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QUADRATURE ROUTINES FOR LADDER VARIABLES[1]

BY ROBERT W. KEENER

University of Michigan

Let $T = \inf\{n \geq 1: S_n > 0\}$ and $H = S_T$ be ladder variables for a random walk $\{S_n\}_{n \geq 1}$ with nonnegative drift. Integral formulas for generating functions and moments of T, H and related quantities are developed. These formulas are suitable for numerical quadrature and should be easier to implement than formulas based on Spitzer's identity when the distribution of S_n is complicated. The approach used makes key use of the Hilbert transform and the main regularity assumption is that some power of the characteristic function for steps of the random walk is integrable.

1. Introduction. In recent years there has been great progress in the science of approximating expectations and probabilities associated with boundary crossing problems. These approximations typically depend on characteristics of the distribution of ladder variables for a random walk. In practice these characteristics can rarely be calculated analytically, and this paper will be concerned with formulas and methods for numerical calculation of these quantities.

Let $X, X_1, X_2, \ldots$ be i.i.d. with common characteristic function ϕ and mean $\mu \geq 0$. Let $S_n = \sum_1^n X_i$ and define

$$T = \inf\{n \geq 1: S_n > 0\} \quad \text{and} \quad H = S_T.$$

The computation of moments and generating functions for T and H and related quantities will be addressed.

The approach used makes essential use of the Hilbert transform, and a key assumption is that $\phi \in \mathscr{L}^q$, that is, $\int |\phi(t)|^q \, dt < \infty$, for some $q \in [1, \infty)$. This assumption implies that X is continuous and absolutely continuous if $q \leq 2$. Examples of singular continuous distributions with $\phi \in \mathscr{L}^q$ and $q > 2$ have been given by Wiener and Wintner (1936, 1938) [see Theorem 13.4.2 of Kawata (1972)]. There are absolutely continuous X with $\phi \notin \mathscr{L}^q$ [see examples (a) and (b), pages 516 and 517 of Feller (1971)], but these variables never seem to arise in practice.

One alternative to calculating some of these quantities is to use Spitzer's identity or related formulas. These tend to involve terms like $E[\exp\{itS_n\}; S_n \leq 0]$, where $E[Y; A] = E[Y 1_A]$. When the distribution of S_n is readily available (as in the normal case), these identities are often practical. In many other cases the distribution of S_n is complicated and the methods presented here will be more practical.

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Section 2 presents background material on the Hilbert transform and passing limits and derivatives inside bounded operators. Section 3 contains the main results about the first two moments of ladder variables for random walks with positive drift. Higher order moments are discussed in Section 4. In Section 5 variables that arise in the study of boundary crossing problems are studied. These variables include $M = \inf\{S_n: n \geq 0\}$ and a variable R with Lebesgue density $P(H > r)/EH$ for $r > 0$. Quadrature formulae for the mean and moment generating function of R have been given by Woodroofe (1979, 1982). The approach used in this paper seems to generalize more easily to higher moments, but the regularity conditions imposed are slightly more stringent. In Section 6 random walks without drift are considered. Formulae for the first three ladder height moments are derived. Formulae for the first two moments are given by Siegmund (1985). Again, the regularity conditions needed here are slightly more stringent, but extension to higher moments is more direct. In Section 7, numerical examples are presented.

2. The Hilbert transform and $\mathscr{L}^q$ differentiation. The Hilbert transform of a function f is defined by

$$\mathscr{H}f(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{|y| > \varepsilon} \frac{f(t-y)}{y}\, dy.$$

If $f \in \mathscr{L}^q$ for some $q \in [1, \infty)$, the limit exists for a.e. t and $\mathscr{H}$ is a bounded linear operator from $\mathscr{L}^q \to \mathscr{L}^q$ for $q > 1$. Stein (1970) has a nice discussion of singular operators like $\mathscr{H}$ including proofs of these assertions. When f is differentiable at t,

$$\mathscr{H}f(t) = \frac{1}{\pi} \int_0^\infty \frac{f(t-y) - f(t+y)}{y}\, dy. \tag{2.1}$$

This form is convenient for numerical quadrature since no limit need be taken. Of course this equation holds under weaker conditions than differentiability—by dominated convergence, (2.1) holds whenever the integrand is absolutely integrable.

Many identities derived later using $\mathscr{L}^q$ theory will only be assured of holding for a.e. t. Often, our primary interest will be for specific values of t. The concern that these values may include some of the almost nowhere points where the identities may fail can often be alleviated by continuity using the following standard lemma, which can be proved by a dominated convergence argument.

LEMMA 2.1. *If $f \in \mathscr{L}^q$ for some $q \in [1, \infty)$ and f is continuously differentiable in some neighborhood of t (locally Lipschitz near t would also be sufficient), then $\mathscr{H}f$ is continuous at t.*

In later results, Lemma 2.1 will provide continuity except at $t = 0$. Lemma 2.3 is specialized to the situations of interest and is more powerful than

Lemma 2.1 in the proper context. As a preliminary, we need the following standard result (called the adjoint relationship for the Hilbert transform).

LEMMA 2.2. *Assume $f \in \mathscr{L}^q$ for some $q \in [1, \infty)$ and that f is bounded. Let w be continuously differentiable with compact support C. If f is continuously differentiable on an open set S containing C, then*

$$\int w(t)\mathscr{H}f(t)\,dt = \int f(y)\mathscr{H}w(-y)\,dy.$$

LEMMA 2.3. *Suppose $f \in \mathscr{L}^q$ for some $q \in [1, \infty)$, f is bounded and f is continuously differentiable except at zero. If $\mathscr{H}f(t)$ has a limit as $t \to 0$ and if $(f(y) - f(-y))/y$ is locally integrable at zero, then $\mathscr{H}f(t)$ is continuous at zero.*

The proof of this result is given in the Appendix.

For $q \in [1, \infty)$, the $\mathscr{L}^q$ norm of a function f is defined as $\|f\|_q = [\int |f(t)|^q\,dt]^{1/q}$. Convergence in this norm will be called convergence in $\mathscr{L}^q$, so $f_h \to f$ in $\mathscr{L}^q$ as $h \to 0$ if $\lim_{h \to 0}\|f_h - f\|_q = 0$. A function f is called *differentiable in $\mathscr{L}^q$* if

$$\frac{f(\cdot + \varepsilon) - f(\cdot)}{\varepsilon} \to g$$

in $\mathscr{L}^q$ as $\varepsilon \to 0$ for some function $g \in \mathscr{L}^q$ called the $\mathscr{L}^q$ derivative of f.

LEMMA 2.4. *Assume $q \in [1, \infty)$ and that $\mathscr{O}\colon \mathscr{L}^q \to \mathscr{L}^q$ is a bounded operator. If $f_n \to f$ in $\mathscr{L}^q$ as $n \to \infty$, then $\mathscr{O}f_n \to \mathscr{O}f$ as $n \to \infty$. If f is differentiable in $\mathscr{L}^q$ with $\mathscr{L}^q$-derivative g, then $\mathscr{O}f$ is differentiable in $\mathscr{L}^q$ with $\mathscr{L}^q$-derivative $\mathscr{O}g$.*

PROOF. The first assertion says that $\mathscr{O}$ is continuous, and boundedness and continuity are equivalent for linear operators [see Theorem 44 of Bochner and Chandrasekharan (1949)]. The second assertion follows using the first assertion to pass difference quotient limits inside the operator. □

The following lemma gives a simple sufficient condition for a function f to be differentiable in $\mathscr{L}^q$. See Bochner and Chandrasekharan (1949) for a proof (they take $q = 2$, but their method of proof works easily for any $q \geq 1$).

LEMMA 2.5. *If $q \in [1, \infty)$, $f \in \mathscr{L}^q$ is absolutely continuous and $f' \in \mathscr{L}^q$, then f is differentiable in $\mathscr{L}^q$ with $\mathscr{L}^q$-derivative f'.*

The final lemma of this section gives conditions which allow differentiation with respect to s of expressions $\mathscr{O}f(t, s)$ to be performed inside the bounded operator $\mathscr{O}$. Here and later, for notational convenience, all functions to be transformed will be considered as functions of t and we will abuse notation writing $\mathscr{O}f(t, s)$ instead of $\mathscr{O}f(\cdot, s)(t)$.

LEMMA 2.6. *Let* $q \in [1,\infty)$ *and let* $\mathscr{O}\colon \mathscr{L}^q \to \mathscr{L}^q$ *be a bounded linear operator. Assume that a.e.* $t \in \mathbb{R}$ *the functions* $f(t,\cdot)$ *are absolutely continuous on* $(s_0 - \delta, s_0]$ *with derivative* $h(t,\cdot)$. *If* $h(t,s_0) = \lim_{s \uparrow s_0} h(t,s)$ *for a.e.* $t \in \mathbb{R}$ *and if* $\sup_{s \in [s_0-\delta, s_0]} |h(\cdot,s)| \in \mathscr{L}^q$, *then*

$$\frac{f(\cdot,s_0) - f(\cdot,s_0-\varepsilon)}{\varepsilon} \to h(\cdot,s_0)$$

in $\mathscr{L}^q$ *as* $\varepsilon \downarrow 0$. *Consequently,*

$$\frac{\mathscr{O}f(\cdot,s_0) - \mathscr{O}f(\cdot,s_0-\varepsilon)}{\varepsilon} \to \mathscr{O}h(\cdot,s_0)$$

in $\mathscr{L}^q$ *as* $\varepsilon \downarrow 0$.

PROOF.

$$\begin{aligned}\int&\left|\frac{f(t,s_0) - f(t,s_0-\varepsilon)}{\varepsilon} - h(t,s_0)\right|^q dt \\ &= \int\left|\int_0^1 [h(t,s_0-\varepsilon y) - h(t,s_0)]\,dy\right|^q dt \\ &\le \int_0^1 \int |h(t,s_0-\varepsilon y) - h(t,s_0)|^q\,dt\,dy.\end{aligned}$$

Since $\sup_{s \in [s_0-\delta,s_0]}|h(\cdot,s)| \in \mathscr{L}^q$, the inner integral goes to zero as $\varepsilon \downarrow 0$ by dominated convergence and the lemma follows. □

3. Ladder variables. Let

$$\beta(t,s) = Es^T e^{itH}$$

for $|s| \le 1$ and $t \in \mathbb{R}$. Our starting point is the following theorem that relates β to the Hilbert transform.

THEOREM 3.1. *If* $0 \le \mu = EX < \infty$ *and* $\phi \in \mathscr{L}^q$ *for some* $q \in [1,\infty)$, *then*

$$\beta(t,s) = 1 - \sqrt{1 - s\phi(t)}\exp\left\{\frac{i}{2}\mathscr{H}\log(1 - s\phi(t))\right\} \tag{3.1}$$

for all $(t,s) \in \mathbb{R} \times [0,1]$, $(t,s) \ne (0,1)$.

This result appears as Lemma 2.5 of Keener (1987). Since this article is not very accessible and the result plays a major role in this paper, a proof has been included in the Appendix.

The expression for β given in Theorem 3.1 is inconvenient for numerical work due to the pole at $t = 0$, $s = 1$ [the Hilbert transform in (3.1) is infinite there]. Let

$$\hat{\phi}(t) = \frac{1}{1 - it\mu},$$

the characteristic function for an exponential variable with mean μ. Also, let

$$\mathscr{A} = (i\mathscr{H} - \mathscr{I})/2,$$

where $\mathscr{I}$ is the identity operator. In the sequel, functions which have a limit as $t \to 0$ but are undefined when $t = 0$ will be extended by continuity. In particular, $[1 - \beta(t, s)]/[1 - s\phi(t)]$ equals $EH/\mu = ET$ at $(t, s) = (0, 1)$ and

$$\log\left[\frac{1 - s\phi(t)}{1 - s\hat{\phi}(t)}\right]$$

is zero at $(t, s) = (0, 1)$.

THEOREM 3.2. *If $\phi \in \mathscr{L}^q$ for some $q \in [1, \infty)$ and if $\mu \in (0, \infty)$, then*

$$\Lambda(t, s) \overset{\text{def}}{=} \log\left[\frac{1 - \beta(t, s)}{1 - s\phi(t)}\right] = \mathscr{A} \log\left[\frac{1 - s\phi(t)}{1 - s\hat{\phi}(t)}\right]$$

for all $(t, s) \in \mathbb{R} \times [0, 1]$. Taking $(t, s) = (0, 1)$,

$$ET = \exp\left\{-\frac{1}{\pi}\int_0^\infty \arg\left[\frac{1 - \phi(y)}{\mu y - i}\right]\frac{dy}{y}\right\}, \tag{3.2}$$

where arg *returns the argument of a complex number.*

PROOF. When X is positive, $(T, H) = (1, X_1)$ and Theorem 3.1 for random walks with $\phi = \hat{\phi}$ gives

$$i\mathscr{H}\log(1 - s\hat{\phi}(t)) = \log(1 - s\hat{\phi}(t)).$$

From this, the formula for Λ must hold for all $(t, s) \in \mathbb{R} \times [0, 1]$, $(t, s) \neq (0, 1)$. Since $\Lambda(t, 1)$ is continuous at $t = 0$ (by convention), the proof will be completed by showing that

$$\mathscr{A} \log\left[\frac{1 - \phi(t)}{1 - \hat{\phi}(t)}\right]$$

is continuous at $t = 0$. Since the contribution from the identity operator is continuous, it is sufficient to show that

$$\mathscr{H}\log\left[\frac{1 - \phi(t)}{1 - \hat{\phi}(t)}\right]$$

is continuous at $t = 0$. Since $\Lambda(t, 1)$ is continuous at $t = 0$, this will follow from Lemma 2.3 provided

$$\begin{aligned} &\left\{\log\left[\frac{1 - \phi(y)}{1 - \hat{\phi}(y)}\right] - \log\left[\frac{1 - \phi(-y)}{1 - \hat{\phi}(-y)}\right]\right\}\frac{1}{y} \\ &\quad = \left\{\log\left[-\frac{1 - \phi(y)}{1 - \bar{\phi}(y)}\right] - \log\left[-\frac{1 - \hat{\phi}(y)}{1 - \bar{\hat{\phi}}(y)}\right]\right\}\frac{1}{y} \end{aligned} \tag{3.3}$$

is locally integrable at zero (here $\bar{\phi}$ and $\bar{\hat{\phi}}$ are the complex conjugates of ϕ and $\hat{\phi}$). Define $R(y)$ by

$$\phi(y) = 1 + i\mu y + yR(y).$$

Since ϕ is differentiable, $R(y) \to 0$ as $y \to 0$. Then

$$\log\left[-\frac{1-\phi(y)}{1-\bar{\phi}(y)}\right] = -\frac{i}{\mu}\int_0^1 \frac{R(y)+\bar{R}(y)}{(1-iuR(y)/\mu)(1+iu\bar{R}(y)/\mu)}\,du.$$

Since the denominator approaches 1 as $y \to 0$, uniformly in u, for some positive constant K,

$$\left|\log\left[-\frac{1-s\phi(y)}{1-s\bar{\phi}(y)}\right]\right| \le K|R(y)+\bar{R}(y)| = 2K|\Re\{R(y)\}|$$

for $|y| < \varepsilon$. Since $\Re\{\phi(y)\} = 1 + y\Re\{R(y)\}$,

$$|\Re\{R(y)\}| = \left|\Re\left(\frac{1-\phi(y)}{y}\right)\right| = \frac{\Re\{1-\phi(y)\}}{y}.$$

Since $\Re\{1-\phi(y)\}/y^2$ is locally integrable at zero [this is a standard result used proving the renewal theorem—see Lemma A.1 of Woodroofe (1982)],

$$\frac{1}{y}\log\left[-\frac{1-\phi(y)}{1-\bar{\phi}(y)}\right]$$

is locally integrable at zero. Using this and the corresponding result with ϕ changed to $\hat{\phi}$, (3.3) is locally integrable at zero, completing the proof. □

In the next two theorems, the formula for Λ in Theorem 3.2 will be differentiated to obtain higher moments of T and H. In the sequel we assume throughout $t \in \mathbb{R}$ and $s \in (0,1]$. Then $|it\mu + s - 1| \ge \min\{\mu, 1\}\|(t, s-1)\|$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^2$. If $EX^2 < \infty$ and $\mu > 0$, then $EH^2 < \infty$ and $ET^2 < \infty$. By Taylor expansion,

$$\begin{aligned}1-\beta(t,s) &= (-it\mu - s + 1)ET + \tfrac{1}{2}t^2EH^2 - it(s-1)ETH\\ &\quad -\tfrac{1}{2}(s-1)^2E[T(T-1)] + o(\|(t,s-1)\|^2)\end{aligned}$$

and

$$1 - s\phi(t) = -it\mu - (s-1) + \tfrac{1}{2}t^2EX^2 - it(s-1)\mu + o(\|(t,s-1)\|^2)$$

as $(t,s) \to (0,1)$. A straightforward (but lengthy) calculation, in which Wald's second identity that $E(H-\mu T)^2 = \sigma^2 ET$ plays an important role, gives

$$\begin{aligned}\frac{1-\beta(t,s)}{1-s\phi(t)} &= ET + \frac{it}{2\mu}(EH^2 - ETEX^2) + \frac{s-1}{2}E[T(T-1)]\\ &\quad + o(\|(t,s-1)\|)\end{aligned}$$

as $(t, s) \to (0, 1)$. Taking logarithms, the expansion for $(1 - \beta)/(1 - s\phi)$ implies

$$\Lambda(t, s) = \log ET + \frac{it}{2\mu ET}(EH^2 - ETEX^2) + \frac{s-1}{2ET}E[T(T-1)] + o(\|(t, s-1)\|)$$

as $(t, s) \to 0$. Hence the partial derivatives of Λ with respect to t and s at $(t, s) = (0, 1)$ are

$$\frac{i}{2\mu ET}(EH^2 - ETEX^2) \tag{3.4}$$

and

$$\frac{1}{2ET}E[T(T-1)], \tag{3.5}$$

respectively.

THEOREM 3.3. *If $\phi \in \mathscr{L}^q$ for some $q \in [1, \infty)$ and if $EX^2 < \infty$ and $\mu > 0$, then*

$$\frac{\partial}{\partial s}\Lambda(t, s) = -\mathscr{A}\left[\frac{\phi(t)}{1 - s\phi(t)} - \frac{\hat{\phi}(t)}{1 - s\hat{\phi}(t)}\right] \tag{3.6}$$

for all $(t, s) \in \mathbb{R} \times (0, 1]$. [The derivative when $s = 1$ is defined as $\lim_{\varepsilon \downarrow 0}(\Lambda(t, s) - \Lambda(t, 1-\varepsilon))/\varepsilon$.] Taking $(t, s) = (0, 1)$,

$$ET^2 = \frac{(\sigma^2 + \mu^2)ET}{2\mu^2} - \frac{2ET}{\pi}\int_0^\infty \left[\Im\left\{\frac{1}{1 - \phi(y)}\right\} - \frac{1}{\mu y}\right]\frac{dy}{y}. \tag{3.7}$$

PROOF. Using Lemma 2.6 to differentiate the identity in Theorem 3.2, for any $s \in (0, 1]$, (3.6) will hold for a.e. $t \in \mathbb{R}$. Since the argument of $\mathscr{A}$ is continuously differentiable in t for all s, unless $t = 0$ and $s = 1$, by Lemma 2.1, (3.6) holds for all $(t, s) \neq (0, 1)$. To show that (3.6) holds at $(t, s) = (0, 1)$ we will use Lemma 2.3. Let

$$M(t) = -\frac{\phi(t)}{1 - \phi(t)} + \frac{\hat{\phi}(t)}{1 - \hat{\phi}(t)} = -\frac{\phi(t) - \hat{\phi}(t)}{(1 - \phi(t))(1 - \hat{\phi}(t))}.$$

By Taylor expansion, the numerator of this expression is asymptotic to

$$\tfrac{1}{2}(\phi''(0) - \hat{\phi}''(0))t^2 = -\tfrac{1}{2}(\sigma^2 - \mu^2)t^2$$

as $t \to 0$ and the denominator is asymptotic to $-\mu^2 t^2$ as $t \to 0$. If we define

$$M(0) = -\frac{\sigma^2 - \mu^2}{2\mu^2}, \tag{3.8}$$

then M is continuous at zero. Since $\partial\Lambda(t,s)/\partial s$ with $s=1$ is a continuous function of t, by Lemma 2.3, (3.6) will hold at $(t,s)=(0,1)$ provided

$$\frac{M(y)-M(-y)}{y}=-\frac{2i}{y}\left[\frac{\Im\{\phi(y)\}}{|1-\phi(y)|^2}-\frac{\Im\{\hat{\phi}(y)\}}{|1-\hat{\phi}(y)|^2}\right] \tag{3.9}$$

is locally integrable at $y=0$. By Fubini's theorem,

$$\begin{aligned}\int_0^\infty|\Im\{\phi(y)\}-\mu y|\frac{dy}{y^3}&=\int_0^\infty|E[\sin(yX)-yX]|\frac{dy}{y^3}\\&\le E\int_0^\infty|\sin(yX)-yX|\frac{dy}{y^3}\\&=EX^2\int_0^\infty|\sin(y)-y|\frac{dy}{y^3}\\&<\infty.\end{aligned} \tag{3.10}$$

After some algebra, (3.9) can be written as the sum of three expressions:

$$-\frac{2i}{y}\left[\frac{\Im\{\phi(y)\}-\mu y}{|1-\phi(y)|^2}-\frac{\Im\{\hat{\phi}(y)\}-\mu y}{|1-\hat{\phi}(y)|^2}\right],$$

$$2i\mu\frac{[\Re\{1-\phi(y)\}]^2-[\Re\{1-\hat{\phi}(y)\}]^2}{|1-\phi(y)|^2|1-\hat{\phi}(y)|^2}$$

and

$$2i\mu\frac{\Im\{\phi(y)+\hat{\phi}(y)\}}{|1-\phi(y)|^2|1-\hat{\phi}(y)|^2}\left[\left(\Im\{\phi(y)\}-\mu y\right)-\left(\Im\{\hat{\phi}(y)\}-\mu y\right)\right].$$

Since $|1-\phi(y)|^2\sim y^2\mu^2$ and $\Re\{1-\phi(y)\}\sim y^2EX^2/2$ (and similar asymptotic relations hold with ϕ changed to $\hat{\phi}$), using (3.10) each of these expressions is locally integrable at zero. □

THEOREM 3.4. *If $\phi\in\mathscr{L}^q$ and $\phi'\in\mathscr{L}^q$ for some $q\in[1,\infty)$ and if $EX^2<\infty$ and $\mu>0$, then*

$$\frac{\partial}{\partial t}\Lambda(t,s)=-\mathscr{A}\left[\frac{s\phi'(t)}{1-s\phi(t)}-\frac{s\hat{\phi}'(t)}{1-s\hat{\phi}(t)}\right] \tag{3.11}$$

for all $(t,s)\in\mathbb{R}\times(0,1]$. Taking $(t,s)=(0,1)$,

$$\begin{aligned}EH^2=&\frac{ET(\sigma^2+3\mu^2)}{2}\\&+\frac{2EH}{\pi}\int_0^\infty\left[\Re\left\{\frac{\phi'(y)}{1-\phi(y)}\right\}+\frac{1}{y(1+\mu^2y^2)}\right]\frac{dy}{y}.\end{aligned} \tag{3.12}$$

PROOF. Using Lemma 2.5, $\log[(1 - s\phi)/(1 - s\hat{\phi})]$ is differentiable in $\mathscr{L}^q$ and using Lemma 2.4 to differentiate the identity in Theorem 3.2, for any $s \in (0,1]$, (3.11) will hold for a.e. $t \in \mathbb{R}$. By Lemma 2.1, (3.11) holds for all $(t, s) \neq (0,1)$ since the argument of $\mathscr{A}$ is then continuously differentiable. To show that (3.11) holds when $(t, s) = (0,1)$ we will use Lemma 2.3. By Fubini's theorem, since $\Im\{\phi'(y)\} = E[X\cos(yX)]$,

$$\begin{aligned}\int_0^\infty \frac{|\Im\{\phi'(y)\} - \mu|}{y^2}\,dy &\le E\int_0^\infty \frac{|X\cos(yX) - X|}{y^2}\,dy \\ &= E\int_0^\infty X^2 \frac{|\cos(y) - 1|}{y^2}\,dy \\ &< \infty.\end{aligned} \tag{3.13}$$

Since $\partial\Lambda(t,1)/\partial t$ is continuous at $t = 0$, by Lemma 2.3, (3.11) will hold at $(t, s) = (0,1)$ provided

$$\left[\frac{\phi'(y)}{1 - \phi(y)} - \frac{\hat{\phi}'(y)}{1 - \hat{\phi}(y)} - \frac{\phi'(-y)}{1 - \phi(-y)} + \frac{\hat{\phi}'(-y)}{1 - \hat{\phi}(-y)}\right]\frac{1}{y}$$

is locally integrable at zero. Since $\phi(-y) = \overline{\phi}(y)$ and $\phi'(-y) = -\overline{\phi}'(y)$, this expression can be written as the sum of

$$\left[\frac{\phi'(y) - i\mu}{1 - \phi(y)} + \frac{\overline{\phi}'(y) + i\mu}{1 - \overline{\phi}(y)}\right]\frac{1}{y} = \frac{2\Re\{((\phi'(y) - i\mu)(1 - \overline{\phi}(y))\}}{y|1 - \phi(y)|^2}, \tag{3.14}$$

$$-\left[\frac{\hat{\phi}'(y) - i\mu}{1 - \hat{\phi}(y)} + \frac{\overline{\hat{\phi}}'(y) + i\mu}{1 - \overline{\hat{\phi}}(y)}\right]\frac{1}{y} \tag{3.15}$$

and

$$-\frac{i\mu}{y}[M(y) - M(-y)]. \tag{3.16}$$

Local integrability of (3.16) was demonstrated in the proof of Theorem 3.3. The numerator of (3.14) is

$$2\Re\{\phi'(y) - i\mu\}\Re\{1 - \overline{\phi}(y)\} - 2\Im\{\phi'(y) - i\mu\}\Im\{1 - \overline{\phi}(y)\}.$$

Using this, local integrability of (3.14) follows from (3.13) and the asymptotic relations

$$\Re\{\phi'(y) - i\mu\} \sim -yEX^2, \qquad \Re\{1 - \overline{\phi}(y)\} \sim \tfrac{1}{2}y^2EX^2,$$
$$\Im\{1 - \overline{\phi}(y)\} \sim y\mu, \qquad |1 - \phi(y)|^2 \sim y^2\mu^2$$

as $y \to 0$. Local integrability of (3.15) follows by changing ϕ to $\hat{\phi}$ in this argument, so (3.11) holds at $(t, s) = (0,1)$. □

4. Higher order moments. Repeated differentiation of the identity in Theorem 3.2 gives higher moments of T and H. In this section, some of the algebra needed to obtain third moments will be detailed. Higher order moments can be obtained similarly, but the effort required will be substantial. Define

$$m_3 = E(X - \mu)^3$$

and

$$\Lambda_{i,j} = \frac{\partial^{i+j}}{\partial t^i\, \partial s^j}\Lambda(t, s)\bigg|_{(t,s)=(0,1)}.$$

Keeping one extra term, the arguments leading to (3.4) and (3.5) give expressions relating $\Lambda_{i,j}$ for $i + j \leq 3$ to moments of T and H of degree 3 or less. The simplification necessary to divide the two Taylor expansions now uses the identity

$$E(H - \mu T)^3 = 3\sigma^2 ET(H - \mu T) + m_3 ET.$$

Straightforward algebra then gives the following formulae:

$$\Lambda_{0,2} = \frac{5}{12} - \frac{ET^2}{2ET} - \frac{(ET^2)^2}{4(ET)^2} + \frac{ET^3}{3ET},$$

$$i\Lambda_{1,1} = \frac{\mu}{12} + \frac{ETHET^2}{2(ET)^2} - \frac{ET^2H}{2ET} - \frac{\mu(ET^2)^2}{4(ET)^2} + \frac{\mu ET^3}{6ET}$$

and

$$\Lambda_{2,0} = \frac{\sigma^2}{2} + \frac{\mu^2}{12} + \frac{(ETH)^2}{(ET)^2} - \frac{ETH^2}{ET} + \frac{\sigma^2 ET^2}{2ET}$$
$$- \frac{\mu ETHET^2}{(ET)^2} + \frac{\mu ET^2H}{ET} + \frac{\mu^2(ET^2)^2}{4(ET)^2} - \frac{\mu^2 ET^3}{3ET}.$$

Let

$$h_{i,j}(t, s) = -\frac{\partial^{i+j}}{\partial t^i\, \partial s^j}\log\left[\frac{1 - s\phi(t)}{1 - s\hat{\phi}(t)}\right],$$

so $h_{0,1}$ and $h_{1,0}$ are the arguments of $\mathscr{A}$ in Theorems 3.3 and 3.4. If $E|X|^3 < \infty$, then $h_{1,1}(\cdot, s)$ and $h_{2,0}(\cdot, s)$ are bounded for all $s \in (0, 1]$ and are continuously differentiable except perhaps at zero when $s = 1$. If ϕ and ϕ' are in $\mathscr{L}^q$ for some $q \in [1\,\infty)$, then $h_{1,1}(\cdot, s) \in \mathscr{L}^q$ and if ϕ, ϕ' and ϕ'' are in $\mathscr{L}^q$ for some $q \in [1, \infty)$, then $h_{2,0}(\cdot, s) \in \mathscr{L}^q$. By Lemma 2.5, $h_{1,1}(\cdot, s)$ and $h_{2,0}(\cdot, s)$ are differentiable in $\mathscr{L}^q$ and using Lemma 2.4 to differentiate the identities in Theorems 3.3 and 3.4,

(4.1)
$$\frac{\partial^2}{\partial t\, \partial s}\Lambda(t, s) = -\mathscr{A}h_{1,1}(t, s)$$

and

(4.2)
$$\frac{\partial^2}{\partial t^2}\Lambda(t, s) = -\mathscr{A}h_{2,0}(t, s)$$

for any $s \in (0, 1]$, for a.e. $t \in \mathbb{R}$. By Lemma 2.1, these equations can fail only at $(t, s) = (0, 1)$. If $EX^4 < \infty$, then $h_{1,1}(t, 1)$ and $h_{2,0}(t, 1)$ are also differentiable at $t = 0$, so (4.1) and (4.2) will also hold at $(t, s) = (0, 1)$. We conjecture that (4.1) and (4.2) hold under the weaker condition that $E|X|^3 < \infty$, and that this can be proved verifying the local integrability condition in Lemma 2.3. A formal proof may be rather delicate. Setting $(t, s) = (0, 1)$ in (4.1) and (4.2) gives

$$(4.3) \qquad i\Lambda_{1,1} = \frac{-2\mu m_3 + \mu^4 + 3\sigma^4}{24\mu^3} - \frac{1}{\pi}\int_0^\infty \Re\left\{\frac{\phi'(y)}{(1-\phi(y))^2}\right\}\frac{dy}{y}$$

and

$$(4.4) \qquad \begin{aligned} \Lambda_{2,0} &= \frac{4\mu m_3 - 11\mu^4 + 6\mu^2\sigma^2 - 3\sigma^4}{24\mu^2} \\ &\quad - \frac{1}{\pi}\int_0^\infty \left[\Im\left\{\frac{\phi''(y)}{1-\phi(y)} + \frac{(\phi'(y))^2}{(1-\phi(y))^2}\right\} - \frac{2y\mu^3}{(1+\mu^2y^2)^2}\right]\frac{dy}{y}. \end{aligned}$$

Differentiating the identity in Theorem 3.3 with respect to s is delicate—$h_{0,2}(\cdot, 1)$ is not locally integrable at zero, so direct application of Lemma 2.6 must fail. By Taylor expansion, $h_{0,1}(t, s) = g(t, s) - (2\mu^2)^{-1}(\mu^2 - \sigma^2) + o(1)$ as $(t, s) \to (0, 1)$, where

$$g(t, s) = \frac{(\mu^2 - \sigma^2)t^2}{2(1 - s - i\mu t)^2} + \frac{\mu^2 - \sigma^2}{2\mu^2}.$$

Direct computations show that $\mathscr{A}g(t, s) = 0$ so, defining $h^*_{0,1}(t, s) = h_{0,1}(t, s) - g(t, s)$, Theorem 3.3 gives

$$(4.5) \qquad \frac{\partial}{\partial s}\Lambda(t, s) = -\mathscr{A}h^*_{0,1}(t, s)$$

for all $(t, s) \in \mathbb{R} \times (0, 1]$. If $E|X|^2 < \infty$ and $\phi \in \mathscr{L}^q$, then $h^*_{0,1}$ satisfies the conditions of Lemma 2.5, so differentiating (4.5) with respect to s gives

$$(4.6) \qquad \frac{\partial^2}{\partial s^2}\Lambda(t, s) = -\mathscr{A}h^*_{0,2}(t, s)$$

for $s \in (0, 1]$, for a.e. $t \in \mathbb{R}$, where $h^*_{0,2}(t, s) = \partial h^*_{0,2}(t, s)/\partial s$. The function $h^*_{0,2}(t, s)$ is differentiable in t unless $(t, s) = (0, 1)$, so by Lemma 2.1, (4.6) can fail only at $(t, s) = (0, 1)$. If $|X|^4 < \infty$, then $h^*_{0,2}(t, 1)$ is also differentiable at $t = 0$ and (4.6) holds for all $(t, s) \in \mathbb{R} \times (0, 1]$. Again we suspect the weaker condition $E|X|^3 < \infty$ is sufficient and that this can be established verifying local integrability necessary to apply Lemma 2.3. Setting $(t, s) = (0,1)$ in (4.6) gives

$$\Lambda_{0,2} = \frac{-4\mu m_3 + 5\mu^4 - 6\mu^2\sigma^2 + 9\sigma^4}{24\mu^4} \tag{4.6}$$
$$-\frac{1}{\pi}\int_0^{\infty}\left[\Im\left\{\frac{\phi^2(y)}{(1-\phi(y))^2}\right\} - \frac{\sigma^2-\mu^2}{\mu^3 y}\right]\frac{dy}{y}.$$

5. Limiting excess and the minimum. Let R be a random variable with Lebesgue density $P(H > r)/EH$ for $r > 0$. This variable plays an important role in many boundary crossing problems. For instance, if $\tau_a = \inf\{n \geq 1: S_n > a\}$, then if the random walk is nonlattice,

$$S_{\tau_a} - a \Rightarrow R$$

as $a \to \infty$. Similar convergence results arise when a random walk crosses a nonlinear boundary or when a perturbed random walk crosses a linear boundary. See Woodroofe (1982) for numerous examples in sequential analysis. After integration by parts,

$$ER = \frac{EH^2}{2EH}, \tag{5.1}$$
$$Ee^{itR} = \frac{\beta(t,1) - 1}{itEH}.$$

Hence the mean and characteristic function for R can be computed directly from the results in Section 3. The quantity Ee^{-R} is important in large deviation approximations in boundary crossing problems [see Theorem 3.1 of Woodroofe (1982) for a simple example with linear boundaries, or later chapters for nonlinear examples]. In these applications there is the following extra structure: $\{S_n\}_{n\geq 1}$ is a random walk under two probability measures, P and P^*, and the likelihood ratio for the restrictions of P and P^* to $\sigma(X_1, \ldots, X_n)$ is

$$\frac{dP^*}{dP}\bigg|\sigma(X_1,\ldots,X_n) = e^{-S_n}.$$

By Wald's fundamental identity [Theorem 1.1 of Woodroofe (1982)],

$$Ee^{-H} = Ee^{-S_T} = P^*(T < \infty) = 1 - \frac{1}{E^*[\overline{T}]},$$

where the last equality follows from a duality argument [see Corollary 2.4 of Woodroofe (1982)]. Integration by parts gives

$$Ee^{-R} = \frac{E[1 - e^{-H}]}{EH} = \frac{1}{E^*[\overline{T}]EH}. \tag{5.2}$$

Here $E^*[\overline{T}]$ can be found numerically by applying Theorem 3.2 to the random walk $\{-S_n\}_{n\geq 1}$ with probability measure P^*.

The random variable

$$M = \inf_{n \geq 0} S_n$$

(where $S_0 = 0$) is important in queuing theory. Since M is the sum of a geometric number of descending ladder heights,

$$Ee^{itM} = \frac{P(\overline{T} = \infty)}{1 - E\left[\exp(it\overline{H}); \overline{T} < \infty\right]}.$$

By Wiener–Hopf factorization [see Corollary 2.3 of Woodroofe (1982)],

$$\{1 - \beta(t, 1)\}\left\{1 - E\left[\exp(it\overline{H}); T < \infty\right]\right\} = 1 - s\phi(t),$$

and since $P(\overline{T} = \infty) = 1/ET$ the formula for the characteristic function of M simplifies to

$$Ee^{itM} = \frac{1}{ET}\,\frac{1 - \beta(t,1)}{1 - \phi(t)} = \frac{e^{\Lambda(t,1)}}{ET}.$$

Differentiating this expression at $t = 0$ and using (3.4),

$$EM = \frac{EH^2 - ET\,EX^2}{2\mu ET}.$$

Hence the mean and characteristic function of M can be computed from the results in Section 3.

6. Walks with $\mu = 0$. When $\mu = 0$, $ET = \infty$, so moments of H are the primary concern. Quadrature formulas for the first two moments of H in this case have been given by Siegmund (1985). Accordingly, this section will be brief and is designed to show how the methods presented here lead directly to similar formulas for the first two moments of H and to formulas for higher order moments. Assume $\sigma^2 = EX^2 < \infty$, so $EH < \infty$ and let

$$\tilde{\phi}(t) = \frac{1}{1 + \sigma^2 t^2/2},$$

the characteristic function for a symmetric double exponential distribution with variance σ^2. When $\phi = \tilde{\phi}$, H has an exponential distribution with mean $\sigma/\sqrt{2}$. Theorem 4.1 with $s = 1$ gives

$$\frac{1}{2}(\mathscr{I} + i\mathscr{H})\log\left(1 - \tilde{\phi}(t)\right) = \log\left(\frac{-it\sigma/\sqrt{2}}{1 - it\sigma/\sqrt{2}}\right)$$

for a.e. $t \in \mathbb{R}$. The same theorem then gives

$$\log\left(\frac{1 - \beta(t,1)}{-it}\right) = \frac{1}{2}(\mathscr{I} + i\mathscr{H})\log\left(\frac{1 - \phi(t)}{1 - \tilde{\phi}(t)}\right) - \log\left(-it + \frac{\sqrt{2}}{\sigma}\right) \tag{6.1}$$

for a.e. $t \in \mathbb{R}$. Using Lemmas 2.1 and 2.3, this identity can be established for all $t \in \mathbb{R}$. Setting $t = 0$ gives

$$EH = \frac{\sigma}{\sqrt{2}} \exp\left\{ \frac{1}{\pi} \int_0^\infty \frac{\arg(1 - \phi(y))}{y}\, dy \right\}.$$

This result appears in Section 10.4 of Siegmund (1985). Under suitable regularity, differentiating (6.1) gives

$$\frac{d^k}{dt^k} \log\left(\frac{1 - \beta(t,1)}{-it} \right) = \frac{1}{2}(\mathscr{I} + i\mathscr{H}) \frac{d^k}{dt^k} \log\left(\frac{1 - \phi(t)}{1 - \tilde{\phi}(t)} \right)$$
$$- \frac{d^k}{dt^k} \log\left(-it + \frac{\sqrt{2}}{\sigma} \right).$$

At $t = 0$, the left-hand side of this equation is

$$\frac{i}{2} \frac{EH^2}{EH}$$

when $k = 1$ and

$$-\frac{1}{3} \frac{EH^3}{EH} + \frac{1}{4} \frac{(EH^2)^2}{(EH)^2}$$

when $k = 2$. Straightforward algebra then gives

$$\begin{aligned} EH^2 &= \sqrt{2}\,\sigma EH + \frac{m_3 EH}{3\sigma^2} \\ &\quad + \frac{2EH}{\pi} \int_0^\infty \left[\Re\left\{ \frac{\phi'(y)}{1 - \phi(y)} \right\} - \frac{2}{y(1 + \sigma^2 y^2/2)} \right] \frac{dy}{y}, \\ EH^3 &= \frac{m_4 EH}{4\sigma^2} - \frac{m_3^2 EH}{6\sigma^4} + \frac{3(EH^2)^2}{4EH} \\ &\quad + \frac{3EH}{\pi} \int_0^\infty \Im\left\{ \frac{\phi''(y)}{1 - \phi(y)} + \frac{(\phi'(y))^2}{(1 - \phi(y))^2} \right\} \frac{dy}{y}, \end{aligned} \tag{6.2}$$

where $m_3 = EX^3$ and $m_4 = EX^4$. Section 10.4 of Siegmund has a formula for EH^2 equivalent to (6.2) after integration by parts. In Siegmund's work, the only smoothness condition imposed is that ϕ is the characteristic function for a continuous distribution, which is weaker than the standing assumption here that $\phi \in \mathscr{L}^q$ for some $q \in [0, \infty)$. In his proof [taken from Hogan (1984)], the result is first established when $\phi \in \mathscr{L}^1$. The more general case is then handled by a limiting argument. It may be possible to similarly relax the assumption that $\phi \in \mathscr{L}^q$ in some of the results in this paper.

7. Numerical examples. The formulas developed in the previous sections will be illustrated in six examples. In five of the examples there were no numerical problems. At zero, all integrands were either continuous or had integrable algebraic singularities, and at infinity the integrands decay exponentially or algebraically. Also, none of the integrands was highly oscillatory. Any good adaptive integration routine should have no trouble handling integrands this regular. In several of these examples, exact analytic answers are available and in other examples certain moments have been computed numerically by other authors. In all of these cases, there is near exact agreement with the answers computed numerically using the formulas developed in this paper.

The first example was chosen because exact answers are available for comparison. In this example, X has density

$$f_X(x) = \begin{cases} \frac{3}{4}e^{-x}, & \text{for } x \geq 0, \\ \frac{3}{4}e^{3x}, & \text{for } x < 0. \end{cases}$$

Then $\mu = 2/3$, $\sigma^2 = 10/9$, $m_3 = 52/27$ and

$$\phi(t) = \frac{3}{(1-it)(3+it)},$$

which is the characteristic function for the convolution of two exponential distributions concentrated on the two half-axes. By example (c), page 608 of Feller (1971),

$$\beta(t,s) = \frac{2-\sqrt{4-3s}}{1-it}.$$

From this, T and H are independent. Taking $s = 1$, H has a standard exponential distribution and so $EH^n = n!$. Taking $t = 0$,

$$Es^T = 2 - \sqrt{4-3s}\,.$$

Differentiating this generating function at $s = 1$ gives $ET = 3/2$, $ET^2 = 15/4$ and $ET^3 = 147/8$. By the independence of T and H, $ET^2H = 15/4$ and $ETH^2 = 3$. Table 1 shows a computer session in which ET, EH, ET^2, EH^2, ET^3, ET^2H, ETH^2 and EH^3 are computed by numerical quadrature using formulas from Sections 3 and 4. These numerical calculations were done in Mathematica, but the code should be fairly transparent to someone not familiar with this package. As a default, the numerical integration routine in Mathematica strives for accuracy to six significant digits after the decimal. In this example, the numerical answers from the program agree completely with the exact answers given above. The numerical values for moments in this and later examples are given in Table 2.

The next example was chosen to see if there are numerical difficulties when higher order moments of X are infinite. The example was chosen

TABLE 1

```
In[1] := phi[y_] := 3 / ((1 − Iy)(3 + Iy));
In[2] := mu = 2 / 3;
In[3] := var = 10 / 9;
In[4] := f1 = Arg[(1 − phi[y]) / (mu y − I)] / y;
In[5] := ET = Exp[NIntegrate[f1, {y, 0, Infinity}] / Pi] / / N
Out[5] = 1.5
In[6] := EH = mu ET
Out[6] = 1.
In[7] := f2 = (Im[1 / (1 − phi[y])] − (1 / mu y))) / y;
In[8] := ETT = (var + mu ^ 2)ET / (2mu ^ 2) − 2ET NIntegrate
  [f2, {y, 0, Infinity}] / Pi / / N
Out[8] = 3.75
In[9] := f3 = (Re[phi '[y] / (1 − phi[y])] + 1 / (y(1 + mu ^ 2 y ^ 2))) / y;
In[10] := EHH = ET(var + 3mu ^ 2) / 2 + mu ET NIntegrate
  [f3, {y, 0, Infinity}] / Pi / / N
Out[10] = 2.
In[11] := m3 = 52 / 27;
In[12] := ETH = (EHH + mu ^ 2 ETT − var ET) / (2 mu);
In[13] := den = 1 − phi[y];
In[14] := f4 = Re[phi '[y] / den ^ 2] / y;
In[15] := i4 = NIntegrate[f4, {y, 0, Infinity}] / Pi / / N;
In[16] := ilam11 = ( − 2 mu m3 + mu ^ 4 + 3 var ^ 2) / (24 mu ^ 3) − i4;
In[17] := f5 = (Im[phi ''[y] / den + phi '[y] ^ 2 / den ^ 2]
  − 2y mu ^ 3 / (1 + mu ^ 2 y ^ 2) ^ 2) / y;
In[18] := i5 = NIntegrate[f5, {y, 0, Infinity}] / Pi / / N;
In[19] := lam20 = (4 mu m3 − 11 mu ^ 4 + 6 mu ^ 2 var − 3 var ^ 2) / (24 mu ^ 2) − i5;
In[20] := f6 = (Im[phi[y] ^ 2 / den ^ 2] − (var − mu ^ 2) / (mu ^ 3 y)) / y;
In[21] := i6 = NIntegrate[f6, {y, 0, Infinity}] / Pi / / N;
In[22] := lam02 = ( − 4mu m3 + 5mu ^ 4 − 6mu ^ 2var + 9var ^ 2) / (24mu ^ 4) − i6;
In[23] := ETTT = 3 ET(lam02 − 5 / 12 + ETT / (2ET) + ETT ^ 2 / (4ET ^ 2))
Out[23] = 18.375
In[24] := ETTH = 2ET( − ilam11 + mu / 12 + ETH ETT / (2ET ^ 2) −
  mu ETT ^ 2 / (4ET ^ 2) + mu ETTT / (6ET))
Out[24] = 3.75
In[25] := ETHH = ET(var / 2 + mu ^ 2 / 12 + ETH ^ 2 / ET ^ 2 + var ETT / (2ET)
  − mu ETH ETT / ET ^ 2 + mu ETTH / ET + mu ^ 2ETT ^ 2 / (4ET ^ 2)
  − mu ^ 2ETTT / (3ET) − lam20)
Out[25] = 3.
In[26] := EHHH = 3mu ETHH − 3mu ^ 2ETTH + mu ^ 3ETTT + 3var ETH − 3mu var ETT + m3 ET
Out[26] = 6.
```

working backward so that certain moments would be known exactly. Specifically, H has a Pareto distribution with density

$$\frac{5}{2(1+x)^{7/2}}$$

for $x > 0$ and $\overline{H}$ (the descending ladder height) has density

$$\tfrac{1}{2}e^{x}$$

TABLE 2

	Example					
	1	2	3	4	5	6
ET	1.5	2.	1.8892	1.46646		
EH	1.	0.666667	0.944599	1.46646	0.707107	1.
ET^2	3.75	20.3016	8.65726	4.59492		
EH^2	2.	2.66668	1.35633	3.90984	0.823917	2.
ET^3	18.375		98.0304			
ET^2H	3.75		6.73664			
ETH^2	3.		2.21671			
EH^3	6.		2.43483		1.25035	6.

for $x < 0$. Integrating the Pareto density, $EH = 2/3$, $EH^2 = 8/3$ and $EH^{5/2} = E|X|^{5/2} = \infty$. Also,

$$E\exp(it\overline{H}) = \frac{1}{2 + 2it}$$

and

$$Ee^{itH} = -\frac{5}{2}t^{5/2}e^{-it}\sqrt{-i}\,\Gamma\left(-\frac{5}{2}, -it\right),$$

where Γ is the incomplete gamma function defined by

$$\Gamma(\alpha, x) = \int_x^\infty y^{\alpha-1}e^{-y}\,dy.$$

By Wiener–Hopf factorization, $1 - \phi(t) = (1 - Ee^{itH})(1 - Ee^{it\overline{H}})$, which gives

$$\phi(t) = \frac{2 - 5(1 + 2it)t^{5/2}e^{-it}\sqrt{-i}\,\Gamma(-5/2, -it)}{4 + 4it}.$$

Differentiating ϕ, $EX = 1/3$ and $\sigma^2 = 17/9$. Initial attempts to compute ET, EH, ET^2 and EH^2 numerically using lines 4–10 in Table 1 failed. The integrands are fairly well behaved: the integrand in the formula for ET is differentiable at zero and the integrands in the formulas for ET^2 and EH^2 have square root singularities near zero. All three integrands decay algebraically near infinity. Plots of the integrand in the formula for ET for very small values of y illustrate the real problem: for y above 0.0006, the numerical values for the integrand behave smoothly, but for y below 0.0006, the values are exceptionally erratic. This problem arises because the routine which computes the incomplete gamma function is not highly accurate, and small errors can lead to large relative errors when ϕ is subtracted from 1. Perhaps the most natural way to proceed would be to use Taylor expansion to evaluate the integral over some small neighborhood of the origin and use the numerical integration routine on the complement of this neighborhood. This did not seem very convenient (in Mathematica), so instead, two changes were made: The integrals were rewritten using the transformation $\int_0^\infty f(y)\,dy = 2\int_0^\infty f(y^2) y\,dy$ and Mathematica was asked to only seek accuracy to four significant digits. The transformation removes the singularities and makes the integrands behave more like polynomials near zero. Both changes should help the routine estimate the integral using a coarser grid of points, a grid that does not sample from regions where the function is erratic. The numerical values obtained for ET, EH, ET^2 and EH^2 are given in Table 2. Once again there is excellent agreement with the exact values $ET = 2$, $EH = 2/3$ and $EH^2 = 8/3$.

In the third example, $X \sim N(1/2, 1)$. In this case, X is the log likelihood ratio between $N(1/2, 1)$ and $N(-1/2, 1)$. Numerical values for ER and Ee^{-R} obtained from equations derived from Spitzer's identity are reported in Table 3.1 of Woodroofe (1982). In this example, the code in Table 1 works fine without modification (only lines 1–3 and 11 were changed to give the correct characteristic function and moments). The computed values for the moments are given in Table 2. Using these values and identities (5.1) and (5.2),

$$ER = \frac{EH^2}{2EH} \approx \frac{1.35633}{2 \times 0.944599} \approx 0.717940$$

and

$$Ee^{-R} = \frac{1}{E^*\overline{T}EH} = \frac{1}{ETEH} \approx \frac{1}{1.8892 \times 0.944599} \approx 0.560370.$$

These values agree with the values obtained by Woodroofe.

In the final example with positive drift, $X - 1 \sim t_3$, so $\mu = 1$, $\sigma^2 = 3$ and $E|X|^3 = \infty$. The characteristic function for the t distribution on n degrees of freedom is

$$2\left(\frac{np^2}{4}\right)^{n/4} \frac{K_{n/2}\left(\sqrt{np^2}\right)}{\Gamma(n/2)},$$

where K is a modified Bessel function. This characteristic function simplifies when n is odd and some algebra gives

$$\phi(t) = \left(1 + \sqrt{3t^2}\right)\exp\left(it - \sqrt{3t^2}\right).$$

Once again the code in Table 1 works find giving values for ET, EH, ET^2 and EH^2 reported in Table 2. Since convolutions of t distributions are not pleasant, computing these moments using formulas based on Spitzer's identity would be rather difficult.

For the final two examples, $\mu = 0$. In the first of these examples (Example 5), $X \sim N(0, 1)$. The first three moments for H were computed numerically using formulas from Section 6. The Mathematica statements for these computations are given in Table 3 and the numerical values for the moments are in Table 2. Due to the symmetry of the normal distribution, the integrands for EH and EH^3 vanish. This simplification in symmetric cases is natural considering a theorem by Spitzer (1964) giving $EH = \sigma/\sqrt{2}$ for symmetric

TABLE 3

```
In[1] := phi[y_] := Exp[ - y ^ 2 / 2]
In[2] := var = 1;
In[3] := f1 = Arg[(1 - phi[y]) / y] / y;
In[4] := EH = Sqrt[var / 2] Exp[NIntegrate[f1, {y, 0, Infinity}] / Pi] / / N
Out[4] = 0.707107
In[5] := m3 = 0;
In[6] := f2 = (Re[phi '[y] / (1 - phi[y])] + 2 / (y(1 + var y ^ 2 / 2))) / y;
In[7] := EHH = Sqrt[2 var] EH + EH m3 / (3 var) +
2 EH NIntegrate[f2, {y, 0, Infinity}] / Pi / / N
Out[7] = 0.823917
In[8] := m4 = 3;
In[9] := f3 = Im[phi ''[y] / (1 - phi[y]) + (phi '[y] / (1 - phi[y])) ^ 2] / y;
In[10] := EHHH = EH m4 / (4 var) - EH m3 ^ 2 / (6 var ^ 2) + 3 EHH ^ 2 / (4 EH) +
3 EH NIntegrate[f3, {y, 0, Infinity}] / Pi / / N
Out[10] = 1.25035
```

walks [see Theorem XVIII.5.1 of Feller (1971)]. The ratio $EH^2/(2EH)$ plays an important role in corrected diffusion approximations [see Chernoff (1965) or Chapter 10 of Siegmund (1985)]. Chernoff gives $EH^2/(2EH) = -\zeta(1/2)/\sqrt{2\pi} = 0.582597$, which agrees (to six digits) with the value computed using the numerical values in Table 2.

In the final example,

$$\phi(t) = \frac{3 + t^2 + 2it^3}{3(1 + t^2)^2},$$

the characteristic function for a mixture of a standard exponential distribution and a negative gamma distribution. Since X has an exponential right tail, H has a standard exponential distribution and $EH^n = n!$. The code in Table 3 (with the obvious modifications) works without modification, and the numerical values (given in Table 2) agree with the exact values.

APPENDIX

PROOF OF LEMMA 2.3. Let w be a symmetric function with a bounded derivative, integrating to one, with support $[-2, -1] \cup [1, 2]$. Let $w_\delta(t) = w(t/\delta)/\delta$. Note that since w_δ is symmetric, $\mathscr{H}w_\delta$ is antisymmetric. Since $\mathscr{H}f(t)$ has a limit as $t \to 0$,

$$\begin{aligned}\lim_{t \to 0} \mathscr{H}f(t) &= \lim_{\delta \downarrow 0} \int w_\delta(t) \mathscr{H}f(t)\, dt \\ &= \lim_{\delta \downarrow 0} \int f(y) \mathscr{H}w_\delta(-y)\, dy \qquad \text{(A.1)} \\ &= \lim_{\delta \downarrow 0} \int_0^\infty \frac{f(-y) - f(y)}{y} [y \mathscr{H}w_\delta(y)]\, dy.\end{aligned}$$

Since w has compact support, when t is large enough, no limit need be taken defining $\mathscr{H}w(t)$, and by dominated convergence

(A.2) $$\lim_{t\to\pm\infty} t\mathscr{H}w(t) = \lim_{t\to\pm\infty} \frac{1}{\pi}\int w(y)\frac{t}{t-y}\,dy = \frac{1}{\pi}.$$

By Lemma 2.1, $\mathscr{H}w$ is continuous and hence

(A.3) $$\sup_{-\infty<t<\infty} |t\mathscr{H}w(t)| < \infty.$$

Since

$$\begin{aligned}\mathscr{H}w_\delta(t) &= \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{|t-y|>\varepsilon}\frac{w(y/\delta)}{\delta(t-y)}\,dy\\ &= \lim_{\varepsilon\downarrow 0}\frac{1}{\delta\pi}\int_{|(t/\delta)-y|>\varepsilon/\delta}\frac{w(y)}{(t/\delta)-y}\,dy\\ &= \frac{1}{\delta}\mathscr{H}w\left(\frac{t}{\delta}\right),\end{aligned}$$

using (A.2), if $y \neq 0$, $y\mathscr{H}w_\delta(y) \to 1/\pi$ as $\delta\downarrow 0$. Hence, by dominated convergence [using (A.3) to bound the integrand] (A.1) converges to

$$\frac{1}{\pi}\int_0^\infty \frac{f(-y)-f(y)}{y}\,dy = \mathscr{H}f(0),$$

proving the lemma. □

PROOF OF THEOREM 3.1. By Lemma 2.1 it is sufficient to show that for fixed $s \in [0,1]$, (3.1) holds for a.e. $t \in \mathbb{R}$. Suppose Q is a probability measure with characteristic function $\chi \in \mathscr{L}^1$ (so Q is absolutely continuous). By Theorem 3, Chapter 2 Stein (1970),

$$\begin{aligned}\int_0^\infty e^{itx}\,dQ(x) &= \tfrac{1}{2}\int e^{itx}[1+\operatorname{sgn}(x)]\,dQ(x)\\ &= \tfrac{1}{2}[\chi(t) + i\mathscr{H}\chi(t)]\end{aligned}$$

for a.e. $t \in \mathbb{R}$. Suppose $\phi \in \mathscr{L}^1$. In this case, the characteristic function for the distribution of S_n is $\phi^n \in \mathscr{L}^1$, so

$$E[\exp(itS_k); S_k > 0] = \tfrac{1}{2}[\phi^k(t) + i\mathscr{H}\phi^k(t)]$$

for a.e. $t \in \mathbb{R}$. Using this in Spitzer's identity [Theorem 2.5 of Woodroofe (1982)], for $s \in (0,1)$,

$$\begin{aligned}\log[1-\beta(t,s)] &\doteq -\sum_{k=1}^\infty \frac{1}{k}s^k E[\exp(itS_k); S_k>0]\\ &= -\frac{1}{2}\sum_{k=1}^\infty \frac{1}{k}s^k[\phi^k(t) + i\mathscr{H}\phi^k(t)]\end{aligned}$$

for a.e. $t \in \mathbb{R}$. If $s \in (0,1)$,

$$\sum_{k=1}^{n} \frac{1}{k} s^k \phi^k \to \log(1 - s\phi)$$

in $\mathscr{L}^2$ as $n \to \infty$, so (3.1) holds for a.e. $t \in \mathbb{R}$ since $\mathscr{H}$ is linear and is a bounded operator from $\mathscr{L}^2$ to $\mathscr{L}^2$. Since $\log(1 - s\phi) \to \log(1 - \phi)$ in $\mathscr{L}^2$ as $s \uparrow 1$ and since $\mathscr{H}$ is a bounded operator, it is easy to show that (3.1) also holds for a.e. $t \in \mathbb{R}$ when $s = 1$. For the general case where $\phi \in \mathscr{L}^q$ but may not lie in $\mathscr{L}^1$, apply the theorem for a random walk with characteristic function $\phi(t)e^{-\varepsilon t^2}$ and take $\mathscr{L}^q$ limits as $\varepsilon \downarrow 0$. □

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DEPARTMENT OF STATISTICS
UNIVERSITY OF MICHIGAN
ANN ARBOR, MICHIGAN 48109