The Annals of Applied Probability
1994, Vol. 4, No. 2, 494–528

LAW OF LARGE NUMBERS FOR A HETEROGENEOUS SYSTEM OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH STRONG LOCAL INTERACTION AND ECONOMIC APPLICATIONS

BY WILLIAM FINNOFF

Siemens AG

A model for the activities of a finite number of agents in an economy is presented as the solution to a system of stochastic differential equations driven by general semimartingales and displaying an extended form of strong local interaction. We demonstrate a law of large numbers for the systems of processes as the number of agents goes to infinity under a weak convergence hypothesis on the triangular array of starting values and driving semimartingales which induces the systems of equations. Further, it is shown that the limit can be uniquely characterized by the distributions of the coordinate processes of the solution to an associated infinite-dimensional stochastic differential equation. Finally, an explicit example describing a currency market is discussed.

0. Introduction. In this paper we will investigate laws of large numbers for systems of stochastic processes with interaction, where the processes describe the activities of agents in an economy. Let $\mathbb{N}$ denote the natural numbers, and let $\mathscr{S} = \mathbb{Z}^v$ be the v-dimensional integer lattice, where $v \in \mathbb{N}$ is arbitrarily chosen but fixed. Further, for $N \in \mathbb{N}$, let $\mathbb{C}_N = [-N, N]^v \cap \mathscr{S}$ denote the N-cube. Then the model we use to describe these activities for an economy of $|\mathbb{C}_N|$ agents is given at the microeconomic level by a vector of stochastic processes $X^N_{\mathbb{C}_N} = (X^N_i)_{i \in \mathbb{C}_N}$, where $X^N_{\mathscr{S}} = (X^N_i)_{i \in \mathscr{S}}$ is the solution to the stochastic differential equation:

$$(N) \qquad \begin{aligned} X_i^N &= 0 \quad \text{for } i \in \mathscr{S} \setminus \mathbb{C}_N, \\ X_i^N(t) &= K_i^N + \int_{[0,t)} g\big(s, \theta_i\big(X^N_{\mathscr{S}}\big)\big) Z_i^N(ds) \quad \text{for } i \in \mathbb{C}_N. \end{aligned}$$

Here $C^d = C([0,\infty), \mathbb{R}^d)$ denotes the space of continuous paths in $\mathbb{R}^d$ with the topology of local uniform convergence, θ_i denotes the i-shift $(f_j)_{j \in \mathscr{S}} \to (f_{i+j})_{j \in \mathscr{S}}$, on $(C^d)^{\mathscr{S}}$ and $g\colon [0,\infty) \times (C^d)^{\mathscr{S}} \to \mathbb{R}^{d \times m}$ is a function such that, for every $(t, f) \in [0,\infty) \times (C^d)^{\mathscr{S}}$, $g(t, f) = g(t, f(\cdot \wedge t))$ and such that g satisfies a Lipschitz condition [see Condition (3.1), (CL)] guaranteeing the existence of a unique solution to the equation. Finally, $Z^N_{\mathbb{C}_N}$ is a vector of square inte-

Received August 1992; revised July 1993.

AMS 1991 *subject classifications*. Primary 60F10, 60H10, 60K35, 62G30, 62P20; secondary 65C05, 65Y05.

Key words and phrases. Interacting random processes, empirical distribution functions, law of large numbers, applications to economics.

grable $\mathbb{R}^m$-valued continuous semimartingales and $K^N_{\mathbb{C}_N}$ is a vector of $\mathbb{R}^d$-valued starting values. At the macroeconomic scale one can described the same economy by functionals of the "empirical measure" $\varphi^N = (1/|\mathbb{C}_N|)\Sigma_{i \in \mathbb{C}_N} \mathscr{E}_{X_i^N}$ belonging to the vector $X^N_{\mathbb{C}_N}$, where $\mathscr{E}_x$ denotes the Dirac measure on the point x. Here, the empirical measure provides a natural formalization of the distribution of activity in the economy. In this paper we try to determine under what circumstances the sequence of empirical measures $(\varphi^N)_{N \in \mathbb{N}}$ will coverge to a deterministic limit measure μ, and if this is the case, how this limit can be characterized. The economic interpretation of this question is whether for a large economy (i.e., for $N \to \infty$) the dynamics of macroeconomic variables are deterministic and can be derived in a rigorous fashion from "realistic" microeconomic models.

There has been considerable discussion in recent years as to whether the observable fluctuations of macroeconomic variables are due to stochastic, unpredictable "shocks" or to complex deterministic dynamics [see Aoki (1980), Feldman and Gilles (1985), Finnoff (1989, 1993), Grandmont and Malgrange (1986), Green (1984, 1989), Jovanovic and Rosenthal (1988) and Judd (1985)]. Föllmer (1974) and Jovanovic (1987) showed that in the case of interaction that is strong and complex enough, randomness at the microeconomic level can propagate to the macroeconomic variables of a large economy. This paper is intended to provide insight into the circumstances under which the deterministic hypothesis can be given a rigorous foundation in the case of interaction over decentralized markets. Our hypotheses on g are such that only the activities of "neighboring agents" will have a strong influence on the activities of a given agent and are a natural extension of the case where the interaction radius is bounded (strong local interaction). Otherwise the hypotheses are general enough so that a wide variety of models of (possibly adaptive) decision making can be taken into account [see, e.g., Blackwell and Girshick (1979), McFadden (1981), Pudney (1989) or Roth (1989)].

A solution $(X^N_{\mathbb{C}_N})$ to (N) can be seen as the finite-dimensional approximation of an infinite system of equations. The infinite-dimensional version is given by

$$(\infty) \qquad X_i(t) = K_i + \int_{[0,t)} g(x, \theta_i(X_{\mathscr{S}}))Z_i(ds) \quad \text{for } i \in \mathscr{S}$$

where, for $i \in \mathscr{S}$, K_i and Z_i are defined in an analogous fashion as before. If, for every $N \in \mathbb{N}$, $K^N_{\mathbb{C}_N} = (K_i)_{i \in \mathbb{C}_N}$, $Z^N_{\mathbb{C}_N} = (Z_i)_{i \in \mathbb{C}_N}$ and the law $\mathscr{L}\{(K_i, Z_i)_{i \in \mathscr{S}}\}$ is shift invariant and ergodic, one would expect by existing results [see Deuschel (1988) and Holley and Stroock (1981)] that the solution to (∞) would be itself ergodic and, as such, that the corresponding sequence of empirical measures would converge to a deterministic limit measure [see, e.g., Ellis (1985)]. Unfortunately, the stochastic elements of microeconomic models cannot generally be assumed to be identically distributed or to have any specific correlation structure such as independence or strong mixing (see Finnoff (1989, 1993), Föllmer (1974)]. There is considerable evidence though to support the assumption that in the case of no observable interaction, the sequence of empirical measures will converge at least in law to a deterministic limit [see Finnoff (1989, 1993), Feldman and Gilles (1985), Green (1984) and Judd (1985)].

This convergence is formalized in a concept we refer to as *point convergence*. Let $\Delta = (Y_i^N)_{i \in \mathbb{C}_N}^{\mathbb{N} \in \mathbb{N}}$ be an "array" of random elements in some topological space E, and let μ be a Borel probability on E. Then we say that Δ is point convergent with limit μ iff the sequence of empirical measures $(\varphi_\Delta^N)_{N \in \mathbb{N}} = (1/|\mathbb{C}_N|)\Sigma_{i \in \mathbb{C}_N} \mathscr{E}_{Y_i^N}$ converges in law to the point μ (i.e., $\mathscr{L}\{\varphi_\Delta^N\} \to \mathscr{E}_\mu$ in the sense of weak convergence of measures). This concept is closley related to "level II" convergence from the theory of large deviations (see Ellis (1985)] and is sometimes referred to as *propagation of chaos* for special types of arrays [see Sznitman (1984a), Dawson (1983) and Example 2.11].

An essentially minimal requirement for the point convergence of $(X_i^N)_{i \in \mathbb{C}_N}^{N \in \mathbb{N}}$ is the point convergence of the "array" of driving processes and starting values. In another paper [see Finnoff (1993)] we investigated this question for similar models displaying weak global interaction, and we were able to show [under further weak regularity conditions, see Condition 3.1 (CP)(ii)] that the array of solutions to comparable equation (N), $N \in \mathbb{N}$, is point convergent. Thus, the property of point convergence can be seen as being in some ways invariant to weak global interaction. In another formulation: One can add certain types of weak global interaction to a point convergent array of processes without losing this property. Based on this result it might be supposed that this property is also preserved under strong local interaction. Unfortunately, this turns out not to be the case (see Example 2.6). To insure the point convergence of the array $(X_i^N)_{i \in \mathbb{C}_N}^{N \in \mathbb{N}}$ of solutions to (N), $N \in N$, we will require a stronger condition on the array of driving processes and starting values than simple point convergence.

Let $\Gamma = (\xi_i^N)_{i \in \mathbb{C}_N}^{N \in \mathbb{N}}$ be an array of random elements in a topological space E. Further, assume that the individual components of a vector $(\xi_i^N)_{i \in \mathbb{C}_N}$ in the array interact in some way with the neighboring components. Then, one might choose as sample variables the "clusters" of random variables in $E^{\mathbb{C}_1}$, $\eta_i^N = (\xi_{i+j}^N)_{j \in \mathbb{C}_1}$, $i \in I_{\mathbb{C}_1}^N = \{i \in \mathscr{S}: i + \mathbb{C}_1 \subset \mathbb{C}_N\}$. Here, the new sample variables consist of the original variable together with all its direct neighbors on the lattice. In the economic interpretation of the model, these "cluster variables" can be seen to represent the activities of an individual agent (the original variable) and a "reference group" of further agents with whom direct information exchanges may take place [see Föllmer (1974)]. If the array formed in this fashion is point convergent, we say that Γ is point convergent for C_1-clusters. Extending this concept to every $\mathbb{H} \in Cl = \{\mathbb{H} \subset \mathscr{S}: |\mathbb{H}| < \infty\}$, we speak of Γ being point convergent for $\mathbb{H}$-clusters. Finally, if Γ is point convergent by $\mathbb{H}$-clusters for every $\mathbb{H} \in Cl$, we say that Γ is *point convergent for clusters*. This concept is closely related to the "level III" convergence from the theory of large deviations [see Ellis (1985)].

We will show that if the array Γ is point convergent for clusters, there exists a shift-invariant measure μ on $E^{\mathscr{S}}$ so that the limit of the $\mathbb{H}$-cluster array coincides with the $\mathbb{H}$ marginal of μ. This measure μ is referred to as the *cluster limit* of Γ.

Assume that Δ is point convergent for clusters with cluster limit $\mu = \mathscr{L}\{(K_i, Z_i)_{i \in \mathscr{I}}\}$. Further assume that the equation (∞) is defined using as starting values and driving processes the corresponding elements of $(K_i, Z_i)_{i \in \mathscr{I}}$ of this cluster limit. It will turn out that this condition on the array Δ (together with the Lipschitz and regularity conditions mentioned above) will be sufficient to insure that the array $(X_i^N)_{i \in \mathbb{C}_N}^{N \in \mathbb{N}}$ is not only point convergent, but also point convergent for clusters where the cluster limit $\mathscr{L}\{(X_i)_{i \in \mathscr{I}}\}$ is given by the solution to the equation (∞). Our method for demonstrating point convergence of processes uses the following program: We first define arrays of approximate solutions by time discretization. Then we show that these arrays are point convergent and converge uniformly in N to the array of genuine solutions. From this we can demonstrate the point convergence of the array of genuine solutions and characterize the limit using the limits of the approximating arrays. The criterion of point convergence for clusters was found while trying to carry out the first step and, considering the examples presented in Section 2, appears to be fairly minimal.

Although the models considered here are motivated by economic considerations, they are general enough so that the results derived may be of interest in other fields. One possible area of application is in the analysis and simulation of "distributed parameter systems" used to model large-scale engineering systems [see Polis (1983) and Tzafestas and Stavroulakis (1983)]. Further, our results may provide insight into the hydrodynamic limiting properties of heterogeneous systems of interacting particles [see Spohn (1980)]. Further, Theorem 3.3 may be of independent interest to those working with discrete time models or performing Monte Carlo simulations and "approximation through simulation" using Monte Carlo methods for certain types of nonlinear partial differential equations [see Babovsky (1994), Engquist and Hou (1989), Griffiths and Mitchell (1988) or Seidman (1988)].

In Section 1 we present a number of technical preliminaries used in the sequel. In Section 2 we investigate point convergence for clusters in detail and introduce a useful criterion for the point convergence for clusters, called point convergence for partitions. We show the relationship between point convergence for clusters, for partitions and simple point convergence and demonstrate the existence of the cluster limit. In Section 3 we carry out the steps of the program given above to demonstrate the point convergence for clusters of the array of solutions to the equations (N), $N \in \mathbb{N}$. Finally, to illustrate the application of these results, in Section 4 we provide an example describing a currency market.

0.1. *Conventions.* Here we list the notation and conventions with regard to topological and probability spaces that we will be using in the sequel. In the following E will always denote a Polish topological space and $(\Omega, \mathscr{A}, (F_t)_{t \in [0,\infty)}, \mathbb{P})$ a fixed filtered probability space on which all random elements are defined and to which all relevant concepts (stopping time, semimartingale, etc.) refer.

1. We denote by $\mathbb{N}$ the natural numbers, by $\mathbb{R}$ the real numbers, by $\mathbb{Z}$ the integers and by $\mathbb{Z}_+$ the positive integers.
2. Let $t \in [0,\infty)$, let I be some finite set and let $\{x_i: i \in I\} \subset E$. Then we will denote by $|I|$ the cardinality of the set I and by x_I the vector $(x_i)_{i \in I}$.
3. Let $\mathbb{H}$ and $\mathbb{G}$ be sets, $\mathbb{H} \subset \mathbb{G}$. (i) For the topological space E, $\mathscr{G}(E)$ denotes the family of open sets and $\mathscr{B}(E)$ denotes the family of Borel-measurable sets in E. (ii) $M(E)$ denotes the space of Borel measures on E equipped with the weak topology, and $M_1(E)$ denotes the subspace of Borel probabilities.(iii) Denote by $\mathscr{C}(E)$ the set of continuous functions f: $E \to \mathbb{R}$ and by $\mathscr{M}(E)$ the set of Borel-measurable functions f: $E \to \mathbb{R}$. Further, define $\mathscr{C}_b(E) = \{f \in \mathscr{C}(E)$: f bounded$\}$ and $\mathscr{M}_b(E) = \{f \in \mathscr{M}(E)$: f bounded$\}$. (iv) $E^{\mathbb{G}}$ denotes the product space equipped with the product topology. Define $\mathrm{pr}_{\mathbb{H}}^{\mathbb{G}}$: $E^{\mathbb{C}} \to E^{\mathbb{H}}$, $x_{\mathbb{G}} \mapsto x_{\mathbb{H}}$ (the projection mapping). (v) $C([0,\infty), E)$ denotes the space of continuous functions f: $[0,\infty) \to E$, $x \to f(x)$ equipped with topology of local uniform convergence. (vi) Finally, $\mathscr{E}_x$ denotes the Dirac measure on the point $x \in E$.
4. Let μ be a signed measure of bounded variation on the Borel sets of E. Then (i) if $f \in \mathscr{M}(E)$ is such that the integral $\int_E f\,d\mu$ exists, we write $\langle f, \mu \rangle = \int_E f\,d\mu$, and (ii) $\|\mu\|_E^V$ denotes the variation norm of μ.
5. Let ξ: $\Omega \to E$ be a Borel-measurable mapping (random element) in E. The Borel probability induced by such ξ is denoted by $\mathscr{L}\{\xi\}$.
6. Let X be another topological space, let $\mu \in M(E)$ and let f: $E \to X$ be a Borel measurable mapping. Then $\tilde{f}\mu$ will denote the image measure of μ under the mapping f.
7. Let $u \in \mathbb{N}$, $f, g \in C^u$, $\lambda, \mu \in M(C^u)$. (i) We denote by m_u the metric on C^u defined by setting
$$m_u(f,g) = \sum_{i \in \mathbb{N}} \frac{1}{2^i}\left(\left(\sup_{t \le i} \| f(t) - g(t) \|\right) \wedge 1\right).$$
(ii) Let $\mathscr{L}_1$ denote the family of $L \in \mathscr{C}_b(C^u)$ so that, for every $x, y \in C^u$, $|L(x) - L(y)| \le m_u(x,y)$ and $|L| \le 1$. Then $\hat{m}_u$ denotes the induced metric on $M(C^u)$ defined by setting $\hat{m}_u(\lambda, \mu) = \sup_{L \in \mathscr{L}_1} |\int L\,d\mu - \int L\,d\lambda|$ (the Kantorovitch–Rubenstein metric).

1. Preliminaries. We first give a precise description of the mathematical objects that we will be the subject of our investigations.

DEFINITION 1.1.

(i) Let I be a finite index set and let ξ_I be a random element in E^I. We then define the continuous mapping φ_E^I: $E^I \to M_1(E)$, $x_I \mapsto (1/|I|)\Sigma_{i \in I} \mathscr{E}_{x_i}$ [see Schief (1986), page 5, resp., Topsøe (1970), pages 68 and 48]. The random element in $M_1(E)$,
$$\frac{1}{|I|} \sum_{i \in I} \mathscr{E}_{\xi_i} = \varphi_E^I(\xi_I),$$
is called *the empirical measure belonging to* ξ_I.

(ii) An *indexing* is a pair consisting of a subset $\tilde{\mathbb{N}}$ of $\mathbb{N}$, $|\mathbb{N} \setminus \tilde{\mathbb{N}}| < \infty$, and a family of finite subsets $\{F^N: N \in \tilde{\mathbb{N}}\}$ of some set H with the property that, for every $N, M \in \tilde{\mathbb{N}}, N \leq M$, $F^N \subset F^M$.

(iii) An *array of random elements on E* (or simply *array*) is a pair (I, Δ) consisting of an indexing $I = (\tilde{\mathbb{N}}, \{F^N: N \in \tilde{\mathbb{N}}\})$ and a family of random elements $\Delta = (\xi^N_{F^N})^{N \in \tilde{\mathbb{N}}}$ $[= (\xi_i^N)^{N \in \tilde{\mathbb{N}}}_{i \in F^N}]$, where $\xi^N_{F^N}$ is a random element in E^{F^N}, for every $N \in \tilde{N}$. (The apparent double indexing with N in $\xi^N_{F^N}$ allows us on the one hand to go over to one of the coordinate variables ξ_i^N, $i \in F^N$, without danger of confusion and on the other hand recalls that generally $\xi_i^N \neq \xi_i^M$ for $N \neq M$.)

Since the indexing I is given (at least implicitly) by the family Δ in the definition of an array, we will not make explicit reference to it in the sequel. Further, $\tilde{\mathbb{N}}$ will always denote some subset of $\mathbb{N}$ such that $|\mathbb{N} \setminus \tilde{\mathbb{N}}| < \infty$. We will be concerning ourselves with the convergence of the sequence of empirical measures induced by an array of random elements. Let $\Delta = (\xi^N_{F^N})^{N \in \tilde{\mathbb{N}}}$ be an array. In the sequel, for every $N \in \tilde{\mathbb{N}}$,

$$\varphi_\Delta^N = \frac{1}{|F^N|} \sum_{i \in F^N} \mathscr{E}_{\xi_i^N}$$

denotes the empirical measure belonging to $\xi^N_{F^N}$.

LEMMA 1.2.

(i) *Let $x \in E$, let $\mathscr{N}$ be a subbase of neighborhoods of x, let $E \in \mathscr{N}$ and let $(\mu^N)^{N \in \tilde{\mathbb{N}}}$ be a sequence of Borel measures on E. Then $(\mu^N)^{N \in \tilde{\mathbb{N}}}$ converges to $\mathscr{E}_x$ iff, for any $\Gamma \in \mathscr{N}$,*

$$\mu^N(\Gamma) \to 1 \quad \text{for } N \to \infty.$$

(ii) *Let Δ be an array in E and let μ be a Borel probability on E. Then (Δ, μ) is point convergent iff for every $G \in \mathscr{G}(E)$ and $\varepsilon > 0$ [resp., $f \in \mathscr{C}_b(E)$],*

$$\lim_{N \in \tilde{\mathbb{N}}} \mathbb{P}\big(\varphi_\Delta^N(G) > \mu(G) - \varepsilon\big) = 1$$

[resp., the net of random variables $(\langle f, \varphi_\Delta^N \rangle)^{N \in \tilde{\mathbb{N}}}$ converges in probability to $\langle f, \mu \rangle$].

(iii) *Let $u \in \mathbb{N}$, let Δ be an array in C^u and let μ be a Borel probability on C^u. Then (Δ, μ) is point convergent iff, for every $f \in \{g \in \mathscr{C}_b(C^u)$: there exists a $t \in [0, \infty)$ so that, for every $x, y \in X$, $x|_{[0,t]} = y|_{[0,t]}$, then $g(x) = g(y)\}$, the net of random variables $(\langle f, \varphi_\Delta^N \rangle)^{N \in \tilde{\mathbb{N}}}$ converges in probability to $\langle f, \mu \rangle$ [resp., $(\hat{m}_u(\varphi_\Delta^N, \mu))^{N \in \tilde{\mathbb{N}}}$ converges in probability to zero].*

PROOF. See Finnoff (1993), Lemmas (1.2) and (1.3). □

Using the transformation formula we have the following corollary as a consequence.

COROLLARY 1.3. *Let X be a further Polish space, let $f\colon E \to X$ be a continuous mapping and let Δ be a point-convergent array in E with limit μ. Then (recalling notation 6 in subsection 0.1) the* image array $f(\Delta) = (f(X_i^N))_{i \in F^N}^{N \in \tilde{\mathbb{N}}}$ *is point convergent with limit $\tilde{f}\mu$.*

The following concept will be used repeatedly in the sequel in situations where one is only concerned with asymptotic properties of an array and it proves more convenient to work with a slightly modified version of the original array.

DEFINITION 1.4. Let $(\varphi_1^N)^{N \in \mathbb{N}}$ and $(\varphi_2^N)^{N \in \tilde{\mathbb{N}}}$ be sequences of discrete random measures on E. Then $(\varphi_1^N)^{N \in \tilde{\mathbb{N}}}$ and $(\varphi_2^N)^{N \in \tilde{\mathbb{N}}}$ are called *asymptotically equivalent* iff the sequence $(\|\varphi_1^N - \varphi_2^N\|_E^V)^{N \in \tilde{\mathbb{N}}}$ converges uniformly to zero. In this case we write $\varphi_1^N \overset{(\infty)}{=} \varphi_2^N$.

The value of this concept in our context is given by the following obvious result.

LEMMA 1.5. *Let $(\varphi_1^N)^{N \in \tilde{\mathbb{N}}}$ and $(\varphi_2^N)^{N \in \tilde{\mathbb{N}}}$ be nets of discrete random measures on E so that $\varphi_1^N \overset{(\infty)}{=} \varphi_2^N$. Further, let μ be a Borel probability on E. Then,*

$$\mathscr{L}\{\varphi_1^N\} \to \mathscr{E}_\mu, \quad \text{as } N \to \infty, \quad \text{iff} \quad \mathscr{L}\{\varphi_2^N\} \to \mathscr{E}_\mu, \quad \text{as } N \to \infty.$$

The final result of this section is a stochastic Gronwall lemma due to Metivier and Pellaumail and is used to show the uniform convergence of the approximating arrays to the array of genuine solutions to (N), $N \in \mathbb{N}$.

LEMMA 1.6. *Let S and R be stopping times, $S \le R$ and $K, \rho, l \in [0,\infty)$. Further, let ϕ and A be adapted increasing continuous processes with $\sup_{\omega \in \Omega} |A(R(\omega)) - A(S(\omega))| < l < \infty$. Finally, assume that, for any stopping time U, $S \le U \le R$,*

$$\mathbb{E}(\phi(U)) \le K + \rho\mathbb{E}\left(\int_{[S,U)} \phi(h)\, dA(h)\right).$$

Then $\mathbb{E}(\phi(R)) \le 2K\sum_{j=0}^{[2\rho l]}(2\rho l)^j$.

PROOF. This lemma is a slightly modified version of Lemma (7.1) in Metivier and Pellaumail (1980). □

2. Point convergence for clusters. In this section we investigate point convergence for clusters. Particular attention is paid to deriving an easily verifiable criterion which insures that an array has this property. This is important since it is rather difficult to verify directly. The situation is similar to that encountered when trying to show that a stationary sequence of random variables is ergodic. There, it is very difficult to demonstrate the ergodic property directly and one usually is forced to show this using some sufficient condition such as strong mixing. We recall the following conventions given in the introduction: Let $v \in \mathbb{N}$ be arbitrarily chosen but fixed, and let $\mathscr{S} = \mathbb{Z}^v$ denote the v-dimensional integer lattice. We will refer to the sets $Cl = \{\mathbb{H} \subset \mathscr{S}\colon |\mathbb{H}| < \infty\}$ as *clusters in* $\mathscr{S}$. For $k \in \mathbb{N}$, let $\mathbb{C}_k = [-k, k]^v \cap S$ denote the k-cube. Finally, for $j \in \mathscr{S}$, we will let $\theta_j^E\colon E^{\mathscr{S}} \to E_j^{\mathscr{S}}$, $(x_i)_{i \in \mathscr{S}} \to (x_{i+j})_{i \in \mathscr{S}}$ denote the j-shift on $E^{\mathscr{S}}$. We will drop the indexing with E (i.e., $\theta_j = \theta_j^E$) whenever the reference is clear.

DEFINITION 2.1. For $\mathbb{H} \in Cl$, $N \in \mathbb{N}$, define $I_{\mathbb{H}}^N = \{l \in \mathbb{C}_N\colon\ l + \mathbb{H} \subset \mathbb{C}_N\}$ and $F_{\mathbb{H}}^N = \{\mathbb{G} \in Cl\colon \mathbb{G} = (\mathbb{H} + l),\ l \in I_{\mathbb{H}}^N\}$. Finally, set $\mathbb{N}_{\mathbb{H}} = \{N \in \mathbb{N}\colon F_{\mathbb{H}}^N \neq \varnothing\}$ and note that $(\mathbb{N}_{\mathbb{H}}, (F_{\mathbb{H}}^N)_{N \in \mathbb{N}}))$ is an indexing. Now let $\Delta = (\xi_{\mathbb{C}_N}^N)^{N \in \mathbb{N}}$ be an array in E. For $\mathbb{H} \in Cl$ define $\Delta_{\mathbb{H}} = (\xi_B^N)_{B \in F_{\mathbb{H}}^N}^{N \in \mathbb{N}}$ *the* $\mathbb{H}$-*cluster array* of Δ.

(i) We then say that Δ is *point convergent for* $\mathbb{H}$-*clusters* iff $\Delta_{\mathbb{H}}$ is point convergent.

(ii) Δ is said to be *point convergent for clusters* iff, for every $\mathbb{H} \in Cl$, Δ is point convergent for $\mathbb{H}$-clusters.

We recall the concept of "asymptotic equivalence" given in Definition 1.4. We will use this concept repeatedly in the following to demonstrate the point convergence of cluster arrays. This is accomplished by showing that these arrays are often only "slightly modified versions" of known point-convergent arrays. The type of modification is given in the following lemma. For two sequences $(a^N)_{N \in \tilde{\mathbb{N}}}$ and $(b^N)_{N \in \tilde{\mathbb{N}}}$ in $\mathbb{N}$ we write $(a^N)_{N \in \tilde{\mathbb{N}}} \sim (b^N)_{N \in \tilde{\mathbb{N}}}$ if $a^N/b^N \to 1$ for $N \to \infty$.

LEMMA 2.2. *Let* $\Delta = (\xi_{F^N}^N)^{N \in \mathbb{N}}$ *and* $\tilde{\Delta} = (\eta_{D^N}^N)^{N \in \mathbb{N}}$ *be arrays in* E, *and let* $(b^N)_{N \in \mathbb{N}}$ *be a sequence in* $\mathbb{N}$ *such that the following hold*:

$$\text{(i)} \qquad \frac{|(F^N \setminus D^N)|}{|F^N|} + \frac{|(D^N \setminus F^N)|}{b^N} \to 0 \quad \text{as } N \to \infty;$$

$$\text{(ii)} \qquad (b^N)_{N \in \mathbb{N}} \sim (|F^N|)_{N \in \mathbb{N}};$$

$$\text{(iii)} \qquad \xi_{F^N \cap D^N}^N = \eta_{F^N \cap D^N}^N \quad \text{for every } N \in \mathbb{N}.$$

Defining for $N \in \mathbb{N}$, $\varphi_1^N = (1/b^N)\sum_{i \in D^N} \mathscr{E}_{\eta_i^N}$, *we have* $\varphi_1^N \overset{(\infty)}{=} \varphi_\Delta^N$.

PROOF. This follows from the fact that

$$\sup_{B \in \mathscr{B}(E)} \left| \varphi_1^N(B) - \frac{1}{|F^N|} \sum_{i \in F^N} \mathscr{E}_{\xi_i^N}(B) \right|$$
$$\leq \left| \frac{b^N - |F^N|}{b^N} \right| + \frac{|D^N \setminus F^N|}{b^N} + \frac{|\mathbb{F}^N \setminus D^N|}{|F^N|} \to 0 \quad \text{for } N \to \infty. \qquad \square$$

In the following lemma we show that one does not really have to consider all the elements of Cl when checking to see whether a given array is point convergent for clusters.

LEMMA 2.3. *Let $\Delta = (\sigma^N_{\mathbb{C}_N})^{N \in \mathbb{N}}$ be an array in E, and let $(\mathbb{H}_n)_{n \in \mathbb{N}}$ be a sequence of clusters that is cofinal in Cl (i.e., for every $\mathbb{H} \in Cl$, there exists an $n \in \mathbb{N}$ so that $\mathbb{H} \subset \mathbb{H}_n$). Then Δ is point convergent for clusters iff, for every $n \in \mathbb{N}$, Δ is point convergent for $\mathbb{H}_n$ clusters. Further, if $\mathbb{H} \in Cl$ and $n \in \mathbb{N}$ is such that $\mathbb{H} \subset \mathbb{H}_n$, then denoting by $\mu_{\mathbb{H}_n}$ the limit of the $\mathbb{H}_n$-cluster array and recalling from subsection* 0.1 *the notation* 3(iv) *and* 6, *the limit of the array $\Delta_{\mathbb{H}}$ is given by $\widetilde{\mathrm{pr}_{\mathbb{H}}^{\mathbb{H}_n}} \mu_{\mathbb{H}_n}$.*

PROOF. For $\mathbb{H} \in Cl$, let $n \in \mathbb{N}$ be such that $\mathbb{H} \subset \mathbb{H}_n$. Then it is straightforward to show that

$$(2.1) \qquad \left(|F^N_{\mathbb{H}_n}|\right)_{N \in \hat{\mathbb{N}}} \sim \left(|F^N_{\mathbb{H}}|\right)_{N \in \hat{\mathbb{N}}},$$

where $\hat{\mathbb{N}} = \mathbb{N}_{\mathbb{H}_n} \cap \mathbb{N}_{\mathbb{H}}$. Define, for $N \in \hat{\mathbb{N}}$,

$$b_N = |F^N_{\mathbb{H}_n}| \quad \text{and} \quad D^N = \left\{ B \in F^N_{\mathbb{H}} : B = j + \mathbb{H}, j \in I^N_{\mathbb{H}_n} \right\}.$$

Since $D^N \subset F^N_{\mathbb{H}}$ and $|D^N| = |F^N_{\mathbb{H}_n}|$, as a consequence of (2.1),

$$(2.2) \qquad \frac{|F^N_{\mathbb{H}} \setminus D^N|}{|F^N_{\mathbb{H}}|} + \frac{|D^N \setminus F^N_{\mathbb{H}}|}{b_N} = \frac{\big|\, |F^N_{\mathbb{H}}| - |F^N_{\mathbb{H}_n}| \,\big|}{|F^N_{\mathbb{H}}|} \to 0 \quad \text{for } N \to \infty.$$

Finally, for $N \in \hat{\mathbb{N}}$ and $B \in D^N$, define $\eta^N_B = \xi^N_B$. Thus, using (2.2) and this definition, it follows from Lemma 2.2 that

$$(2.3) \qquad \begin{aligned} \varphi^N_{\Delta_{\mathbb{H}}} &= \frac{1}{|F^N_{\mathbb{H}}|} \sum_{B \in F^N_{\mathbb{H}}} \mathscr{E}_{\xi^N_B} \overset{(\infty)}{=} \frac{1}{|F^N_{\mathbb{H}_n}|} \sum_{B \in D^N} \mathscr{E}_{\eta^N_B} \\ &= \frac{1}{|F^N_{\mathbb{H}_n}|} \sum_{I \in F^N_{\mathbb{H}_n}} \mathscr{E}_{\mathrm{pr}_{\mathbb{H}}^{\mathbb{H}_n}(\xi^N_I)} = \varphi^N_{\mathrm{pr}_{\mathbb{H}}^{\mathbb{H}_n}(\Delta_{\mathbb{H}_n})}. \end{aligned}$$

If $\Delta_{\mathbb{H}_n}$ is point convergent with limit $\mu_{\mathbb{H}_n}$, then by the continuity of $\mathrm{pr}_{\mathbb{H}}^{\mathbb{H}_n}$ and Corollary 1.3, $\Delta_{\mathbb{H}}$ is point convergent with limit $\widetilde{\mathrm{pr}_{\mathbb{H}}^{\mathbb{H}_n}} \mu_{\mathbb{H}_n}$. The result is then immediate. $\square$

DEFINITION 2.4. For every $\mathbb{H} \in Cl$, let $\mu_{\mathbb{H}}$ be a Borel probability on $E^{\mathbb{H}}$. Then (i) the family $(\mu_{\mathbb{H}})_{\mathbb{H} \in Cl}$ is called a *projective family of measures* (on E) iff, for every $\mathbb{H}, \mathbb{K} \in Cl$, $\mathbb{H} \subset \mathbb{K}$,

$$\widetilde{p_{\mathbb{H}}^{\mathbb{K}}} \mu_{\mathbb{K}} = \mu_{\mathbb{H}}.$$

Let μ be a Borel probability on $E^{\mathscr{S}}$, and let $(\mu_{\mathbb{H}})_{\mathbb{H} \in Cl}$ be a projective family of measures. Then (ii) μ is called the *projective limit of* $(\mu_{\mathbb{H}})_{\mathbb{H} \in Cl}$ iff, for every $\mathbb{H} \in Cl$,

$$\widetilde{p_{\mathbb{H}}^{\mathscr{S}}} \mu = \mu_{\mathbb{H}}.$$

Using this definition and Lemma 2.3, we have the following lemma as an immediate consequence.

LEMMA 2.5. *Let $\Delta = (\xi_{\mathbb{C}_N}^N)^{N \in \mathbb{N}}$ be an array in E that is point convergent for clusters. For every $\mathbb{H} \in Cl$, denote by $\mu_{\mathbb{H}}$ the limit of $\Delta_{\mathbb{H}}$. Then the family of measures $(\mu_{\mathbb{H}})_{\mathbb{H} \in Cl}$ is a projective system.*

We return for a moment to the problem that led us to the concept of point convergence for clusters. Recall the array $(X_{\mathbb{C}_N}^N)^{N \in \mathbb{N}}$ given in the introductory remarks, where $(X_{\mathbb{C}_N}^N)$ is a solution to (N), for $N \in \mathbb{N}$.

What we were looking for was a condition on the array $((K_i^N, Y_i^N))_{i \in \mathbb{C}_N}^{N \in \mathbb{N}}$ of starting values and driving processes under which a unique solution to (N), for every $N \in \mathbb{N}$, exists and is such that the array of these solutions would be point convergent. We were particularly interested in the situation given when the array of starting values and driving processes are neither identically distributed nor fulfill any strong independence conditions.

To find some essentially minimal criteria we reduced the equations to the discrete time case, restricted the interaction to that with the value of the immediately neighboring processes in the previous time step, dropped the semimartingales and discovered that even under very strong conditions on the array of starting values that point convergence can be destroyed through local interaction. That is the subject of the following.

EXAMPLE 2.6. Let the time scale be discrete (i.e., $t = 0, 1, 2, \dots$) and let $v = 1$. The only activity considered is whether the agents purchase a good at time t. Then we can define the activity space to be $E = \{0, 1\}$ (for purchasing or not) and assume that an agent i will purchase a good at time t only if his purchasing activities in the previous period did not coincide with that of the neighboring agent with the following index $(i + 1)$. Assume $\Delta = (K_{\mathbb{C}_N}^N)^{N \in \mathbb{N}}$ to be an array of starting values for the activity at time $t = 0$. This behavioral rule can be formalized using the function $g\colon \{0,1\}^2 \to \{0,1\}$, $(x_1, x_2) \mapsto |x_1 - x_2|$, by setting for $N \in \mathbb{N}$, $Y_i^N(t) = 0$ if $i \in \mathscr{S} \setminus \mathbb{C}_N$ and if $i \in \mathbb{C}_N$,

$$Y_i^N(t) = \begin{cases} K_i^N, & \text{for } t = 0, \\ g\big((Y_i^N(t-1), Y_{i+1}^N(t-1))\big), & \text{for } t > 0. \end{cases}$$

We now define a specific array of starting values $\Delta = (K_{\mathbb{C}_N}^N)^{N \in \mathbb{N}}$ by setting $K_i^N = x_i$, for every $i \in \mathbb{C}_N$, $N \in \mathbb{N}$, where the sequence $(x_i)_{i \in \mathbb{Z}}$ is defined as follows: We start by constructing two sequences $(x_i^h)_{i \in \mathbb{Z}}$, $h = 1, 2$, in E. For $i \in 4\mathbb{Z} = \{4m\colon m \in \mathbb{Z}\}$, set

$$x^1_{\{i, i+1, i+2, i+3\}} = (0, 0, 1, 1)$$

and

$$x^2_{\{i, i+1, i+2, i+3\}} = (0, 1, 0, 1).$$

Define inductively the family of sets $A_0 = \varnothing$ and, for $k > 0$, $A_k = \{-4^k, -4^k + 1, \dots, 4^k - 1\} \setminus A_{k-1}$. Then $\mathbb{Z}$ is equal to the disjoint union of the family $(A_k)_{k \in \mathbb{Z}_+}$. Therefore, if we set

$$x_i = \begin{cases} x_i^1, & \text{for } i \in A_k \text{ and } k \text{ even}, \\ x_i^2, & \text{otherwise}, \end{cases}$$

the sequence $(x_i)_{i \in \mathbb{Z}}$ is well defined. Further,

$$\varphi_\Delta^N \to \tfrac{1}{2}(\mathscr{E}_0 + \mathscr{E}_1) \quad \text{for } N \to \infty.$$

Therefore, the array is point convergent (actually the convergence is in this case much stronger). Further, the array is deterministic and, as such, the variables of the array are all independent. Now consider the array $\tilde{\Gamma} = (Y_{\mathbb{C}_N}^N(1))_{N \in \mathbb{N}}$, and assume that $\tilde{\Gamma}$ is point convergent. Since $f\colon \{0,1\} \to \mathbb{R}$, $x \mapsto x$ is a bounded continuous function, by Lemma 1.2(iii), the variable $\langle f, \varphi_\Gamma^N \rangle$ must converge to some constant for $N \to \infty$. We note then that

$$g(x_i, x_{i+1}) = \begin{cases} 0, & \text{if } i \in A_k \text{ and both } i \text{ and } k \text{ are even}, \\ 1, & \text{otherwise}. \end{cases}$$

Further, noting that $|A_k| = 2(4^k - 4^{k-1})$ we have, for $N = 4^k$ and k even,

$$\langle f, \varphi_\Gamma^N \rangle \le \frac{4^k - 4^{k-1}}{2(4^k) + 1} + \frac{1 + 2(4^{k-1})}{2(4^k) + 1}.$$

Since the first term in the last expression converges to $\frac{3}{8}$ and the last term converges to $\frac{1}{4}$, for $k \to \infty$ we have $\lim_{l \to \infty} \langle f, \varphi_\Gamma^{4^{2l}} \rangle \le \frac{5}{8}$.

On the other hand, for $N = 4^k$ and k odd,

$$\langle f, \varphi_{\Gamma}^{N} \rangle \geq \frac{2(4^k - 4^{k-1})}{2(4^k) + 1} \to \frac{3}{4} \quad \text{as } k \to \infty.$$

Therefore $\lim_{l \to \infty} \langle f, \varphi_{\Gamma}^{4^{2l+1}} \rangle \geq \frac{3}{4}$ in contradiction to the assumed convergence.

Setting $\mathbb{H} = \{0, 1\}$, one sees in the previous example that the array $\tilde{\Gamma} = (Y_{\mathbb{C}_N}^N(1))^{N \in \mathbb{N}_{\mathbb{H}}}$ is (up to a slight modification) the image of the $\mathbb{H}$-cluster array of $\tilde{\Delta}$ under the mapping g. If $\tilde{\Gamma}$ is to be point convergent and one does not want to restrict the function g to a particular case, then one will probably have to require Δ to be point convergent by $\mathbb{H}$-clusters. Extending these considerations for the following time steps, one comes to the conclusion that even in this remarkably trivial situation, to insure the point convergence of $(Y_{\mathbb{C}_N}^N)^{N \in \mathbb{N}}$ one must demand that Δ be point convergent for clusters.

To provide some description of the limit for an infinite or continuous time scale, it will be necessary to collect the limits of the various cluster arrays in a fashion that permits a unified description. In Lemma 2.5 we have seen that the limits of the cluster arrays $(\mu_{\mathbb{H}})_{\mathbb{H} \in Cl}$ form a projective family of measures. We recall the classical result of topological measure theory stating that to any projective family of measures $(\mu_{\mathbb{H}})_{\mathbb{H} \in Cl}$ on the Polish space E there exists a uniquely defined projective limit measure. This will provide the unified description that we need.

THEOREM AND DEFINITION 2.7. *Let* $\Delta = (\xi_{\mathbb{C}_N}^N)^{N \in \mathbb{N}}$ *be an array in E that is point convergent for clusters. Denote by* $(\mu_{\mathbb{H}})_{\mathbb{H} \in Cl}$ *the projective system of measures given by Lemma 2.5. Then there exists a unique, shift-invariant measure* μ *on* $E^{\mathscr{S}}$ *that is the projective limit of the family* $(\mu_{\mathbb{H}})_{\mathbb{H} \in Cl}$. *The measure* μ *is called the cluster limit of* Δ.

PROOF. What remains to be shown is the shift invariance of μ. It is sufficient to demonstrate for any $\mathbb{H} \in Cl$, $A_{\mathbb{H}} \in \mathscr{B}(E^{\mathbb{H}})$, that, for the cylinder set $A = A_{\mathbb{H}} \times E^{\mathscr{S} \setminus \mathbb{H}}$ and $j \in \mathscr{S}$, $\tilde{\theta}_j \mu(A) = \mu(A)$. Assume that $\mathbb{H}$, $A_{\mathbb{H}}$, A and j are given as above. Set $\mathbb{K} = \mathbb{H} \cup (\mathbb{H} - j)$ and $A_{\mathbb{H}-j} = \{(x_i)_{i \in (\mathbb{H}-j)} \in E^{\mathbb{H}-j}: (x_{i+j})_{i \in \mathbb{H}} \in A_{\mathbb{H}}\}$. Then, by definition,

$$\tilde{\theta}_j \mu(A) = \mu\left(A_{(\mathbb{H}-j)} \times E^{\mathscr{S} \setminus (\mathbb{H}-j)}\right) = \mu_{\mathbb{K}}\left(A_{\mathbb{H}-j} \times E^{\mathbb{K} \setminus (\mathbb{H}-j)}\right).$$

Define the continuous mapping $f: E^{\mathbb{K}} \to E^{(\mathbb{H}-j)}$, $(x_i)_{i \in \mathbb{K}} \mapsto (x_{i+j})_{i \in (\mathbb{H}-j)}$. Then consider the image array of $\Delta_{\mathbb{K}}$ under the mapping f. Define $D_{\mathbb{K}}^N = \{B \in F_{\mathbb{K}}^N: j + B \in F_{\mathbb{K}}^N\}$. Then $(|D_{\mathbb{K}}^N|)_{N \in \mathbb{N}_{\mathbb{K}}} \sim (|F_{\mathbb{K}}^N|)_{N \in \mathbb{N}_{\mathbb{K}}}$ (see the proof of Lemma 2.3). We have by the definition of $\Delta_{\mathbb{K}}$ and Lemma 2.2,

$$\varphi_{f(\Delta_{\mathbb{K}})}^N = \frac{1}{|F_{\mathbb{K}}^N|} \sum_{B \in F_{\mathbb{K}}^N} \mathscr{E}_{f(\xi_B^N)} \overset{(\infty)}{=} \frac{1}{|F_{\mathbb{K}}^N|} \sum_{B \in D_{\mathbb{K}}^N} \mathscr{E}_{\mathrm{pr}_{(\mathbb{H}-j)}^{\mathbb{K}}(\xi_B^N)}$$

$$\overset{(\infty)}{=} \frac{1}{|F_{\mathbb{K}}^N|} \sum_{B \in F_{\mathbb{K}}^N} \mathscr{E}_{\mathrm{pr}_{(\mathbb{H}-j)}^{\mathbb{K}}(\xi_S^N)} = \varphi_{\mathrm{pr}_{(\mathbb{H}-j)}^{\mathbb{K}}(\Delta_{\mathbb{K}})}^{N_{\mathbb{K}}}.$$

Lemma 1.5 and Corollary 1.3 imply $\tilde{f}\mu_{\mathbb{K}} = \widetilde{\mathrm{pr}^{\mathbb{K}}_{(\mathbb{H}-j)}\mu_{\mathbb{K}}}$. Using the fact that $(\mu_{\mathbb{H}})_{\mathbb{H} \in Cl}$ is a projective family, we have

$$\begin{aligned}\mu_{\mathbb{K}}\left(A_{(\mathbb{H}-j)} \times E^{\mathbb{K}\setminus(\mathbb{H}-j)}\right) &= \mu_{(\mathbb{H}-j)}\left(A_{(\mathbb{H}-j)}\right) = \widetilde{\mathrm{pr}^{\mathbb{K}}_{(\mathbb{H}-j)}\mu_{\mathbb{K}}}\left(A_{(\mathbb{H}-j)}\right)\\ &= \tilde{f}\mu_{\mathbb{K}}\left(A_{(\mathbb{H}-j)}\right) = \mu_{\mathbb{K}}\left(A_{\mathbb{H}} \times E^{\mathbb{K}\setminus\mathbb{H}}\right)\\ &= \mu\left(A_{\mathbb{H}} \times E^{\mathscr{S}\setminus\mathbb{H}}\right) = \mu(A).\end{aligned}$$

This completes the proof. □

The property of an array Δ in E that it be point convergent for clusters is sufficient for our purposes. It can, though, be rather difficult to verify directly for a given array. The following property, which is strictly stronger, is often simpler to verify (see Examples 2.11 and 2.12). First we introduce some further notation.

NOTATION.

(i) For $k \in \mathbb{N}$ denote by $\mathbb{M}_k = \mathbb{C}_{2k+1} \setminus \{-(2k+1), 2k+1\}^v$ the $(2k+1)$-cube without corners.

(ii) For $j \in \mathscr{S}$ and $k, N \in \mathbb{N}$, set

$$F^N_{k,j} = \{\mathbb{H} \in Cl : \mathbb{H} = (\mathbb{C}_k + l + j),\, l \in (2k+1)\mathscr{S},\, \mathbb{H} \subset \mathbb{C}_N\},$$

and define $N^k_j = \{N \in \mathbb{N}: F^N_{k,j} \neq \varnothing\}$.

The sets $F^N_{k,j}$ are formed by first partitioning $\mathscr{S}$ into cubes congruent to $\mathbb{C}_k$ (with the original $\mathbb{C}_k$ serving as center and reference for the partition) then shifting by j and taking those that are contained in the cube $\mathbb{C}_N$.

We note that, for N, $k \in \mathbb{N}$, $j \in \mathbb{C}_k$ and $\mathbb{H} \in Cl$, that $F^N_{k,j} \subset F^{N+1}_{k,j}$ and $F^N_{\mathbb{H}} \subset F^{N+1}_{\mathbb{H}}$. Therefore $(\mathbb{N}^k_j, (F^N_{k,j})_{N \in \mathbb{N}})$ is an indexing.

DEFINITION 2.8. Let $\Delta = (\xi^N_{\mathbb{C}_N})^{N \in \mathbb{N}}$ be an array in E, and let $k \in \mathbb{N}$. Define, for every $j \in \mathbb{M}_k$, $\Delta^k_j = (\xi^N_B)^{N \in \mathbb{N}^k_j}_{B \in F^N_{k,j}}$ *the* k, j *partition array*. We then say the following:

(i) Δ *is point convergent for k-partitions* iff, for every $j \in \mathbb{M}_k$, Δ^k_j is point convergent.

(ii) Δ *is point convergent for partitions* iff, for every $k \in \mathbb{N}$, Δ is point convergent for k-partitions.

This concept is perhaps a little difficult to digest at first glance, but the idea behind it is quite rudimentary since one derives the partition arrays simply by repartitioning the vectors in the original arrays, but without the overlaps one has in the cluster arrays.

The next theorem describes the relationship between point convergence for partitions and for clusters.

THEOREM 2.9. *Let $k \in \mathbb{N}$, and let $\Delta = (\sigma_{\mathbb{C}_k}^N)^{N \in \mathbb{N}}$ be an array in E that is point convergent for k-partitions. For every $j \in \mathbb{M}_k$, denote by μ_j^k the limit of the k, j partition array. Then Δ is point convergent for $\mathbb{C}_k$-clusters, and the limit μ^k of the $\mathbb{C}_k$-cluster array is given by $\mu^k = (1/|\mathbb{M}_k|)\sum_{j \in \mathbb{M}_k} \mu_j^k$.*

PROOF. First we note that

$$\varphi_{\Delta_{\mathbb{C}_k}}^N = \frac{1}{|F_{\mathbb{C}_k}^N|} \sum_{B \in F_{\mathbb{C}_k}^N} \mathscr{E}_{\xi_B^N}$$

$$= \frac{1}{|\mathbb{M}_k|} \sum_{j \in \mathbb{M}_k} \frac{|\mathbb{M}_k|}{|F_{\mathbb{C}_k}^N|} \sum_{B \in F_{k,j}^N} \mathscr{E}_{\xi_B^N}.$$

Define, for $N \in \mathbb{N}$, $b_N = |F_{\mathbb{C}_k}^N|/|\mathbb{M}_k|$. Trivially, for every $j, \tilde{j} \in \mathbb{M}_k$, $(|F_{k,j}^N|)_{N \in \tilde{\mathbb{N}}} \sim (|F_{k,\tilde{j}}^N|)_{N \in \tilde{\mathbb{N}}}$, where $\tilde{\mathbb{N}} = \mathbb{N}_j^k \cap \mathbb{N}_{\tilde{j}}^k$. Further, for every $N \in \tilde{\mathbb{N}}$, it follows from the definitions that $F_{\mathbb{C}_k}^N$ is equal to the disjoint union of the sets $F_{k,j}^N$, $j \in M_k$. Therefore, for every $j \in M_k$, $(|F_{k,j}^N|)_{N \in \tilde{\mathbb{N}}} \sim (b_N)_{N \in \tilde{\mathbb{N}}}$. We then have, by Lemma 2.2,

$$\varphi_{\Delta_j^k}^N \overset{(\infty)}{=} \frac{|\mathbb{M}_k|}{|F_{\mathbb{C}_k}^N|} \sum_{B \in F_{k,j}^N} \mathscr{E}_{\xi_B^N},$$

which completes the proof using the hypothesis of the theorem. □

By Theorem 2.9 and the fact that $(C_k)_{k \in \mathbb{N}}$ is cofinal in $\mathscr{S}$, it then follows that an array that is point convergent for partitions is point convergent for clusters. To illustrate how useful the criterion of point convergence for partitions is, we will show that several natural examples have this property. For the first example we introduce the following concepts.

DEFINITION 2.10. Let $\Delta = (\xi_{\mathbb{C}_N}^N)^{N \in \mathbb{N}}$ be an array in E, and let μ be a Borel probability on E. Then Δ is said to be *symmetric* if, for every $N \in \mathbb{N}$, $\xi_{\mathbb{C}_N}^N$ is symmetrically distributed (i.e., the law $\mathscr{L}\{\xi_{\mathbb{C}_N}^N\}$ is invariant to permutations of the indexes $i, j \in \mathbb{C}_N$, $i \neq j$). Further, Δ is said to be *μ-chaotic* iff Δ is symmetric and for every $\mathbb{H} \in Cl$,

$$\mathscr{L}\{\xi_{\mathbb{H}}^N\} \to \bigotimes_{i \in \mathbb{H}} \mu \quad \text{for } N \to \infty,$$

where $\otimes_{i \in \mathbb{H}} \mu$ denotes the product measure of μ on $E^{\mathbb{H}}$.

Symmetric arrays arise frequently when one considers systems with weak global interaction, since the factor which provides the interaction is the (permutation-invariant) empirical measure. In fact, most papers investigating point convergence for arrays of processes with weak global interaction assume that the arrays of starting values and driving processes are symmetric [see Sznitman (1984a, b, 1986), Graham (1988) and Nappo and Orlandi (1988)]. One of the notable properties of symmetric arrays in a Polish space lies in the fact that such an array Δ is point convergent with limit μ iff Δ is μ-chaotic [see Sznitman (1984a), page 580].

EXAMPLE 2.11. Let $\Delta = (\xi^N_{\mathbb{C}_N})^{N \in \mathbb{N}}$ be a symmetric array in E that is point convergent with limit μ. Then Δ is point convergent for partitions and, for every $k \in \mathbb{N}$, $j \in \mathbb{C}_k$, the limit of Δ^k_j is given by $\otimes_{i \in \mathbb{C}_k} \mu$.

PROOF. For any $k \in \mathbb{N}$, $j \in M_k$, the k,j partition array is, with Δ, symmetric. We will show that Δ^k_j is $(\otimes_{i \in \mathbb{C}_k} \mu)$-chaotic. Define $X = E^{\mathbb{C}_k}$. Let $\tilde{F} \subset F^N_{j,k}$ for some $N \in N^k_j$ and $f \in \mathscr{C}_b(X^{\tilde{F}})$. Set $\hat{F} = \bigcup_{B \in \tilde{F}} B$ and $\hat{f}\colon E^{\hat{F}} \to \mathbb{R}$, $(x_i)_{i \in \hat{F}} \mapsto f((x_B)_{B \in \tilde{F}})$. Then it follows that, for any $\tilde{N} \geq \tilde{N}$,

$$\int_{X^{F^N_{k,j}}} f((x_B)_{B \in \tilde{F}})\, d\mathscr{L}\{(\xi^N_B)_{B \in F^N_{k,j}}\} = \int_{E^{\hat{F}}} \hat{f}((x_i)_{i \in \hat{F}})\, d\mathscr{L}\{(\xi^N_{\hat{F}})\}.$$

As Δ is μ-chaotic, the last term above converges to $\int_{E^{\hat{F}}} \hat{f}\, d(\otimes_{i \in \hat{F}} \mu)$, for $N \to \infty$. Since

$$\int_{E^{\hat{F}}} \hat{f}\, d\left(\bigotimes_{i \in \hat{F}} \mu\right) = \int_{X^{\tilde{F}}} f\, d\left(\bigotimes_{B \in \tilde{F}} \left(\bigotimes_{i \in \mathbb{C}_k} \mu\right)\right),$$

this completes the proof. □

The following example is typical for numerical applications (e.g., numerical integration).

EXAMPLE 2.12. Let $v = 1$ and let λ denote the normalized Lebesgue measure on $E = [-1, 1]$. We define an array $\Delta = (\eta^N_{\mathbb{C}_N})^{N \in \mathbb{N}}$ in E by setting, for $i = -N, \dots, N$, $N \in \mathbb{N}$,

$$\eta^N_i = \frac{i}{2N + 1}.$$

Define further, for every $j \in \mathbb{N}$, $f_j = \mathrm{id}_{[-1,1]}$ and

$$f^k\colon E \to E^{\mathbb{C}_k}, \qquad x \mapsto (f_j(x))_{j \in \mathbb{C}_k}.$$

Then Δ is point convergent for partitions and, for every $k \in \mathbb{N}$, $j \in \mathbb{M}_k$, the limit μ^k_j of Δ^k_j is given by $\widetilde{f^k}\lambda$.

PROOF. This follows directly from the basic theory of the Riemann integral. □

We know by Theorem 2.9 that an array which is point convergent for partitions is point convergent for clusters. By definition, an array that is poitn convergent for clusters is point convergent. It is natural to ask whether either of the reverse implications hold. The answer is, generally speaking, no. That is the subject of the following example.

EXAMPLE 2.13. Once again assume $v = 1$. First define two sequences, $(\gamma_i)_{i \in \mathbb{Z}}$ and $(\tilde{\gamma}_i)_{i \in \mathbb{Z}}$, in the space $E = \{0, 1\}$. For every $i \in 6\mathbb{Z}$, set $\tilde{\gamma}(i + \mathbb{C}_i) = (0, 0, 0)$ and $\tilde{\gamma}_{(i+3+\mathbb{C}_i)} = (1, 1, 1)$. For $i \le 0$, set $\gamma_i = 0$ and, for $i > 0$, $\gamma_i = 1$. We now define two arrays $\Delta = (\xi^N_{\mathbb{C}_N})^{N \in \mathbb{N}}$ and $\tilde{\Delta} = (\eta^N_{\mathbb{C}_N})^{N \in \mathbb{N}}$ in E. For $N \in 2\mathbb{N}$, set $\eta^N_{\mathbb{C}_N} = \tilde{\gamma}_{\mathbb{C}_N}$ and, for $N \in 2\mathbb{N} + 1$, $\eta_{\mathbb{C}_N} = \tilde{\gamma}_{(\mathbb{C}_N - 1)}$. Finally, for $N \in \mathbb{N}$, let $\xi^N_{\mathbb{C}_N} = \gamma_{\mathbb{C}_N}$. Then we have

$$\varphi^N_{\Delta} \overset{(\infty)}{=} \varphi^N_{\tilde{\Delta}} \overset{(\infty)}{=} \tfrac{1}{2}(\mathscr{E}_0 + \mathscr{E}_1).$$

Thus Δ and $\tilde{\Delta}$ are both point convergent with limit $\frac{1}{2}(\mathscr{E}_0 + \mathscr{E}_1)$. Consider the 1, 0 partition array of $\tilde{\Delta}$. It follows from the definition that

$$\varphi^{2N}_{\tilde{\Delta}^1_0} \overset{(\infty)}{=} \tfrac{1}{2}\big(\mathscr{E}_{(0,0,0)} + \mathscr{E}_{(1,1,1)}\big) \quad \text{and} \quad \varphi^{2N+1}_{\tilde{\Delta}^1_0} \overset{(\infty)}{=} \tfrac{1}{2}\big(\mathscr{E}_{(1,0,0)} + \mathscr{E}_{(0,1,1)}\big).$$

Obviously $\tilde{\Delta}$ is not point convergent for 1-partitions. We will show that Δ and $\tilde{\Delta}$ are point convergent for clusters. Fix $k \in \mathbb{N}$, and set

$$0_k = \underbrace{(0, \dots, 0)}_{2k+1 \text{ times}} \quad \text{and} \quad 1_k = \underbrace{(1, \dots, 1)}_{2k+1 \text{ times}}.$$

Further, for $i = 1, \dots, 6$, define $\beta_i = (\tilde{\gamma}_{i+\mathbb{C}_k})$. One then has

$$\varphi^N_{\tilde{\Delta}_{\mathbb{C}_i}} \overset{(\infty)}{=} \frac{1}{6} \sum_{i=1}^{6} \mathscr{E}_{\beta_i}$$

and

$$\varphi^N_{\Delta_{\mathbb{C}_k}} \overset{(\infty)}{=} \frac{1}{2N+1}\left(\sum_{i=-N}^{0} \mathscr{E}_{0_k} + \sum_{i=1}^{N} \mathscr{E}_{1_k}\right) \to \frac{1}{2}\big(\mathscr{E}_{0_k} + \mathscr{E}_{1_k}\big), \quad \text{for } N \to \infty.$$

Since $k \in \mathbb{N}$ was arbitrarily chosen, the proposition follows from Lemma 2.3.

Having seen that point convergence for partitions is generally strictly stronger than point convergence for clusters, the only question that remains is whether, for arbitrarily chosen $\mathbb{H} \in Cl$, simple point convergence implies point convergence for $\mathbb{H}$-clusters. Define the array $\Delta^* = (\rho^N_{\mathbb{C}_N})^{N \in \mathbb{N}}$ by setting, for $N \in 2\mathbb{N}$, $\rho^N_{\mathbb{C}_N} = \eta^N_{\mathbb{C}_N}$ and, for $N \in 2\mathbb{N} + 1$, $\rho^N_{\mathbb{C}_N} = \xi^N_{\mathbb{C}_N}$. It follows from the preceding that Δ^* is point convergent with limit $\frac{1}{2}(\mathscr{E}_0 + \mathscr{E}_1)$. The sequence

$$\Big(\mathscr{L}\Big\{\varphi^{2N}_{\Delta^*_{\mathbb{C}_1}}\Big\}\Big)_{N \in \mathbb{N}} \qquad \Big[\text{resp.}, \Big(\mathscr{L}\Big\{\varphi^{2N+1}_{\Delta^*_{\mathbb{C}_1}}\Big\}\Big)_{N \in \mathbb{N}}\Big]$$

has as limit

$$\tfrac{1}{6}\big(\mathscr{E}_{(0,0,0)} + \mathscr{E}_{(0,0,1)} + \mathscr{E}_{(0,1,1)} + \mathscr{E}_{(1,1,1)} + \mathscr{E}_{(1,1,0)} + \mathscr{E}_{(1,0,0)}\big)$$

$$\Big[\text{resp. } \tfrac{1}{2}\big(\mathscr{E}_{(0,0,0)} + \mathscr{E}_{(1,1,1)}\big)\Big].$$

As such, Δ^* is not point convergent for $\mathbb{C}_1$ clusters.

It is perhaps worth noting that these examples also show the divergence of the various point convergence concepts in the deterministic area.

3. Point convergence for systems of interacting processes. In this paragraph we will consider the point convergence for clusters of systems of processes displaying extended strong local interaction. Recall the system of equations defined in the introduction by setting, for every $N \in \mathbb{N}$,

$$(N) \qquad \begin{aligned} X_i^N &= 0 \quad \text{for } i \in \mathscr{S} \setminus \mathbb{C}_N, \\ X_i^N(t) &= K_i^N + \int_0^t g(s, \theta_i(X_{\mathscr{S}}^N)) Z_i^N(ds) \quad \text{for } i \in \mathbb{C}_N. \end{aligned}$$

As noted there, the solution to (N) can be seen as a finite-dimensional approximation of an infinite system of equations. The infinite-dimensional version is given by

$$(\infty) \qquad X_i(t) = K_i + \int_0^t g(s, \theta_i(X_{\mathscr{S}})) Z_i(ds) \quad \text{for } i \in \mathscr{S}.$$

We will view (∞) as a single equation in a Hilbert space of square-summable sequences $\mathscr{X} = \{(x_i)_{i \in \mathscr{S}} \in (\mathbb{R}^d)^{\mathscr{S}} : \sum_{i \in \mathscr{S}} \gamma^2 \|x_i\|^2 < \infty\}$, with norm

$$\|(x_i)_{i \in \mathscr{S}}\|_{\mathscr{X}} = \sqrt{\sum_{i \in \mathscr{S}} \gamma^2 \|x_i\|^2}$$

where $\gamma = (\gamma_i)_{i \in \mathscr{S}}$ is a square-summable sequence of strictly positive real numbers.

Assume that $\tilde{\gamma} = (\tilde{\gamma}_i)_{i \in \mathscr{S}}$ is another square-summable sequence of strictly positive real numbers and let

$$\mathscr{Z} = \left\{ (x_i)_{i \in \mathscr{S}} \in (\mathbb{R}^m)^{\mathscr{S}} : \sum_{i \in \mathscr{S}} \tilde{\gamma}^2 \|x_i\|^2 < \infty \right\}$$

be the induced Hilbert space. We recall that an adapted, $\mathscr{Z}$-valued process Z is a semimartingale iff Z admits a *control process* A. A positive, increasing adapted process A is called *a control process for* Z iff, for every further separable Hilbert space $\mathscr{Y}$, $\mathrm{Op}(\mathscr{Z}, \mathscr{Y})$-valued elementary predictable process Y and stopping time τ, one has

$$\mathbb{E}\left(\sup_{t < \tau} \left\| \int_{[0,t)} Y \, dZ \right\|_{\mathscr{Z}}^2 \right) \le \mathbb{E}\left(A(\tau -) \int_{[0,\tau)} \|Y\|_{\mathrm{Op}(\mathscr{Z},\mathscr{Y})}^2 \, dA(s) \right),$$

where $\mathrm{Op}(\mathscr{Z}, \mathscr{Y})$ denotes the space of bounded linear operators from $\mathscr{Z}$ into $\mathscr{Y}$ and $\|\cdot\|_{\mathrm{Op}(\mathscr{Z},\mathscr{Y})}$ denotes the operator norm [see Metivier (1982), page 157].

CONDITIONS 3.1. In the sequel we will assume the following to be given:

(i) a continuous, positive, increasing, adapted $\mathbb{R}$-valued process A, an array $(Z_{\mathbb{C}_N}^N)^{N \in \mathbb{N}}$ [resp., $(K_{\mathbb{C}_N}^N)^{N \in \mathbb{N}}$] of continuous $\mathbb{R}^m$-valued square-integrable semimartingales (resp., of $\mathbb{R}^d$-valued square-integrable F_0-measurable random variables) and a continuous $\mathscr{Z}$-valued square-integrable semimartingale $(Z_i)_{i \in \mathscr{S}}$ [resp., an $\mathscr{X}$-valued F_0-measurable random element $(K_i)_{i \in \mathscr{S}}$];

(ii) a continuous function $g: [0,\infty) \times (C^d)^{\mathscr{S}} \to \mathbb{R}^{d\times m}$ and the induced family of linear operators defined for every $(t, x_{\mathscr{S}}) \in [0,\infty) \times (C^d)^{\mathscr{S}}$ by setting $G(t, x_{\mathscr{S}})\colon (\mathbb{C}^m)^{\mathscr{S}} \to (\mathbb{C}^d)^{\mathscr{S}}, (z_i)_{i\in\mathscr{S}} \mapsto (g(t, \theta_i(x_{\mathscr{S}}))z_i)_{i\in\mathscr{S}}$.

The array $\Delta = ((Z^N_{\mathbb{C}_N}))^{N\in\mathbb{N}}$ and the functions g and G fulfill the following conditions.

(CL) (Lipschitz and integration condition). There exist constants $I, L > 0$ so that the following hold:

(i) For every $N \in \mathbb{N}$, $\mathbb{E}((1/|\mathbb{C}_N|)\Sigma_{i\in\mathbb{C}_N}\|K^N_i\|^2) < I$.

(ii) For every $f, \tilde{f} \in C([0,\infty), \mathscr{X})$, $s, t \in [0,\infty)$, $g(t,f) = g(t, f(\cdot \wedge t))$. Further, $G(t,f), G(s,\tilde{f}) \in \mathrm{Op}(\mathscr{Z},\mathscr{X})$ and

$$\left\|g(t,f) - g(s,\tilde{f})\right\|^2_{\mathrm{Op}(\mathbb{R}^m,\mathbb{R}^d)} \le L \sup_{h<s\vee t} \left\|f(h\wedge t) - \tilde{f}(h\wedge s)\right\|^2_{\mathscr{X}},$$

$$\left\|G(t,f) - G(s,\tilde{f})\right\|^2_{\mathrm{Op}(\mathscr{Z},\mathscr{X})} \le L \sup_{h<s\vee t} \left\|f(h\wedge t) - \tilde{f}(h\wedge s)\right\|^2_{\mathscr{X}}.$$

(CP) (Point convergence condition).

(i) The array $(K^N_i, Z^N_i)^{N\in\mathbb{N}}_{i\in\mathbb{C}_N}$ is point convergent for clusters in $\mathbb{R}^d \times C^m$ with cluster limit $\mathscr{L}\{(K_i, Z_i)_{i\in\mathscr{S}}\}$.

(ii) For every $i \in \mathbb{C}_N$, $N \in \mathbb{N}$, A is a control process for Z^N_i.

For the economic motivation of (CP)(i), consult Finnoff (1989). Condition (CP)(ii) can be interpreted as saying that there exists a common bound to the expected maximal growth of the processes $(Z^N_i)^{N\in\mathbb{N}}_{i\in\mathbb{C}_N}$. The most widely investigated example of an infinite-dimensional semimartingale on a countable space of summable sequences such as $\mathscr{Z}$ is given by a system of i.i.d. Brownian motions $(B_i)_{i\in\mathscr{S}}$ [see Holley and Stroock (1981) or Leha and Ritter (1984)]. For further examples of functions g and G and systems of processes $(Z^N_{\mathbb{C}_N})^{N\in\mathbb{N}}$ [resp., $(Z_i)_{i\in\mathscr{S}}$] that satisfy Conditions 3.1, consult Metivier (1982), Metivier and Pellaumail (1980), Shiga and Shimizu (1980) and the references given above.

A (strong) solution to the equation (∞) is defined as an adapted continuous $\mathscr{X}$-valued process $(X_i)_{i\in\mathscr{S}}$ so that, for $\mathbb{P}$ a.e. $\omega \in \Omega$, $(X_i(\omega))_{i\in\mathscr{S}}$ satisfies equation (∞). [Solutions to equations (N) are defined in an analogous fashion.] The following known results are an immediate consequence of (CL).

LEMMA 3.2.

(i) *For every* $N \in \mathbb{N}$, *there exists a unique solution* $(X^N_{\mathbb{C}_N})$ *to* (N).

(ii) *There exists a unique solution* $(X_i)_{i\in\mathscr{S}}$ *to* (∞).

PROOF. See Metivier (1982), Theorem 34.7. □

We now present the result which will permit us to demonstrate the point convergence of arrays of approximate solutions. We note that none of the martingale-theoretic, integrability or Lipschitz hypotheses given above are required for this result.

THEOREM 3.3. *Let* $n \in \mathbb{N}$, *let* $\Delta = ((\tilde{K}_i^N, \tilde{Z}_i^N))_{i \in \mathbb{C}_N}^{N \in \mathbb{N}}$ *be an array in* $R^d \times C^m$ *that is point convergent for clusters with cluster limit* $\mu = \mathscr{L}\{((\tilde{K}_i, \tilde{Z}_i))_{i \in \mathscr{S}}\}$ *and let* $h\colon [0,\infty) \times (C^d)^{\mathbb{C}_n} \to \mathbb{R}^{d \times m}$ *be a continuous function with* $h(t,f) = h(t, f(\cdot \wedge t))$, *for every* $(t,f) \in [0,\infty) \times (C^d)^{\mathbb{C}_n}$. *For every* $N \in \mathbb{N}$, *define* $(Y_i^N)_{i \in \mathscr{S}}$ *by setting* $Y_i^N = 0$, *for* $i \in \mathscr{S} \setminus \mathbb{C}_N$, *and, for every* $i \in \mathbb{C}_N$ *and* $t \in [0,\infty)$,

$$Y_i^N(t) = \begin{cases} \tilde{K}_i^N, & \text{for } t \in \left[0, \frac{1}{n}\right], \\ Y_i^N\left(\frac{k}{n}\right) + h\left(\frac{k}{n}, Y^N_{(\mathbb{C}_n + i)}\right)\left(\tilde{Z}_i^N(t) - \tilde{Z}_i^N\left(\frac{k}{n}\right)\right), & \\ & \text{for } t \in \left(\frac{k}{n}, \frac{k+1}{n}\right], k \in \mathbb{N}. \end{cases}$$

Further, for every $i \in \mathscr{S}$ *and* $t \in [0,\infty)$, *set*

$$Y_i(t) = \begin{cases} \tilde{K}_i, & \text{for } t \in \left[0, \frac{1}{n}\right], \\ Y_i\left(\frac{k}{n}\right) + h\left(\frac{k}{n}, Y_{(\mathbb{C}_n + i)}\right)\left(\tilde{Z}_i(t) - \tilde{Z}_i\left(\frac{k}{n}\right)\right), & \\ & \text{for } t \in \left(\frac{k}{n}, \frac{k+1}{n}\right], k \in \mathbb{N}. \end{cases}$$

Then $\tilde{\Delta} = (Y_{\mathbb{C}_N}^N)^{N \in \mathbb{N}}$ *is point convergent for clusters with cluster limit* $\mathscr{L}\{(Y_i)_{i \in \mathscr{S}}\}$.

PROOF. Let $k \in \mathbb{Z}_+$. For a process Y we define

$${}^kY = Y\left(\cdot \wedge \frac{k}{n}\right).$$

For any $N \in \mathbb{N}$ and $i \in \mathbb{C}_N$,

$${}^{k+1}Y_i^N = {}^kY_i^N + h\left(\frac{k}{n}, Y^N_{(\mathbb{C}_n + i)}\right)\left(\tilde{Z}_i^N\left(\left(\cdot \vee \frac{k}{n}\right) \wedge \frac{k+1}{n}\right) - \tilde{Z}_i^N\left(\frac{k}{n}\right)\right)$$

and, for every $i \in \mathscr{S}$,

$${}^{k+1}Y_i = {}^kY_i + h\left(\frac{k}{n}, Y_{\mathbb{C}_n + i}\left(\frac{k}{n}\right)\right)\left(\tilde{Z}_i\left(\left(\cdot \vee \frac{k}{n}\right) \wedge \frac{k+1}{n}\right) - \tilde{Z}_i\left(\frac{k}{n}\right)\right).$$

For $k \in \mathbb{Z}_+$, define

$$^k\Delta = \left(\left({}^kY_i^N, \tilde{Z}_i^N\right)\right)_{i\in\mathbb{C}_N}^{N\in\mathbb{N}}$$

and

$$^k\mu = \mathscr{L}\left\{\left(\left({}^kY_i, \tilde{Z}_i\right)\right)_{i\in\mathscr{S}}\right\}.$$

Once we have shown that $^k\Delta$ is a point convergent for clusters with cluster limit $^k\mu$ for every $k \in \mathbb{Z}_+$, we are finished, since it is sufficient to consider $\tilde{\Delta}$ restricted to bounded intervals of $[0, \infty)$ by Lemma 1.2(iii). We accomplish this by induction over k.

For every $N \in \mathbb{N}$ and $i \in \mathbb{C}_N$ (resp., $i \in \mathscr{S}$), interpreting $\tilde{K}_i^N$ (resp., $\tilde{K}_i$) as a (constant) process (and, as such, as a random element in C^d), then for $k = 0$ the proposition follows from the fact that $((\tilde{K}_i^N, \tilde{Z}_i^N))_{i\in\mathbb{C}_N}^{N\in\mathbb{N}}$ is point convergent for clusters. Let $k \in \mathbb{N}$ and assume that $^k\Delta$ is point convergent for clusters with cluster limit $^k\mu$. Fix $l \in \mathbb{N}$. Then define

$$r: \left(C^d \times C^m\right)^{\mathbb{C}_{l+n}} \to \left(C^d \times C^m\right)^{\mathbb{C}_l},$$
$$\left(\left(x_i, z_i\right)\right)_{i\in\mathbb{C}_{l+n}} \mapsto \left(\left(y_i, z_i\right)\right)_{i\in\mathbb{C}_l},$$

with

$$y_i = x_i + h\left(\frac{k}{n}, x_{(\mathbb{C}_n+i)}\right)\left(z_i\left(\left(\cdot \vee \frac{k}{n}\right) \wedge \frac{k+1}{n}\right) - z_i\left(\frac{k}{n}\right)\right) \quad \text{for } i \in \mathbb{C}_l.$$

The mapping r is continuous. Recall the definition of $I_{\mathbb{C}_{l+n}}^N = \{i \in \mathscr{S}\colon \mathbb{C}_n + i \in F_{\mathbb{C}_{l+n}}^N\}$. We note that, for every $i \in I_{\mathbb{C}_{l+n}}^N$,

$$\left(\left({}^{k+1}Y_j^N, \tilde{Z}_j^N\right)\right)_{j\in(\mathbb{C}_l+i)} = r\left(\left(\left({}^kY_j^N, \tilde{Z}_j^N\right)\right)_{j\in(\mathbb{C}_{l+n}+i)}\right)$$

and, for every $i \in \mathscr{S}$,

$$\left(\left({}^{k+1}Y_j, \tilde{Z}_j\right)\right)_{j\in(\mathbb{C}_l+i)} = r\left(\left(\left({}^kY_j, \tilde{Z}_j\right)\right)_{j\in(\mathbb{C}_{l+n}+i)}\right).$$

Further we have that

$$\left(|I_{\mathbb{C}_{n+l}}^N|\right)_{N\in\mathbb{N}} = \left(|F_{\mathbb{C}_{n+l}}^N|\right)_{N\in\mathbb{N}} \sim \left(|F_{\mathbb{C}_l}^N|\right)_{N\in\mathbb{N}} = \left(|I_{\mathbb{C}_l}^N|\right)_{N\in\mathbb{N}}.$$

Therefore, it follows from Lemma 2.2 that

$$\begin{aligned}
\varphi^N_{^{k+1}\Delta_{\mathbb{C}_l}} &= \frac{1}{|F_{\mathbb{C}_l}^N|}\sum_{i\in I_{\mathbb{C}_l}^N}\mathscr{E}_{((^{k+1}Y_j^N, \tilde{Z}_j^N))_{j\in(\mathbb{C}_l+i)}}\\
&\overset{(\infty)}{=} \frac{1}{|F_{\mathbb{C}_l}^N|}\sum_{i\in I_{\mathbb{C}_{l+n}}^N}\mathscr{E}_{r(((^kY_j^N, \tilde{Z}_j^N))_{j\in(\mathbb{C}_{l+n}+i)})}\\
&\overset{(\infty)}{=} \frac{1}{|F_{\mathbb{C}_{l+n}}^N|}\sum_{B\in F_{\mathbb{C}_{l+n}}^N}\mathscr{E}_{r(((^kY_j^N, \tilde{Z}_j^N))_{j\in B})} = \varphi^N_{r(^k\Delta_{\mathbb{C}_{l+n}})}.
\end{aligned}$$

For $B \subset \mathscr{S}$, let p_B denote the projection

$$p_B\colon \left(C^d \times C^m\right)^{\mathscr{S}} \to \left(C^d\right)^B, \qquad \left(\left(x_i, z_i\right)\right)_{i\in S} \mapsto x_B.$$

The preceding, together with Corollary 1.3 and Lemma 1.5, implies that ${}^{k+1}\Delta_{\mathbb{C}_l}$ is point convergent with limit

$$\left(r\circ\widetilde{p_{\mathbb{C}_{l+n}}}\right)\left({}^{k}\mu\right)=\left(\widetilde{p_{\mathbb{C}_l}}\right)\left({}^{k+1}\mu\right).$$

Since l was arbitrarily chosen and the sequence $(\mathbb{C}_l)_{l\in\mathbb{N}}$ is cofinal in $\mathscr{S}$, by Lemma 2.3, this completes the proof. □

DEFINITION 3.4 (The approximating arrays). For every $n\in\mathbb{N}$, define p^n: $(C^d)^{\mathscr{S}}\to(C^d)^{\mathscr{S}}$, $(x_i)_{i\in\mathscr{S}}\mapsto(y_i)_{i\in\mathscr{S}}$, where $y_i=x_i$ for $i\in\mathbb{C}_n$ and $y_i=0$ otherwise, and define g^n: $[0,\infty)\times(C^d)^{\mathscr{S}}\to\mathbb{R}^{d\times m}$, $(t,(x_i)_{i\in\mathscr{S}})\mapsto g(t^n,p^n((x_i)_{i\in\mathscr{S}}))$, with $t^n=\max\{k/n\colon k\in\mathbb{Z}_+,\,k/n\le t\}$. If, for every $N\in\mathbb{N}$ and $i\in\mathscr{S}\setminus\mathbb{C}_N$, one sets $X_i^N=0$, then one can define recursively as in Theorem 3.3 for every $N\in\mathbb{N}$ a solution $({}^nX^N_{\mathbb{C}_N})$ to the following equation:

$$(n,N)\quad {}^nX_i^N(t)=K_i^N+\int_{[0,t)}g^n\left(s,\theta_i\left({}^nX^N_{\mathscr{S}}\right)\right)dZ_i^N(s),\qquad i\in\mathbb{C}_N.$$

Analogously, one defines a solution $(X_i^n)_{i\in\mathscr{S}}$ to the equation

$$(n,\infty)\qquad X_i^n(t)=K_i+\int_{[0,t)}g^n\left(s,\theta_i(X^n_{\mathscr{S}})\right)dZ_i(s),\qquad i\in\mathscr{S}.$$

[This uses on the one hand the fact that $(Z_i)_{i\in\mathscr{S}}$ [resp., $(K_i)_{i\in\mathscr{S}}$] takes its values in $\mathscr{Z}$ (resp., $\mathscr{K}$) and on the other that, for every $(t,x_{\mathscr{S}})\in[0,\infty)\times C([0,\infty),\mathscr{K})$, $G(t,x_{\mathscr{S}})\in\mathrm{Op}(\mathscr{Z},\mathscr{K})$.] We then have the following corollary.

COROLLARY 3.5. *For every* $n\in\mathbb{N}$, $({}^nX^N_{\mathbb{C}_N})^{N\in\mathbb{N}}$ *is point convergent for clusters with cluster limit* $\mathscr{L}\{(X_i^n)_{i\in\mathscr{S}}\}$.

PROOF. For $n\in\mathbb{N}$, define c^n: $(C^d)^{\mathbb{C}_n}\to(C^d)^{\mathscr{S}}$, $(x_i)_{i\in\mathbb{C}_n}\mapsto(y_i)_{i\in\mathscr{S}}$, with $y_i=x_i$ for $i\in\mathbb{C}_n$ and $y_i=0$ otherwise, and use h: $[0,\infty)\times(C^d)^{\mathbb{C}_n}$, $(t,(x_i)_{i\in\mathbb{C}_n})\mapsto g(t,c^n((x_i)_{i\in\mathbb{C}_n}))$. □

To show that the approximating arrays converge uniformly in N to the array of genuine solutions, we need some technical results. These are closely related to Lemmas (3.6)–(3.8) in Finnoff (1993). We carry out the proofs here for the sake of completeness and because the modification for the current situation is perhaps not completely straightforward.

DEFINITION 3.6. Let $T,\ \delta\in[0,\infty)$, $\mathscr{K}\in\mathscr{A}$.

(i) Set

$$\eta_\delta(\mathscr{K},T)=\sup\{|A(\omega,s)-A(\omega,t)|\colon \omega\in\mathscr{K},\,s,t\in[0,T],\,|s-t|\le\delta\}.$$

We now define a number of stopping times.

(ii) $\hat{T}(\mathscr{K})=\inf\{t\in[0,\infty]\colon A(t)>\eta_T(\mathscr{K},T)\}\wedge T$.

(iii) For every $n\in\mathbb{N}$ and $k\in\mathbb{Z}_+$, set

$$\tau_k^n(\mathscr{K},T)=\inf\left\{t\in\left[\frac{k}{n},\infty\right]\colon A(t)-A\left(\frac{k}{n}\right)>\eta_{1/n}(\mathscr{K},T)\right\}\wedge\hat{T}(\mathscr{K})\wedge\frac{k+1}{n}.$$

(iv) For every $n \in \mathbb{N}$ and $s \in [0,\infty)$, set $s^n = \max\{k/n: k \in \mathbb{Z}_+, k/n \le s\}$. Noting that $n(T^n) = \max\{k \in \mathbb{N}: k/n \le T\}$, we define the adapted process

$$V^n = \sum_{k=0}^{n(T^n)} 1_{[k/n, \tau_k^n(\mathscr{H},T))}$$

and stopping time

$$\mathscr{T}^n = \inf\{t \in [0,\infty): V^n(t) = 0\}.$$

In the following, the indexes $\mathscr{H}$ and T will be dropped whenever the reference is clear.

LEMMA 3.7. *For every T, $\varepsilon > 0$, there exists a set $\mathscr{H}_\varepsilon^T \in \mathscr{A}$ with the following:*

(i) $\mathbb{P}(\overline{\mathscr{H}_\varepsilon^T}) < \varepsilon$.
(ii) $\eta_T(\mathscr{H}_\varepsilon^T, T) < \infty$ *and* $\eta_{1/n}(\mathscr{H}_\varepsilon^T, T) \to 0$ *for* $n \to \infty$.

PROOF. The process $A|_{[0,T]}$ is realized in the Polish space $C([0,T],\mathbb{R})$. Therefore, for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset C([0,T],\mathbb{R})$ so that $\mathscr{L}\{A|_{[0,T]}\}(\overline{K_\varepsilon}) < \varepsilon$. Set $\mathscr{H}_\varepsilon^T = (A|_{[0,T]})^{-1}(K_\varepsilon)$. By the Arzela–Ascoli theorem, the set K_ε is uniformly bounded and equicontinuous. The result is then immediate. □

In the following two lemmas we will assume that T, $\varepsilon \in [0,\infty)$ are arbitrarily chosen, but fixed, and $\mathscr{H} = \mathscr{H}_\varepsilon^T \in \mathscr{A}$ is the set given in Lemma 3.7. Further, any of the symbols η_δ, τ_k^n, $\hat{T}$ and so on which appear will be assumed to refer to T and $\mathscr{H}$.

LEMMA 3.8. *There exists a constant W such that, for enery n, $N \in \mathbb{N}$,*

$$\mathbb{E}\left(\frac{1}{|\mathbb{C}_N|}\sum_{i\in\mathbb{C}_N}\int_{[0,\mathscr{T}^n)}\sup_{h<s}\left\|X_i^N(h\wedge s^n) - X_i^N(h)\right\|^2 dA(s)\right) < W\eta_{1/n}.$$

PROOF. Define $G = \sup_{t\in[0,T]} g(t,0)$ and (recalling the definition of the norm $\|\cdot\|_{\mathscr{X}}$), set $\sum_{i\in\mathscr{S}}\gamma_i^2 = J < \infty$. Then, for any $t \in [0,\infty)$ and $x_{\mathscr{S}} \in C([0,\infty),\mathscr{X})$, with $x_{\mathscr{S}\setminus\mathbb{C}_N} = 0$,

$$\begin{aligned}\sum_{i\in\mathbb{C}_N}\sup_{h<t}\left\|\theta_i(x_{\mathscr{S}}(h))\right\|_{\mathscr{X}}^2 &\le \sum_{i\in\mathbb{C}_N}\sum_{j\in(\mathbb{C}_N-i)}\gamma_j^2\sup_{h<t}\left\|x_{i+j}(h)\right\|^2\\ (*)\qquad &\le \sum_{q\in\mathbb{C}_N}\sum_{i\in\mathbb{C}_N}\gamma_{(q-i)}^2\sup_{h<t}\left\|x_q(h)\right\|^2 \qquad (q=i+j)\\ &\le J\sum_{q\in\mathbb{C}_N}\sup_{h<t}\left\|x_q(h)\right\|^2.\end{aligned}$$

Step 1. Let U be some stopping time, $U \le \hat{T}$. Then, for every $N \in \mathbb{N}$,

$$\mathbb{E}\left(\frac{1}{|\mathbb{C}_N|}\sum_{i\in\mathbb{C}_N}\sup_{h<U}\left\|X_i^N(h)-K_i^N\right\|^2\right)$$

$$\le \mathbb{E}\left(\frac{1}{|\mathbb{C}_N|}\sum_{i\in\mathbb{C}_N}A(U)\int_{[0,U)}\left\|g\left(t,\theta_i\left(X_{\mathscr{S}}^N\right)\right)\right\|^2_{\mathrm{Op}(\mathbb{R}^m,\mathbb{R}^d)}dA(t)\right) \quad [\text{by (CP)(ii)}]$$

$$\le 2\mathbb{E}\left(\frac{1}{|\mathbb{C}_N|}\sum_{i\in\mathbb{C}_N}A(U)\int_{[0,U)}\left\|g\left(t,\theta_i\left(X_{\mathscr{S}}^N\right)\right)-g(t,0)\right\|^2_{Op(\mathbb{R}^m,\mathbb{R}^d)}dA(t)\right)$$

$$+\,2\mathbb{E}\left(\frac{1}{|\mathbb{C}_N|}\sum_{i\in\mathbb{C}_N}A(U)\int_{[0,U)}\left\|g(t,0)\right\|^2_{\mathrm{Op}(\mathbb{R}^m,\mathbb{R}^d)}dA(t)\right)$$

$$\le 2L\eta_T\mathbb{E}\left(\frac{1}{|\mathbb{C}_N|}\sum_{i\in\mathbb{C}_N}\int_{[0,U)}\sup_{h<t}\left\|\theta_i\left(X_{\mathscr{S}}^N(h)\right)\right\|^2_{\mathscr{X}}dA(t)\right)$$

$$+\,2G\mathbb{E}\left(A(U)^2\right) \quad [\text{by (CL)}]$$

$$\le 2JL\eta_T\mathbb{E}\left(\int_{[0,U)}\frac{1}{|\mathbb{C}_N|}\sum_{q\in\mathbb{C}_N}\sup_{h<t}\left\|X_q^N(h)\right\|^2dA(t)\right)$$

$$+\,2G\mathbb{E}\left(A(U)^2\right) \quad [\text{by } (*)].$$

Step 2. For any $t \in [0,\infty)$, define $\phi^N(t) = (1/|\mathbb{C}_N|)\Sigma_{i\in\mathbb{C}_N}\sup_{h<t}\|X_i^N(h)\|^2$. Recalling (CL)(i), we have, for any stopping time $U \le \hat{T}$,

$$\mathbb{E}\left(\phi^N(U)\right) \le 2I + 2\mathbb{E}\left(\frac{1}{|\mathbb{C}_N|}\sum_{i\in\mathbb{C}_N}\sup_{h<U}\left\|X_i^N(h)-K_i^N\right\|^2\right)$$

$$\le 4JL\eta_T E\left(\int_{[0,U)}\phi^N(t)\,dA(t)\right) + 2I + 4G\eta_T^2.$$

Setting $K = 2I + 4G\eta_T^2$, $\rho = 4JL\eta_T$, it follows from Lemma 1.6 that there exists a constant $W_1 \ge 0$, such that

$$(**) \qquad \mathbb{E}\left(\phi^N(\hat{T})\right) \le W_1.$$

Now, using the second and third inequalities in Step 1, it follows, for any $N \in \mathbb{N}$,

$$\mathbb{E}\left(\frac{1}{|\mathbb{C}_N|}\sum_{i\in\mathbb{C}_N}A(\hat{T})\int_{[0,\hat{T})}\left\|g\left(t,\theta_i\left(X_{\mathscr{S}}^N\right)\right)\right\|^2_{\mathrm{Op}(\mathbb{R}^m,\mathbb{R}^d)}dA(t)\right)$$

$$\le 2JL\eta_T W_1 + 2G\eta_T^2 = W.$$

Step 3. Let $n \in \mathbb{N}$ be arbitrarily chosen but fixed. Then,

$$\mathbb{E}\left(\frac{1}{|\mathbb{C}_N|}\sum_{i\in\mathbb{C}_N}\int_{[0,\mathscr{T}^n)}\sup_{h<s}\left\|X_i^N(h\wedge s^n)-X_i^N(h)\right\|^2 dA(s)\right)$$

$$\leq \mathbb{E}\left(\frac{1}{|\mathbb{C}_N|}\sum_{i\in\mathbb{C}_N}\sum_{k=0}^{n(T^n)}\int_{[k/n,\tau_k^n)}\sup_{k/n\leq h<s}\left\|X_i^N(h)-X_i^N\left(\frac{k}{n}\right)\right\|^2 dA(s)\right)$$

(by the definition of $\mathscr{T}^n$)

$$\leq \mathbb{E}\left(\frac{1}{|\mathbb{C}_N|}\sum_{i\in\mathbb{C}_N}\sum_{k=0}^{n(T^n)}\sup_{k/n\leq h<\tau_k^n}\left\|X_i^N(h)-X_i^N\left(\frac{k}{n}\right)\right\|^2\left(A(\tau_k^n)-A\left(\frac{k}{n}\right)\right)\right)$$

$$\leq \eta_{1/n}\mathbb{E}\left(\frac{1}{|\mathbb{C}_N|}\sum_{i\in\mathbb{C}_N}\sum_{K=0}^{n(T^n)}\sup_{k/n\leq s<\tau_k^n}\left\|\int_{[k/n,s)}g\left(t,\theta_i\left(X_{\mathscr{S}}^N\right)\right)dZ_i^N(t)\right\|^2\right)$$

(by the definition of τ_k^n)

$$\leq \eta_{1/n}\mathbb{E}\left(\frac{1}{|\mathbb{C}_N|}\sum_{i\in\mathbb{C}_N}\sum_{k=0}^{n(T^n)}A(\tau_k^n)\int_{[k/n,\tau_k^n)}\left\|g\left(t,\theta_i\left(X_{\mathscr{S}}^N\right)\right)\right\|^2_{\mathrm{Op}(\mathbb{R}^m,\mathbb{R}^d)}dA(t)\right)$$

[by (CP)(ii)]

$$\leq \eta_{1/n}\mathbb{E}\left(\frac{1}{|\mathbb{C}_N|}\sum_{i\in\mathbb{C}_N}A(\hat{T})\int_{[0,\hat{T})}\left\|g\left(t,\theta_i\left(X^N\mathscr{S}\right)\right)\right\|^2_{\mathrm{Op}(\mathbb{R}^m,\mathbb{R}^d)}dA(t)\right)$$

$$\leq \eta_{1/n}W \quad \text{(by step 2)}.$$ □

LEMMA 3.9. *For every* $n \in \mathbb{N}$, *define*

$$\beta^n = \sup_{N\in\mathbb{N}}\mathbb{E}\left(\frac{1}{|\mathbb{C}_N|}\sum_{i\in\mathbb{C}_N}\sup_{[0,T)}\left\|{}^nX_i^N(t)-X_i^N(t)\right\|^2 1_{\mathscr{K}}\right).$$

Then we have $\beta^n \to 0$ *for* $n \to \infty$.

PROOF. By the definition of the stopping time $\mathscr{T}^n$, we have, for every $n \in \mathbb{N}$, $1_{\mathscr{K}\times[0,T)} \leq 1_{[0,\mathscr{T}^n)}$. Therefore, for every $n, N \in \mathbb{N}$,

$$(***)\qquad \begin{aligned}&\frac{1}{|\mathbb{C}_N|}\sum_{i\in\mathbb{C}_N}\sup_{h<T}\left\|{}^nX_i^N(h)-X_i^N(h)\right\|^2 1_{\mathscr{K}}\\ &\quad\leq \frac{1}{|\mathbb{C}_N|}\sum_{i\in\mathbb{C}_N}\sup_{h<\mathscr{T}^n}\left\|{}^nX_i^N(h)-X_i^N(h)\right\|^2.\end{aligned}$$

For every $n \in \mathbb{N}$, define $\alpha^n = \sum_{i \in \mathscr{S} \setminus \mathbb{C}_n} \gamma_i^2$. Now let U be any stopping time, $U \leq \mathscr{T}^n$. Then, for every $n, N \in \mathbb{N}$,

$$\mathbb{E}\left(\frac{1}{|\mathbb{C}_N|} \sum_{i \in \mathbb{C}_N} \sup_{h < U} \left\| {}^nX_i^N(h) - X_i^N(h) \right\|^2 \right)$$

$$= \mathbb{E}\left(\frac{1}{|C_N|} \sum_{i \in \mathbb{C}_N} \sup_{h < U} \left\| \int_{[0,h)} g^n(s, \theta_i({}^nX_S^N)) - g(s, \theta_i(X_S^N))\, dZ_i^N(s) \right\|^2 \right)$$

$$\leq 2\eta_T \mathbb{E}\left(\frac{1}{|\mathbb{C}_N|} \sum_{i \in \mathbb{C}_N} \int_{[0,U)} \left[\left\| g^n(s, \theta_i({}^nX_S^N)) - g^n(s, \theta_i(X_S^N)) \right\|^2_{\mathrm{Op}(R^m, R^d)} \right.\right.$$

$$\left.\left. + \left\| g^n(s, \theta_i(X_S^N)) - g(s, \theta_i(X_S^N)) \right\|^2_{\mathrm{Op}(R^m, R^d)} \right] dA(s) \right)$$

[by (CP)(ii)]

$$\leq 2\eta_T L \mathbb{E}\left(\int_{[0,U)} \frac{1}{|\mathbb{C}_N|} \sum_{i \in \mathbb{C}_N} \sup_{t < s} \left\| \theta_i({}^nX_S^N(t)) - \theta_i(X_S^N(t)) \right\|_\chi^2 dA(s) \right)$$

$$+ 4\eta_T \mathbb{E}\left(\frac{1}{|\mathbb{C}_N|} \sum_{i \in \mathbb{C}_N} \int_{[0,U)} \| g(s^n, p^n(\theta_i(X_S^N))) \right.$$

$$\left. - g(s^n, \theta_i(X_S^N)) \|^2_{\mathrm{Op}(R^m, R^d)} dA(s) \right)$$

$$+ 4\eta_T \mathbb{E}\left(\frac{1}{|\mathbb{C}_N|} \sum_{i \in \mathbb{C}_N} \int_{[0,U)} \| g(s^n, \theta_i(X_S^N)) \right.$$

$$\left. - g(s, \theta_i(X_S^N)) \|^2_{\mathrm{Op}(R^m, R^d)} dA(s) \right)$$

[by (CL) and the definition of g^n]

$$\leq H \mathbb{E}\left(\int_{[0,U)} \frac{1}{|\mathbb{C}_N|} \sum_{i \in \mathbb{C}_N} \sup_{h < s} \| {}^nX_i^N(h) - X_i^N(h) \|^2 dA(s) \right)$$

$$+ H \mathbb{E}\left(\frac{1}{|\mathbb{C}_N|} \sum_{i \in \mathbb{C}_N} \int_{[0,U)} \sum_{j \in ((\mathbb{C}_N \setminus \mathbb{C}_n) - i)} \gamma_j^2 \left(\sup_{h < s} \| X_{i+j}^N(h) \|^2 \right) dA(s) \right)$$

$$+ H \mathbb{E}\left(\frac{1}{|\mathbb{C}_N|} \sum_{i \in \mathbb{C}_N} \int_{[0,U)} \left(\sup_{h < s} \| X_i^N(h \wedge s^n) - X_i^N(h) \|^2 \right) dA(s) \right)$$

[by (CL) and $(*)$]

$$\leq H\mathbb{E}\left(\int_{[0,U)} \frac{1}{|\mathbb{C}_N|} \sum_{i \in \mathbb{C}_N} \sup_{h<s} \|{}^nX_i^N(h) - X_i^N(h)\|^2 \, dA(s)\right)$$
$$+ H\eta_T W_1 \alpha^n + HW\eta_{1/n}$$

[by the definition of T^n, Lemma 3.8 and $(***)$],

where $H = 4\eta_T LJ$. Then define, for $t \in [0,\infty)$,

$$\phi_n^N(t) = \frac{1}{|\mathbb{C}_N|} \sum_{i \in \mathbb{C}_N} \sup_{s<t} \left\|{}^nX_i^N(s) - X_i^N(s)\right\|^2.$$

Further, for $n \in \mathbb{N}$, set $K(n) = H\eta_T W_1 \alpha^n + HW\eta_{1/n}$. Finally, define $\rho = H$ and $l = \eta_T$. Then, by Lemma 1.6,

$$\mathbb{E}\big(\phi_n^N(\mathscr{T}^n)\big) \leq K(n) \sum_{i=0}^{[2\rho l]} (2\rho l)^i.$$

By Lemma 3.7, $\eta_{1/n} \to 0$ for $n \to \infty$. Therefore, $K(n) \to 0$ for $n \to \infty$. The proposition then follows directly from $(***)$. □

COROLLARY 3.10. *Let $T \in (0,\infty)$, $\varepsilon > 0$ and $N \in \mathbb{N}$. Then there exists a set $\tilde{\mathscr{H}} = \tilde{\mathscr{H}}_\varepsilon^T \in \mathscr{A}$ so that $\mathbb{P}(\Omega \setminus \tilde{\mathscr{H}}_\varepsilon^T) < \varepsilon$ and*

$$\mathbb{E}\left(\sup_{h<T} \sum_{i \in \mathbb{C}_N} \left\|X_i^n(h) - X_i(h)\right\|^2 1_{\tilde{\mathscr{H}}}\right) \to 0 \quad \textit{for } n \to \infty.$$

PROOF. Taking a continuous control process $\tilde{A}$ for $(Z_i)_{i \in \mathscr{S}}$, one defines the set $\tilde{\mathscr{H}}$ as in Lemma 3.7. The remainder of the proof follows immediately by substituting ${}^nX^N$ and X^N in the proof of Lemma 3.9. □

For our next result we recall the definition given in 7 of subsection 0.1 of the metric m_u (resp., induced metric $\hat{m}_u$) on C^u [resp., $M(C^u)$] for $u \in \mathbb{N}$ and define, for any random variable ξ, the L_0 pseudonorm $\|\xi\|_0 = \mathbb{E}(|\xi| \wedge 1)$.

LEMMA 3.11. *Define $\Delta^\infty = (X_{\mathbb{C}_N}^N)^{N \in \mathbb{N}}$ and, for every $n \in \mathbb{N}$, $\Delta^n = ({}^nX_{\mathbb{C}_N}^N)^{N \in \mathbb{N}}$. Let $l \in \mathbb{N}$ and set $u = d \cdot |\mathbb{C}_l|$. Then the following hold*:

(i) $$\sup_{N \in \mathbb{N}_{\mathbb{C}_l}} \left\|\hat{m}_u\left(\varphi_{\Delta^n_{\mathbb{C}_l}}^N, \varphi_{\Delta^\infty_{\mathbb{C}_l}}^N\right)\right\|_0 \to 0 \quad \textit{for } n \to \infty;$$

(ii) $$\hat{m}_u\big(\mathscr{L}\{(X_i^n)_{i \in \mathbb{C}_l}\}, \mathscr{L}\{(X_i)_{i \in \mathbb{C}_l}\}\big) \to 0 \quad \textit{for } n \to \infty.$$

PROOF. (i) By the definition of the metric $\hat{m}_u$ it follows that, for any $N, T \in \mathbb{N}$, $x_i, y_i \in C^u$ and $i \in \mathbb{C}_N$,

$$\hat{m}_u\left(\frac{1}{|\mathbb{C}_N|} \sum_{i \in \mathbb{C}_N} \mathscr{E}_{x_i}, \frac{1}{|\mathbb{C}_N|} \sum_{i \in \mathbb{C}_N} \mathscr{E}_{y_i}\right)$$
$$\leq \frac{1}{|\mathbb{C}_N|} \sum_{i \in \mathbb{C}_N} m_u(x_i, y_i)$$
$$\leq \frac{1}{|\mathbb{C}_N|} \sum_{i \in \mathbb{C}_N} \sup_{t \leq T} \|x_i(t) - y_i(t)\| + \frac{1}{2^T}.$$

Let ε be artibtarily chosen. Choose some $T \in \mathbb{N}$ large enough that $1/2^r < \varepsilon$, and let $\mathscr{K}_\varepsilon^T \in \mathscr{A}$ be the set given in Lemma 3.7. Noting that $m_u \le 1$, it follows that

$$\sup_{N \in \mathbb{N}_{\mathbb{C}_l}} \left\| \hat{m}_u\left(\varphi_{\Delta^n_{\mathbb{C}_l}}^N, \varphi_{\Delta^\infty_{\mathbb{C}_l}}^N\right) \right\|_0$$

$$\le \sup_{N \in \mathbb{N}_{\mathbb{C}_l}} \mathbb{E}\left(\hat{m}_u\left(\varphi_{\Delta^n_{\mathbb{C}_l}}^N, \varphi_{\Delta^\infty_{\mathbb{C}_l}}^N\right) 1_{\mathscr{K}_\varepsilon^T} \right) + \varepsilon$$

$$\le \sup_{N \in \mathbb{N}_{\mathbb{C}_l}} \mathbb{E}\left(\frac{1}{|F_{\mathbb{C}_l}^N|} \sum_{B \in F_{\mathbb{C}_l}^N} \sup_{h<T} \left\| {}^n X_B^N(h) - X_B^N(h) \right\| 1_{\mathscr{K}_\varepsilon^T} \right) + 2\varepsilon$$

$$\le \sup_{N \in \mathbb{N}_{\mathbb{C}_l}} \mathbb{E}\left(\left(\frac{1}{|F_{\mathbb{C}_l}^N|} \sum_{B \in F_{\mathbb{C}_l}^N} \sup_{h<T} \left\| {}^n X_B^N(h) - X_B^N(h) \right\|^2 1_{\mathscr{K}_\varepsilon^T} \right)^{1/2} \right) + 2\varepsilon$$

(by the Cauchy–Schwarz inequality)

$$\le \sup_{N \in \mathbb{N}_{\mathbb{C}_l}} \left(\mathbb{E}\left(\frac{1}{|F_{\mathbb{C}_l}^N|} \sum_{B \in F_{\mathbb{C}_l}^N} \sup_{h<T} \left\| {}^n X_B^N(h) - X_B^N(h) \right\|^2 1_{\mathscr{K}_\varepsilon^T} \right) \right)^{1/2} + 2\varepsilon$$

(by Jensen's inequality)

$$\le \sup_{N \in \mathbb{N}_{\mathbb{C}_l}} \left(\mathbb{E}\left(\frac{1}{|F_{\mathbb{C}_l}^N|} \sum_{j \in I_{\mathbb{C}_l}^N} \sum_{i \in \mathbb{C}_l} \sup_{h<T} \left\| {}^n X_{i+j}^N(h) - X_{i+j}^N(h) \right\|^2 1_{\mathscr{K}_\varepsilon^T} \right) \right)^{1/2} + 2\varepsilon$$

$$\le \sup_{N \in \mathbb{N}_{\mathbb{C}_l}} \sqrt{\frac{|\mathbb{C}_l|\,|\mathbb{C}_N|}{|F_{\mathbb{C}_l}^N|}} \left(\mathbb{E}\left(\frac{1}{|\mathbb{C}_N|} \sum_{i \in \mathbb{C}_N} \sup_{h<T} \left\| {}^n X_i^N(h) - X_i^N(h) \right\|^2 1_{\mathscr{K}_\varepsilon^T} \right) \right)^{1/2} + 2\varepsilon.$$

Recalling that $F_{\mathbb{C}_l}^N = \mathbb{C}_{N-l}$ for every $N \in \mathbb{N}_{\mathbb{C}_l}$, it follows that $|\mathbb{C}_N|/|F_{\mathbb{C}_l}^N| \to 1$, for $N \to \infty$. As such,

$$\sup_{N \in \mathbb{N}_{\mathbb{C}_l}} \sqrt{\frac{|\mathbb{C}_l|\,|\mathbb{C}_N|}{|F_{\mathbb{C}_l}^N|}} = C < \infty.$$

Thus,

$$\sup_{N \in \mathbb{N}_{\mathbb{C}_l}} \left\| \hat{m}_u\left(\varphi_{\Delta^n_{\mathbb{C}_l}}^N, \varphi_{\Delta^\infty_{\mathbb{C}_l}}^N\right) \right\|_0$$

$$\le C \sup_{N \in \mathbb{N}_{\mathbb{C}_l}} \left(\mathbb{E}\left(\frac{1}{|\mathbb{C}_N|} \sum_{i \in \mathbb{C}_N} \sup_{h<T} \left\| {}^n X_i^N(h) - X_i^N(h) \right\|^2 1_{\mathscr{K}_\varepsilon^T} \right) \right)^{1/2} + 2\varepsilon.$$

Using the monotonicity and continuity of the square root function and Lemma 3.9, the first term above converges to zero for $n \to \infty$. Since ε was arbitrarily chosen, this completes the proof of (i).

(ii) Using the fact that convergence in probability implies convergence in distribution, this follows immediately from Corollary 3.10. □

Now we can complete the last step of our program.

THEOREM 3.12. *The array* $\Delta^\infty = (X^N_{\mathbb{C}_N})^{N \in \mathbb{N}}$ *is point convergent for clusters with cluster limit* $\mathscr{L}\{(X_j)_{j \in \mathscr{S}}\}$.

PROOF. We will show that Δ^∞ is point convergent for $\mathbb{C}_l$ clusters, for every $l \in \mathbb{N}$. Let $l \in \mathbb{N}$ and set, as in Lemma 3.9, $u = d|\mathbb{C}_l|$. Let $\varepsilon > 0$ be arbitrarily chosen. By Lemma 3.9 there exists an $n_\varepsilon \in \mathbb{N}$ such that, for every $n \geq n_\varepsilon$,

$$\sup_{N \in \mathbb{N}_{\mathbb{C}_l}} \left\| \hat{m}_u\left(\varphi^N_{\Delta^n_{\mathbb{C}_l}}, \varphi^N_{\Delta^\infty_{\mathbb{C}_l}}\right)\right\|_0 + \hat{m}_u\left(\mathscr{L}\left\{(X^n_j)_{j \in \mathbb{C}_l}\right\}, \mathscr{L}\left\{(X_j)_{j \in \mathbb{C}_l}\right\}\right) \leq \varepsilon.$$

Further, by Corollary 3.5, $\Delta^n_{\mathbb{C}_l}$ is point convergent with limit $\mathscr{L}\{(X^n_j)_{j \in \mathbb{C}_l}\}$, for every $n \in \mathbb{N}$. Using Lemma 1.2(iii), it follows that

$$\left\| \hat{m}_u\left(\varphi^N_{\Delta^n_{\mathbb{C}_l}}, \mathscr{L}\left\{(X^n_i)_{i \in \mathbb{C}_l}\right\}\right)\right\|_0 \to 0 \quad \text{for } N \to \infty.$$

Therefore,

$$\begin{aligned}
&\left\| \hat{m}_u\left(\varphi^N_{\Delta^\infty_{\mathbb{C}_l}}, \mathscr{L}\left\{(X_i)_{i \in \mathbb{C}_l}\right\}\right)\right\|_0 \\
&\quad \leq \sup_{N \in \mathbb{N}_{\mathbb{C}_l}} \left\| \hat{m}_u\left(\varphi^N_{\Delta^{n_\varepsilon}_{\mathbb{C}_l}}, \varphi^N_{\Delta^\infty_{\mathbb{C}_l}}\right)\right\|_0 + \hat{m}_u\left(\mathscr{L}\left\{(X^{n_\varepsilon}_j)_{j \in \mathbb{C}_l}\right\}, \mathscr{L}\left\{(X_j)_{j \in \mathbb{C}_l}\right\}\right) \\
&\qquad + \left\| \hat{m}_u\left(\varphi^N_{\Delta^{n_\varepsilon}_{\mathbb{C}_l}}, \mathscr{L}\left\{(X^{n_\varepsilon}_j)_{j \in \mathbb{C}_l}\right\}\right)\right\|_0 \\
&\quad \leq \varepsilon + \left\| \hat{m}_u\left(\varphi^N_{\Delta^{n_\varepsilon}_{\mathbb{C}_l}}, \mathscr{L}\left\{(X^{n_\varepsilon}_j)_{j \in \mathbb{C}_l}\right\}\right)\right\|_0 \to \varepsilon \quad \text{for } N \to \infty.
\end{aligned}$$

A second application of Lemma 1.2(iii) delivers the promised point convergence of $\Delta_{\mathbb{C}_l}$. Used together with Lemma 2.3 this completes the proof. □

4. Model of a financial market. As motivation for the results presented above, we provide the following example describing a currency market. In contrast to many financial markets, such as stock exchanges, in whch trading takes place at a single location, currency trading is organized in a decentralized fashion among a large group of traders who communicate and perform their transactions either electronically or by telephone.

Typically each of the individual agents will only communicate and trade with a small subgroup (reference group) of all potential trading partners, due to the limits of his information processing ability and the organizational aspects of the transaction process. A typical assumption on the speculative behavior of traders is that they adapt their individual demand based on deviations between the current rates and estimated long-term equilibrium rates [see Beja and Goldman (1980)]. The classical view is that this occurs instantaneously based on unpredictable information "shocks" observed simultaneously by all traders.

A more realistic view is that the updating of each agent's estimate of long-term equilibria is based to a large degree on weighted averages of the expectations signaled by a reference group of traders and other individuals that serve as information sources, for example, traders and analysts within the same financial institution dealing in other assets, and that the updating occurs at a rather slow rate to filter out the "noise" in the communication channels.

This can be formalized in the following fashion: Let $N \in \mathbb{N}$. We describe the activities of each agent i, $i \in C_N$, in a currency market with $|C_N|$ agents by the stochastic process X_i. For the moment we drop the indexing with N to reduce notation. Here, $X_i = (P_i, S_i, R_i, D_i, PC_i)$, where P_i denotes a vector of individual parameters such as risk aversion and weighting factors for the signals received from other agents, S_i the signals agent i sends to other agents in his reference group, R_i the filtered noisy signals received from other agents, D_i bid and offer rates for the currencies being traded and PC_i the agent's portfolio of currencies. Further, we assume that the interaction radius is bounded to the immediate neighbors on the lattice and (by our usual convention) set $X_i = 0$ for $i \notin \mathbb{C}_N$.

We assume that the agents estimate the long-term equilibrium exchange rates by filtering and then forming weighted averages of the noisy signals received from other agents as to their expectations of the logarithmic equilibrium rates. Denote by $REC_i(t)$ signals received by agent $i \in C_N$ from other agents as to their expectations. Let $c \in \mathbb{N}$ be the number of currencies being traded. Then the signals received from agent $i + j$, $j \in \mathbb{C}_1$, are $\mathbb{R}^{c \times c}$-valued processes such that, for $k, l \in \{1, \dots, c\}$,

$$rec_{i+j,i}^{l,k}(t) = s_{i+j,i}^{l,k}(t) + \text{noise} = s_{i+j,i}^{l,k}(t) + db_{j,i}^{l,k}(t),$$

where $b_{j,i}^{l,k}$ is a Brownian motion with variance $\sigma_{j,i}^{l,k}$. Here we have used a notational convention that we will follow throughout. Subscripts give the index of agents involved in trading or exchange of information, where the first ($i + j$, above) denotes the source and the second the receiver. There is a slight abuse of notation here, since the signals are actually indexed only by the agent's index $(i + j)$ and the relevant coordinate within $\mathbb{C}_1$ (i.e., $i - j$ rather than i). This convention is useful for expository purposes though, and is removed in the strictly formal analysis following equation (4.3) of this section. Superscripts refer to the relevant currencies. Further, we denote with uppercase letters $(REC_i, S_i, \dots)$ the entire vector, while lowercase letters $(rec_{i+j,i}^{l,k}, s_{i+j,i}^{l,k}, \dots)$ refer to the elements in the vector. Finally, in the following we denote by various lowercase Greek letters $(\alpha_{j,i}^{l,k}, \lambda_i, \dots)$ elements of the vector of individual parameters.

Denoting with $\hat{d}_i^{l,k}(t)$ the direct (not logarithmic) estimate of the long-term equilibrium rate for every $t \in [0, \infty)$, then

$$\hat{d}_i^{l,k}(t) = \exp\left(\sum_{j \in \mathbb{C}_1} \alpha_{j,i}^{l,k} r(t)_{i+j,i}^{l,k} \right) \wedge U.$$

Here, $(\alpha_{j,i}^{l,k})_{j \in \mathbb{C}_1}$ is a vector of positive weighting factors for the signals from the other agents and, for every $i \in \mathbb{C}_N$ and $j \in \mathbb{C}_1$,

$$\begin{aligned} r_{i+j,i}^{l,k}(t) &= \int_0^t \lambda_i \big(s_{i+j,i}^{l,k}(s) - r_{i+j,i}^{l,k}(s) \big)\, ds \\ &= \int_0^t \lambda_i \big(s_{i+j,i}^{l,k}(s) - r_{i+j,i}^{l,k}(s) \big)\, ds - \int_0^t \lambda_i \, db_{j,i}^{l,k}(s) \\ &= \int_0^t \exp((s-t)\lambda_i) s_{i,i+j}^{l,k}(s)\, ds + \int_0^t \exp((s-t)\lambda_i)\, db_{j,i}^{l,k}(t) \end{aligned}$$

is the filtered estimate of the true signal found by exponentially smoothing the noisy signal received by agent i from agent $i+j$, with smoothing parameter $\lambda_i > 0$. Finally, using LP_b to denote the family of monotonically increasing bounded Lipschitz continuous functions $\mathbb{R} \to \mathbb{R}$, the function $\exp(\cdot) \wedge U \in LP_b$ transforms the estimates of the logarithmic rates into direct rates, and U is a constant representing an upper bound that the agents place on the estimates.

It is assumed that the rate at which a trader offers to exchange one currency for another is based on the difference between the estimate of the long-term equilibrium rate and the current rate offered. So that the rates are directly comparable, it is assumed that for $k < l$ the rate is given in units of currency k. As such, at time t, an agent will offer one unit of currency l in exchange for $d_i^{l,k}(t)$ units of currency k, while bidding $d_i^{k,l}(t)$ units of k for one unit of l, and the difference $d_i^{l,k}(t) - d_i^{k,l}(t)$ (spread) will be positive for agents that are not attempting financial suicide. The dynamics of this adaptation are given by

$$\frac{dd_i^{l,k}}{dt} = \phi\Big(\eta_i^{l,k} \big(\hat{d}_i^{l,k}(t) - d_i^{l,k}(t) \big) \Big).$$

The function $\phi \in LP_b$ is such that $\phi(0) = 0$ and the individual weighting factor $\eta_i^{l,k}$ is greather than or equal to 0. Thus, the agent adapts his rate toward the (currently estimated) long-term equilibrum rate at a speed which is controlled by weighting factor $\eta_i^{l,k}$ which describes the agent's confidence in his estimate. Furthermore, assuming that the starting value $d_i^{l,k}(0)$ is in $[0, U]$, then as a consequence of the fact that (by construction) $\hat{d}_i^{l,k}(t) \in [0, U]$ for every $t \in [0, \infty)$, it follows that $d_i^{l,k}$ cannot leave this interval either.

The portfolio dynamics of agent i are then described by the sum of the exchanges between i and the traders in his reference group. Trading can take place whenever the offer rate of an agent goes below the bid rate of one of his potential trading partners. For $l = 1, \dots, c$, denote by $pc_i^l(t)$ the amount of currency l in agent i's portfolio at time $t \in [0, \infty)$. Then these exchanges occur in a continuous fashion between i and $i+j$, $j \in \mathbb{C}_1 \setminus \{0\}$ and the speed of the flow of currency l from i to $i+j$, $tr_{i,i+j}^{l,k}(t)$ in exchange for currency k,

$k < l$, is described by

$$(4.1) \quad \frac{d\,tr_{i,i+j}^{l,k}(t)}{dt} = \varrho\Big(\psi\big(\gamma_i^l pc_i^l(t)\big)u\Big(\beta_i^{l,k}\big(d_i^{l,k}(t) - d_{i+j}^{k,l}(t)\big)\Big) \times \psi\big(\gamma_{i+j}^k pc_{i+j}^k(t)\big)u\Big(-\beta_{i+j}^{k,l}\big(d_i^{l,k}(t) - d_{i+j}^{k,l}(t)\big)\Big)\Big),$$

where $\psi, u, \varrho \in LP_b$ are such that $\psi(x) = u(x) = 0$, for $x \le 0$, and ϱ is an approximation of the square root function on the interval $[0, \max_{x \in \mathbb{R}}\{\psi(x) \vee u(x)\}]$, with $\varrho(0) = 0$. This can be interpreted as follows. The product

$$\psi\big(\gamma_i^l pc_i^l(t)\big)u\Big(\beta_i^{l,k}\big(d_i^{l,k}(t) - d_{i+j}^{k,l}(t)\big)\Big)$$

gives agent i's propensity to trade based on the amount of currency l available and the size of the difference between his offer rate and his counterpart's bid rate, weighted with the individual parameters γ_i^l, $\alpha_i^{l,k} \ge 0$ which captures the effects of the agent's utility function and his attitude towards risk, particularly that of reducing his portfolio position in currency l to zero [see Machina (1987)]. It is then assumed that a consensus on the trading rate is reached by building (an approximation of) the geometric mean between the two agents' propensities. The conditions on ψ, ϱ and u insure that trading only occurs when both agents have strictly positive amounts of the relevant currencies available (no short trading is permitted) and that the difference between bid and offer rates has the appropriate sign.

The speed of the return flow $tr_{i+j,i}^{k,l}$ is then given by

$$(4.2) \quad \frac{d\,tr_{i+j,i}^{k,l}}{dt} = \left(\frac{1}{2}\big(d_i^{l,k}(t) + d_{i+j}^{k,l}(t)\big)\right)\frac{d\,tr_{i,i+j}^{l,k}}{dt},$$

where it is assumed that a consensus on the return rate is reached by building the arithmetic mean of the bid and offer rates of the agents involved. The changes in agent i's portfolio position in currency l over time are then given by

$$(4.3) \quad \frac{d\,pc_i^l(t)}{dt} = \sum_{j \in \mathbb{C}_1 \setminus \{0\}} \left(\sum_{k \in \{1,\dots,m\} \setminus \{l\}} \frac{d\,tr_{i+j,i}^{l,k}}{dt} - \frac{d\,tr_{i,i+j}^{l,k}(t)}{dt} \right).$$

Write $X_i = ({}^1X_i, {}^2X_i, {}^3X_i, {}^4X_i, {}^5X_i) = (P_i, S_i, R_i, D_i, PC_i)$ and $X_i(0) = ({}^1K_i, {}^2K_i, {}^3K_i, {}^4K_i, {}^5K_i)$. Assuming that the signal process S_i is a square-integrable semimartingale, we can write

$$ {}^1X_i(t) = {}^1K_i, \qquad {}^1Z_i, {}^1g = 0,$$

$$ {}^2X_i(t) = {}^2K_i + \int_0^t dS_i(s), \qquad {}^2Z_i = S_i, {}^2g = 1.$$

Further, for $j \in \mathbb{C}_1 \setminus \{0\}$ and $l, k = 1, \dots, m$,

$$
{}^3x_{j,i}^{l,k}(t) = {}^3k_{j,i}^{l,k} + \int_0^t \underbrace{\lambda_i\left({}^2x_{i+j,i-j}^{l,k}(s) - {}^3x_{j,i}^{l,k}(s)\right)}_{{}^3_1g_j^{l,k}(\cdots)} ds
$$

$$
+ \int_0^t \underbrace{\lambda_i}_{{}^2_3g(\cdots)} db_{j,i}^{l,k}(s), \qquad {}^3z_{j,i}^{l,k}(t) = \left(t, b_{j,i}^{l,k}(t)\right),
$$

where we note that λ_i is an element of the vector ${}^1X_i(t)$. Finally,

$$
{}^4x_i^{l,k}(t) = {}^4k_i^{l,k} + \int_0^t {}^4g\left(\underbrace{\left(\left(\alpha_{\mathbb{C}_1,i}^{l,k}\right), \eta_i^{l,k}\right)}_{\text{part of } {}^1X_i(t)}, {}^3x_{i+\mathbb{C}_1}^{l,k}(t), {}^4x_i^{l,k}(t)\right) ds
$$

and, for $l = 1, \dots, c$,

$$
{}^5x_i^l(t) = {}^5k_i^l + \int_0^t {}^5g^l\left(\underbrace{\left(\gamma_i^l, \gamma_{i+\mathbb{C}_1}^k, \beta_i^{l,k}, \beta_{i+\mathbb{C}_1}^{k,l}\right)}_{\text{part of } {}^1X_{i+\mathbb{C}_1}(t)}, {}^4X_{i+\mathbb{C}_1}(t), {}^5X_{i+\mathbb{C}_1}(t)\right) ds,
$$

where ${}^4z_{i+j,i}^{l,k}(t) = {}^5z_{i+j,i}^{l,k}(t) = t$ and the functions ${}^4g$ and ${}^5g^{l,k}$ can be derived using equations (4.1), (4.2) and (4.3).

Now we consider a sequence of such markets of ever increasing size. Assume that, for every $N \in \mathbb{N}$, a vector of starting values and driving semimartingales $((K_i^N, Z_i^N))_{i \in \mathbb{C}_N}$ of the type described above is given. Using the function g defined above, this induces for every $N \in \mathbb{N}$ a vector of processes $(X_i^N)_{i \in \mathbb{C}_N}$ describing the activities of the agents in the market. Assume that there exists a common bound $\mathbb{K} > 0$ on all of the elements in the array of starting values $(K_{\mathbb{C}_N}^N)^{N \in \mathbb{N}}$ and that the array $((K_i^N, Z_i^N))_{i \in \mathbb{C}_N}^{N \in \mathbb{N}}$ is point convergent for clusters with cluster limit $\mathscr{L}\{(K_i, Z_i)_{i \in \mathscr{S}}\}$ such that the remaining provisions of Condition 3.1 (CP) are satisfied.

If we assume that summable sequences $(\gamma_i)_{i \in \mathscr{S}}$ and $(\tilde{\gamma}_i)_{i \in \mathscr{S}}$ used to define the spaces $\mathscr{X}$ and $\mathscr{Z}$ are such that $\tilde{\gamma}_i \le \gamma^2$, for every $i \in \mathscr{S}$, and that $\gamma_i^2/\gamma_{i+j}^2$ is uniformly bounded, for every $i \in \mathscr{S}$ and $j \in \mathbb{C}_1$, then it is tedious but straightforward to show that the function g satisfies the Lipschitz condition [Condition 3.1 (CL)]. This is accomplished using the fact that the sum and superposition of Lipschitz continuous functions is Lipschitz continuous, as is the product of bounded Lipschitz continuous functions. Further, since not only the starting values but also the bid and offer rates are uniformly bounded, one may assume that these values are evaluated using $x \to x \vee (-\hat{\mathbb{K}} \wedge \hat{\mathbb{K}})$, for some $\hat{\mathbb{K}} > 0$, wherever these values appear in the equations above.

Denoting by $\Delta = (X_{\mathbb{C}_N}^N)^{N \in \mathbb{N}}$ the array of processes describing the agents' activities, we have by Theorem 3.12 that Δ is point convergent for clusters with cluster limit $\mu = \mathscr{L}\{(X_i)_{i \in \mathscr{S}}\}$, where $(X_i)_{i \in \mathscr{S}}$ is the solution to the infinite-dimensional version of the equations given above, induced by the function g and the cluster limit of the array of starting values and driving semimartingales.

There are a number of interesting facets to this result from an economic point of view. First of all, the dynamics of aggregate (macroeconomic) variables such as average bid rates will be essentially deterministic if the market is large enough. Second, these dynamics cannot be characterized by a model utilizing a "typical agent" approach commonly used in the microeconomic derivation of macroeconomic models [see e.g., Ramser (1988)]. Rather, one must consider entire clusters of typical agents whose (average) activities are described by the coordinate distributions of the cluster limit μ.

Further, although the decisions of the agents are influenced by information shocks from their reference group, the aggregate information process (given by the cluster limit of the signals transmitted) is deterministic as well, but can only be characterized by a complex distribution of information that will be unavailable to an individual agent. The individual information shocks will only have a small influence on the total market aggregates due to the dampening effects of filtering and the negligible (in the limit) influence of individual agents.

Another interesting point is that the structure of the model contains a large number of nonlinearities caused by individual risk aversion and the limits of the traders' portfolios, which will not "aggregate out" as is often assumed in the construction of linear econometric models commonly used in the analysis and forecasting of economic time series [see Phillips (1988)]. Finally, all of the nonlinear functions used in g are bounded and monotonically increasing, producing a structure quite similar to that found in certain types of artificial neural network models [specifically, higher-order Hopfield networks; see Schürmann (1989) and Giles, Griffin and Maxwell (1988)]. As such, some of the formal analysis carried out for these network models with regard to stability, number of attractors, chaotic behavior and so on may be relevant for the model presented above.

In conclusion we note that a market of the type described above will not be "efficient" in the sense commonly used in the theory of financial markets (i.e., that the logarithmic aggregate price process is a martingale), but will indeed be very complicated. As such, an identification of the system dynamics based on a time series of market aggregates may still be an intractable problem [see McCurdy and Morgan (1988) or Meese and Rogoff (1983)].

REFERENCES

AOKI, M. (1980). Dynamics and control of systems composed of a large number of subsystems. In *Dynamic Optimization and Mathematical Economics* (P. T. Liu, ed.). Plenum, New York.

BABOVSKY, H. (1994). A convergence proof for Nanbu's Boltzmann simulation scheme. *J. Méc. Théor. Appl.* To appear.

BEJA, A. and GOLDMAN, M. B. (1980). On the dynamic behavior of prices in disequilibrium. *J. Finance* **35** 235–248.

BLACKWELL, D. and GIRSHICK, M. A. (1979). *Theory of Games and Statistical Decisions*. Dover, New York.

DAWSON, D. A. (1983). Critical dynamics and fluctuations for a mean field model of cooperative behavior. *J. Statist. Phys.* **31** 29–85.

DEUSCHEL, J. D. (1988). Central limit theorem for an infinite lattice system of interacting diffusion processes. *Ann. Probab.* **16** 700–716.

ELLIS, R. (1985). *Entropy, Large Deviations and Statistical Mechanics*. Springer, Berlin.

ENGQUIST, B. and HOU, T. (1989). Particle method approximation of oscillatory solutions to hyperbolic differential equations. *SIAM J. Numer. Anal.* **26** 289–319.

FELDMAN, M. and GILLES, C. (1985). An expository note on individual risk without aggregate uncertainty. *J. Econom. Theory* **35** 26–32.

FINNOFF, W. (1989). Derivation of deterministic macroeconomic models as limits of stochastic microeconomic models. Ifo–Ökonometrie Arbeitspapier7, Ifo Institut für Wirtschaftsforschung, Munich.

FINNOFF, W. (1993). Law of large numbers for a general system of stochastic differential equations with global interaction. *Stochastic Process. Appl.* **46** 153–182.

FÖLLMER, H. (1974). Random economies with many interacting agents. *J. Math. Econom.* **3** 50–62.

GILES, G. L., GRIFFIN, R. D. and MAXWELL, T. (1988). Encoding geometric invariances in higher order neural networks. *Amer. Inst. Phys. Conf. Proc.* **58** 301–314.

GRANDMONT, J. M. and MALGRANGE, P. (1986). Nonlinear economic dynamics. *J. Econom. Theory* **40** 3–12.

GRAHAM, C. (1988). The martingale problem with sticky reflection conditions and a system of particles interacting at the boundary. *Ann. Inst. H. Poincaré* **24** 45–72.

GREEN, E. (1984). Continuum and finite-player noncooperative models of competition. *Econometrica* **52** 50–62.

GREEN, E. (1989). Individual-level randomness in a nonatomic population. Working Paper 227, Dept. Economics, Univ. Pittsburgh.

GRIFFITHS, D. F. and MITCHELL, A. R. (1988). Stable periodic bifurcations of an explicit discretization of a nonlinear partial differential equation in reaction diffusion. *IMA J. Numer. Anal.* **8** 435–454.

HOLLEY, R. and STROOCK, D. (1981). Diffusions on the infinite dimensional torus. *J. Funct. Anal.* **42** 29–63.

JACOD, J. and SHIRYAEV, A. N. (1987). *Limit Theorems for Stochastic Processes*. Springer, Berlin.

JOVANOVIC, B. (1987). Micro shocks and aggregate risk. *Quart. J. Econom.* 395–409.

JOVANOVIC, B. and ROSENTHAL, R. (1988). Anonymous sequential games. *J. Math. Econom.* **17** 77–87.

JUDD, K. (1985). The law of large numbers with a continuum of iid random variables. *J. Econom. Theory* **35** 19–25.

LÉCOT, C. (1994). A direct simulation Monte Carlo scheme and uniformly distributed sequences for solving the Boltzmann equation. *Computing*. To appear.

LEHA, G. and RITTER, G. (1984). On diffusion processes and their semigroups in Hilbert spaces with an application to interacting stochastic systems. *Ann. Probab.* **12** 1077–1112.

MACHINA, M. J. (1987). Choice under uncertainty: Problems solved and unsolved. *Econom. Perspectives* **1** 121–154.

MCCURDY, T. H. and MORGAN, I. G. (1988). Testing the martingale hypothesis in the Deutsch Mark futures with models specifying the form of heteroskedacity. *J. Appl. Econometrics* **3** 187–203.

MCFADDEN, D. (1981). Econometric models of probabilistic choice. In *Structural Analysis of Discrete Data with Econometric Applications* (C. Manski and D. McFadden, eds.). MIT Press.

MEESE, R. A. and ROGOFF, K. (1983). Empirical exchange rate models of the seventies: Do they fit out of sample? *J. Internat. Econom.* **14** 3–24.

MERTON, R. C. (1971). Optimum consumption and portfolio rules in a continuous time model. *J. Econom. Theory* **3** 373–413.

METIVIER, M. (1982). *Semimartingales*. de Gruyter, Berlin.

METIVIER, M. and PELLAUMAIL, J. (1980). *Stochastic Integration*. Academic, London.

NAPPO, G. and ORLANDI, E. (1988). Limit laws for a coagulation model of interacting random particles. *Ann. Inst. H. Poincaré* **24** 319–344.

PHILLIPS, A. W. (1988). Reflections on econometric methodology. Cowles Foundation Discussion Paper 893, Yale Univ.

POLIS, M. F. (1983). The distributed parameter identification problem: A survey of recent results. In *Control of Distributed Parameter Systems* (J. P. Babary and L. LeLetty, eds.) 45–58. Pergamon, Oxford.

PUDNEY, S. (1988). *Modelling Individual Choice*. Cambridge Univ. Press.

RAMSER, H. J. (1988). *Neuere Beiträge zur Konjunkturtheorie: ein Überblick*. In *Ifo Studien* **29**. Ifo Institut für Wirtschaftsforschung, Munich.

ROTH, T. P. (1989). *The Current State of Consumer Theory*. Univ. Press of America, New York.

SCHIEF, A. (1986). Topologische Eigenshaften masswertiger Abbildungen und stochastische Anwendungen. Dissertation, Univ. München.

SCHÜRMANN, B. (1989). Stability and adaptation in artificial neural systems. *Phys. Rev. A* **40** 2681–2692.

SEIDMAN, T. (1988). A new algorithmic model for the transient semiconductor problem. In *Mathematical Modelling and Simulation of Electrical Circuits and Semiconductor Devices* (R. Bank, R. Bulirsch and K. Merten, eds.) 141–152. Birkhäuser, Berlin.

SHIGA, T. and SHIMIZU, A. (1980). Infinite dimensional stochastic differential equations and applications. *J. Kyoto Univ.* **20** 213–242.

SPOHN, H. (1980). Kinetic equations from Hamiltonian dynamics: Markovian limits. *Rev. Modern Phys.* **53** 569–615.

SZNITMAN, A. S. (1984a). Equations de type Boltzmann spaticielement homogenés. *Z. Wahrsch. Verw. Gebiete* **66** 559–593.

SZNITMAN, A. S. (1984b). Nonlinear reflecting diffusion processes and the propagation of chaos and fluctuations associated. *J. Funct. Anal.* **56** 311–336.

SZNITMAN, A. S. (1986). A propagation of chaos result for Burger's equation. *Probab. Theory Related Fields* **71** 581–613.

TOPSØE, F. (1970). *Topology and Measure*. *Lecture Notes in Math.* **133**. Springer, Berlin.

TZAFESTAS, S. G. and STAVROULAKIS, P. (1983). Recent advances in the study of distributed parameter systems. *J. Franklin Inst.* **315** 285–305.

SIEMENS AG
CORPORATE RESEARCH AND DEVELOPMENT
OTTO-HAHN-RING 6
D-8000 MUNICH 83
GERMANY