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A NEW MARTINGALE IN BRANCHING RANDOM WALK[1]

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Dedicated to the memory of Michel Métivier

Martingale methods have played an important role in the theory of Galton-Watson processes and branching random walks. The (random) Fourier transform of the position of the particles in the nth generation, normalized by its mean, is a martingale. Under second moments assumptions on the branching this has been very useful to study the asymptotics of the branching random walk. Using a different normalization, we obtain a new martingale which is in L^2 under weak assumptions on the displacement of the particles and strong assumptions on the branching.

1. Introduction. The branching random walk can be described briefly as follows: We consider the random tree $\mathscr{T}$ generated by a Galton–Watson process and a family of independent identically distributed random variables X_τ indexed by the nodes τ of $\mathscr{T}$. Let $\mathscr{T}_n$ denote the nodes (individuals) of the nth generation and $\leq$ denote the partial ordering on the tree ($\tau_1 \leq \tau_2$ if τ_1 is an ancestor of τ_2). Then the position of the node τ will be given by

$$S_\tau = \sum_{\xi \leq \tau} X_\xi .$$

The family $\{S_\tau\}_{\tau \in \mathscr{T}}$ describes the branching random walk. For more details, the reader is referred to Joffe and Moncayo [6] or Neveu [10], who provided a complete description of the probability space of the process with the notion of marked tree, the marks here being the position of the particles.

Let Z_n denote the cardinality of $\mathscr{T}_n$. Then the model is completely described by the law $\{p_n\}$ of the number of children of each individual and the law of X_τ. Let m be the mean of the number of children and assume that it is finite. (In order to avoid conditioning on nonextinction, we will assume that p_0 is null and of course $p_1 \neq 1$.) $\phi(\theta)$ will denote the characteristic function of X_τ. The natural filtration is given by $\mathscr{F}_n$, the σ-field generated by (X_τ, τ) $\tau \in \mathscr{T}_k$, $k \leq n$. We are concerned mainly with the study of the asymptotic behaviour of the random measure

$$\mu_n(\cdot) \doteq \sum_{\tau \in \mathscr{T}_n} \delta_{S_\tau}(\cdot), \tag{1.1}$$

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where δ_x is the Dirac measure at the point x. Through the Fourier transform, Theorem 1.3 describes the asymptotics of the positions of the particles of the nth generation.

Martingale methods have played an important role in the theory of Galton–Watson processes: Z_n/m^n was seen to be a martingale by Doob and its limit W has been studied by many authors. The interested reader may consult Athreya and Ney [1] and Grey [4].

In 1967, S. Watanabe and A. Joffe noticed independently that

$$W_n(\theta) = \frac{1}{m^n \phi^n(\theta)} \sum_{\tau \in \mathscr{T}_n} e^{i\theta S_\tau} \tag{1.2}$$

is, for each θ such that $\phi(\theta) \neq 0$, a martingale. The convergence of this martingale for fixed θ has been studied by many authors, but it is much more useful to establish the preceding results with some uniformity in θ. It was conjectured in [6] and proved in [7] that such a result holds for $W_n(\theta)$. Under a second moment assumption on Z_1 and a Lipschitz condition on $\phi(\theta)$ (which is satisfied in particular if X_τ has second moment), $W_n(\theta)$ converges almost surely uniformly in some neighborhood of zero. For continuous time models Uchiyama [13] obtained similar results under different assumptions. Very recently Biggins [2], [3], for a slightly more general model, showed that such a result holds under the weaker assumption on Z_1: $EZ_1 \log Z_1 < \infty$ in dimension one and $EZ_1^{1+\alpha} < \infty$, $\alpha > 0$ otherwise, and a stronger assumption on X_τ, namely, that its Laplace transform exists. Moreover, he obtains the uniform convergence of $W_n(\theta)$ for θ belonging to some compact set of the complex plane. This phenomenon of balancing assumptions between the branching and the motion has been noticed in [6] in the more general setting of nonrandom trees. In [2] and [3], Biggins used methods of complex variables based on the Cauchy integral formula. This explains the assumption about the existence of the Laplace transform of the X. The method used in Joffe, Le Cam and Neveu [7] relies on L_2 techniques applied to martingales taking values in a Banach space of continuous functions. This explains the assumption on the second moment of Z_1. In the present work, we will obtain the almost sure uniform convergence by introducing a new martingale whose L_2 norm will be finite under the existence of the $(1 + \alpha)$ moment of Z_1, $\alpha > 0$.

2. A new martingale. It is easy to see that if in (1.2) one normalizes by the total population instead of its average, then

$$V_n(\theta) = \frac{1}{Z_n \phi^n(\theta)} \sum_{\tau \in \mathscr{T}_n} e^{i\theta S_\tau} \tag{2.1}$$

is still a martingale. Indeed

$$V_{n+1}(\theta) = \frac{1}{Z_n \phi^n(\theta)} \sum_{\tau \in \mathscr{T}_n} e^{i\theta S_\tau} \frac{Z_n}{Z_{n+1}} \sum_{\xi \in \tau^+} \frac{e^{i\theta X_\xi}}{\phi(\theta)},$$

where τ^+ denotes the set of children of τ and $|\tau^+|$ denotes its cardinality. Because $\Sigma_{\tau \in \mathscr{T}_n}|\tau^+| = Z_{n+1}$, we get by symmetry

$$E^{\mathscr{T}_n}\frac{|\tau|^+}{Z_{n+1}} = \frac{1}{Z_n}.$$

Taking conditional expectations first with respect to $\mathscr{T}_n$ and $(|\tau^+|)$, $\tau \in \mathscr{T}_n$, then with respect to $\mathscr{T}_n$, we obtain $E^{\mathscr{T}_n}V_{n+1}(\theta) = V_n(\theta)$.

The next step is to show that (2.1) is bounded in L^2. This follows from Lemma 1, where the genealogy structure of the Galton–Watson tree plays a crucial role. We recall from [6] the definition of $\alpha(n,k)$, the number of pairs of individuals of the nth generation whose first common ancestor is in the kth generation:

$$\alpha(n,k) = \text{card}\{(\tau,\tau') \in \mathscr{T}_n \times \mathscr{T}_n : \tau \wedge \tau' \in \mathscr{T}_k\}, \tag{2.2}$$

where $\wedge$ denotes the inf. Note that $\alpha(n,n) = Z_n$. Let ξ_i denote a sequence of i.i.d. random variables with distribution p_n. Define γ_n by

$$\gamma_n = \frac{\Sigma_1^n \xi_i^2}{(\Sigma_1^n \xi_i)^2}, \quad \text{where } \gamma_n \le \frac{1}{n}. \tag{2.3}$$

Then Lemma 1 is easy to establish.

LEMMA 1. *The following relations hold for $k \le n-1$:*

$$\alpha(n+1,k) = \sum_{\{(\tau,\tau') \in \mathscr{T}_n \times \mathscr{T}_n : \tau \wedge \tau' \in \mathscr{T}_k\}} |\tau^+|\,|\tau'^+|, \tag{2.4}$$

$$\alpha(n+1,n) = \sum_{\tau \in \mathscr{T}_n} |\tau^+|(|\tau^+| - 1), \tag{2.5}$$

$$\alpha(n+1,n+1) = Z_{n+1}, \tag{2.6}$$

$$E^{\mathscr{T}_n}\frac{\alpha(n+1,k)}{Z_{n+1}^2} = \frac{1-\gamma_{Z_n}}{Z_n(Z_n-1)}\alpha(n,k) \le \frac{\alpha(n,k)}{Z_n^2}, \tag{2.7}$$

$$E^{\mathscr{T}_n}\frac{\alpha(n+1,n)}{Z_{n+1}^2} = \gamma_{Z_n} - E^{Z_n}\frac{1}{Z_{n+1}}, \tag{2.8}$$

$$E^{\mathscr{T}_n}\frac{\alpha(n+1,n+1)}{Z_{n+1}^2} \doteq E^{Z_n}\frac{1}{Z_{n+1}}. \tag{2.9}$$

In particular $(\alpha(n,k))/Z_n^2$ is, for k fixed, a nonnegative supermartingale (for $n \ge k+1$) that converges, as n goes to infinity, to a limit denoted by π_k.

Let r_k be its expectation. Then

$$\lim E\frac{\alpha(n,k)}{Z_n^2} = r_k.$$

REMARK. The sequence $(\alpha(n,k))/m^{2n}$, $k \le n-1$, is itself a martingale under the assumption $E(Z_1^2) < \infty$. This explains why the role of the genealogy of the tree is not apparent in the study of W_n.

The covariance of $V_n(\cdot)$ is given by

$$EV_n(\theta_1)\overline{V_n(\theta_2)} = \sum_{k=0}^{n} E\frac{\alpha(n,k)}{Z_n^2}\left[\frac{\phi(\theta_1-\theta_2)}{\phi(\theta_1)\phi(-\theta_2)}\right]^k. \tag{2.10}$$

To study its asymptotics, we need more information on the behaviour of $E(\alpha(n,k))/Z_n^2$. From (2.7), we have $E(\alpha(n,k))/Z_n^2 \le E(\alpha(k+1,k))/Z_{k+1}^2$, but from (2.5) it follows that

$$E\frac{\alpha(k+1,k)}{Z_{k+1}^2} \le E\frac{\sum_{j=0}^{Z_k}\xi_j^2}{\left(\sum_{j=0}^{Z_k}\xi_j\right)^2}.$$

Now if we assume that $E\xi_j^p$ is finite for $p > 1$, it follows from [9] that $E\gamma_n$ is $O(1/n^{p-1})$. Therefore the last expression behaves as $E(1/Z_k^{p-1})$ for large k. Also it is shown in [5] that for any $\varepsilon > 0$, $E(1/Z_n^p)$ is $O([\max(p_1, 1/m^p)] + \varepsilon)^n$. We can summarize the preceding remarks by the following lemma.

LEMMA 2. *If one assumes that $EZ_1^{1+\alpha}$ is finite for $\alpha > 0$, then for any $\varepsilon > 0$, as n goes to ∞, $E(\alpha(n,k))/Z_n^2$ is $O(c^k + \varepsilon)$, where c is given by $c = \max(p_1, 1/m^{p-1})$.*

3. Limit theorems. From Lemma 2 the following theorem is easily established.

THEOREM 1. *If one assumes that $EZ_1^{1+\alpha}$ is finite for $\alpha > 0$, then on the set D defined by*

$$\left\{\theta : c\frac{1}{|\Phi(\theta)|^2} < 1\right\}$$

the limit, as n goes to infinity, of $EV_n(\theta_1)\overline{V_n(\theta_2)}$ exists and is given by

$$\sum_{k=0}^{\infty} r_k\left[\frac{\phi(\theta_1-\theta_2)}{\phi(\theta_1)\phi(-\theta_2)}\right]^k. \tag{3.1}$$

On D the martingale $V_n(\theta)$ converges almost surely, as well as in L^2. The covariance of this limit V is given by (3.1).

Now one can proceed as in [7] to establish the almost sure uniform convergence of V_n on any compact subset K of D. We summarize the argument as follows: By Theorem 1, we can find a countable dense subset of K for which there is almost sure convergence of V_n to V. If one assumes a Lipschitz condition on Φ' of order β, in particular, if $EX^{1+\beta}$ is finite, Theorem 1 yields an expression for $E|V(\theta + h) - V(\theta)|^2$ that can be easily seen to be $O(h^{1+\beta})$. Then by Kolmogorov's criterion there is a version of V that is almost surely continuous on K and $E\sup\{|V(\theta)|, \theta \in K\}$ is finite (see, e.g., [12], page 25). On the Banach space of complex valued continuous functions endowed with the sup norm, the theorem of vectorial martingales shows that $E_n^F V(\cdot)$ converges almost surely uniformly on K (see, e.g., [11], page 104). Because $V_n(\cdot)$ is continuous on K one must have $E^{F_n}V(\cdot) = V_n(\cdot)$. This establishes the following theorem:

THEOREM 2. *Under the assumptions of Theorem* 1, *if* $\Phi'(\cdot)$ *satisfies a Lipschitz condition, then the martingale* $V_n(\cdot)$ *converges almost surely uniformly on any compact subset* K *of* D *to a continuous process* $V(\cdot)$ *whose covariance function is given by*

$$EV(\theta_1)\overline{V(\theta_2)} = \sum_{k=0}^{\infty} r_k \left[\frac{\phi(\theta_1 - \theta_2)}{\phi(\theta_1)\phi(-\theta_2)} \right]^k.$$

Of course one gets a similar result for the convergence of W_n because $W_n = V_n(Z_n/m^n)$. It is well known that the uniform convergence of W_n in a neighbourhood of zero yields (using the Fourier inversion formula and the Fubini theorem) a proof of a Harris-type theorem (Joffe and Moncayo [6], Kaplan and Asmussen [8] and Biggins [3]).

THEOREM 3. *If* $EZ_1^{1+\alpha} < \infty$, $\alpha > 0$, *and* $\phi(\theta)$, *with mean* 0, *is in the domain of attraction of a stable law of order* $1 + \beta$, $\beta > 0$, *then the sequence* $\nu_n(B) = (1/Z_n)\mu_n(c_n B)$ *converges almost surely weakly to that stable law. The* c_n *are the normalizing constants for* $\phi(\theta)$.

However, the real challenge is to prove the statement in Theorem 2 under minimal conditions such as $EZ \log Z$ finite. Our limitation is due to the use of L^2 techniques.

NOTE. The preceding techniques do not extend to higher dimension d because the exponent required in the Kolmogorov criterion is $d + \beta$. For the same reason, we cannot work in the complex plane, but of course we can mimic the foregoing techniques with the Laplace transform in the real domain.

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