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AN INEQUALITY FOR GREEDY LATTICE ANIMALS

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Let $\{X_v: v \in \mathbb{Z}^d\}$ be i.i.d. positive random variables with

$$E\left\{X_0^d (\log^+ X_0)^{d+\varepsilon}\right\} < \infty$$

for some $\varepsilon > 0$ and $d \geq 2$. Define M_n and N_n by

$$M_n = \max\left\{ \sum_{v \in \pi} X_v : \pi \text{ a self-avoiding path of length } n \text{ starting at the origin} \right\},$$

$$N_n = \max\left\{ \sum_{v \in \xi} X_v : \xi \text{ a lattice animal of size } n \text{ containing the origin} \right\}.$$

Then it has been shown that there exist $M < \infty$ and $N < \infty$ such that

$$\frac{M_n}{n} \to M \quad \text{and} \quad \frac{N_n}{n} \to N \quad \text{a.s. and in } L^1.$$

In this paper we show that $M = N$ if and only if X_0 has bounded support and $P\{X_0 = R\} \geq p_c$, where R is the right end point of support of X_0 and p_c is the critical probability for site percolation on $\mathbb{Z}^d$.

1. Introduction and statement of results. Let $\mathbb{Z}^d$ be a d-dimensional cubic lattice. $x \in \mathbb{Z}^d$ is called a *vertex* and the origin is denoted by 0. The *distance* between $x \in \mathbb{Z}^d$ and $y \in \mathbb{Z}^d$ is defined by

$$\|x - y\| = \sum_{i=1}^{d} |x_i - y_i|.$$

π, a sequence $(v_1, \ldots, v_n)$ in $\mathbb{Z}^d$, is a *path* if $\|v_{i+1} - v_i\| = 1$ for $i = 1, \ldots, n-1$, and n is the *length* of the path $\pi = (v_1, \ldots, v_n)$ and denoted by $|\pi|$. Note that the length $|\pi|$ of a path π is not defined in a usual way because we count not the edges but the vertices that are contained in the path. If a path $\pi = (v_1, \ldots, v_n)$ satisfies $v_i \neq v_j$ for all $i \neq j$, it is called *self-avoiding*.

ξ, a subset of $\mathbb{Z}^d$, is a *lattice animal* (or *connected*) if there is a path $\pi = (v_1 = x, v_2, \ldots, v_{n-1}, v_n = y)$ in ξ for any $x \in \xi$ and $y \in \xi$, and the *size* of ξ is the cardinality of the lattice animal ξ and denoted by $|\xi|$.

$x \in \mathbb{Z}^d$ is *adjacent* to $W \subset \mathbb{Z}^d$ if x is not in W but there exists $y \in W$ such that $\|x - y\| = 1$.

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Let $\{Y_v: v \in \mathbb{Z}^d\}$ be i.i.d. Bernoulli random variables with a parameter p; that is,

$$Y_0 = \begin{cases} 1, & \text{with probability } p, \\ 0, & \text{with probability } 1 - p. \end{cases}$$

We take $d \geq 2$ to avoid trivialities, because percolation theory is trivial for the case $d = 1$. Consider the random subset of $\mathbb{Z}^d$ that is obtained by deleting all vertices v for which $Y_v = 0$. The connected component of this subset that contains the origin is denoted by C. The fundamental theorem of percolation theory is that there exists $0 < p_c < 1$ such that

$$P_p\{|C| = \infty\} = \begin{cases} 0, & \text{if } p < p_c, \\ > 0, & \text{if } p > p_c, \end{cases}$$

where p_c is called the *critical probability* for site percolation on $\mathbb{Z}^d$ [see Grimmett (1989), Chapter 1, for more details].

Let $\{X_v: v \in \mathbb{Z}^d\}$ be i.i.d. positive random variables, where again we take $d \geq 2$ to avoid trivialities, because greedy lattice animal theory, which was developed by Cox, Gandolfi, Griffin and Kesten (1993) and by Gandolfi and Kesten (1993), reduces to the strong law of large numbers for the case $d = 1$. Cox, Gandolfi, Griffin and Kesten (1993) introduce

$$M_n = \max\{S(\pi): \pi \text{ a self-avoiding path of length } n \text{ starting at the origin}\},$$

$$N_n = \max\{S(\xi): \xi \text{ a lattice animal of size } n \text{ containing the origin}\},$$

where $S(\pi) = \sum_{v \in \pi} X_v$ and $S(\xi) = \sum_{v \in \xi} X_v$, and Gandolfi and Kesten (1994) show that there exist $M < \infty$ and $N < \infty$ such that

$$\frac{M_n}{n} \to M \quad \text{and} \quad \frac{N_n}{n} \to N \quad \text{a.s. and in } L^1 \tag{1}$$

under the moment condition $E\{X_0^d(\log^+ X_0)^{d+\varepsilon}\} < \infty$ for some $\varepsilon > 0$. They also point out that the argument of Theorem 7.4 in Smythe and Wierman (1978) shows that

$$EX_0 < M \leq N \tag{2}$$

when X_0 is not concentrated on one point. In the same paper they mention the problem "Do there exist $\{X_v: v \in \mathbb{Z}^d\}$ such that $M < N$?"

In this paper we give the answer to this problem. Our results are as follows.

THEOREM 1. *If X_0 has unbounded support, then $M < N$.*

THEOREM 2. *Let X_0 have bounded support and let $R = \inf\{r \geq 0: X_0 \leq r$ a.s.$\}$. If $P\{X_0 = R\} < p_c$, then $M < N < R$.*

THEOREM 3. *Let X_0 have bounded support and let R be as in Theorem 2. If $P\{X_0 = R\} \geq p_c$, then $M = N = R$.*

In the following sections we examine the three cases separately. Before we start, it is worthwhile to point out that during the argument for Theorem 2 we get a strengthening of the following result of Kesten [(1980b), Theorem 1].

THEOREM 4 (Kesten). *Let $\{Y_v: v \in \mathbb{Z}^d\}$ be i.i.d. positive random variables with $P\{Y_0 = 0\} < p_c$, where p_c is the critical probability for site percolation on $\mathbb{Z}^d$. Then there exist two constants $a > 0$ and $C_5 > 0$ such that*

$$P\left\{\begin{array}{l}\text{there is a self-avoiding path } \pi \text{ of length } |\pi| \geq n \\ \text{starting at the origin such that } \sum_{v\in\pi} Y_v \leq an\end{array}\right\} \leq 2e^{-C_5 n}.$$

Our stronger form is as follows.

THEOREM 5. *Let $\{Y_v: v \in \mathbb{Z}^d\}$ be i.i.d. positive random variables with $P\{Y_0 = 0\} < p_c$, where p_c is the critical probability for site percolation on $\mathbb{Z}^d$. Then there exist two constants $b > 0$ and $C_6 > 0$ such that*

$$P\left\{\begin{array}{l}\text{there is a lattice animal } \xi \text{ of size } |\xi| \geq n \\ \text{containing the origin such that} \\ \sum_{v\in\xi} Y_v \leq bn\end{array}\right\} \leq 4e^{-C_6 n}.$$

2. Proof of Theorem 1. In this section we assume that X_0 has unbounded support. Before starting our work, we need some notation. Let π_n be a self-avoiding path of length n starting at the origin, for which $S(\pi_n)$ achieves M_n. There may be several such self-avoiding paths. However, by giving a deterministic order to the set of all self-avoiding paths of length n starting at the origin, we can choose π_n uniquely as the first self-avoiding path in the given order, of length n and starting at the origin for which $S(\pi_n)$ achieves M_n. From now on when we say π_n, we mean this optimal path.

The basic idea of the proof is the following. Consider π_n. There may be several vertices v, adjacent to π_n, for which X_v has a high value. If there is a strictly positive frequency (at least in expectation) of such vertices along π_n, we construct a lattice animal ζ_n from the optimal path π_n by attaching vertices v, adjacent to π_n, for which X_v has a high value. On the other hand, there may be several vertices $v \in \pi_n$, for which X_v has a low value and such that $\pi_n \setminus \{v\}$ is still connected. If there is a strictly positive frequency (at least in expectation) of such vertices along π_n, we construct a lattice animal ζ_n from the optimal path π_n by removing vertices $v \in \pi_n$, for which X_v has a low value and $\pi_n \setminus \{v\}$ is still connected. Note that ζ_n is not of size n. However, this surgery has a strictly positive impact on N because we attach only vertices v for which X_v has a high value to π_n and we remove only vertices v for which X_v has a low value from π_n, and from this one can easily see $M < N$. So the major step in the proof is to show that there is a strictly positive frequency along π_n (at least in expectation) of vertices v, adjacent to π_n, for which X_v has a high value or of vertices $v \in \pi_n$, for which X_v has a low value and for which $\pi_n \setminus \{v\}$ is still connected. We use a block construction technique for this.

We start with a large deviation estimate for the binomial distribution.

LEMMA 1. *There exists an $s > 0$ such that*

$$P\left\{\begin{array}{l}\textit{there is a self-avoiding path } \pi \textit{ of length } n \textit{ starting at the}\\ \textit{origin in which there are more than } (1/10)(1/5^d)\, n\\ \textit{vertices } v \textit{ for which } X_v > s\end{array}\right\} \le e^{-n}. \tag{3}$$

PROOF. For a fixed self-avoiding path π of length n starting at the origin, Chebyshev's inequality gives

$$\begin{aligned}&\exp\left(t\frac{1}{10}\frac{1}{5^d}n\right)P\left\{\begin{array}{l}\text{there are more than } (1/10)(1/5^d)n\\ \text{vertices } v \in \pi \text{ for which } X_v > s\end{array}\right\}\\ &\quad\le Ee^{tS_n} = \left(Ee^{tI_1}\right)^n,\end{aligned} \tag{4}$$

where $I_1, I_2, \dots$ are i.i.d. with the common distribution

$$I_1 = \begin{cases}1, & \text{with probability } P\{X_0 > s\},\\ 0, & \text{with probability } P\{X_0 \le s\},\end{cases} \tag{5}$$

and $t > 0$ is chosen explicitly.

Because there are at most $(2d)^n$ distinct self-avoiding paths π of length n starting at the origin,

$$\begin{aligned}&P\left\{\begin{array}{l}\text{there is a self-avoiding path } \pi \text{ of length } n \text{ starting at the}\\ \text{origin in which there are more than } (1/10)(1/5^d)n\\ \text{vertices } v \text{ for which } X_v > s\end{array}\right\}\\ &\quad\le (2d)^n e^{-t(1/10)(1/5^d)n}\left(Ee^{tI_1}\right)^n\\ &\quad= \exp\left\{-n\left[\frac{1}{10}\frac{1}{5^d}t - \log 2d - \log\left(e^tP\{X_0 > s\} + P\{X_0 \le s\}\right)\right]\right\},\end{aligned} \tag{6}$$

by (4). We choose $t > 0$ such that $(1/10)(1/5^d)t - \log 2d - \log 2 = 1$, and then we choose $s > 0$ so large that $e^tP\{X_0 > s\} \le 1$. For these choices of t and s, the lemma follows from (6). □

To carry out the proof we need a slightly different large deviation estimate for the binomial distribution that can be easily justified by the argument of Lemma 1 with certain changes for the choices of t and s. We need some definitions to state the next lemma in an appropriate form.

Let B_x be the $5 \times \cdots \times 5$ box of the form

$$B_x = \left\{v = (v_1, \dots, v_d) \in \mathbb{Z}^d : 5x_i \le v_i < 5(x_i + 1) \text{ for } 1 \le i \le d\right\}.$$

We call $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ the *corner vertex* of the box B_x. If the choice of the corner vertex is unimportant, we abbreviate B_x by B. It is important to note that the corner vertices have to lie on $(5\mathbb{Z})^d$ and that these boxes are pairwise disjoint; that is, $B_x \cap B_y = \varnothing$ if $x \ne y$.

For each box B we define

$$\partial B = \{v \in B \colon v_i = 5x_i \text{ or } v_i = 5(x_i + 1) - 1 \text{ for some } i\},$$
$$\dot{B} = \{v = (v_1 = 5x_1 + 1, v_2 = 5x_2 + 2, \ldots, v_d = 5x_d + 2)\},$$
$$\mathring{B} = B \setminus (\partial B \cup \dot{B})$$

and call these the *boundary*, the *peak*, and the *interior* of the box B, respectively. Note that the interior $\mathring{B}$ of the box B is not defined in the usual way because we exclude one vertex, the peak $\dot{B}$ of the box B, from the "common" definition of the interior. The peak $\dot{B}$ (see Figure 1) plays a special role in our proof: If there exist a path segment of π_n from ∂B to $\dot{B}$ and another path segment of π_n from $\dot{B}$ to ∂B, then π_n must contain a vertex v in $\mathring{B}$. That is why we distinguish the peak vertex from the rest of the vertices in the "common" interior.

Let π be a self-avoiding path. We define π_B by

$$\pi_B = \cup \{B_x \colon B_x \cap \pi \neq \varnothing\}.$$

LEMMA 2. *There exists an $s > 0$ such that*

$$(7)\quad P\left\{\begin{matrix}\textit{there is a self-avoiding path } \pi \textit{ of length } n \textit{ starting at the}\\ \textit{origin such that there are more than } (1/10)(1/5^d)n\\ \textit{vertices } v \in \pi_B \textit{ for which } X_v > s\end{matrix}\right\} \le e^{-n}.$$

PROOF. For a fixed self-avoiding path π of length n starting at the origin and $t > 0$, Chebyshev's inequality gives

$$\exp\left(t\frac{1}{10}\frac{1}{5^d}n\right)P\left\{\begin{matrix}\text{there are more than } (1/10)(1/5^d)n\\ \text{vertices } v \in \pi_B \text{ for which } X_v > s\end{matrix}\right\} \le (Ee^{tI_1})^{5^d n}$$

because $|\pi_B| \le 5^d n$ and $Ee^{tI_1} \ge 1$, where I_1 is as in (5). Now the argument is exactly the same as that of Lemma 1 except for the choices of t and s. This

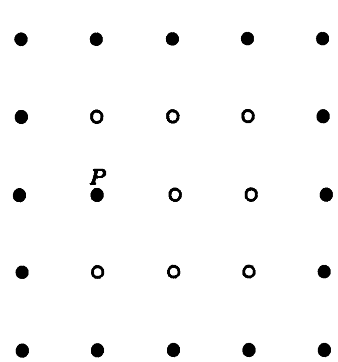

FIG. 1. *An illustration of a good box for $d = 2$. P is the peak, ● represents a high value of X_v $[> (5^d s_0) \vee (5^d N)]$ and ○ represents a low value of X_v $(\le E\{X_0\})$. Note that we require a high value at the peak.*

time we choose $t > 0$ such that $(1/10)(1/5^d)t - \log 2d - 5^d \log 2 = 1$, and then we choose $s > 0$ so large that $e^t P\{X_0 > s\} \leq 1$. $\square$

We next define *good* boxes B, and use Lemma 2 to show that there is a strictly positive frequency (at least in expectation) of good boxes that π_n meets.

Fix an $s_0 > 0$ that satisfies (7). Let B be a $5 \times \cdots \times 5$ box. We say that X_v has a *good configuration* on B (or B is a *good box*) if

$$\begin{aligned} &X_v > (5^d s_0) \vee (5^d N) && \text{for } v \in \partial B \cup \dot{B}, \\ &X_v \leq E\{X_0\} && \text{for } v \in \mathring{B}. \end{aligned}$$

LEMMA 3. *There exists a constant $C_1 > 0$ such that*

$$\sum_B P\{\pi_n \text{ meets } B \text{ and } X_v \leq s_0 \text{ for all } v \in B\} \geq C_1 n. \tag{8}$$

PROOF. For large n, say $n \geq n_0$,

$$P\left\{\begin{array}{l} \text{there is a self-avoiding path } \pi \text{ of length } n \\ \text{starting at the origin such that there are} \\ \text{more than } (1/10)(1/5^d)n \text{ vertices} \\ v \in \pi_B \text{ for which } X_v > s_0 \end{array}\right\} \leq \frac{1}{2}, \tag{9}$$

by Lemma 2. Because $\sum_B P\{\pi_n$ meets B and $X_v \leq s_0$ for all $v \in B\}$ is the expected number of $5 \times \cdots \times 5$ boxes that π_n meets and in which $X_v \leq s_0$ for all v, and because π_n meets at least $n/5^d$ boxes B, we have for $n \geq n_0$,

$$\sum_B P\{\pi_n \text{ meets } B \text{ and } X_v \leq s_0 \text{ for all } v \in B\} \geq \frac{1}{2}\left(\frac{n}{5^d} - \frac{1}{10}\frac{1}{5^d}n\right), \tag{10}$$

by (9). Clearly there exists a constant $C_2 > 0$ such that

$$\sum_B P\{\pi_n \text{ meets } B \text{ and } X_v \leq s_0 \text{ for all } v \in B\} \geq C_2 n \tag{11}$$

for $n < n_0$. The lemma follows from (10) and (11) with $C_1 = (1/2)(9/10) \times (1/5^d) \wedge C_2$. $\square$

LEMMA 4. *There exists a constant $C_3 > 0$ such that*

$$P\left\{\begin{array}{l} \pi_n \text{ meets } B \text{ and } B \text{ has a} \\ \text{good configuration} \end{array}\right\} \geq C_3 P\left\{\begin{array}{l} \pi_n \text{ meets } B \text{ and } X_v \leq s_0 \\ \text{for all } v \in B \end{array}\right\} \tag{12}$$

for any box B. Moreover, there exists a constant $C_4 > 0$ such that

$$\sum_B P\{\pi_n \text{ meets } B \text{ and } B \text{ has a good configuration}\} \geq C_4 n. \tag{13}$$

PROOF. Fix n and B, and let $\{X'_v\colon v \in \mathbb{Z}^d\}$ be i.i.d. positive random variables that are also independent of $\{X_v\colon v \in \mathbb{Z}^d\}$ and that have the same distribution as $\{X_v\colon v \in \mathbb{Z}^d\}$. Define $\{X^*_v\colon v \in \mathbb{Z}^d\}$ by

$$X^*_v = \begin{cases} X_v, & \text{for } v \in (\mathbb{Z}^d \setminus B), \\ X'_v, & \text{for } v \in B. \end{cases} \tag{14}$$

Let π^*_n be the optimal path for the X^*_v values; that is, the first self-avoiding path in our given order, of length n, starting at the origin and for which $S^*(\pi^*_n) := \sum_{v \in \pi^*_n} X^*_v$ achieves $M^*_n := \max\{\sum_{v \in \pi} X^*_v\colon \pi$ a self-avoiding path of length n starting at the origin$\}$. Assume that π_n meets B and $X_v \le s_0$ for all $v \in B$ for the X_v values. Moreover, assume that B is a good box for the X'_v values; that is,

$$X'_v > (5^d s_0) \wedge (5^d N) \quad \text{for } v \in \partial B \cup \dot{B},$$
$$X'_v \le E\{X_0\} \qquad \text{for } v \in \mathring{B}.$$

We claim that in this situation π^*_n still meets B. To prove this, consider any self-avoiding path π of length n starting at the origin that does not meet B. By (14) and by the definition of π_n,

$$\sum_{v \in \pi} X^*_v = \sum_{v \in \pi} X_v \le \sum_{v \in \pi_n} X_v. \tag{15}$$

On the other hand, because π_n meets B, $X_v \le s_0$ for $v \in B$ and $X'_v > 5^d s_0$ for $v \in \partial B$,

$$\begin{aligned} \sum_{v \in \pi_n} X_v &= \sum_{v \in \pi_n \setminus B} X_v + \sum_{v \in \pi_n \cap B} X_v \\ &\le \sum_{v \in \pi_n \setminus B} X_v + 5^d s_0 \\ &< \sum_{v \in \pi_n \setminus B} X^*_v + \sum_{v \in \pi_n \cap B} X^*_v \\ &= \sum_{v \in \pi_n} X^*_v, \end{aligned} \tag{16}$$

by (14). Combining (15) and (16), we see that π^*_n meets B. Therefore, we get (12):

$$\begin{aligned} &P\{\pi_n \text{ meets } B \text{ and } B \text{ is a good box for the } X_v \text{ values}\} \\ &\quad = P\{\pi^*_n \text{ meets } B \text{ and } B \text{ is a good box for the } X^*_v \text{ values}\} \\ &\quad \ge P\left\{\begin{matrix} \pi_n \text{ meets } B \text{ and } X_v \le s_0 \text{ for } v \in B \text{ for the } X_v \text{ values,} \\ \text{and } B \text{ is a good box for the } X'_v \text{ values} \end{matrix}\right\} \\ &\quad = C_3 P\{\pi_n \text{ meets } B \text{ and } X_v \le s_0 \text{ for } v \in B\}, \end{aligned}$$

where $C_3 = P\{X_0 > (5^d s_0) \vee (5^d N)\}^{5^d - 3^d + 1} P\{X_0 \le E\{X_0\}\}^{3^d - 1}$. Equation (13) follows from Lemma 3 and (12) with $C_4 = C_1 C_3$. □

Lemma 4 shows that there is a strictly positive frequency in expectation of good boxes B that π_n meets. When π_n meets B and B is a good box, two possible cases arise. The first case is that π_n does not contain all vertices in $\partial B \cup \dot{B}$, and the second case is that π_n does. Note that if π_n contains all vertices in $\partial B \cup \dot{B}$, if π_n does not start from $\mathring{B}$ and if π_n does not end in $\mathring{B}$, then there exist a path segment of π_n from ∂B to $\dot{B}$ and another path segment of π_n from $\dot{B}$ to ∂B and π_n must contain a vertex v in $\mathring{B}$, and hence must pick up a low value of X_v at such a vertex v. In the first case we construct a lattice animal η_n from the optimal path π_n by attaching one vertex v in $\partial B \cup \dot{B}$ with a high value of X_v, and in the second case by removing one vertex v in $\mathring{B}$ with a low value of X_v. After surgery it should be clear that $N > M$ because we attach only high-valued vertices to π_n and remove only low-valued vertices from π_n.

PROOF OF THEOREM 1. For each n we say B *is type* 1_n if π_n meets B, B is a good box and $\pi_n \not\supset \partial B \cup \dot{B}$. We say B *is type* 2_n if π_n meets B, B is a good box and $\pi_n \supset \partial B \cup \dot{B}$. We denote by $K_i^{(n)}$ the number of boxes of type i_n that π_n meets. Then Chebyshev's inequality gives

$$\left(n - \tfrac{2}{3}C_4 n\right)P\left\{K_1^{(n)} < \tfrac{1}{3}C_4 n \text{ and } K_2^{(n)} < \tfrac{1}{3}C_4 n\right\} \le E\left\{n - \left(K_1^{(n)} + K_2^{(n)}\right)\right\}$$
$$\le n - C_4 n$$

for $n \ge 1$ by (13). Consequently, for $n \ge 1$,

$$P\left\{K_1^{(n)} < \frac{1}{3}C_4 n \text{ and } K_2^{(n)} < \frac{1}{3}C_4 n\right\} \le \frac{1 - C_4}{1 - \frac{2}{3}C_4}. \tag{17}$$

Now let us do surgery as outlined just before starting the proof. If type 1_n is dominant, that is, $K_1^{(n)} \ge \frac{1}{3}C_4 n$, then choose the first $\lceil \frac{1}{3}C_4 n \rceil$ boxes B of type 1_n appearing as we travel through π_n. For each such box B, give a deterministic order to the vertices in B and choose the first vertex $v \in (\partial B \cup \dot{B} \setminus \pi_n)$ in this order that is adjacent to π_n, and attach this vertex v to π_n. If type 1_n is not dominant but type 2_n is, that is, $K_1^{(n)} < \frac{1}{3}C_4 n$ and $K_2^{(n)} \ge \frac{1}{3}C_4 n$, then choose the first $\lceil \frac{1}{3}C_4 n \rceil - 1$ boxes B of type 2_n appearing as we travel through π_n. For each such box B there exists a vertex $v \in (\mathring{B} \cap \pi_n)$ such that, when we remove it from π_n, $\pi_n \setminus \{v\}$ is still connected. To see this, observe that because $\dot{B}$ is at the middle of the path π_n (note that we choose not the $\lceil \frac{1}{3}C_4 n \rceil$ boxes B of type 2_n but the $\lceil \frac{1}{3}C_4 n \rceil - 1$ boxes B of type 2_n, and that π_n does not start from $\mathring{B}$ and π_n does not end in $\mathring{B}$), that is, there exist a path segment of π_n from ∂B to $\dot{B}$ and another path segment of π_n from $\dot{B}$ to ∂B, and because there is only one vertex in ∂B that is adjacent to $\dot{B}$, there is a vertex $v \in (\mathring{B} \cap \pi_n)$ just before or just after $\dot{B}$ along π_n. If we remove this vertex v from π_n, $\pi_n \setminus \{v\}$ is still connected because $\pi_n \supset \partial B \cup \dot{B}$. For each such box B, give a deterministic order to the vertices in B and choose the first vertex $v \in (\mathring{B} \cap \pi_n)$ in this order such that, when we remove it from π_n, $\pi_n \setminus \{v\}$ is still connected, and remove this vertex v from π_n. If neither type 1_n nor type 2_n is dominant, then just leave π_n alone. After surgery we get a lattice animal ξ_n from the optimal path π_n. Note that ξ_n is not of size n in general.

Choose ε and δ such that

$$(18)\qquad \begin{gathered} \frac{(M-\varepsilon)+(5^d s_0 \vee 5^d N)\frac{1}{3}C_4}{1+\frac{1}{3}C_4} \wedge \frac{(M-\varepsilon)-E\{X_0\}\frac{1}{3}C_4}{1-\frac{1}{3}C_4} > M+\varepsilon, \\ \frac{1-C_4}{1-\frac{2}{3}C_4}+\delta < 1, \end{gathered}$$

and then choose n_1 such that

$$(19)\qquad P\left\{\left|\frac{M_n}{n}-M\right| < \varepsilon \text{ and } \left|\frac{N_n}{n}-N\right| < \varepsilon\right\} > \frac{1-C_4}{1-\frac{2}{3}C_4}+\delta$$

for $n \ge n_1$. The choices of such ε and δ are possible because

$$(20)\qquad \begin{gathered} \frac{M+(5^d s_0 \vee 5^d N)\frac{1}{3}C_4}{1+\frac{1}{3}C_4} \wedge \frac{M-E\{X_0\}\frac{1}{3}C_4}{1-\frac{1}{3}C_4} > M, \\ \frac{1-C_4}{1-\frac{2}{3}C_4} < 1. \end{gathered}$$

In (20) we use $E\{X_0\} < M$ and $M < 5^d N$ [see (2)]. Also the choice of such n_1 is possible because of (1). By (17) and (19),

$$(21)\qquad P\left\{\begin{array}{l} |M_n/n - M| < \varepsilon, |N_n/n - N| < \varepsilon \text{ and} \\ \text{either } K_1^{(n)} \ge \frac{1}{3}C_4 n \text{ or } K_2^{(n)} \ge \frac{1}{3}C_4 n \end{array}\right\} > \delta$$

for $n \ge n_1$. Now choose $n_2 \ge n_1$ such that

$$(22)\qquad \begin{aligned} &\frac{(M-\varepsilon)n+(5^d s_0 \vee 5^d N)\lceil\frac{1}{3}C_4 n\rceil}{n+\lceil\frac{1}{3}C_4 n\rceil} \\ &\quad\wedge \frac{(M-\varepsilon)n - E\{X_0\}(\lceil\frac{1}{3}C_4 n\rceil - 1)}{n-(\lceil\frac{1}{3}C_4 n\rceil - 1)} > M+\varepsilon \end{aligned}$$

for $n \ge n_2$. The choice of such n_2 is possible because of (18).

Now fix $n \ge n_2$. If $|M_n/n - M| < \varepsilon$, $|N_n/n - N| < \varepsilon$ and $K_1^{(n)} \ge \frac{1}{3}C_4 n$, then by (22),

$$(23)\qquad \begin{aligned} \frac{N_{|\xi_n|}}{|\xi_n|} &\ge \frac{\sum_{v\in\xi_n} X_v}{|\xi_n|} \\ &= \frac{\sum_{v\in\pi_n} X_v + \sum_{v\in\xi_n\setminus\pi_n} X_v}{|\xi_n|} \\ &> \frac{(M-\varepsilon)n+(5^d s_0 \vee 5^d N)\lceil\frac{1}{3}C_4 n\rceil}{n+\lceil\frac{1}{3}C_4 n\rceil} \\ &> M+\varepsilon. \end{aligned}$$

If $|M_n/n - M| < \varepsilon$, $|N_n/n - N| < \varepsilon$, $K_1^{(n)} < \frac{1}{3}C_4 n$ and $K_2^{(n)} \geq \frac{1}{3}C_4 n$, then again by (22),

$$\begin{aligned} \frac{N_{|\xi_n|}}{|\xi_n|} &\geq \frac{\sum_{v \in \xi_n} X_v}{|\xi_n|} \\ &= \frac{\sum_{v \in \pi_n} X_v - \sum_{v \in \pi_n \setminus \xi_n} X_v}{|\xi_n|} \\ &> \frac{(M - \varepsilon)n - E\{X_0\}(\lceil \frac{1}{3}C_4 n \rceil - 1)}{n - (\lceil \frac{1}{3}C_4 n \rceil - 1)} \\ &> M + \varepsilon. \end{aligned} \tag{24}$$

Combining (21) with (23) and (24), we see that $P\{N_n/n > M + \varepsilon\} > \delta/2$ for infinitely many n, because either $P\{N_{n+\lceil(1/3)C_4 n\rceil}/(n + \lceil \frac{1}{3}C_4 n \rceil) > M + \varepsilon\} > \delta/2$ or $P\{N_{n-\lceil(1/3)C_4 n\rceil+1}/(n - \lceil \frac{1}{3}C_4 n \rceil + 1) > M + \varepsilon\} > \delta/2$ for $n \geq n_2$. This is impossible if $M = N$ because of (1). Therefore, $M < N$. □

3. Proof of Theorem 2. In this section we assume that X_0 has bounded support and $P\{X_0 = R\} < p_c$, where $R = \inf\{r \geq 0\colon X_0 \leq r \text{ a.s.}\}$. If $P\{X_0 = R\} = 0$, we get Lemma 2 with no extra work. However, to get Lemma 4 we need some additional conditions that are not important in our further discussions. So if $P\{X_0 = R\} = 0$ with the additional conditions, we can apply the argument of Theorem 1 and reach the conclusion by Theorem 5 and Lemma 5. To get rid of the additional conditions we try to modify the argument of Theorem 1 by replacing a $5 \times \cdots \times 5$ box by a $k \times \cdots \times k$ box, for some large k, in which we require high values on the thick boundary and the peak, and low values on the interior. However, the argument of Theorem 1 does not work well for the general case of Theorem 2 because there may be a positive mass $P\{X_0 = R\} > 0$ at the right end point R of support of X_0 and Lemma 2 is not true in this case. It turns out that a suitable modification for the general case is the replacement of a $5 \times \cdots \times 5$ box by a $5 \times \cdots \times 5$ box with a path attached in which we require high values on the boundary, the peak and the attached path, and low values on the interior (see Figure 2). Let us start with proving Theorem 5, which is used in showing $M \leq N < R$.

Let B_x^L be the $L \times \cdots \times L$ box of the form

$$B_x^L = \{v = (v_1, \dots, v_d) \in \mathbb{Z}^d\colon Lx_i \leq v_i < L(x_i + 1) \text{ for } 1 \leq i \leq d\},$$

and let D_x^{3L} be the $3L \times \cdots \times 3L$ box of the form

$$D_x^{3L} = \bigcup_{|y_i - x_i| \leq 1} B_y^L.$$

So B_x, which is defined in Section 2, is B_x^5 in this new notation, and D_x^{3L} is the disjoint union of B_y^L boxes with the center box B_x^L that forms a box of size $(3L)^d$. We call $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ the *corner vertex* of the L-box B_x^L. If the choice of the corner vertex is unimportant, we abbreviate B_x^L by B^L.

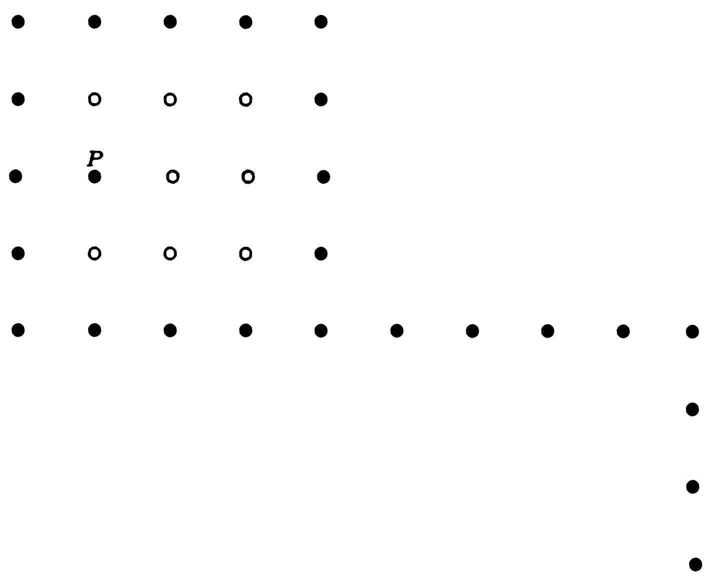

FIG. 2. *An illustration of an excellent box with an attached excellent path for $d = 2$. P is the peak, ● represents a high value of X_v ($> r_0$) and ○ represents a low value of X_v ($\le E\{X_0\}$). Note that we require a high value at the peak.*

Again we note that $B_x^L \cap B_y^L = \emptyset$ for $x \neq y$. The distance between B_x^L and B_y^L is defined by

$$\|B_x^L - B_y^L\| = \|x - y\| = \sum_{i=1}^{d} |x_i - y_i|.$$

PROOF OF THEOREM 5. For a fixed lattice animal ξ of size $|\xi| \ge n$ containing the origin, we cover ξ by ρ ($\ge n/L^d$) boxes B_x^L of size L^d. $\{x \in \mathbb{Z}^d: B_x^L \cap \xi \neq \emptyset\}$ form a lattice animal $\bar{\xi}$ of size ρ. Construct a spanning tree τ for $\bar{\xi}$ with a root 0. Then τ has ρ vertices and hence $\rho - 1$ edges. Construct a path $\pi = (v_1 = 0, v_2, \dots, v_k)$ (which is not necessarily self-avoiding) starting at the origin that contains all vertices in τ with (at most twice as many edges as τ, and hence) at most $2\rho - 1$ vertices [see Durrett, Kesten and Waymire (1991), Section 2, for the explicit construction of such π]. Because π visits all ρ vertices in $\bar{\xi}$ and any box D_x^3 contains exactly 3^d vertices, we can deterministically construct a sequence $x_1, \dots, x_m$, $m = \lfloor \rho/3^d \rfloor$, such that $B_{x_i}^L \cap \xi \neq \emptyset$ for $i = 1, \dots, m$, and $D_{x_i}^{3L} \cap D_{x_j}^{3L} = \emptyset$ for $i \neq j$, in the following way: Choose $x_1 = v_1 = 0$. Assume $x_i = v_{i_0}$ is chosen. Then x_{i+1} is the first vertex in π after v_{i_0} such that $D_{x_{i+1}}^{3L} \cap \bigcup_{\alpha=1}^{i} D_{x_\alpha}^{3L} = \emptyset$. So for large n, say $n \ge n_3$ with $\lceil (n_3/L^d)/6^d \rceil \le \lfloor (n_3/L^d)/3^d \rfloor$, we can deterministically construct a sequence $x_1, \dots, x_l$, $l = \lceil \rho/6^d \rceil$, such that $B_{x_i}^L \cap \xi \neq \emptyset$ for $i = 1, \dots, l$ and

$D_{x_i}^{3L} \cap D_{x_j}^{3L} = \varnothing$ for $i \neq j$ by taking the initial piece of the sequence $x_1, \dots, x_m$ of length $\lceil \rho/6^d \rceil$. Define $t(\partial B_x^L, \partial D_x^{3L})$ by

$$t(\partial B_x^L, \partial D_x^{3L}) = \inf\{S(\pi) \colon \pi \text{ a self-avoiding path from } \partial B_x^L \text{ to } \partial D_x^{3L} \text{ in } D_x^{3L}\}.$$

Because there is a self-avoiding path η_{x_i} from $\partial B_{x_i}^L$ to $\partial D_{x_i}^{3L}$ in $\xi \cap D_{x_i}^{3L}$, and because $\sum_{v \in \xi} Y_v \le bn$ implies $\sum_{i=1}^{l} t(\partial B_{x_i}^L, \partial D_{x_i}^{3L}) \le bn$ by $D_{x_i}^{3L} \cap D_{x_j}^{3L} = \varnothing$ for $i \neq j$, we have for $n \ge n_3$,

$$P\left\{\begin{array}{l}\text{there is a lattice animal } \xi \text{ of size } |\xi| \ge n\\ \text{containing the origin such that}\\ \sum_{v \in \xi} Y_v \le bn\end{array}\right\} \le \sum_{\rho \ge n/L^d} \sum_{x_1, \dots, x_l} P\left\{\sum_{i=1}^{l} t(\partial B_{x_i}^L, \partial D_{x_i}^{3L}) \le bn\right\}.$$

Furthermore, because $\{t(\partial B_{x_i}^L, \partial D_{x_i}^{3L}) \colon i = 1, \dots, l\}$ are i.i.d. random variables with the common distribution $t(\partial B_0^L, \partial D_0^{3L})$, for $n \ge n_3$,

$$P\left\{\begin{array}{l}\text{there is a lattice animal } \xi \text{ of size } |\xi| \ge n\\ \text{containing the origin such that}\\ \sum_{v \in \xi} Y_v \le bn\end{array}\right\}$$

$$\begin{aligned}
&\le \sum_{\rho \ge n/L^d} \sum_{x_1, \dots, x_l} P\left\{\sum_{i=1}^{l} t(\partial B_{x_i}^L, \partial D_{x_i}^{3L}) \le bn\right\}\\
&\le \sum_{\rho \ge n/L^d} \sum_{x_1, \dots, x_l} \exp\{\lambda bn\} E \exp\left\{-\lambda \sum_{i=1}^{l} t(\partial B_{x_i}^L, \partial D_{x_i}^{3L})\right\}\\
(25) \qquad &\le \sum_{\rho \ge n/L^d} \sum_{x_1, \dots, x_l} \exp\{\lambda bn\} \left[E \exp\{-\lambda t(\partial B_0^L, \partial D_0^{3L})\}\right]^l\\
&\le \sum_{\rho \ge n/L^d} (2d)^{2\rho - 2} \exp\{\lambda bn\} \left[E \exp\{-\lambda t(\partial B_0^L, \partial D_0^{3L})\}\right]^{\lceil \rho/6^d \rceil}\\
&\le \sum_{\rho \ge n/L^d} (4d^2)^{\rho} \exp\{\lambda b L^d \rho\} \left[E \exp\{-\lambda t(\partial B_0^L, \partial D_0^{3L})\}\right]^{\rho/6^d}\\
&= \sum_{\rho \ge n/L^d} \left[4d^2 \exp\{\lambda b L^d\} \left[E \exp\{-\lambda t(\partial B_0^L, \partial D_0^{3L})\}\right]^{1/6^d}\right]^{\rho}.
\end{aligned}$$

Because $P\{Y_0 = 0\} < p_c$ it follows from Menshikov (1986) or Aizenman and Barsky (1987) that P {there exists a path η from 0 to ∂D_0^{3L} with $Y_v = 0$, $v \in \eta$} decreases exponentially in L [see Grimmett (1989), Chapter 3, for more details]. Therefore, we can choose L so large that

$$4d^2 \left[P\{t(\partial B_0^L, \partial D_0^{3L}) = 0\}\right]^{1/6^d} \le \tfrac{1}{4},$$

and by the dominated convergence theorem we can choose λ so large that

$$4d^2\left[E\exp\{-\lambda t(\partial B_0^L, \partial D_0^{3L})\}\right]^{1/6^d} \le \tfrac{1}{2}.$$

Finally, we can choose $b > 0$ so small that

$$4d^2\exp\{\lambda b L^d\}\left[E\exp\{-\lambda t(\partial B_0^L, \partial D_0^{3L})\}\right]^{1/6^d} \le \tfrac{3}{4}.$$

For these choices of L, λ and b, we have for $n \ge n_3$,

$$P\left\{\begin{array}{l}\text{there is a lattice animal } \xi \text{ of size } |\xi| \ge n\\ \text{containing the origin such that}\\ \sum_{v\in\xi} Y_v \le bn\end{array}\right\} \tag{26}$$
$$\le 4\exp\left\{-\frac{\log 4/3}{L^d}n\right\},$$

by (25). Clearly there exists $C_7 > 0$ such that

$$P\left\{\begin{array}{l}\text{there is a lattice animal } \xi \text{ of size } |\xi| \ge n\\ \text{containing the origin such that}\\ \sum_{v\in\xi} Y_v \le bn\end{array}\right\} \le 4e^{-C_7 n} \tag{27}$$

for $n < n_3$. Theorem 5 follows from (26) and (27) with $C_6 = (\log 4/3)/L^d \wedge C_7$. □

LEMMA 5. $M \le N < R$.

PROOF. Choose $0 < r < R$ such that $P\{r < X_0 \le R\} < p_c$. Define $\{Y_v: v \in \mathbb{Z}^d\}$ by

$$Y_v = \begin{cases} 0, & \text{if } r < X_v \le R,\\ 1, & \text{otherwise.}\end{cases} \tag{28}$$

By Theorem 5, there exist $0 < c < 1$, $C_8 > 0$ such that

$$P\left\{\begin{array}{l}\text{there is a lattice animal } \xi \text{ of size } |\xi| \ge n\\ \text{containing the origin such that}\\ \sum_{v\in\pi} Y_v \le cn\end{array}\right\} \le 4e^{-C_8 n} \tag{29}$$

for $n \ge 1$. So for large n, say $n \ge n_4$,

$$\begin{aligned} P&\left\{N_n \ge \frac{(n-\lfloor cn\rfloor)R + \lfloor cn\rfloor r}{n}\right\}\\ &= P\left\{\begin{array}{l}\text{there is a lattice animal } \xi \text{ of size } n \text{ containing}\\ \text{the origin such that } \sum_{v\in\xi} X_v \ge\\ (n-\lfloor cn\rfloor)R + \lfloor cn\rfloor r\end{array}\right\}\\ &\le P\left\{\begin{array}{l}\text{there is a lattice animal } \xi \text{ of size } n \text{ containing}\\ \text{the origin such that } \sum_{v\in\xi} Y_v \le cn\end{array}\right\}\\ &\le 4e^{-C_8 n}\\ &\le \tfrac{1}{2}\end{aligned}$$

by (28) and (29), because for each $v \in \xi$ with $Y_v = 1$ we can only pick up a contribution $X_v \le r$ to $\sum_{v \in \xi} X_v$. By (1), this proves $N \le (1-c)R + cr$ and hence $N < R$, because $0 < c < 1$ and $0 < r < R$. □

Lemma 6 corresponds to van den Berg and Kesten [(1993), Lemma (5.2)] with a slight modification, and it can be easily verified by an argument of Peierls. We skip the proof.

LEMMA 6. *Suppose that for each L all L-boxes are randomly colored black or white in such a way that the process* (*colors of* B_x^L, $x \in \mathbb{Z}^d$) *is translation invariant. Moreover, suppose that there is* $C_9 > 0$ (*independent of L*) *such that for each L and x the color of* B_x^L *is completely determined by the values of* X_v *on* $\cup\{B_y^L\colon \|x-y\| \le C_9\}$. *Finally, suppose that*

$$\lim_{L\to\infty} P\{B_0^L \text{ is black}\} = 1.$$

Then, for sufficiently large L there exist $\varepsilon = \varepsilon(L) > 0$ *and* $D = D(L) > 0$ *such that*

$$P\left\{\begin{array}{l}\text{there is a self-avoiding path } \pi \text{ of length } n \text{ starting at the}\\ \text{origin that visits at most } \varepsilon n \text{ distinct black } L\text{-boxes}\end{array}\right\} \le e^{-Dn}. \tag{30}$$

We next use Lemma 5 to define *excellent* boxes B and *excellent* paths π, and use Lemma 6 to show that there is a strictly positive frequency (at least in expectation) of excellent boxes with an attached excellent path that meets π_n.

Fix $M \le N < r_0 < R$; this can be done by Lemma 5. Let B be a $5 \times \cdots \times 5$ box. We say that X_v has an *excellent configuration* on B (or B is an excellent box) if

$$X_v > r_0 \quad \text{for } v \in \partial B \cup \dot{B},$$
$$X_v \le E\{X_0\} \quad \text{for } v \in \mathring{B}.$$

Also we say that X_v has an *excellent configuration* on π (or π is an excellent path) if

$$X_v > r_0 \quad \text{for } v \in \pi.$$

LEMMA 7. *There exist* L_0, n_5 *and* $C_{10} > 0$ *such that*

$$\sum_{B^{L_0}} P\{\pi_n \text{ meets } B^{L_0} \text{ and } B^{L_0} \text{ contains an excellent box } B\} \ge C_{10} n \tag{31}$$

for $n \ge n_5$.

PROOF. We color B_x^L black if B_x^L contains an excellent box B. Clearly

$$\lim_{L\to\infty} P\{B_0^L \text{ is black}\} = 1.$$

So by Lemma 6 there exist L_0, $\varepsilon = \varepsilon(L_0) > 0$ and $D = D(L_0) > 0$ such that

$$P\left\{\begin{array}{l}\text{there is a self-avoiding path } \pi \text{ of length } n \text{ starting at the}\\ \text{origin that visits at most } \varepsilon n \text{ distinct black } L_0\text{-boxes}\end{array}\right\} \le e^{-Dn}$$

for $n \ge 1$, and hence the lemma follows with $e^{-Dn_5} \le \frac{1}{2}$ and $C_{10} = \frac{1}{2}\varepsilon$. □

LEMMA 8. *Let L_0 and n_5 be as in Lemma 7. There exists a constant $C_{11} > 0$ such that*

$$(32)\qquad \begin{aligned} &P\left\{\begin{array}{l}\pi_n \text{ meets } B^{L_0}, B^{L_0} \text{ contains an excellent box } B\\ \text{and there is an excellent path } \pi \text{ in } B^{L_0}\\ \text{from } B \text{ to } \pi_n\end{array}\right\}\\ &\qquad \ge C_{11} P\left\{\begin{array}{l}\pi_n \text{ meets } B^{L_0} \text{ and } B^{L_0}\\ \text{contains an excellent}\\ \text{box } B\end{array}\right\}\end{aligned}$$

for $n \ge n_5$ and for any L_0-box B^{L_0}. Moreover, there exists a constant $C_{12} > 0$ such that

$$(33)\qquad \begin{aligned} &\sum_{B^{L_0}} P\left\{\begin{array}{l}\pi_n \text{ meets } B^{L_0}, B^{L_0} \text{ contains an excellent}\\ \text{box } B \text{ and there is an excellent path } \pi\\ \text{in } B^{L_0} \text{ from } B \text{ to } \pi_n\end{array}\right\}\\ &\qquad \ge C_{12} n\end{aligned}$$

for $n \ge n_5$.

PROOF. Fix $n \ge n_5$ and B^{L_0}. Assume that π_n meets B^{L_0} and B^{L_0} contains an excellent box B. There may be several such excellent boxes in B^{L_0}. However, by giving a deterministic order to the set of all boxes in B^{L_0}, we can choose B uniquely as the first excellent box in the given order. After the choice of B we choose vertices $x \in \partial B$ and $y \in \pi_n \cap B^{L_0} \setminus (\dot{B} \cup \mathring{B})$ such that

$$\|x - y\| = \min\left\{\|u - v\|: u \in \partial B \text{ and } v \in \pi_n \cap B^{L_0} \setminus \left(\dot{B} \cup \mathring{B}\right)\right\}.$$

There may be several such pairs. However, by giving a deterministic order to the set of all pairs in $B^{L_0} \times B^{L_0}$ we can choose (x, y) uniquely as the first such pair in the given order. Note that $x = y \in \partial B$ is possible. Finally we choose the path $\pi = (u_1 = x, u_2, \dots, u_{m-1}, u_m = y)$ of length $m = \|x - y\| + 1$ from x to y in which we first move $|x_1 - y_1|$ steps parallel to the first coordinate axis, then $|x_2 - y_2|$ steps parallel to the second coordinate axis and so on (see Figure 3). Note that π consists of the single vertex x if $x = y \in \partial B$. Also note that $B \cap \pi = \{x\}$: If there is $z \in B \cap \pi$ with $z \neq x$, then $\|z - y\| < |\pi| - 1 = \|x - y\|$ and so $z \in (\dot{B} \cup \mathring{B})$ by the choice of (x, y). Furthermore, the path segment of π from z to y must meet ∂B because $z \in (\dot{B} \cup \mathring{B})$ and $y \in B^{L_0} \setminus (\dot{B} \cup \mathring{B})$. Therefore, there exists $w \in \partial B$ such that $\|w - y\| < \|z - y\| < |\pi| - 1 = \|x - y\|$. This contradicts our choice of

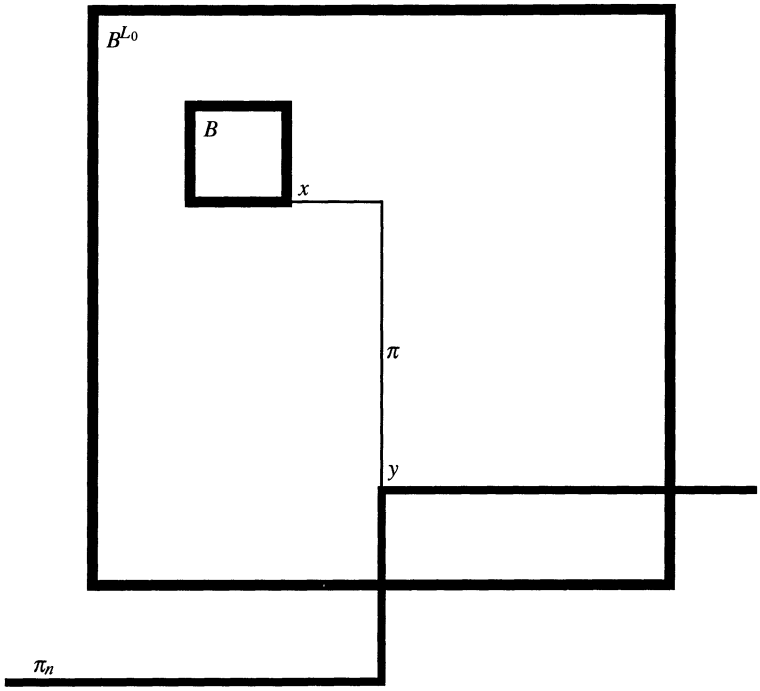

FIG. 3. *An illustration of a type $(B, \pi, W)_n$ L_0-box B^{L_0} for $d = 2$. The small box represents an excellent box B, and the big box represents B^{L_0} which contains B. The thick line represents π_n and the thin line represents an excellent path π from B to π_n. Remember the string W of 0's and 1's is attached to π.*

(x, y). For the chosen path $\pi = (u_1 = x, u_2, \dots, u_{m-1}, u_m = y)$, we define $W = (w_1, \dots, w_m)$ by

$$w_i = w_i(X_{u_i}) = \begin{cases} 1, & \text{if } X_{u_i} > r_0, \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, \dots, m$.

We say that B^{L_0} has *type* $(B, \pi, W)_n$ if π_n meets B^{L_0} and B^{L_0} contains an excellent box, and if B, π and W are the uniquely chosen excellent box, excellent path and string of random variables, respectively, as described before.

There exists a type $(B, \pi, W)_n$ such that

$$(34) \qquad P\left\{ \begin{matrix} B^{L_0} \text{ has type} \\ (B, \pi, W)_n \end{matrix} \right\} \geq \frac{1}{(L_0^d)^3 2^{L_0^d}} P\left\{ \begin{matrix} \pi_n \text{ meets } B^{L_0} \text{ and } B^{L_0} \\ \text{contains an excellent box} \end{matrix} \right\},$$

because there are at most $(L_0^d)^3 2^{L_0^d}$ distinct types.

For our given $n \geq n_5$ and B^{L_0} fix a type $(B_0, \pi_0, W_0)_n$ that satisfies (34) and assume that B^{L_0} has type $(B_0, \pi_0, W_0)_n$. Let $\{X'_v \colon v \in \mathbb{Z}^d\}$ be i.i.d. positive random variables that are also independent of $\{X_v: v \in \mathbb{Z}^d\}$ and that have the same distribution as $\{X_v: v \in \mathbb{Z}^d\}$. Define $\{X_v^*: v \in \mathbb{Z}^d\}$ by

$$X_v^* = \begin{cases} X_v', & \text{if } v \in \pi_0 \text{ and } X_v \le r_0, \\ X_v, & \text{otherwise.} \end{cases} \tag{35}$$

Note that $\{X_v^*: v \in B_0\} = \{X_v: v \in B_0\}$ because $B_0 \cap \pi_0 = \{x_0\}$, $x_0 \in \partial B_0$ and B_0 is an excellent box.

Let π_n^* be the optimal path for the X_v^* values; that is, the first self-avoiding path in our given order, of length n, starting at the origin and for which $S^*(\pi_n^*) := \sum_{v \in \pi_n^*} X_v^*$ achieves $M_n^* := \max\{\sum_{v \in \pi} X_v^*: \pi$ a self-avoiding path of length n starting at the origin$\}$. Assume that B^{L_0} has type $(B_0, \pi_0, W_0)_n$ for the X_v values. Moreover, assume that $X_v' > r_0$ for $v \in \pi_0$ and $X_v \le r_0$. Then in this situation π_n^* still meets $B_0 \cup \pi_0$ by the same argument as that of Lemma 4. Furthermore, $B_0 \cup \pi_0$ is an excellent box with an attached excellent path for the X_v^* values. Therefore,

$$\begin{aligned} &P\left\{\begin{array}{l}\pi_n \text{ meets } B^{L_0}, B^{L_0} \text{ contains an excellent box } B \text{ and there is} \\ \text{an excellent path } \pi \text{ in } B^{L_0} \text{ from } B \text{ to } \pi_n \text{ for the } X_v \text{ values}\end{array}\right\} \\ &\quad = P\left\{\begin{array}{l}\pi_n^* \text{ meets } B^{L_0}, B^{L_0} \text{ contains an excellent box } B \text{ and there is} \\ \text{an excellent path } \pi \text{ in } B^{L_0} \text{ from } B \text{ to } \pi_n^* \text{ for the } X_v^* \text{ values}\end{array}\right\} \\ &\quad \ge P\left\{\begin{array}{l}B^{L_0} \text{ has type } (B_0, \pi_0, W_0)_n \text{ for the } X_v \text{ values} \\ \text{and } X_v' > r_0 \text{ for } v \in \pi_0 \text{ and } X_v \le r_0\end{array}\right\} \qquad (36) \\ &\quad \ge P\{X_0 > r_0\}^{L_0^d} P\{B^{L_0} \text{ has type } (B_0, \pi_0, W_0)_n\} \\ &\quad \ge \frac{P\{X_0 > r_0\}^{L_0^d}}{(L_0^d)^3 2^{L_0^d}} P\left\{\begin{array}{l}\pi_n \text{ meets } B^{L_0} \text{ and } B^{L_0} \\ \text{contains an excellent} \\ \textit{box } \text{B}\end{array}\right\}, \end{aligned}$$

by (34) and (35). Equation (32) follows from (36) with $C_{11} = (P\{X_0 > r_0\}^{L_0^d})/((L_0^d)^3 2^{L_0^d})$. Equation (33) follows from Lemma 7 and (32) with $C_{12} = C_{10}C_{11}$. □

PROOF OF THEOREM 2. Fix L_0 and n_5 as in Lemma 7 and Lemma 8 and let $n \ge n_5$. Assume that π_n meets B^{L_0}, B^{L_0} contains an excellent box B and there is an excellent path π in B^{L_0} from B to π_n. Again we can pick a unique excellent pair (B, π) by ordering the pairs in some deterministic way. We say (B, π) has *type* 1_n if π_n does not contain all vertexes in $\partial B \cup \dot{B} \cup \pi$, and we say (B, π) has *type* 2_n if π_n does. We denote by $K_i^{(n)}$ the number of pairs of type i_n that π_n meets. If type 1_n is dominant, that is, $K_1^{(n)} \ge \frac{1}{3}C_{12}n$, then choose the first $\lceil \frac{1}{3} C_{12} n \rceil$ pairs (B, π) of type 1_n appearing as we travel through π_n. For each such pair (B, π), give a deterministic order to the vertices in $B \cup \pi$ and choose the first vertex $v \in (\partial B \cup \dot{B} \cup \pi \setminus \pi_n)$ in the given order that is adjacent to π_n, and attach this vertex v to π_n. If type 1_n is not dominant but type 2_n is, that is, $K_1^{(n)} < \frac{1}{3} C_{12} n$ and $K_2^{(n)} \geq \frac{1}{3} C_{12} n$, then choose the first $\lceil \frac{1}{3} C_{12} n \rceil - 1$ pairs (B, π) of type 2_n appearing as we travel through π_n. For each such pair (B, π), give a deterministic order to the vertices in $B \cup \pi$ and choose the first vertex $v \in (\mathring{B} \cap \pi_n)$ in the given order such that, when we remove it from π_n, $\pi_n \setminus \{v\}$ is still connected, and remove this vertex v from π_n (note that we choose not the $\lceil \frac{1}{3} C_{12} n \rceil$ pairs (B, π) of type 2_n but the $\lceil \frac{1}{3} C_{12} n \rceil - 1$ pairs (B, π) of type 2_n, and that this choice guarantees the existence of such v; see the argument of Theorem 1). If neither type 1_n nor type 2_n is dominant, then just leave π_n alone. After surgery the rest is the same as that of Theorem 1. □

4. Proof of Theorem 3. In this section we assume that X_0 has bounded support and $P\{X_0 = R\} \geq p_c$, where $R = \inf\{r \geq 0: X_0 \leq r \text{ a.s.}\}$. The proof is a straightforward application of percolation theory and first-passage percolation theory.

PROOF OF THEOREM 3. If $P\{X_0 = R\} > p_c$ hold, there is an infinite cluster, that is, a connected subset of $\mathbb{Z}^d$, in which $X_v = R$ for v with probability 1. So we can construct an infinite self-avoiding path ϖ starting at the origin that stays in the infinite cluster after a fixed finite number of steps with probability 1. Therefore, $M = N = R$.

If $P\{X_0 = R\} = p_c$, define $\{Y_v: v \in \mathbb{Z}^d\}$ by

$$Y_v = R - X_v. \tag{37}$$

The time constant μ corresponding to $\{Y_v: v \in \mathbb{Z}^d\}$ is 0 by Kesten [(1980a), Theorem (6.1)] and by the equality $p_c = p_T$ because $P\{Y_0 = 0\} = P\{X_0 = R\} = p_c$. Now fix $\delta > 0$. Because $\mu = 0$, there exists with probability 1 a finite (but random) n_6 such that for each $n \geq n_6$ there exists a self-avoiding path π of length n, starting at the origin, with

$$\frac{\sum_{v \in \pi} Y_v}{n} < \delta. \tag{38}$$

Indeed, by definition of the time constant μ, there is for large n a path $\tilde{\pi}_n$ from the origin to $(n, 0, \dots, 0)$ with $\sum_{v \in \tilde{\pi}_n} Y_v < \delta n$, and we can take for π the initial piece of length n of $\tilde{\pi}_n$. From (37) and (38) we get

$$M_n \geq \frac{\sum_{v \in \pi} X_v}{n} = \frac{\sum_{v \in \pi} R - Y_v}{n} > R - \delta, \tag{39}$$

for $n \geq n_6$. Now let $n \to \infty$ in (39). Because δ is arbitrary, we get $M \geq R$ and hence $M = N = R$. □

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REFERENCES

AIZENMAN, M. and BARSKY, D. J. (1987). Sharpness of the phase transition in percolation models. *Comm. Math. Phys.* **108** 489–526.

COX, J. T., GANDOLFI, A., GRIFFIN, PH. S. and KESTEN, H. (1993). Greedy lattice animals I: Upper bounds. *Ann. Appl. Probab.* **3** 1151–1169.

DURRETT, R., KESTEN, H. and WAYMIRE, E. (1991). On weighted heights of random trees. *J. Theoret. Probab.* **4** 223–237.

GANDOLFI, A. and KESTEN, H. (1994). Greedy lattice animals II: Linear growth. *Ann. Appl. Probab.* **4**(1).

GRIMMETT, G. (1989). *Percolation.* Springer, New York.

KESTEN, H. (1980a). Aspects of first passage percolation. *École d'Été de Probabilités de Saint-Flour XIV. Lecture Notes in Math.* **1180** 125–264. Springer, New York.

KESTEN, H. (1980b). On the time constant and path length of first-passage percolation. *Adv. in Appl. Probab.* **12** 848–863.

MENSHIKOV, M. V. (1986). Coincidence of critical points in percolation problems. *Soviet Math. Dokl.* **33** 856–859.

SMYTHE, R. T. and WIERMAN, J. C. (1978). *First-Passage Percolation on the Square Lattice VIII. Lecture Notes in Math.* **671**. Springer, New York.

VAN DEN BERG, J. and KESTEN, H. (1993). Inequalities for the time constant in first-passage percolation. *Ann. Appl. Probab.* **3** 56–80.

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