M}),$$

where

$$\mathbf{M} \equiv \begin{bmatrix} \frac{1}{2}(1+s) & \frac{1}{2}w \\ 1 & 0 \end{bmatrix}.$$

9.2. *The general case.* Even when the exchangeability condition (9.2) is not assumed, we can still define a Markovian shape process $\{\tilde{\sigma}_n,\ n = 0, 1, \dots\}$ for which the asymptotic behaviour of $\mathscr{L}(\tilde{\sigma}_n)$ is the same as that of $\mathscr{L}$(shape $(\mathbf{Z}_n)$), $n \to \infty$. To avoid a lengthy discussion, some of the formal details are omitted.

The method is to define a transition between shapes $\sigma_0 \to \sigma_1$ that includes the following intermediate steps:

$$\sigma_0 \to \mathbf{Z}_0 \to \mathbf{G}_1\mathbf{Z}_0 \to \sigma_1,$$

where $\mathbf{Z}_0$ is a matrix with shape σ_0 and where shape$(\mathbf{G}_1\mathbf{Z}_0) = \sigma_1$. The last two steps $\mathbf{Z}_0 \to \mathbf{G}_1\mathbf{Z}_0$ and $\mathbf{G}_1\mathbf{Z}_0 \to \sigma_1$ are well defined. The first step is not: there are many $\mathbf{Z}_0$ (with the given shape σ_0) to choose from. We can restrict the choice of $\mathbf{Z}_0$ to the 12 matrices:

$$\tilde{\mathbf{Z}}^{(j)} \equiv \begin{bmatrix} x^{(j)} & y^{(j)} \\ 1 & 0 \end{bmatrix}, \qquad j = 1, 2, \dots, 12,$$

where

$$\left(x^{(j)}, y^{(j)}\right) = \left(x^{(j)}(\sigma_0), y^{(j)}(\sigma_0)\right).$$

The obvious thing to do is to define a random variable J that takes values in $\{1, 2, \dots, 12\}$, where $\Pr[J = j] = P_j(\sigma_0)$ depends only on σ_0, and then choose $\tilde{\mathbf{Z}}^{(j)}$ with probability $P_j(\sigma_0)$. We show that $P_j(\sigma_0)$ can be chosen so that

$$\lim_{n\to\infty} \mathscr{L}(\tilde{\sigma}_n) = \lim_{n\to\infty} \mathscr{L}(\text{shape}(\mathbf{Z}_n)). \tag{9.4}$$

In general, however, it is *not* the case that

$$\mathscr{L}(\tilde{\sigma}_0, \tilde{\sigma}_1, \dots, \tilde{\sigma}_n) = \mathscr{L}(\text{shape}(\mathbf{Z}_0), \text{shape}(\mathbf{Z}_1), \dots, \text{shape}(\mathbf{Z}_n)), \quad \text{for each } n.$$

Recall the definition (3.3):

$$\tilde{\mathbf{Z}}_n \equiv \begin{bmatrix} x_n & y_n \\ 1 & 0 \end{bmatrix}.$$

We know that if condition (2.5) holds, then (Theorem 3.2) $x_n \to \tau$ in probability and $y_n \to 0$ a.s. as $n \to \infty$. A triangle whose vertices are collinear with coordinates $(0,0)$, $(\tau, 0)$ and $(1,0)$ has shape $(\tau^*, 0)$, where

$$\tau^* = \begin{cases} |(1+\tau)|/(1-\tau), & \text{if } \tau < 0, \\ |2\tau - 1|, & \text{if } 0 \le \tau \le 1, \\ |(2-\tau)|/\tau, & \text{if } \tau > 1. \end{cases}$$

Because $\sigma_n = (s_n, w_n) = \text{shape}(\mathbf{Z}_n) = \text{shape}(\tilde{\mathbf{Z}}_n)$, it follows that $s_n \to \tau^*$ in probability and $w_n \to 0$ a.s. Recall that ψ is the probability measure of τ and let ψ^* be the probability measure of τ^*. ψ^* is a probability measure on the unit interval, the space of shapes of triangles whose vertices are collinear.

For $0 \le s \le 1$, let

$$x_1(s) \equiv \frac{1+s}{2}, \qquad x_2(s) \equiv \frac{1-s}{2}, \qquad x_3(s) \equiv \frac{2}{1+s},$$
$$x_4(s) \equiv -\frac{1-s}{1+s}, \qquad x_5(s) \equiv \frac{2}{1-s}, \qquad x_6(s) \equiv -\frac{1+s}{1-s}.$$

For $\delta > 0$, let $N \equiv (s-\delta, s+\delta)$ be a neighborhood of s and let $N_i \equiv x_i(N)$ be the image of N under x_i. We suppose that the following limits exist:

$$p_i(s) \equiv \lim_{\delta \downarrow 0} \psi(N_i)/\psi^*(N), \qquad i = 1, 2, \dots, 6.$$

If $0 < s < 1$ and if δ is small enough to ensure that N is included in the unit interval, then

$$\psi^*(N) = \sum_{i=1}^{6} \psi(N_i).$$

It follows that

$$\sum_{i=1}^{6} p_i(s) = 1.$$

We suppose that the limits $\lim_{w \downarrow 0} P_j(\sigma)$ exist, where $\sigma = (s, w)$, and we choose

$$\lim_{w \downarrow 0} P_{2j-1}(\sigma) = p_j(s), \qquad j = 1, 2, \dots, 6,$$
$$\lim_{w \downarrow 0} P_{2j}(\sigma) = p_j(s), \qquad j = 1, 2, \dots, 6.$$

Let B be a Borel subset of the unit interval and let $\boldsymbol{Q}^*(s, B)$ be the probability measure of the shape of the triangle whose vertices are collinear and whose coordinates are $(0,0)$, $(as+b, 0)$ and $(cs+d, 0)$. We define

$$\check{\boldsymbol{Q}}(s, B) \equiv \sum_{i=1}^{6} p_i(s) \boldsymbol{Q}^*(s_i, B),$$

where $s_i = x_i(s)$. We choose $\check{\boldsymbol{Q}}(s, B)$ as the transition probability measure for the "collinear" shape process. [Under exchangeability, $\boldsymbol{Q}^*(s_i, B) = \boldsymbol{Q}^*(s_1, B)$,

$i = 2, 3, \ldots, 6$]. It is easily checked that ψ^* is the unique probability measure that satisfies

$$\psi^*(B) = \int_0^1 \psi^*(ds)\check{\boldsymbol{Q}}(s, B). \tag{9.5}$$

(9.4) follows from (9.5).

Under exchangeability, the formula for the Lyapounov exponent γ, written in terms of ψ^*, becomes

$$\gamma = \int_0^1 H(t)\,\psi^*(dt).$$

Acknowledgments. We wish to thank the two referees for their careful reading of the manuscript, their helpful comments and suggestions for a number of improvements.

REFERENCES

[1] BILLINGSLEY, P. (1979). *Probability and Measure.* Wiley, New York.

[2] BOUGEROL, P. and LACROIX, J. (1985). *Products of Random Matrices with Applications to Schrödinger Operators.* Birkhäuser, Boston.

[3] CHASSAING, P., LETAC, G. and MORA, M. (1984). Brocot sequences and random walks on Sl(2, $\boldsymbol{R}$). *Probability Measures on Groups VII. Lecture Notes in Math.* **1064** 36–48. Springer, Berlin.

[4] COHEN, J. E. and NEWMAN, C. M. (1984). The stability of large random matrices and their products. *Ann. Probab.* **12** 283–310.

[5] FURSTENBERG, H. (1963). Non-commuting random products. *Trans. Amer. Math. Soc.* **108** 377–428.

[6] GOODALL, C. (1991). Eigenshape analysis of a cut-grow mapping for triangles, and its application to phyllotaxis in plants. *SIAM J. Appl. Math.* **51** 735–798.

[7] GUIVARC'H, Y. and RAUGI, A. (1985). Frontière de Furstenberg, propriétés de contraction et théorèmes de convergence. *Z. Wahrsch. Verw. Gebiete* **69** 187–242.

[8] KENDALL, D. G. (1984). Shape manifolds, procrustean metrics, and complex projective spaces. *Bull. London Math. Soc.* **16** 81–121.

[9] KENDALL, D. G. (1985). Exact distributions for shapes of random triangles in convex sets. *Adv. in Appl. Probab.* **17** 308–329.

[10] KENDALL, D. G. and LE, H.-L. (1986). Exact shape-densities for random triangles in convex polygons. *Adv. in Appl. Probab.* **18** 59–72.

[11] KENDALL, D. G. (1989). The statistical theory of shape. *Statist. Sci.* **4** 87–120.

[12] KEY, E. (1987). Computable examples of the maximal Lyapunov exponent. *Probab. Theory Related Fields* **75** 97–107.

[13] MANNION, D. (1988). A Markov chain of triangle shapes. *Adv. in Appl. Probab.* **20** 348–370.

[14] MANNION, D. (1990). Convergence to collinearity of a sequence of random triangle shapes. *Adv. in Appl. Probab.* **22** 831–844.

[15] MANNION, D. (1990). The invariant distribution of a sequence of random collinear triangle shapes. *Adv. in Appl. Probab.* **22** 845–865.

[16] POLLARD, D. (1984). *Convergence of Stochastic Processes.* Springer, New York.

DEPARTMENT OF MATHEMATICS
ROYAL HOLLOWAY
UNIVERSITY OF LONDON
EGHAM, SURREY TW20 0EX
ENGLAND