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A LIMIT THEOREM FOR LINEAR BOUNDARY VALUE PROBLEMS IN RANDOM MEDIA[1]

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The asymptotic behavior of the solutions of linear equations with random coefficients, random external forces and with affine boundary conditions is studied, motivated by a transmission–reflection problem for a one-dimensional wave equation in a random slab. The fluctuations of the coefficients are on a small scale in such a way that our problem is a diffusion-approximation problem except that we impose boundary conditions which force the solution to be anticipating. In the limit we obtain linear stochastic differential equations with affine boundary conditions, studied by Ocone and Pardoux. Our main tools are diffusion approximation results (Papanicolaou, Stroock and Varadhan or Ethier and Kurtz) and the properties of the limiting equations involving generalized Stratonovich integrals (Ocone and Pardoux). As an application, the transmission–reflection problem is discussed. We prove that the solution has a density with respect to the Lebesgue measure and satisfies the Markov field property.

1. Introduction. A one-dimensional monochromatic wave equation defined on an interval can be written as a finite-dimensional linear equation with boundary conditions corresponding to a two-point boundary value problem. Moreover, if the coefficients of the equation are random processes fluctuating on a small scale, one can rescale the problem in such a way that it is in the diffusion-approximation regime. (Details will be given in Section 4.)

The aim of this paper is to study the asymptotic behavior of the solutions of linear equations with random coefficients, random external forces and with affine boundary conditions. Clearly, these solutions are anticipating. The main tools used here are diffusion-approximation results, following Papanicolaou, Stroock and Varadhan [9], or Ethier and Kurtz [3], and properties of the limiting equation developed by Ocone and Pardoux [7].

In Section 2 we study a general linear diffusion-approximation problem with boundary conditions, corresponding to an initial value problem. Using classical diffusion-approximation results, we identify the limiting diffusion process and give some examples.

Section 3 is devoted to the same problem, with boundary conditions corresponding to a two-point boundary value problem. The limiting process is

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identified as the unique solution of a linear stochastic differential equation with affine boundary conditions studied in [7]. The stochastic integrals involved here are generalized Stratonovich integrals.

In Section 4, as an application, a problem studied by G. Papanicolaou is discussed. We show how a transmission–reflection problem for a one-dimensional wave equation in a random slab is a particular case of the general result obtained previously. We then apply a result of [7] to prove the existence of a probability density for the limiting wave field.

Section 5 deals with the Markov property. We show that the limiting field is not Markovian, and we extend a result of [7] to prove that it is a Markov field.

2. Notations and the initial value problem. The coefficients of the equations studied in this paper will be random as functions of a "driving" Markov process $(Z_t)_{t \geq 0}$ defined in $\mathbb{R}^n$ or in a compact subset of $\mathbb{R}^n$.

We shall assume that this process has a unique invariant probability measure μ, under which it is ergodic, and that the Fredholm alternative holds for its infinitesimal generator $L(Z)$, which then has an inverse on functions centered under the probability μ. We will denote by $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, P)$ a stochastic basis for (Z_t) such that under P the law of Z_t is μ for every $t \geq 0$. All the expectations will be with respect to P. As examples we may think of a Markov process on a compact space satisfying the Doeblin condition or the Markov diffusion processes in $\mathbb{R}^n$ studied in [1].

This process Z_t will be varying on a small scale ε^2 with $0 < \varepsilon \ll 1$. We shall denote by (Z_t^ε) the rescaled process (Z_{t/ε^2}).

We study the following differential equation in $\mathbb{R}^d$ for $0 \leq t \leq 1$:

$$(1) \qquad \frac{dX_t^\varepsilon}{dt} = \left\{ \frac{1}{\varepsilon} B\left(\frac{t}{\varepsilon}, Z_t^\varepsilon\right) + A\left(\frac{t}{\varepsilon}, Z_t^\varepsilon\right) \right\} X_t^\varepsilon + \frac{1}{\varepsilon} b(t, Z_t^\varepsilon) + a(t, Z_t^\varepsilon)$$

with the following *centering condition*:

$$E\{B(\tau, Z_0)\} = E\{b(t, Z_0)\} = 0 \quad \text{for every } \tau \geq 0,\ t \geq 0,$$

where A and B are bounded smooth functions from $\mathbb{R}_+ \times \mathbb{R}^n$ into $\mathscr{M}_{d \times d}$, the space of real $d \times d$ matrices, and a and b are bounded smooth functions from $\mathbb{R}_+ \times \mathbb{R}^n$ into $\mathbb{R}^d$.

We assume that $A(\tau, z)$ and $B(\tau, z)$ are periodic in τ with period T independent of z.

We shall use the following notations:

$$\frac{1}{T} \int_0^T E\{A(\tau, Z_0)\}\, d\tau = \bar{A} \quad \text{and} \quad E\{a(t, Z_0)\} = \bar{a}(t).$$

Note that a fast-varying variable $\tau = t/\varepsilon$ appears in A and B, while the slowly-varying variable t appears in the "random external force" $(1/\varepsilon) b(t, Z_t^\varepsilon) + a(t, Z_t^\varepsilon)$ added to the linear part of (1).

In order to describe the limiting equation, we shall assume that the following integrals are well-defined and finite:

$$\int_0^\infty \frac{1}{T}\int_0^T E\{B_{i,j}(\tau, Z_0)B_{k,l}(\tau, Z_s)\}\,d\tau\,ds,$$

$$\int_0^\infty \frac{1}{T}\int_0^T E\{B_{i,j}(\tau, Z_0)b_k(t, Z_s)\}\,d\tau\,ds,$$

$$\int_0^\infty \frac{1}{T}\int_0^T E\{b_i(t, Z_0)B_{k,l}(\tau, Z_s)\}\,d\tau\,ds$$

and

$$\int_0^\infty E\{b_i(t, Z_0)b_j(t, Z_s)\}\,ds$$

for every $i, j, k, l = 0, 1, \dots, d$. We also assume that the last two expressions are smooth as functions of the variable t for $0 \le t \le 1$. To avoid painful lists of hypotheses we assume, as much as needed, regularity and boundedness properties on the coefficients.

Equation (1) will be studied with the following affine boundary condition:

$$F_0 X_0^\varepsilon + F_1 X_1^\varepsilon = f, \tag{1a}$$

where $F_0, F_1 \in \mathscr{M}_{d\times d}$, $f \in \mathbb{R}^d$ are given and $\operatorname{rank}(F_0{:}F_1) = d$. In particular (1a) includes the case of periodic solutions by choosing $F_0 = -F_1 = I_{d\times d}$ and $f = O_{\mathbb{R}^d}$.

In our application we shall be interested in the case of a two-point boundary value problem: for $0 < l < d$, writing $F_0 = \binom{F_0'}{0}$, $F_1 = \binom{0}{F_1'}$, with $F_0' \in \mathscr{M}_{l\times d}$ with rank l, and $F_1' = \in \mathscr{M}_{(d-l)\times d}$ with rank $(d-l)$, if $f = \binom{f_0}{f_1}$ with $f_0 \in \mathbb{R}^l$ and $f_1 \in \mathbb{R}^{d-l}$, (1a) becomes

$$F_0' X_0^\varepsilon = f_0, \qquad F_1' X_1^\varepsilon = f_1, \tag{1b}$$

with rank $(F_0{:}F_1) = d$ being satisfied.

Before studying (1) with (1a), we shall recall the classical result for the initial value problem, with

$$X_0^\varepsilon = x_0 \in \mathbb{R}^d. \tag{1c}$$

We first notice that (1), with the initial value condition, has a unique solution (X_t^ε); an explicit form will be given in Section 3 by using the variation-of-constants formula. We shall consider this solution as a continuous process in $\mathbb{R}^d$.

The pair $(X_t^\varepsilon, Z_t^\varepsilon)$ is a nonhomogeneous Markov process in $\mathbb{R}^d \times \mathbb{R}^n$ with infinitesimal generator $L_t(X^\varepsilon, Z^\varepsilon)$ given by

$$L_t(X^\varepsilon, Z^\varepsilon) = \frac{1}{\varepsilon^2}L(Z) + \left(\left\{\frac{1}{\varepsilon}B\left(\frac{t}{\varepsilon}, z\right) + A\left(\frac{t}{\varepsilon}, z\right)\right\}x + \frac{1}{\varepsilon}b(t, z) + a(t, z)\right)\cdot\nabla_x. \tag{2}$$

Using a general result of Papanicolaou, Stroock and Varadhan (Theorem 2.8 in [9]), we have that (X_t^{ε}) converges in distribution, as $\varepsilon \searrow 0$, to a Markov diffusion (X_t) on $\mathbb{R}^d$ with infinitesimal generator $L_t(X)$ given by

$$
\begin{aligned}
L_t(X) = \int_0^\infty \frac{1}{T}\int_0^T &E\{((B(\tau, Z_0)x + b(t, Z_0)) \cdot \nabla_x) \\
&\times ((B(\tau, Z_s)x + b(t, Z_s)) \cdot \nabla_x)\}\, d\tau\, ds \\
&+ (\bar{A}x + \bar{a}(t)) \cdot \nabla_x.
\end{aligned} \tag{3}
$$

This can be written in the form

$$
L_t(X) = \frac{1}{2}\sum_{i=1}^d \sum_{j=1}^d \alpha_{i,j}(t, x)\frac{\partial^2}{\partial x_i\, \partial x_j} + \sum_{j=1}^d \beta_j(t, x)\frac{\partial}{\partial x_j} \tag{4}
$$

with

$$
\begin{aligned}
\alpha(t, x) = \int_0^\infty \frac{1}{T}\int_0^T &E\{(B(\tau, Z_0)x + b(t, Z_0)) \\
&\times (B(\tau, Z_s)x + b(t, Z_s))^*\}\, d\tau\, ds \\
+\int_0^\infty \frac{1}{T}\int_0^T &E\{(B(\tau, Z_s)x + b(t, Z_s)) \\
&\times (B(\tau, Z_0)x + b(t, Z_0))^*\}\, d\tau\, ds,
\end{aligned} \tag{4'}
$$

$$
\begin{aligned}
\beta(t, x) = {} & \bar{A}x + \bar{a}(t) \\
&+ \int_0^\infty \frac{1}{T}\int_0^T E\{B(\tau, Z_s)(B(\tau, Z_0)x + b(t, Z_0))\}\, d\tau\, ds,
\end{aligned} \tag{4''}
$$

where the asterisk (*) denotes transposition.

We want to represent the limiting diffusion (X_t) as the solution of a stochastic differential equation. In order to specify this equation, let us observe that for every c in $\mathbb{R}^d$,

$$
\begin{aligned}
&\sum_{i=1}^d \sum_{j=1}^d c_i c_j \alpha_{i,j}(t, x) \\
&\quad = 2\int_0^\infty \frac{1}{T}\int_0^T E\{[c^*(B(\tau, Z_0)x + b(t, Z_0))] \\
&\qquad\qquad \times [c^*(B(\tau, Z_s)x + b(t, Z_s))]\}\, d\tau\, ds,
\end{aligned} \tag{5}
$$

which is nonnegative for every x and c in $\mathbb{R}^d$. Its quadratic part

$$
2\int_0^\infty \frac{1}{T}\int_0^T E\{[c^*B(\tau, Z_0)x][c^*B(\tau, Z_s)x]\}\, d\tau\, ds
$$

can be decomposed in $\mathbb{R}^d \otimes \mathbb{R}^d$ into $\sum_{k=1}^{r}(c^* Q_k x)^2$ with at most d^2 matrices (Q_k) in $\mathscr{M}_{d\times d}$: $r \le d^2$. A change of origin in x enables us to include first order terms in x in such a way that

$$\sum_{i=1}^{d}\sum_{j=1}^{d} c_i c_j \alpha_{i,j}(t,x) - \sum_{k=1}^{r}\left(c^*(Q_k x + q_k(t))\right)^2$$

is a nonnegative quadratic form in c, with $q_k(t)$ being an $\mathbb{R}^d$-valued smooth deterministic function of t. Decomposing this nonnegative quadratic form in c into $\sum_{k=r+1}^{r+p}(c^* q_k(t))^2$ with $p \le d$, we obtain

$$(5')\quad \sum_{i=1}^{d}\sum_{j=1}^{d} c_i c_j \alpha_{i,j}(t,x) = \sum_{k=1}^{r}\left(c^*(Q_k x + q_k(t))\right)^2 + \sum_{k=r+1}^{r+p}\left(c^* q_k(t)\right)^2$$

with $r \le d^2$, $Q_1,\dots,Q_r \in \mathscr{M}_{d\times d}$ and $q_1(t),\dots,q_{r+p}(t) \in \mathbb{R}^d$, smooth in t.

REMARK. The matrices Q_k, $k = 1,\dots,r$, are independent of b; in particular, for $b = 0$ we have $q_k = 0$. This remark will be important in Section 3.

This decomposition (5′) enables us to define

$$\sigma(t,x) = \left(Q_1 x + q_1(t) : \cdots : Q_r x + q_r(t) : q_{r+1}(t) : \cdots : q_{r+p}(t)\right)$$

as a $d \times (r+p)$ matrix satisfying $\sigma(t,x)\sigma(t,x)^* = \alpha(t,x)$.

Now let $(W_t^{(1)} \cdots W_t^{(r+p)})$ be $(r+p)$ real-valued independent standard Brownian motions defined on a stochastic basis $(\Omega_0, \mathscr{G}, (\mathscr{G}_t)_{t\ge 0}, W)$. We have thus obtained the following proposition.

PROPOSITION 1. *The continuous solution* $(X_t^{\varepsilon})_{0\le t\le 1}$ *of* (1) *and* (1c) *converges in distribution to the unique continuous solution* $(X_t)_{0\le t\le 1}$ *of the stochastic differential equation*

$$\text{(6a)}\quad \begin{aligned} dX_t = {} & \sum_{k=1}^{r}\left(Q_k X_t + q_k(t)\right) dW_t^{(k)} \\ & + \sum_{k=r+1}^{r+p} q_k(t)\, dW_t^{(k)} + \bar{A}X_t\, dt + QX_t\, dt + \bar{a}(t)\, dt + q(t)\, dt, \end{aligned}$$

$$\text{(6b)}\quad X_0 = x_0 \in \mathbb{R}^d,$$

with

$$Q = \int_0^{\infty} \frac{1}{T}\int_0^T E\{B(\tau, Z_s)B(\tau, Z_0)\}\, d\tau\, ds$$

and

$$q(t) = \int_0^{\infty} \frac{1}{T}\int_0^T E\{B(\tau, Z_s)b(t, Z_0)\}\, d\tau\, ds.$$

Since (1) with boundary conditions (1a) will lead us to (6) with boundary conditions which will force (X_t) to be anticipative, in order to give a meaning to the stochastic integrals, we rewrite (6a) in the Stratonovich form:

$$(6a') \quad dX_t = \sum_{k=1}^{r} (Q_k X_t) \circ dW_t^{(k)} + \sum_{k=1}^{r+p} q_k(t)\, dW_t^{(k)} + \tilde{A}\, X_t\, dt + \tilde{a}(t)\, dt$$

with

$$\tilde{A} = \bar{A} + Q - \tfrac{1}{2} \sum_{k=1}^{r} Q_k^2 \quad \text{and} \quad \tilde{a}(t) = \bar{a}(t) + q(t).$$

To end this section, let us consider three examples with $A = a = b = 0$, $d = 2$ and $B(\tau, z) = B(z) \in \mathscr{M}_{2\times 2}$.

EXAMPLE 1. Let $B(z) = F(z)B$ with $F(z)$ real such that $E\{F(Z_0)\} = 0$ and $B \neq 0$. Then $\alpha(x) = (\sqrt{2\alpha}Bx)(\sqrt{2\alpha}Bx)^*$ with $\alpha = \int_0^\infty E\{F(Z_0)F(Z_s)\}\, ds$ (we assume $\alpha \neq 0$ and $\alpha < +\infty$ and, consequently, $0 < \alpha < +\infty$), $r = 1$ and $Q = \alpha B^2$, which implies

$$dX_t = \sqrt{2\alpha}\,(BX_t) \circ dW_t^{(1)}.$$

EXAMPLE 2. Assume $B_{1,1}(Z_t)$, $B_{2,1}(Z_t)$, $B_{1,2}(Z_t)$ and $B_{2,2}(Z_t)$ are independent processes. Then $\alpha(x)$ is diagonal with $\alpha_{i,i}(x) = 2\sum_{k=1}^{2} \alpha_{i,k} x_k^2$, where $\alpha_{i,k} = \int_0^\infty E\{B_{i,k}(Z_0)B_{i,k}(Z_s)\}\, ds$ $(0 < \alpha_{i,k} < +\infty)$. In this case, $r = 4$ and

$$dX_t = \sqrt{2\alpha_{1,1}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X_t \circ dW_t^{(1)} + \sqrt{2\alpha_{1,2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} X_t \circ dW_t^{(2)}$$
$$+ \sqrt{2\alpha_{2,1}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} X_t \circ dW_t^{(3)} + \sqrt{2\alpha_{2,2}}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} X_t \circ dW_t^{(4)}.$$

EXAMPLE 3. Let $B_{1,1}(z) = B_{2,2}(z) = 0$. A simple computation shows that

$$dX_t = \begin{pmatrix} 0 & m_1 \\ n_1 & 0 \end{pmatrix} X_t \circ dW_t^{(1)} + \begin{pmatrix} 0 & m_2 \\ n_2 & 0 \end{pmatrix} X_t \circ dW_t^{(2)}$$
$$+ \left\{ \begin{pmatrix} \alpha' & 0 \\ 0 & \alpha'' \end{pmatrix} - \tfrac{1}{2}(\alpha' + \alpha'') I \right\} X_t\, dt$$

with

$$m_1^2 + m_2^2 = 2\int_0^\infty E\{B_{1,2}(Z_0)B_{1,2}(Z_s)\}\, ds,$$

$$n_1^2 + n_2^2 = 2\int_0^\infty E\{B_{2,1}(Z_0)B_{2,1}(Z_s)\}\, ds,$$

$$\alpha' = \int_0^\infty E\{B_{2,1}(Z_0)B_{1,2}(Z_s)\}\, ds,$$

$$\alpha'' = \int_0^\infty E\{B_{1,2}(Z_0)B_{2,1}(Z_s)\}\, ds,$$

$$m_1 n_1 + m_2 n_2 = \alpha' + \alpha''.$$

We notice that, in general, $\alpha' \neq \alpha''$.

3. The boundary value problem. Our goal is to obtain a result similar to Proposition 1 when (1c) is replaced by (1a) and when the stochastic integrals in (6a′) are generalized Stratonovich integrals (with anticipative integrands) studied in [7], Section 3.

We run into two problems:

1. Does problem (1) with (1a) have a solution?
2. How do we establish the convergence result for that solution?

Because the equation is linear, we shall use variation of constants to answer the first question and to characterize the solution (X_t^ε). This characterization will then help us to proceed with the second question.

Let (ϕ_t^ε) be the fundamental $(d \times d)$ matrix-valued solution of the linear equation with initial value

$$\frac{d\phi_t^\varepsilon}{dt} = \left\{\frac{1}{\varepsilon}B\left(\frac{t}{\varepsilon}, Z_t^\varepsilon\right) + A\left(\frac{t}{\varepsilon}, Z_t^\varepsilon\right)\right\}\phi_t^\varepsilon, \qquad \phi_0^\varepsilon = I_{d\times d}, \quad 0 \le t \le 1. \tag{7}$$

We define $\phi^\varepsilon(t,s) = \phi_t^\varepsilon(\phi_s^\varepsilon)^{-1}$ for $s, t \in [0,1]$ and

$$V_t^\varepsilon = \int_0^t a(s, Z_s^\varepsilon)\,ds + \frac{1}{\varepsilon}\int_0^t b(s, Z_s^\varepsilon)\,ds. \tag{8}$$

By the variation of constants formula we get

$$X_t^\varepsilon = \phi^\varepsilon(t,0)X_0^\varepsilon + \int_0^t \phi^\varepsilon(t,s)\,dV_s^\varepsilon \tag{9}$$

and the boundary condition (1a) becomes

$$(F_0 + F_1\phi^\varepsilon(1,0))X_0^\varepsilon = f - F_1\int_0^1 \phi^\varepsilon(1,s)\,dV_s^\varepsilon. \tag{10}$$

If $M^\varepsilon = F_0 + F_1\phi^\varepsilon(1,0)$ is *invertible*, then X_0^ε is well-defined and X_t^ε, given by (9), is the unique solution to our problem [(1) + (1a)].

Let D_ε be the subset of Ω such that M^ε is invertible: $D_\varepsilon\{w \in \Omega \mid \det(M^\varepsilon) \neq 0\}$. Identifying $\mathscr{M}_{d\times d}$ with $\mathbb{R}^{d\times d}$, $\{\phi \in \mathscr{M}_{d\times d} \mid \det(F_0 + F_1\phi) \neq 0\}$ is an open subset of $\mathbb{R}^{d\times d}$ and, therefore, D_ε is $\mathscr{F}$-measurable. On D_ε, $X_0^\varepsilon = (M^\varepsilon)^{-1}(f - F_1\int_0^1\phi^\varepsilon(1,s)\,dV_s^\varepsilon)$ and $(X_t^\varepsilon, 0 \le t \le 1)$ is given by (9).

In the sequel, the following assumption will be made:

(10′) there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, $P(D_\varepsilon) = 1$.

This will be the case in our application. After our convergence result, we shall indicate what to do if this is not the case.

Using the techniques recalled in Section 2, it is possible to prove the convergence in distribution of $(\phi_t^\varepsilon, V_t^\varepsilon)$ defined by (7) and (8); this is again a classical diffusion approximation. Unfortunately, this is not enough to handle the convergence of the integral $\int_0^t\phi^\varepsilon(t,s)\,dV_s^\varepsilon = \phi_t^\varepsilon\int_0^t(\phi_s^\varepsilon)^{-1}\,dV_s^\varepsilon$ involved for

$t = 1$ in (10) defining X_0^ε. The difficulty is due to the fact that the mapping $(\phi, V) \to \int \phi \, dV$ is not continuous on the set of pairs of continuous processes, for which V is a semimartingale.

In order to handle this integral, we shall study the convergence of the pair $(\phi_t^\varepsilon, Y_t^\varepsilon)$, where (ϕ_t^ε) is defined by (7) and (Y_t^ε) is the solution of (1) with zero initial condition. Using (9) we have $Y_t^\varepsilon = \int_0^t \phi^\varepsilon(t, s) \, dV_s^\varepsilon$.

Under our hypothesis (10′), the solution (X_t^ε) of [(1) + (1a)] can be written as

$$X_t^\varepsilon = \phi_t^\varepsilon (F_0 + F_1 \phi^\varepsilon(1,0))^{-1} (f - F_1 Y_1^\varepsilon) + Y_t^\varepsilon, \tag{11}$$

which converges in distribution as a continuous functional of $(\phi_t^\varepsilon, Y_t^\varepsilon)_{0 \le t \le 1}$, as the following result demonstrates.

PROPOSITION 2. *The pair $(\phi_t^\varepsilon, Y_t^\varepsilon)_{t \ge 0}$ converges in distribution, as $\varepsilon \searrow 0$, to the Markov diffusion $(\phi_t, Y_t)_{t \ge 0}$ on $\mathscr{M}_{d \times d} \times \mathbb{R}^d$ with infinitesimal generator given by*

$$\begin{aligned} L_t(\phi, Y) = \int_0^\infty \frac{1}{T} \int_0^T E\{ & [(B(\tau, Z_0) y + b(t, Z_0)) \cdot \nabla y + (B(\tau, Z_0) \varphi) \cdot \nabla_\varphi] \\ & \times [(B(\tau, Z_s) y + b(t, Z_s)) \cdot \nabla y \\ & \qquad + (B(\tau, Z_s) \varphi) \cdot \nabla_\varphi] \} \, d\tau \, ds \\ & + (\bar{A} y + \bar{a}(t)) \cdot \nabla y + (\bar{A} \varphi) \cdot \nabla \varphi \end{aligned} \tag{12}$$

and initial values $(\phi_0, Y_0) = (I_{d \times d}, O_{\mathbb{R}^d})$.

PROOF. The process $(\phi_t^\varepsilon, Y_t^\varepsilon, Z_t^\varepsilon)_{t \ge 0}$ is a Markov process in $\mathscr{M}_{d \times d} \times \mathbb{R}^d \times \mathbb{R}^n$ with initial value $(I_{d \times d}, O_{\mathbb{R}^d}, Z_0)$ and infinitesimal generator given by

$$\begin{aligned} L_t(\phi^\varepsilon, Y^\varepsilon, Z^\varepsilon) = & \frac{1}{\varepsilon^2} L(Z) + \left(A\left(\frac{t}{\varepsilon}, z \right) y + a(t, z) \right) \cdot \nabla y \\ & + \left(A\left(\frac{t}{\varepsilon}, z \right) \varphi \right) \cdot \nabla \varphi \\ & + \frac{1}{\varepsilon} \left\{ \left(B\left(\frac{t}{\varepsilon}, z \right) y + b(t, z) \right) \cdot \nabla y \right. \\ & \qquad \left. + B\left(\frac{t}{\varepsilon}, z' \right) \varphi \cdot \nabla \varphi \right\}. \end{aligned} \tag{13}$$

We then apply the general result of Papanicolaou, Stroock and Varadhan (Theorem 2.8 in [9]). □

REMARK. The process $(\phi_t x_0 + Y_t)$ is the limiting solution of [(1) + (1c)] obtained in Proposition 1.

In order to pass to the limit as $\varepsilon \searrow 0$ in (11), we need the invertibility of $F_0 + F_1\phi_1$. For that we shall characterize the first component (ϕ_t) of the pair (ϕ_t, Z_t) in Proposition 2 as the solution of a stochastic differential equation, and impose conditions on the coefficients of this equation. The infinitesimal generator of (ϕ_t) is given by

$$(14)\quad L(\phi) = \int_0^\infty \frac{1}{T}\int_0^T E\{(B(\tau, Z_0)\varphi\cdot\nabla\varphi)(B(\tau, Z_s)\varphi\cdot\nabla\varphi)\}\,d\tau\,ds + \bar{A}\varphi\cdot\nabla\varphi.$$

Using the remark following (5), one can identify (ϕ_t) as the unqiue solution of the linear stochastic differential equation

$$(15)\quad \begin{aligned} d\phi_t &= \sum_{k=1}^{r}(Q_k\phi_t)\,dW_t^{(k)} + (\bar{A} + Q)\phi_t\,dt,\\ \phi_0 &= I_{d\times d}, \end{aligned}$$

where $Q_1, \dots, Q_k$ and Q have been defined in (5′) and Proposition 1. Equation (15) is the matrix version of (6a) in the case $b = 0$. We recall that $(\Omega_0, \mathscr{G}, (\mathscr{G}_t), W)$ is a stochastic basis for Brownian motion.

We shall assume that $F_0 + F_1\phi_1$ is W-a.s. invertible:

$$(15')\quad W\{\det(F_0 + F_1\phi_1) \neq 0\} = 1.$$

Following Ocone and Pardoux [7], this hypothesis can be verified for the coefficients of (15). In the *hypoelliptic case* (i.e., when the ideal generated by $Q_1, \dots, Q_r$ in the Lie algebra of matrices generated by $\tilde{A} = \bar{A} + Q - \frac{1}{2}\sum_{k=1}^{r}(Q_k)^2$, $Q_1, \dots, Q_r$ has rank d^2) we have

$$\operatorname{rank}(F_0{:}F_1) = d \quad\Leftrightarrow\quad W\{\det(F_0 + F_1\phi_1) \neq 0\} = 1.$$

In the *general case*, the condition $\operatorname{rank}(F_0{:}F_1) = d$ does not imply (15′), and the verification of (15′) amounts to computing a finite number of determinants which are expressible in terms of $\tilde{A}$, $Q_1, \dots, Q_r$, F_0 and F_1 (see Theorem 4.8 in [7]).

PROPOSITION 3. *Assuming* (10′) *and hypothesis* (15′), *the unique continuous solution* (*on* D_ε) *of* [(1) + (1a)] $(X_t^\varepsilon;\ 0 \le t \le 1)$, *converges in distribution, as* $\varepsilon \searrow 0$, *to the continuous process* $(X_t = \phi_t(F_0 + F_1\phi_1)^{-1}(f - F_1Y_1) + Y_t;\ 0 \le t \le 1)$, *where* (ϕ_t, Y_t) *is the diffusion defined by* (12).

PROOF. The only difficulty comes from the fact that (X_t^ε) and (X_t) are only defined almost surely on their respective bases. It is enough to prove the convergence along sequences $(\varepsilon_n)_{n\ge1}$ such that $0 < \varepsilon_n < \varepsilon_0$ for every $n \ge 1$ and $\lim_n \varepsilon_n = 0$; we have $P(\bigcap_{n\ge1} D_{\varepsilon_n}) = 1$. Using *Skorohod's representation theorem* (Theorem 1.8 in [3]), we can find a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ where $(\tilde{\phi}_t^{\varepsilon_n}, \tilde{Y}_t^{\varepsilon_n}, \tilde{\phi}_t, \tilde{Y}_t;\ 0 \le t \le 1,\ n \ge 1)$ are defined in such a way that $\mathrm{Law}((\tilde{\phi}_t^{\varepsilon_n}, \tilde{Y}_t^{\varepsilon_n})_{0\le t\le1}) = \mathrm{Law}((\phi_t^{\varepsilon_n}, Y_t^{\varepsilon_n})_{0\le t\le1})$ for every $n \ge 1$, $\mathrm{Law}((\tilde{\phi}_t, \tilde{Y}_t)_{0\le t\le1}) = \mathrm{Law}((\phi_t, Y_t)_{0\le t\le1})$, $\det(F_0 + F_1\tilde{\phi}_1^{\varepsilon_n}) \neq 0$, $\det(F_0 + F_1\tilde{\phi}_1) \neq 0$ and $(\tilde{\phi}_t^{\varepsilon_n}, \tilde{Y}_t^{\varepsilon_n})_{0\le t\le1}$ converges, as $n \uparrow +\infty$, $\tilde{P}$-a.s. to $(\tilde{\phi}_t, \tilde{Y}_t)_{0\le t\le1}$ for the sup norm over $[0,1]$. Defining $\tilde{P}$-a.s. $(\tilde{X}_t^{\varepsilon_n})_{0\le t\le 1}$ [resp. $(\tilde{X}_t)_{0\le t\le 1}$] as a continuous functional of $(\tilde{\phi}_t^{\varepsilon_n}, \tilde{Y}_t^{\varepsilon_n})_{0\le t\le 1}$ [resp. $(\tilde{X}_t, \tilde{Y}_t)_{0\le t\le 1}$] through (11), we see that $(\tilde{X}_t^{\varepsilon_n})_{0\le t\le 1}$ converges $\tilde{P}$-a.s., as $n\uparrow +\infty$, to $(\tilde{X}_t)_{0\le t\le 1}\cdot(\tilde{X}_t^{\varepsilon_n})_{0\le t\le 1}$ [resp. $(\tilde{X}_t)_{0\le t\le 1}$] having the same law as $(X_t^{\varepsilon_n})_{0\le t\le 1}$ [resp. $(X_t)_{0\le t\le 1}$]. We conclude that $(X_t^{\varepsilon_n})_{0\le t\le 1}$ converges in distribution, as $n\uparrow +\infty$, to $(X_t)_{0\le t\le 1}$. □

REMARK. If we only assume hypothesis (15′), we still have that $\lim_{\varepsilon\searrow 0} P(D_\varepsilon) = 1$. The set $\{\phi \in \mathscr{M}_{d\times d}/\det(F_0 + F_1\phi_1)\neq 0\}$, identified as a subset of $\mathbb{R}^{d\times d}$, is open and the convergence in distribution of ϕ_1^ε to ϕ_1 implies

$$\liminf_{\varepsilon\searrow 0} P\big(\det(F_0 + F_1\phi_1^\varepsilon)\neq 0\big) \ge W\big(\det(F_0 + F_1\phi_1)\neq 0\big) = 1$$

by (15′). Using again Skorohod's representation theorem, $(\tilde{X}_t^{\varepsilon_n})_{0\le t\le 1}$, defined through (11) on $\tilde{D}_{\varepsilon_n} = \{\det(F_0 + F_1\tilde{\phi}_1^{\varepsilon_n})\neq 0\}$, and 0 otherwise, converges in the sup norm over $[0,1]$ in probability to $(\tilde{X}_t)_{0\le t\le 1}$. This gives a meaning to the statement that $(X_t^\varepsilon)_{0\le t\le 1}$, defined on D_ε, converges in distribution to $(X_t)_{0\le t\le 1}$.

We now give our main result:

THEOREM 1. *Assume that* (10′) *and* (15′) *hold. Then* $(X_t^\varepsilon)_{0\le t\le 1}$, *the unique continuous solution of* [(1) + (1a)] *defined on* D_ε, *converges in distribution as* $\varepsilon\searrow 0$ *to the unique continuous solution of*

$$(16)\qquad \begin{aligned} dX_t &= \sum_{k=1}^{r} (Q_k X_t)\circ dW_t^{(k)} + \sum_{k=1}^{r+p} q_k(t)\, dW_t^{(k)} + \tilde{A}X_t\, dt + \tilde{a}(t)\, dt,\\ &\qquad F_0X_0 + F_1X_1 = f, \qquad 0\le t\le 1, \end{aligned}$$

where

(i) $Q_1,\dots,Q_k$, $q_1,\dots,q_{r+p}$, $\tilde{A}$ *and* $\tilde{a}$ *have been defined in* (5) *and* (6a′).
(ii) *The stochastic integrals are generalized Stratonovich integrals* (*see* [7]).
(iii) *Convergence in distribution is as given in Proposition* 3.

PROOF. Using Theorem 4.2 of [7] and property (15′), (16) has a unique continuous solution. On the other hand, it follows from Proposition 3 that $(X_t^\varepsilon)_{0\le t\le 1}$ converges in distribution as $\varepsilon\searrow 0$ to $(X_t)_{0\le t\le 1}$ given by $X_t = \phi_t(F_0 + F_1\phi_1)^{-1}(f - F_1Y_1) + Y_t$.

Therefore, the only thing that remains to be proved is that $(X_t)_{0\le t\le 1}$ is a solution to (16). Indeed (16) is satisfied. We have

$$\phi_t = I + \sum_{k=1}^{r}\int_0^t (Q_k\phi_s)\circ dW_s^{(k)} + \int_0^t \tilde{A}\phi_s\, ds.$$

In order to compute $\phi_t X_0$, we use Proposition 3.3 of [7] and enter X_0 in the Stratonovich integrals, which are then defined as generalized Stratonovich integrals [X_0 is not $(\mathscr{G}_t)$-adapted]. Therefore, we obtain

$$\phi_t X_0 = X_0 + \sum_{k=1}^{r} \int_0^t (Q_k \phi_s X_0) \circ dW_s^{(k)} + \int_0^t \tilde{A} \phi_s X_0 \, ds.$$

We then have

$$\begin{aligned} X_t &= \phi_t X_0 + Y_t \\ &= X_0 + \sum_{k=1}^{r} \int_0^t (Q_k \phi_s X_0) \circ dW_s^{(k)} + \int_0^t \tilde{A} \phi_s X_0 \, ds \\ &\quad + \sum_{k=1}^{r} \int_0^t (Q_k Y_s) \circ dW_s^{(k)} + \sum_{k=1}^{r+p} \int_0^t q_k(s) \, dW_s^{(k)} \\ &\quad + \int_0^t \tilde{A} Y_s \, ds + \int_0^t \tilde{a}(s) \, ds \\ &= X_0 + \sum_{k=1}^{r} \int_0^t (Q_k X_s) \circ dW_s^{(k)} + \int_0^t \tilde{A} X_s \, ds \\ &\quad + \sum_{k=1}^{r+p} \int_0^t q_k(s) \, dW_s^{(k)} + \int_0^t \tilde{a}(s) \, ds, \end{aligned}$$

which is the integral form of (16).

The boundary condition (1a) is also satisfied:

$$\begin{aligned} F_0 X_0 + F_1 X_1 &= F_0 \big(\phi_0 (F_0 + F_1 \phi_1)^{-1} (f - F_1 Y_1) + Y_0 \big) \\ &\quad + F_1 \big(\phi_1 (F_0 + F_1 \phi_1)^{-1} (f - F_1 Y_1) + Y_1 \big) \\ &= F_0 (F_0 + F_1 \phi_1)^{-1} (f - F_1 Y_1) \\ &\quad + F_1 \phi_1 (F_0 + F_1 \phi_1)^{-1} (f - F_1 Y_1) + F_1 Y_1 \\ &= f. \end{aligned}$$

□

Before giving our example, which was our original motivation, we make some comments:

1. Proposition 3 gives a "classical" characterization of the limit (X_t), in the sense that (X_t) is obtained via classical diffusion approximation results on $(\phi_t^\varepsilon, Y_t^\varepsilon)$. Theorem 1 gives a "nonclassical" characterization of (X_t) and is interesting in the sense that it enables us to apply the results of [7] to study the properties of (X_t) such as the existence of probability densities or Markov properties.
2. We have presented here a model of an equation "driven" by a Markov process. Obviously, the results of this paper can be extended to much more general situations by using diffusion approximation results, such as in [4], [3] or [5], provided we have enough mixing in our model.

3. Recent results on nonlinear situations with boundary conditions [2], [6], or [10] give some hope to treat nonlinearity in (1).

4. Application to waves in random media. We come back to a problem studied in [8] (and references therein). We look at this problem from the point of view of the result obtained in Section 3.

Let us describe the simplest form of the problem.

For a fixed $L > 0$, we consider the one-dimensional acoustic wave equation in the interval $0 \le x \le L$:

$$\begin{aligned} \rho(x)\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} &= 0, \\ \frac{1}{K(x)}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} &= 0, \end{aligned} \tag{17}$$

with boundary conditions to be specified later. Here $u \equiv u(x,t)$ is the velocity, $p \equiv p(x,t)$ is the pressure, $\rho(x)$ is the density of $K(x)$ is the bulk modulus.

Let $(Z_x)_{x \ge 0}$ be a Markov process like the one considered in this paper. We assume that $\rho(x) = 1 + \eta(Z_x^\varepsilon)$ for some smooth function η taking its values in $[-c, c]$ with $0 < c < 1$ [we need $\rho(x)$ to be positive]; we assume the centering condition $E\{\eta(Z_0)\} = 0$ and recall that $Z_x^\varepsilon = Z_{x/\varepsilon^2}$ for $0 < \varepsilon \ll 1$. For simplicity, we take $K(x)$ to be constant equal to 1. This is a homogeneous model in the sense that $\rho(x)$ is stationary.

Without fluctuation ($\eta = 0$), (17) can be rewritten with $A = u + p$ and $B = u - p$ in a very simple form: A satisfies $\partial A/\partial x = -\partial A/\partial t$ and will then be called the *right going wave*; similarly, B satisfies $\partial B/\partial x = \partial B/\partial t$ and will called the *left going wave*.

We rewrite (17) for the right going wave $A = u + p$ and the left going wave $B = u - p$:

$$\frac{\partial}{\partial x}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} A \\ B \end{pmatrix} + \frac{1}{2}\eta(Z_x^\varepsilon)\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} A \\ B \end{pmatrix}. \tag{17'}$$

Our boundary conditions will correspond to a pulse entering the interval (at $x = 0$) at time $t = 0$ with nothing entering the interval at its end $x = L$. We assume $\rho(x) = K(x) = 1$ for $x \notin [0, L]$ and continuity for the solution at $x = 0$ and $x = L$. The boundary conditions are

$$A(0,t) = \delta_0(t) \quad \text{and} \quad B(L,t) = 0, \qquad \forall t. \tag{17''}$$

Considering $\binom{A}{B}$ as a stochastic process indexed by x for $0 \le x \le L$, the problem (17′) with (17″) is an infinite-dimensional problem due to the variable t (of course x will play the role of a time for the solution seen as a process in x).

In a more realistic way, we should have a pulse of the form $f(t)$ for some C^∞ function f with compact support in $\mathbb{R}_+$. This means that with (17″) we are studying the Green's function for our problem (17′). Taking the Fourier transform in time at the rescaled frequency w/ε, we define

$$\hat{A}^{\varepsilon}(x,w) = \int e^{iwt/\varepsilon} A(x,t)\,dt \quad \text{and} \quad \hat{B}^{\varepsilon} = \int e^{iwt/\varepsilon} B(x,t)\,dt.$$

In order to center the equation, we change the phase by setting

$$\tilde{A}^{\varepsilon}(x,w) = \hat{A}^{\varepsilon}(x,w) e^{iwx/\varepsilon} \quad \text{and} \quad \tilde{B}^{\varepsilon}(x,w) = \hat{B}^{\varepsilon}(x,w) e^{-iwx/\varepsilon}$$

and, therefore, (17′) with (17″) can be rewritten in the following form, when w is considered as a parameter:

$$(18) \quad \frac{d}{dx}\begin{pmatrix} \tilde{A}^{\varepsilon} \\ \tilde{B}^{\varepsilon} \end{pmatrix} = \frac{iw}{2\varepsilon}\eta(Z_x^{\varepsilon})\begin{pmatrix} 1 & e^{-2iwx/\varepsilon} \\ -e^{2iwx/\varepsilon} & -1 \end{pmatrix}\begin{pmatrix} \tilde{A}^{\varepsilon} \\ \tilde{B}^{\varepsilon} \end{pmatrix}, \qquad 0 \le x \le L$$

with

$$(18') \qquad \tilde{A}^{\varepsilon}(0,w) = 1 \quad \text{and} \quad \tilde{B}^{\varepsilon}(L,w) = 0.$$

Of course, our problem is still infinite dimensional since we have (18) for all frequencies w. Nevertheless, the study of this problem for a finite number of frequencies gives information on the global solution, as explained in [8]. We shall study (18) and (18′) for one fixed frequency $w \neq 0$ (monochromatic wave) and we shall comment at the end on the multifrequency analysis.

In [8], Papanicolaou studied the reflected coefficient $\tilde{B}^{\varepsilon}(y)$ for (18) with $0 \le y \le x \le L$ and the two-point boundary value condition $\tilde{A}^{\varepsilon}(y,w) = 1$ and $\tilde{B}^{\varepsilon}(L,w) = 0$. At $y = 0$, this is $\tilde{B}^{\varepsilon}(0,w) = \hat{B}^{\varepsilon}(0,w)$ for our problem [this quantity is denoted $R^{\varepsilon}(w)$ or $R_L^{\varepsilon}(w)$ to indicate the dependence upon L].

Here we keep the interval $[0,L]$ fixed and study the unique solution $(\tilde{A}^{\varepsilon}(x,w),\ \tilde{B}^{\varepsilon}(x,w))_{0 \le x \le L}$ of the problem [(18) + (18′)]. Replacing x by t and $[0,L]$ by $[0,1]$, this problem is like [(1) + (1b)] studied in this paper, except that the solution is complex ($\mathbb{C}^2$-valued).

Using Theorem 1 (Section 3), we shall prove that $(\tilde{A}^{\varepsilon}, \tilde{B}^{\varepsilon})$ converges in distribution, as $\varepsilon \searrow 0$, and identify the limit as the unique solution of problem (16).

We shall then show that for every x in $(0,L)$, the limit has a probability density with respect to Lebesgue measure. Finally, we shall show that the limit has the Markov field property. For that, we rely strongly on results from Ocone and Pardoux [7] for R^d-valued processes. Without any doubt, these results can be extended to $\mathbb{C}^d$-valued processes, but to stay in a reasonable length we shall work with $X^{\varepsilon} = (\mathrm{Re}(\tilde{A}^{\varepsilon}), \mathrm{Im}(\tilde{A}^{\varepsilon}), (\mathrm{Re}(\tilde{B}^{\varepsilon}), (\mathrm{Im}(\tilde{B}^{\varepsilon}))$ in $\mathbb{R}^4$ (at the expense of having to work in four dimensions instead of two: see the remark following Theorem 2).

For $(X_x^{\varepsilon})_{0 \le x \le L}$, [(18) + (18′)] can be rewritten as

$$(19) \qquad \frac{dX_x^{\varepsilon}}{dx} = \frac{1}{\varepsilon}B\left(\frac{x}{\varepsilon}, Z_x^{\varepsilon}\right)X_x^{\varepsilon}, \qquad 0 \le x \le L,$$

and

$$(19') \qquad F_0' X_0^{\varepsilon} = f_0, \qquad F_L' X_L^{\varepsilon} = f_L,$$

where

$$
(20)\quad
\begin{aligned}
B(\tau, z) &= \frac{w}{2}\eta(z)\begin{pmatrix} 0 & -1 & \sin 2w\tau & -\cos 2w\tau \\ 1 & 0 & \cos 2w\tau & \sin 2w\tau \\ \sin 2w\tau & \cos 2w\tau & 0 & 1 \\ -\cos 2w\tau & \sin 2w\tau & -1 & 0 \end{pmatrix}, \\
F_0' &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad f_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
F_L' &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad f_L = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned}
$$

Since [(17′) + (17″)] is well posed, condition (10′) holds.

One can easily compute the limiting infinitesimal generator given by (4) and obtain:

$$
(21)\quad \alpha(x) = \frac{w^2\alpha}{4}\begin{pmatrix} 2x_2^2 + x_3^2 + x_4^2 & -2x_1x_2 & -3x_2x_4 + x_3x_1 & 3x_2x_3 + x_1x_4 \\ \cdot & 2x_1^2 + x_3^2 + x_4^2 & 3x_1x_4 + x_2x_3 & -3x_1x_3 + x_2x_4 \\ \cdot & \cdot & 2x_4^2 + x_1^2 + x_2^2 & -2x_3x_4 \\ \cdot & \cdot & \cdot & 2x_3^2 + x_1^2 + x_2^2 \end{pmatrix},
$$

where $\alpha(x)$ is symmetric, and

$$
\alpha = \int_0^\infty E\{\eta(Z_0)\eta(Z_s)\}\, ds.
$$

We assume $0 < \alpha < +\infty$ ($\alpha < +\infty$ has been assumed in the diffusion-approximation result and $\alpha \neq 0$ means the presence of randomness in our problem) and

$$
\beta(x) = 0.
$$

From now on x will be in the state space $\mathbb{R}^4$ and t will denote the variable in $[0, L]$.

One can check that $\sigma(x)\sigma(x)^* = \alpha(x)$ for

$$
\sigma(x) = (Q_1 x : Q_2 x : Q_3 x)
$$

with

$$
(22)\quad
\begin{aligned}
Q_1 &= \frac{w\sqrt{\alpha}}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\
Q_2 &= \frac{w\sqrt{\alpha}}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
Q_3 &= \frac{w\sqrt{\alpha}}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

By Proposition 1, the solution of [(19) + (1c)] has a limit in distribution, as $\varepsilon \searrow 0$, which is the unique solution of the Itô equation

$$(23)\qquad \begin{aligned} dX_t &= \sum_{k=1}^{3} (Q_k X_t)\, dW_t^{(k)} \qquad 0 \le t \le L,\\ X_0 &= x_0 \in \mathbb{R}^4. \end{aligned}$$

One can easily check that $\Sigma_{k=1}^{3} Q_k^2 = 0$, which implies that Ito integrals can be replaced by Stratonovich integrals. We then get the following theorem.

THEOREM 2. *The unique solution* $(X_t^{\varepsilon})_{0 \le t \le L}$ *of* [(19) + (19′)] *converges in distribution, as* $\varepsilon \searrow 0$, *to the unique solution of*

$$(24)\qquad dX_t = \sum_{k=1}^{3} (Q_k X_t) \circ dW_t^{(k)}, \qquad 0 \le t \le L,$$

with

$$(24')\qquad F_0' X_0 = f_0, \qquad F_L' X_L = f_L.$$

PROOF. One can check that $F_0 + F_L I_{4\times 4}$ is invertible with $F_0 = \begin{pmatrix} F_0' \\ \mathbf{0}_{2\times 4} \end{pmatrix}$ and $F_L = \begin{pmatrix} \mathbf{0}_{2\times 4} \\ F_L' \end{pmatrix}$. By Theorem 4.8(ii) in [7], [(23) + (24′)] has a unique continuous solution, the stochastic integrals being generalized Stratonovich integrals. Theorem 2 is then a consequence of Theorem 1. □

REMARK. Reintroducing $\binom{A}{B}$ in $\mathbb{C}^2$ such that $A = X^1 + iX^2$ and $B = X^3 + iX^4$, we get

$$(24'')\qquad d\begin{pmatrix} A_t \\ B_t \end{pmatrix} = \sum_{k=1}^{3} P_k \begin{pmatrix} A_t \\ B_t \end{pmatrix} \circ dW_t^{(k)}, \qquad 0 \le t \le L,$$

with

$$(24''')\qquad A_0 = 1, \qquad B_L = 0,$$

where

$$P_1 = \frac{w\sqrt{\alpha}}{\alpha\sqrt{2}} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad P_2 = \frac{w\sqrt{\alpha}}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad P_3 = \frac{w\sqrt{\alpha}}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Problem [(24″) + (24‴)] is the expected equation with boundary condition if we worked from the beginning in $\mathbb{C}^2$ with problem [(18) + (18′)] instead of $\mathbb{R}^4$ with problem [(19) + (19′)].

In order to study the existence of a probability density, with respect to Lebesgue measure on $\mathbb{R}^4$, for the law of X_t, $0 < t < L$, we first discuss the problem

$$(25)\qquad dY_t = \sum_{k=1}^{3} (Q_k Y_t)\, dW_t^{(k)} = \sum_{k=1}^{3} (Q_k Y_t) \circ dW_t^{(k)}$$

with

$$Y_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{25'}$$

[Because $\Sigma_{k=1}^3 Q_k^2 = 0$ in (25′) we can replace Stratonovich integrals by Itô integrals.]

It can be easily checked that the following three matrices R_1, R_2 and R_3 commute with Q_1, Q_2 and Q_3:

$$R_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$R_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $(R_1 Y_t)$ [resp. $(R_2 Y_t)$, $(R_3 Y_t)$] is the solution of (25) with initial value

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \left[\text{resp.} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right].$$

This gives us a simple representation for the fundamental solution (ϕ_t) with initial value $I_{4\times 4}$:

$$\phi_t = \begin{pmatrix} Y_t^1 & -Y_t^2 & Y_t^3 & Y_t^4 \\ Y_t^2 & Y_t^1 & -Y_t^4 & Y_t^3 \\ Y_t^3 & -Y_t^4 & Y_t^1 & Y_t^2 \\ Y_t^4 & Y_t^3 & -Y_t^2 & Y_t^1 \end{pmatrix}. \tag{26}$$

Since $\mathrm{tr}(Q_k) = 0$ for $k = 1, 2, 3$, we have $\det(\phi_t) = \det(\phi_0) = 1$ for every $t \geq 0$. This implies

$$\left((Y_t^1)^2 + (Y_t^2)^2 - (Y_t^3)^2 - (Y_t^4)^2 \right)^2 = 1.$$

By continuity with respect to t and the value at $t = 0$, we get

$$(Y_t^1)^2 + (Y_t^2)^2 = 1 + (Y_t^3)^2 + (Y_t^4)^2 \quad \text{for every } t \geq 0. \tag{27}$$

One can observe that this is nothing but the conservation of energy in the interval $[0, t]$: $(Y_t^1)^2 + (Y_t^2)^2 + (Y_0^3)^2 + (Y_0^4)^2$ is the energy going out $(|A_t|^2 + |B_0|^2)$ and $(Y_0^1)^2 + (Y_0^2)^2 + (Y_t^3)^2 + (Y_t^4)^2$ is the energy entering $[0, t]$ $(|A_0|^2 + |B_t|^2)$, where we have our initial value problem (25′): $Y_0^1 = 1$, $Y_0^2 = Y_0^3 = Y_0^4 = 0$ $(A_0 = 1,\ B_0 = 0)$.

Not surprisingly, this tells us that Y_t does not have a density with respect to Lebesgue measure on $\mathbb{R}^4$, since it belongs to the submanifold $(y_1)^2 + (y_2)^2 = 1 + (y_3)^2 + (y_4)^2$, which has dimension 3.

Using the notation of [7], (ϕ_t) belongs to G, the connected component containing the identity of the matrix Lie group generated by $\mathscr{G}$, the Lie algebra of matrices generated by Q_1, Q_2 and Q_3.

One can check that in our case

$$(28)\quad G = \left\{ \phi = \begin{pmatrix} a & -b & c & d \\ b & a & -d & c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} \in \mathscr{M}_{4\times 4}(\mathbb{R});\ a^2 + b^2 = 1 + c^2 + d^2 \right\}.$$

REMARK. On $\mathbb{C}$, G is simply

$$SU(1,1) = \left\{ \phi = \begin{pmatrix} A & \overline{B} \\ B & \overline{A} \end{pmatrix} \in \mathscr{M}_{2\times 2}(\mathbb{C});\ |A|^2 - |B|^2 = 1 \right\}.$$

By Proposition 4.6 of [7], for every $t > 0$, the law of ϕ_t admits a C^∞ density with respect to ν, the induced (from $\mathbb{R}^{4\times 4}$) volume measure on G.

Coming back to our problem [(24) + (24′)] and its unique continuous solution $(X_t,\ 0 \le t \le L)$, we have the following result.

THEOREM 3. *For every t in $(0, L)$, the law of X_t has a density with respect to Lebesgue measure on* $\mathbb{R}^4$.

PROOF. We use Proposition 5.12 in [7]; with similar notations, the existence of a density is equivalent to the existence of a point (u, v) in $\mathbb{R}^4 \times \mathbb{R}^4$ such that:

(i) $u_1 = 1$, $u_2 = 0$, $v_3 = 0$, $v_4 = 0$ (that is, $F_0 u + F_L v = f$).
(ii) There exists T in G such that $v = Tu$ and $F_0 + F_L T$ is invertible.
(iii) $\mathrm{Span}\{F_0 Q_k u;\ k = 1,2,3\} + \mathrm{Span}\{F_L Q_k v;\ k = 1,2,3\} = \mathbb{R}^4$.

REMARK. It can be checked that $[Q_1, Q_2]$ (resp. $[Q_1, Q_3]$, $[Q_2, Q_3]$) is proportional to Q_3 (resp. Q_2, Q_1). That is why, in (iii), we only need Q_k with $k = 1, 2, 3$. For (u, v) satisfying (i), we have

$$u = \begin{pmatrix} 1 \\ 0 \\ u_3 \\ u_4 \end{pmatrix}, \qquad v = \begin{pmatrix} v_1 \\ v_2 \\ 0 \\ 0 \end{pmatrix}$$

and, with $c = w\sqrt{\alpha}/2 \neq 0$,

$$F_0 Q_0 u = c\begin{pmatrix} -u_4 \\ u_3 \\ 0 \\ 0 \end{pmatrix}, \qquad F_L Q_2 v = c\begin{pmatrix} 0 \\ 0 \\ v_1 \\ v_2 \end{pmatrix} \quad \text{and} \quad F_L Q_3 v = c\begin{pmatrix} 0 \\ 0 \\ v_2 \\ -v_1 \end{pmatrix}.$$

Choosing $u_4 = 0$ and $v_2 = 0$, it is easily seen that $v = Tu$ has a solution T in G provided that $0 < u_3^2 < 1$. We then have $v_1^2 = 1 - u_3^2$ and if T is as in (26),

$$a = \frac{1}{v_1}, \qquad b = 0, \qquad c = -\frac{u_3}{v_1} \quad \text{and} \quad d = 0.$$

Because $F_0 + F_L T$ is invertible for every T in G, the three conditions are satisfied. □

5. The Markov field property. We shall first observe that under our two-point boundary value problem, there is a one-to-one correspondence between (X_0, X_L) and ϕ_L, the fundamental solution of (25), at $t = L$. Also we recall that (ϕ_t) is related to (Y_t) through (26).

We have $\phi_L X_0 = X_L$, which can be solved either in $(X_0^3, X_0^4, X_L^1, X_L^2)$ or in $(Y_L^1, Y_L^2, Y_L^3, Y_L^4)$. Using also $(Y_L^1)^2 + (Y_L^2)^2 = 1 + (Y_L^3)^2 + (Y_L^4)^2$, which is (27) at $t = L$, we can get

$$\begin{pmatrix} X_L^1 \\ X_L^2 \\ X_0^3 \\ X_0^4 \end{pmatrix} = \frac{1}{\left(Y_L^1\right)^2 + \left(Y_L^2\right)^2}\begin{pmatrix} Y_L^1 \\ Y_L^2 \\ Y_L^2 Y_L^4 - Y_L^1 Y_L^3 \\ -Y_L^3 Y_L^2 - Y_L^1 Y_L^4 \end{pmatrix} \tag{29}$$

where $\left(Y_L^1\right)^2 + \left(Y_L^2\right)^2 \geq 1$.

From this we get

$$\left(X_L^1\right)^2 + \left(X_L^2\right)^2 + \left(X_0^3\right)^2 + \left(X_0^4\right)^2 = 1, \tag{30}$$

which is again the conservation of energy in $[0, L]$ under (24′). It follows that $(X_0^3)^2 + (X_0^4)^2 = |B_0|^2 < 1$ since, if not, $X_L^1 = X_L^2 = 0$ and then $X_L = 0$, which implies $X_0 = 0$ and contradicts (24′):

$$T(X_0) Y_L = X_L$$

where

$$T(X_0) = \begin{pmatrix} 1 & 0 & X_0^3 & X_0^4 \\ 0 & 1 & X_0^4 & -X_0^3 \\ X_0^3 & X_0^4 & 1 & 0 \\ X_0^4 & -X_0^3 & 0 & 1 \end{pmatrix} \quad \text{is invertible} \tag{31}$$

since its determinant is equal to $(1-(X_0^3)^2-(X_0^4)^2)^2$ and $(X_0^3)^2+(X_0^4)^2<1$ by the previous argument.

The relation (30) shows that the process $(X_t)_{0\le t\le L}$ is not Markovian [X_0 and X_L are not conditionally independent given X_t for t in $(0, L)$].

Nevertheless we have the following theorem.

THEOREM 4. *$(X_t)_{0\le t\le L}$ satisfies the Markov field property: for any $0\le s<t\le L$, the σ-algebras $\sigma(X_v;\ s\le v\le t)$, $\sigma(X_u;\ 0\le u\le s)\vee\sigma(X_w;\ t\le w\le L)$ are conditionally independent, given (X_s, X_t).*

PROOF. The situation is similar to Example 6.2 in Section 6 of [7], except that it is four-dimensional instead of two-dimensional.

Nevertheless, the proof given in [7] can be generalized to our situation because of the following remark: (31) can easily be generalized into

$$T(X_0)Y_t = X_t \quad \text{for every } 0\le t\le L, \tag{31'}$$

with $T(X_0)$ invertible and commuting with G defined in (28). This last point can easily be checked from the definitions of $T(X_0)$ in (31) and G in (28). It implies that $T(X_0)$ commutes with any $\phi(t,s)=\phi_t(\phi_s)^{-1}$.

It follows that for every $0\le r<s<t\le L$,

$$\sigma(X_u;\, u\notin(r,t)) = \sigma(Y_u;\, u\notin(r,t)) \tag{32}$$

[which is also equal to $\sigma(\phi_u;\ u\notin(r,t))$ by (26)].

We want to prove that for any φ in $C_0(\mathbb{R}^4)$,

$$E\{\varphi(X_s)|X_u;\, u\notin(r,t)\} = E\{\varphi(X_s)|X_r, X_t\}. \tag{33}$$

We shall first prove that for any Ψ, bounded and Borel, on $\mathbb{R}^{4\times4}$,

(34) $\{\Psi(\phi(s,r))|Y_r, Y_t\}$ is $\sigma(X_r, X_t)$-measurable up to sets of measure 0.

From the independence of Y_r and $(\phi(s,r), \phi(t,r))$ we get

$$E\{\Psi(\phi(s,r))|Y_r,Y_t\} = E\{E\{\Psi(\phi(s,r))|\phi(t,r)\}|Y_r,Y_t\}.$$

Hence it suffices to prove that $E\{\rho(\phi(t,r))|Y_r,Y_t\}$ is $\sigma(X_r,X_t)$-measurable up to sets of measure 0 for any bounded Borel ρ on $\mathbb{R}^{4\times4}$. The conditional expectation $E\{\rho(\phi(t,r))|\phi(t,r)x\}$ is a measurable function of $\phi(t,r)x$, which can be written $H(x,\phi(t,r)x)$ for a measureable function H. For every invertible α in $\mathscr{M}_{4\times4}(G)$ commuting with G we have that

$$\begin{aligned} E\{\rho(\phi(t,r))|\alpha\phi(t,r)x\} &= E\{\rho(\phi(t,r))|\phi(t,r)x\}\\ &= E\{\rho(\phi(t,r))|\phi(t,r)\alpha x\}. \end{aligned}$$

Consequently, we may assume $H(\alpha x, \alpha y)=H(x,y)$. Since Y_r and $\phi(t,r)$ are independent, one can choose H so that $H(Y_r,Y_t)$ is a version of $E\{\rho(\phi(t,r))|Y_r,Y_t\}$, which gives the desired result by choosing $\alpha=T(X_0)$.

To end the proof of (33), one can write

$$E\{\varphi(X_s)|X_u;\, u \notin (r,t)\} = E\{\varphi(T(X_0)\phi(s,r)Y_r)|Y_u;\, u \notin (r,t)\} \quad \text{[by (32)]}$$
$$= E\{\varphi(\alpha\phi(s,r)Y_r)|Y_u;\, u \notin (r,t)\}|_{\alpha=T(X_0)}$$
$$= E\{\varphi(\alpha\phi(s,r)Y_r)|Y_r, Y_t\}|_{\alpha=T(X_0)}$$

by the Markov field property of $(Y_t)_{0\le t\le L}$ [(Y_t) is Markovian]. This last quantity is equal to

$$E\{\varphi(\alpha\phi(s,r)y)|Y_r, Y_t\}|_{\alpha=T(X_0),\, y=Y_r} = E\{\varphi(\phi(s,r)\alpha y)|Y_r, Y_t\}|_{\alpha=T(X_0),\, y=Y_r}$$
$$= E\{\varphi(\phi(s,r)x)|Y_r, Y_t\}|_{x=X_r},$$

which is $\sigma(X_r, X_t)$-measurable up to sets of measure 0 by (34). This gives (33) and, therefore, the desired Markov field property for $(X_t)_{0\le t\le L}$. □

We end this paper with two remarks:

1. One quantity of interest in this problem is the reflected energy

$$|B_0|^2 = (X_0^3)^2 + (X_0^4)^2 = S(L)$$

to indicate the dependency upon the length L. From (29) we get

$$S(L) = \frac{(Y_L^3)^2 + (Y_L^4)^2}{(Y_L^1)^2 + (Y_L^2)^2} = 1 - \frac{1}{(Y_L^1)^2 + (Y_L^2)^2}.$$

If we denote by R_t the (adapted) process $(Y_t^1)^2 + (Y_t^2)^2$, we have $S(L) = 1/R_L$. Using standard stochastic calculus, one can easily show that if $Z_t = \log R_t$, we have $Z_t = M_t + (w^2\alpha/2)t$, where the martingale M_t is given by

$$M_t = \frac{w\sqrt{\alpha}}{R_t}\left((Y_t^1Y_t^3 + Y_t^2Y_t^4)\, dW_t^{(2)} + (Y_t^2Y_t^3 + Y_t^1Y_t^4)\, dW_t^{(3)}\right).$$

A simple computation shows that $\langle M\rangle_t = w^2\alpha(1 - 1/R_t)$, which implies that

$$E\left\{\left(\frac{M_t}{t}\right)^2\right\} = \frac{1}{t^2}E\{\langle M\rangle_t\} = \frac{w^2\alpha}{t^2}\left(1 - E\left\{\frac{1}{R_t}\right\}\right) < \frac{w^2\alpha}{t^2}.$$

Therefore, $(1/t)Z_t$ converges in L^2 to $w^2\alpha/2$ as $t \nearrow +\infty$ and we obtain

$$\text{(35)} \qquad \frac{1}{L}\log(1 - S(L)) \text{ converges in } L^2 \text{ to } -\frac{w^2\alpha}{2} \text{ as } L \nearrow +\infty.$$

Since $1 - S(L) = (X_t^1)^2 + (X_t^2)^2 = T(L)$ is the transmitted energy, we get $T(L) \sim \exp(-(w^2\alpha/2)L)$ [as $L \nearrow +\infty$ in the sense of (35)]. This is much weaker than the exponential decay obtained in the localization theory since a limit in distribution, as $\varepsilon \searrow 0$, has been taken first.

2. Everything we have done here for one frequency w can be generalized to $(X^\varepsilon(w_1), \dots, X^\varepsilon(w_n))$ for n frequencies $(w_1, w_2, \dots, w_n)$.

It is more interesting (see [8]) to study this problem for two close frequencies such as $w_1 = w + \varepsilon h/2$ and $w_2 = w - \varepsilon h/2$. This analysis will be carried out elsewhere.

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