The Annals of Applied Probability
1994, Vol. 4, No. 2, 287–346

AN OPTIMAL CONTROL FORMULATION AND RELATED NUMERICAL METHODS FOR A PROBLEM IN SHAPE RECONSTRUCTION

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The main problem considered in this paper is the construction of numerical methods and proofs of their convergence for the problem of "shape from shading." In the first part of the paper, it is assumed that the height function that describes the surface to be reconstructed is known at all local minima (or maxima). These points are a subset of the singular points, which are the brightest points in the image. A pair of optimal control problems are defined that provide representations for the height function. Numerical schemes based on these representations are then constructed. While both schemes lead to the same approximation, one yields a more efficient algorithm, while the other is more convenient in the convergence analysis. The proof of convergence is based on a representation of the approximation to the height as a functional of a controlled Markov chain. In a later part of the paper the assumption that the height must be known at all local minima (or maxima) is dropped. An extension of the algorithm is described that is capable of reconstruction without this information. Numerical experiments for both algorithms on synthetic and real data are included.

1. Introduction. Shape from shading has been a central problem in computer vision for the entire history of the field. Initial work dates back to the 1960s with the mapping of the lunar surface from visual images [25]. In the early 1970s, Horn's work on this problem [8] pioneered the new quantitative focus on low level, precognitive vision, which has since dominated vision research. The problem has continued to be of interest over the years, and recently has been dramatic progress [1, 4, 10, 19–22, 27–30].

The problem of shape from shading is to reconstruct the three dimensional shape of a surface from the brightness or intensity variation in a black-and-white photographic image of the surface. If the illumination is mainly unidirectional and the surface is uniform and untextured, then the brightness at an image point provides information about the surface orientation imaged at that point. However, in general it does not fully specify the surface orien-

Received January 1993; revised May 1993.
[1]Research supported in part by NSF Grant DMS-91-15762.
[2]Research supported in part by NSF Grants IRI-91-13690 and CDA-89-22572 and by DARPA Grant DAAE07-91-C-R035 via TACOM.
AMS 1991 *subject classifications*. 35F30, 49N55, 60J10, 49L20, 93E20.
Key words and phrases. Shape-from-shading, Markov chain approximation, optimal control representation, shape reconstruction algorithms.

tation, even if the illumination and surface reflectance properties are completely known, but only constrains it. Consequently, the task of reconstructing the surface is difficult. It is for this reason that, for much of its history, shape from shading was believed to be an ill-posed problem. However, it has been argued more recently, and is reillustrated here, that the problem is actually well posed under a wide range of conditions [19, 20].

Shape from shading is something that humans do well [17]. It is clearly an essential ingredient in visual interpretation for the many contexts where other cues for shape reconstruction are lacking, for example, for still photographs, or for distant objects where stereo or motion parallax are not effective. It is an important aid in reconstructing shape corresponding to regions of smoothly varying intensity in images, where the absence of distinguishable features makes stereo or motion reconstruction difficult. Finally, the problem in its idealized form is precise and well-defined mathematically: It amounts to finding the "proper" solution to a particular nonlinear first order partial differential equation.

We consider here the idealized problem. Thus we make the standard assumptions that the surface is matte, rather than mirrorlike, and reflects light evenly in all directions ("Lambertian" reflectance), that the illuminating light is from a single known direction and that the surface is distant from the camera ("orthographic" projection). Moreover, we make the standard but stronger assumptions that the surface is uniform and untextured, that there are no cast shadows and, finally, that all portions of the surface are visible (no "occlusion"). Although there has been some work on the problem relaxing one or another of these assumptions [2, 7, 10, 16, 18, 23, 28], most work has focused on the problem in this form. Understanding this idealized problem is a prerequisite for attacking the larger problem of how shading, in combination with other information (e.g., from stereo), can be used for image understanding.

Under the stated assumptions, the intensity $I(\cdot)$ registered on the image plane at coordinates $x \equiv (x_1, x_2)$ is given by

$$I(x) = \langle \gamma, \hat{n}(x) \rangle, \tag{1.1}$$

where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is the fixed direction from which the light is coming and $\hat{n}(\cdot)$ is the surface normal at the corresponding surface point (x_1, x_2, z). Such an equation relating $I(\cdot)$ to the corresponding surface orientation is known as the image irradiance equation [12]. We will always assume here that the surface height $z(x_1, x_2)$ is continuously differentiable (though this is not essential). Then the surface normal is given in terms of $z(\cdot)$ by

$$\hat{n}(x) = \frac{(-z_x(x), 1)}{\left(1 + \|z_x(x)\|^2\right)^{1/2}}, \tag{1.2}$$

and the image irradiance equation is a first order nonlinear partial differential equation. In general, this equation can have multiple classical sense solutions. The problem of shape from shading is to reconstruct $z(\cdot)$ from the given data $I(\cdot)$.

We briefly summarize previous work on this problem. Most of this work is based on the assumption that the reconstruction problem is not well posed in any reasonable sense. In the early work of Horn [8], the characteristic strip method was used to give a numerical scheme for shape reconstruction. Horn later formulated the problem in the idealized form considered in most subsequent work [9]. So-called variational methods were introduced and studied in, for example, [2, 10, 11, 13, and 31]. Let the image irradiance equation be written in the form $H(x, z_x(x)) = 0$, and let $V(\cdot)$ be an approximation for the height function. In the variational approach, a functional of this approximation $\mathscr{F}(V) \equiv \int_{\mathscr{D}} H^2(x, V_x(x))\,dx$ is defined, where $\mathscr{D}$ is the domain of definition of the intensity function. Since $z(\cdot)$ obeys $H(x, z_x(x)) = 0$ it minimizes this functional. In practice, $V(\cdot)$ is approximated by a function $V^h(\cdot)$ defined on a discrete grid $\mathscr{D}^h$, and the functional is approximated by $\mathscr{F}^h(V^h) \equiv \Sigma_{\mathscr{D}^h} H(x, V_x^h(x))$, where V_x^h is a discrete approximation of the derivative V_x. It is assumed that $z^h(\cdot)$, defined to minimize $\mathscr{F}^h$, is a good approximation to $z(\cdot)$.

The variational approach attempts to locate $z^h(\cdot)$ by letting an initial estimate $V^h(\cdot)$ evolve in the steepest descent direction $-\partial\mathscr{F}^h(V^h)/\partial V^h$. This procedure guarantees convergence to a local minimum of $\mathscr{F}^h$. In practice, however, this often produces a spurious local minimum rather than the desired approximation $z^h(\cdot)$. To ameliorate this problem, a "regularization" term proportional to $\int_{\mathscr{D}} \Sigma_{ij \in \{1,2\}} V^2_{x_i, x_j}\,dx$ is included in $\mathscr{F}(V)$ and discretely approximated in $\mathscr{F}^h$ as before. Because it penalizes those functions $V(\cdot)$ with high curvature, this term destabilizes some of the spurious local minima of $\mathscr{F}^h$, which tend to have multiple regions of high "curvature." Unfortunately, this term also has the effect of perturbing the global minimum of $\mathscr{F}^h$ away from the best approximation to $z(\cdot)$. A potential solution to this problem was discussed in [10].

In a second approach [6, 16, 23], the surface is reconstructed separately in small image patches based on restrictive assumptions about the surface shape there—for instance, that the surface is spherical in each patch. The partial reconstructions are then patched together to give an overall reconstruction.

More recently, Pentland [24] presented an algorithm that replaces the shape from shading problem with a linearized approximation, which can then be solved by direct integration. However, boundary conditions are required to make the approximating linearized reconstruction unambiguous. In general, proofs of convergence are lacking in all of these approaches.

Finally, in [3, 19, 20, 29 and 30], the uniqueness of the solution $z(\cdot)$ was discussed, under the assumption that $z(\cdot)$ was at least C^2. The constraint on $z(\cdot)$ provided by a visible "occluding boundary," that is, the set of points on the surface where the derivative of the map from the surface to the image plane is singular was also considered in [19].

In this paper we view the problem of shape-from-shading in an entirely different way. The cornerstone to our approach is a representation for the height function $z(\cdot)$ as the infimal cost for a deterministic optimal control problem [21]. Convergent numerical methods can then be constructed on the basis of this representation. Work by other authors that is closest to ours is [27] and [28], which also make use of the connection to an optimal control problem in order to derive a numerical scheme. The main mathematical tool used in these papers is the theory of viscosity solutions of first order nonlinear partial differential equations. However, the types of assumptions made here on the data available for reconstruction differ from those in [27] and [28], as do the particular results, algorithms and methods of analysis.

We conclude this introduction by loosely summarizing the main results of the paper. In the idealized framework mentioned previously, the "brightest" points in the image have value $I(x) = 1$. Furthermore, these brightest points include all local maxima, minima and saddle points of the height $z(\cdot)$ (in the case of oblique light, this is with respect to a coordinate system in which the "vertical" direction is the direction from which the light arrives). It turns out that specification of the height function on either the set of local maxima or minima allows, via an optimal control formulation, a representation for $z(\cdot)$ in a region around these points. A more precise statement is as follows. Let an isolated local minimum point x of the surface be given, and define the "domain of attraction" of this point to be a set of points y such that any steepest descent trajectory [with respect to $z(\cdot)$] that starts at y terminates at x. One can then construct an optimal control problem such that the infimal cost for this control problem has $z(\cdot)$ as its value for all points in the domain of attraction. Knowledge of $z(\cdot)$ at a number of local minima allows reconstruction in the union of their domains of attraction. This indicates in a precise way what information is needed to reconstruct a given part of the surface. The optimal control problem itself is easily (formally) derived from the irradiance equation via a dynamic programming equation and convex duality. In fact, different ways of rewriting the irradiance equation give rise to different optimal control representations: two that are particularly convenient to work with are discussed in Sections 2.3 and 3. The proof that the representation is valid for the first control problem is given in Section 5.

In general, the data available for shape reconstruction are not the complete intensity function $I(\cdot)$, but rather a restriction $I^h(\cdot)$ of $I(\cdot)$ to a regular grid of points $\mathscr{D}^h$, where h is the spacing between points. Our schemes for computing approximations to $z(\cdot)$ are based on replacing the original optimal control problems by control problems defined on $\mathscr{D}^h$. It is clearly desirable to preserve as far as possible the characteristics of the original control problems. Thus we leave the cost structure essentially unchanged. However, we must approximate the original deterministic dynamical model by one whose state space is the grid $\mathscr{D}^h$. It turns out that the best model to use is a controlled Markov chain. The reason for this is discussed in Section 2. Thus the approximating control problem takes the form of a stochastic optimal control problem whose state space is the grid $\mathscr{D}^h$. By applying the principle of dynamic programming to this stochastic control problem, we obtain a nonlinear iteration that defines a monotonically nonincreasing sequence of functions defined on $\mathscr{D}^h$. The approximation to $z(\cdot)$ is defined as the limit of these functions, which is a fixed point of the iteration. This approximation scheme is an example of a widely applicable method of approximation due to Kushner and known as the Markov chain approximation method [14]. The basis for the proof of convergence is a representation of the approximation as a functional of a controlled Markov chain.

In Section 2, we define a deterministic optimal control problem. The fact that solving this control problem is equivalent to solving for the height function is the content of Theorem 2.1. We then build an algorithm for computing an approximation to the solution to the control problem, and state the convergence properties in Theorem 2.3. In Section 3 we introduce a second control problem and, following the procedure of Section 2, construct a second algorithm. We also show that the two algorithms yield the same approximation. Because the control problem of Section 3 has quadratic costs, it gives rise to an efficient algorithm that is simple to construct and implement. However, it is a more difficult problem to work with as far as the representation of the height function and the proof of convergence of the scheme are concerned. This is the reason for also considering the control problem of Section 2. Although the algorithm of Section 2 is not as computationally attractive as that of Section 3, proofs of the representation and convergence theorems are much easier. Thus the equivalence of the approximations is a key observation.

There are a number of nonstandard features associated to these control problems and numerical schemes that make the proofs much more difficult than usual. For example, the nonlinear iterations that define the approximations will, in general, have multiple fixed points, and we must be careful to pick out the "correct" fixed point. The nonuniqueness of fixed points to the approximating discrete equations, and also the nonuniqueness of solutions to the irradiance equation itself, are tied to the lack of a positive lower bound on the running cost in the control representation of the height function. By coincidence the running cost is nonzero at precisely the points where the orientation of the surface is specified, which play an extremely important role in our approach. In the "standard" theory for the type of control problem we deal with, a strictly positive lower bound on the running cost is assumed. Because of this one can produce an a priori upper bound on how long the process will be controlled before it is stopped. This has many technical advantages, including the fact that it "compactifies" in the time variable. In our problem there is a basic lack of compactness due to the fact that the running costs are not bounded away from zero and the control problem is of interest over a potentially unbounded time interval. Heuristically, without such a lower bound the controlled process can "hang around" the neighborhood of a zero cost point, with no incentive to move to places where the process is stopped and boundary data values are learned. As a result, this problem falls outside the category of problems to which the existing control and approximation theories apply.

In Section 4.1 we present some experimental results and further discussion of the algorithms described in the first part of the paper. For these algorithms it is assumed that the values of $z(\cdot)$ are given on either the set of local maxima or minima in order to obtain an approximation. In Section 4.2 we describe a method which eliminates the need for such data. Suppose that the extreme points of $z(\cdot)$ are isolated. As noted before, the optimal control representation tells us that the correct specification of the height at either a local maximum or minimum identifies the height at all points in the domain of attraction and by continuity in the closure of this domain as well. Suppose that the only datum available besides the intensity function is the location of one local minimum point. Under conditions stated in Section 5, there is always at least one local maximum on the boundary of the domain of attraction. An application of the algorithm of Section 3 yields an approximation to $z(\cdot)$, which, on the basis of results stated in Section 2, should be good in the closure of the domain of attraction. Given this approximation, a method is described for identifying from among all the remaining "brightest" points, one that is a local maximum on the boundary of the domain of attraction. Since an approximation to the height at this maximum point has already been generated, the algorithm can be applied again (although this time we would use the algorithm that is appropriate when data are given at the local maxima) to obtain an even larger region on which the reconstruction can be expected to be good. The procedure is then repeated until all possible extrema have been identified and their heights approximated. A scheme is also discussed for identifying a starting local minimum or maximum. Thus approximations can be constructed that are based only on the image intensity function. Besides a description of this method, we present experimental results obtained using it.

2. A representation and an algorithm for the shape from shading problem. In this section we give the first of two optimal control representations for the unknown height function in the shape from shading problem. The numerical procedures that are constructed later will be based on these control problems.

All the schemes we consider in this section and the next will take the following form. For a scalar h and set A, let $hA = \{ha\colon a \in A\}$. Given $h \in (0, \infty)$, let $\mathscr{D}^h = h\mathbb{Z}^2 \cap \mathscr{D}$, where $\mathscr{D}$ is some open bounded subset of $\mathbb{R}^2$. Assume that we are given as data for the reconstruction problem the restriction I^h of the intensity function I to $\mathscr{D}^h$. Then our goal is to produce an approximation to the restriction of $z(\cdot)$ to $\mathscr{D}^h$ that is based on I^h. The schemes for the approximations V^h will always be iterative: given an initial condition V_0^h (to be described), we define in a recursive fashion

$$V_{n+1}^h = F^h\left(V_n^h, I^h\right),$$

and then set $V^h = \lim_{n \to \infty} V_n^h$. In all cases, the iterates V_n^h will be monotonically nonincreasing and bounded from below. This implies V^h will always exist and be a fixed point of the iteration $V^h \to F^h(V^h, I^h)$. It will often be the case that the iteration itself has multiple fixed points. However, if the initial condition V_0^h is chosen sufficiently large, then the iteration will be shown to converge to the "correct" fixed point V^h, which gives the desired approximation. Also, though various forms of F^h will be suggested by the different control representations, it will be shown below that the limit V^h will be the same for any of the schemes we define.

As noted in the introduction, the reason we consider two control problems is that each representation gives rise to a different numerical procedure, and each scheme has some attractive features. Our reason for introducing a second control problem in the next section is that, due to its simple dynamics and cost structure, it results in an algorithm that is easy to construct and iterate. On the other hand, besides computational efficiency, we are also interested in demonstrating the convergence of the approximation $V^h(x) \to z(x)$. This turns out to be simpler using the control problem defined in this section, which, as already stated, also yields the same approximation $V^h(x)$. Our proof of convergence will be based on interpreting $V^h(x)$ as the minimal (or maximal, depending on the case) cost for a stochastic control problem whose dynamics "mimic" those of the controlled process. For technical reasons the proof turns out to be substantially simpler for the control problem that is considered in this section.

The remainder of this section is divided as follows. In Section 2.1 we formulate the problem. The next subsection presents an optimal control representation for the height function. Section 2.3 constructs an approximation scheme that is based on this representation and states the convergence theorem for the approximations.

2.1. *Problem formulation.* Recall that $\mathscr{D}$ describes the subset of $\mathbb{R}^2$ on which the image data are recorded and that $z(\cdot)$ is the height function to be recovered. We assume that the surface $S = \{(x_1, x_2, x_3): x_3 = z(x_1, x_2)\}$ is illuminated from the direction γ by a point light source that is infinitely far away, and that the reflected light is recorded in an imaging plane that is parallel to the plane $\{(x_1, x_2, x_3): x_3 = 0\}$. The vector $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is assumed to be a unit vector with $\gamma_3 > 0$. We take the reflected light as being characterized in terms of the deterministic intensity function $I(x)$, where x identifies the (x_1, x_2) coordinates of a point on the imaging plane. Under a number of additional assumptions that were given in the introduction (including the assumption that the surface is "Lambertian" [12]), the height function $z(\cdot)$ and the intensity function $I(\cdot)$ are related by the equations (1.1) and (1.2). Eliminating $\hat{n}$ from these equations yields

$$I(x) = \left\langle \gamma, \frac{\left(-z_{x_1}(x), -z_{x_2}(x), 1\right)}{\left(1 + \|z_x(x)\|^2\right)^{1/2}} \right\rangle \tag{2.1}$$

in regions where $z(\cdot)$ is continuously differentiable. Note that by the Cauchy–Schwarz inequality $I(x) \leq 1$ for all $x \in \mathscr{D}$. We will refer to the points in $\mathscr{D}$ where $I(x) = 1$ as the *singular points* of $I(\cdot)$ [or $z(\cdot)$]. The singular points play a key role because [as is evident from (1.1)] it is only at the singular points that the irradiance equation alone specifies the orientation of the surface S. Let $\mathscr{S}$ denote the set of all singular points. Owing to the significance of the singular points, it is best to study the reconstruction problem in the new coordinate system defined by $x_1' = x_1$, $x_2' = x_2$, and $x_3' = x_1\gamma_1 + x_2\gamma_2 + x_3\gamma_3$. Thus we define a new height function $f(x_1, x_2)$ that measures height along the light (rather than camera) direction:

$$f(x_1, x_2) = x_1\gamma_1 + x_2\gamma_2 + z(x_1, x_2)\gamma_3.$$

Of course a reconstruction of $f(\cdot)$ will also give a reconstruction of $z(\cdot)$. In terms of $f(\cdot)$, (2.1) becomes

$$I(x) = \frac{1 - \langle f_x(x), (\gamma_1, \gamma_2)\rangle}{\left(1 + \|f_x(x)\|^2 - 2\langle f_x(x), (\gamma_1, \gamma_2)\rangle\right)^{1/2}}. \tag{2.2}$$

The function $f(\cdot)$ is the natural object of study, in that its extreme points are contained in the singular points of $I(\cdot)$. Of particular importance is the fact that all local maximum and minimum points of $f(\cdot)$ in $\mathscr{D}$ are contained in $\mathscr{S}$.

We will consider only the case where $\gamma_1 = 0$. In practice, the case $\gamma_1 \neq 0$ is handled by a coordinate transformation, that is, by rotating and reinterpolating the data. It is also possible to develop algorithms and a convergence theory that apply directly to this case. However, it has been the authors' experience that the resulting algorithms are somewhat cumbersome and that the change of coordinates method is preferable.

Terminology. In the following development we will want to specify precisely what data are needed to reconstruct a given part of the surface. Because of this, we need to introduce some nonstandard terminology. We will usually assume that the set of singular points consists of a finite collection of connected sets and that the function f is constant on each connected subset (recall that $f_x = 0$ at a singular point). We will refer to such a connected subset $\mathscr{S}_C$ as a set of local minima, and also refer to all points in the subset as local minimum points, if there exists $\varepsilon > 0$ such that $d(x, \mathscr{S}_C) < \varepsilon$ and $x \in \mathscr{D}$ imply $f(x) \geq f(y)$ for $y \in \mathscr{S}_C$. An analogous definition is used for local maxima. A connected subset that is neither a set of local maxima nor local minima is called a set of saddle points.

2.2. *An optimal control representation.* We next describe an optimal control representation for $f(\cdot)$ that is based on knowing $I(x)$ for $x \in \mathscr{D}$ and $f(x)$ only on a subset of $\mathscr{S}$. A result analogous to the one stated in the following text is possible when the set of local minima is replaced by the set of local maxima. In order to state the representation theorem, we first give a few definitions and introduce some new notation. After this has been done we state the assumption that is needed for the representation to be valid. As a means of motivating these definitions and conditions, a heuristic proof of the representation theorem is provided after the statement of the assumption.

Equation (2.2) can be rewritten as

$$H(x, f_x(x)) = 0, \tag{2.3}$$

where

$$H(x,\alpha) = I(x)\left(1 + \|\alpha\|^2 - 2\alpha_2\gamma_2\right)^{1/2} + \alpha_2\gamma_2 - 1.$$

Note that $H(x, \alpha)$ is convex in α and that in writing this equation we have used the assumption $\gamma_1 = 0$. We define the function $L(x, \beta)$ to be the Legendre transform of $H(x, \alpha)$ in α:

$$L(x,\beta) = \sup_{\alpha\in\mathbb{R}^2}\left[-\langle\alpha,\beta\rangle - H(x,\alpha)\right]. \tag{2.4}$$

(Note that our sign convention results in a slight deviation from the usual definition of the Legendre transform.) Evaluating the supremum in (2.4) yields

$$L(x,\beta) = \gamma_3^2 - \gamma_2\beta_2 - \gamma_3\left(I(x)^2 - |\beta_1|^2 - |\beta_2+\gamma_2|^2\right)^{1/2} \tag{2.5}$$

if $|\beta_1|^2 + |\beta_2 + \gamma_2|^2 \le I^2(x)$ and ∞ if $|\beta_1|^2 + |\beta_2+\gamma_2|^2 > I^2(x)$. It is easily checked that $L(x, \beta) \ge 0$ and that $L(x, \beta) = 0$ if and only if $x \in \mathscr{S}$ and $\beta = 0$.

Define

$$\mathscr{U}(x) = \left\{(u_1,u_2)\colon |u_1|^2 + |u_2+\gamma_2|^2 \le I^2(x)\right\}.$$

Thus $\mathscr{U}(x)$ is the domain on which $L(x,\cdot)$ is finite. The functions $H(x,\cdot)$ and $L(x,\cdot)$ are strictly convex on $\mathbb{R}^2$ and $\mathscr{U}(x)$, respectively. By convex duality,

$$H(x,\alpha) = \sup_{\beta\in\mathscr{U}(x)}\left[-\langle\alpha,\beta\rangle - L(x,\beta)\right], \tag{2.6}$$

and for each $\alpha \in \mathbb{R}^2$, there exists a unique vector $u(x, \alpha) \in \mathscr{U}(x)$ such that

$$H(x,\alpha) = -\langle\alpha, u(x,\alpha)\rangle - L(x,u(x,\alpha)).$$

Define $\bar{u}(x)$ for $x \in \mathscr{D}$ by

$$0 = H(x, f_x(x)) = -\langle f_x(x), \bar{u}(x)\rangle - L(x,\bar{u}(x)). \tag{2.7}$$

Equation (2.4) implies that $\bar{u}(x)$ is given by

$$\bar{u}(x) = -\nabla_\alpha H(x,\alpha)|_{f_x(x)}.$$

If (as we assume later) $f_x(x)$ is continuous, then the continuity of $\nabla_\alpha H(x, \alpha)$ implies $\bar{u}(x)$ is continuous on $\mathscr{D}$. The function $\bar{u}(x)$ plays an important role in determining exactly what data are needed to identify f in any given subset of $\mathscr{D}$. In the special case of vertical light [$\gamma = (0,0,1)$ and $f = z$], $\bar{u}(x)$ is a positive multiple of the steepest descent direction $-z_x(x)$ on the surface described by the function z. In the general case, $\bar{u}(x)$ is proportional to the projection in the (x_1, x_2) plane of the steepest descent direction on the surface, where "steepest descent" is defined with respect to the light direction γ, rather than the vertical direction $(0, 0, 1)$.

We are now ready to give the assumption used in the representation theorem. We say that a set $A \subset \mathbb{R}^2$ is smoothly connected if given any two points x and y in A, there is an absolutely continuous path ϕ: $[0,1] \to A$ such that $\phi(0) = x$, $\phi(1) = y$.

ASSUMPTION 2.1. Assume that $\mathscr{S}$ consists of a finite collection of disjoint, compact, smoothly connected sets and that $f_x(\cdot)$ is continuous on the closure of $\mathscr{D}$. Let $\mathscr{G} \subset \mathscr{D}$ be a compact set and assume $\mathscr{G}$ is of the form $\mathscr{G} = \bigcap_{j=1}^{J} \mathscr{G}_j$, $J < \infty$, where each $\mathscr{G}_j$ has a continuously differentiable boundary. Let $\mathscr{M}$ be the set of local minima of $f(\cdot)$ inside $\mathscr{G}$. Then we assume that the value of $f(\cdot)$ is known at all points in $\mathscr{M}$. Let $\bar{u}$ denote the "steepest descent" direction given by (2.7). Define $n_j(x)$ to be the inward (with respect to $\mathscr{G}$) normal to $\partial\mathscr{G}_j$ at x. Then we also assume that $\langle \bar{u}(x), n_j(x)\rangle > 0$ for all $x \in \partial\mathscr{G} \cap \partial\mathscr{G}_j$, $j = 1, \ldots, J$.

The minimizing trajectories for the control problem we define later are essentially the two dimensional projections of the paths of steepest descent on the surface represented by the height function. Thus, the conditions that are placed on $\mathscr{G}$ in this assumption guarantee that any minimizing trajectory that starts in $\mathscr{G}$ stays in $\mathscr{G}$. If we consider an initial point $x \in \mathscr{D}$ such that the minimizing trajectory exits $\mathscr{D}$, then we cannot construct $f(\cdot)$ at x as the infimal cost in the control representation (2.9), since this requires $I(x)$ for values of x outside $\mathscr{D}$. However, if we assume that the height function is specified at the local maximum points, then $f(\cdot)$ may be constructible at x by using an analogous control problem with a maximization.

We next define the control problem. Let B be an upper bound for $\{f(x): x \in \mathscr{G}\}$. Define

$$g(x) = \begin{cases} f(x), & \text{for } x \in \mathscr{M}, \\ B, & \text{for } x \notin \mathscr{M}, \end{cases} \tag{2.8}$$

and define L by (2.5). As an admissible control, we will consider any integrable function u: $[0,\infty) \to \mathbb{R}^2$. Given such a control, the dynamics of the controlled process are simply $\dot{\phi}(s) = u(s)$ and $\phi(0) = x$. Let

$$V(x) = \inf\left[\int_0^{\rho\wedge\tau} L(\phi(s), u(s))\, ds + g(\phi(\rho\wedge\tau))\right], \tag{2.9}$$

where $\tau = \inf\{t: \phi(t) \in \partial\mathscr{D} \cup \mathscr{M}\}$ and the infimum is over all $\rho \in [0,\infty)$ and admissible controls. We follow the convention of defining the integral to be ∞ if the integrand is not integrable.

A formal derivation of the equality $V(x) = f(x)$ for $x \in \mathscr{G}$ is as follows. Let $u(\cdot)$ be any admissible control, $\phi(t) = x + \int_0^t u(s)\,ds$, and let $\rho \in [0,\infty)$. If $g(\phi(\rho\wedge\tau)) = B$, then $V(x) \geq f(x)$ is automatic. If $g(\phi(\rho\wedge\tau)) < B$, then $\tau \leq \rho$ and $\phi(\tau) \in \mathscr{M}$. Also, by (2.3) and (2.4),

$$\begin{aligned} L(\phi(t), u(t)) &\geq -\langle f_x(\phi(t)), u(t)\rangle - H(\phi(t), f_x(\phi(t))) \\ &= -\langle f_x(\phi(t)), u(t)\rangle \end{aligned}$$

and hence

$$\int_0^\tau L(\phi(t), u(t))\,dt + g(\phi(\tau))$$
$$\geq \int_0^\tau -\langle f_x(\phi(t)), \dot{\phi}(t)\rangle\,dt + f(\phi(\tau)) = f(x).$$

Thus $V(x) \geq f(x)$.

Next consider a solution $\phi(t)$ to $\dot{\phi} = \bar{u}(\phi)$, $\phi(0) = x$. Note that Assumption 2.1 implies that $\phi(t)$ remains in $\mathscr{G}$ for all $t \geq 0$. Since $L(x, \beta) \geq 0$, (2.7) implies that $f(\phi(t))$ is decreasing with time. Thus $\phi(t)$ must tend to the set $\mathscr{S}$ as $t \to \tau$, where τ may be infinite. Assume that $\phi(t)$ actually tends to a point in $\mathscr{M}$. If $\tau < \infty$, then we can take $\rho > \tau$ and $u(t) = \bar{u}(\phi(t))$ to achieve

$$\begin{aligned} V(x) &\leq \int_0^\tau L(\phi(t), u(t))\,dt + g(\phi(\tau)) \\ &= \int_0^\tau -\langle f_x(\phi(t)), u(t)\rangle\,dt + f(\phi(\tau)) \\ &= \int_0^\tau -\langle f_x(\phi(t)), \dot{\phi}(t)\rangle\,dt + f(\phi(\tau)) \\ &= f(x). \end{aligned}$$

The first inequality is due to the definition of $V(x)$ as an infimum. The following equality is due to (2.7). If $\tau = \infty$, then we can use the same $u(\cdot)$ to get $\phi(\cdot)$ into an arbitrarily small neighborhood of the limit point in a finite amount of time. Once we have reached a nearby point we modify the control to move $\phi(\cdot)$ to the limit point in a finite time with small running cost. Although this results in a control that incurs greater running cost than the control that takes infinite time, it can be shown that the additional cost can be made arbitrarily small. The proof as given so far is fairly standard. The only issue not dealt with is what happens if $\phi(t) \to \mathscr{S} \backslash \mathscr{M}$. This requires a somewhat elaborate construction, the details of which are given in the proof of Theorem 2.1.

Note that the main facts used in this formal derivation were that $H(x, f_x(x)) = 0$ is equivalent to the basic equation (2.1), and that the running cost $L(x, \beta)$ is related to $H(x, \alpha)$ via the Legendre transform (2.4). Thus we expect that various ways of rewriting (2.1) would give rise to various control representations for $f(x)$, with the different running costs defined via (2.4). This observation will be exploited in the next section.

We next state the representation theorem. The proof is given in Section 5.

THEOREM 2.1. *Suppose Assumption* 2.1 *is true and that B is an upper bound for $f(\cdot)$ on $\mathscr{G}$. Define $L(\cdot, \cdot)$ by* (2.5), *$g(\cdot)$ by* (2.8) *and $V(x)$ by* (2.9). *Then $V(x) = f(x)$ for all $x \in \mathscr{G}$.*

REMARKS. Because $L(x, \beta) > 0$ whenever $(x, \beta) \in (\mathscr{D} \setminus \mathscr{S}) \times \mathbb{R}^2$ or $(x, \beta) \in \mathscr{D} \times (\mathbb{R}^2 \setminus \{0\})$, (2.7) implies that $\bar{u}(x) = 0$ if and only if $x \in \mathscr{S}$, that is, if and only if $f_x(x) = 0$. Let y be an isolated local minimum of $f(\cdot)$ and let O be the domain of attraction of y with respect to $\bar{u}(\cdot)$, that is, if $\dot{\phi} = \bar{u}(\phi)$, $\phi(0) = x$ and $x \in O$, then $\phi(t) \to y$ as $t \to \infty$. Assume that $O \subset \mathscr{D}$. Further suppose that $f(\cdot)$ is C^2, so that $\bar{u}(\cdot)$ is C^1. In this case O is open and one can find an increasing sequence of sets $\mathscr{G}_i$ such that each $\mathscr{G}_i$ satisfies Assumption 2.1 and $\mathscr{G}_i \uparrow O$ as $i \to \infty$. Hence the conclusion of Theorem 2.1 will hold with $\mathscr{G}$ replaced by O. If $V(x)$ is continuous on the closure of O (which is usually easy to verify and holds automatically if the closure of O is contained in a domain that satisfies Assumption 2.1), then the conclusion of Theorem 2.1 will hold with $\mathscr{G}$ replaced by the closure of O. Similar remarks will apply to the conclusion of Theorem 2.3 of the next subsection. These remarks will find application in the scheme described in Section 4 for eliminating the assumption that the height function must be known at all local minima (or local maxima).

2.3. *An approximation scheme.* In this subsection we consider the construction of numerical schemes. It will be useful to give a description of what we have in mind before actually constructing the scheme. There are two basic ingredients in the control problem just described. The first is the dynamical equation, which is simply $\dot{\phi} = u$. The second is the cost structure, which includes a running cost and a stopping cost. Since the data in the reconstruction problem are really only given on the discrete set $\mathscr{D}^h$, the best we can hope for is to build an approximation $V^h(x)$ to $V(x)$. To do this, we will pose another control problem that will be defined in terms of the data actually available, which is I^h. Of course there are many ways in which this could be done. Ideally, for the alternative control problem it should be easy to solve for $V^h(x)$, but we also desire the convergence $V^h(x) \to V(x)$ as $h \to 0$ (and perhaps even a fast rate of convergence). In our approach, we approximate the dynamics $\dot{\phi} = u$ by suitably choosing a controlled Markov chain whose state space is $\mathscr{D}^h$, while keeping essentially the same cost structure as that of $V(x)$. As will be seen, the Markov property allows one to easily solve for $V^h(x)$, while the "stochasticity" in the dynamics of this new control problem is desirable if we want the approximation to the deterministic dynamics to be of sufficient quality that $V^h(x) \to V(x)$. In fact, if the scheme considered in this section is extended to the general oblique light case ($\gamma_1 \neq 0$ and $\gamma_2 \neq 0$), this "stochasticity" can be shown to be necessary for the convergence. Further remarks on this point will be given later on. However, we emphasize that $V^h(x)$ will not be random; it will simply have a representation as a functional of a controlled Markov chain.

Dynamics. For each $h > 0$ and $u \in \mathbb{R}^2$, let $p^h(x, y|u)$ be a probability transition function on $h\mathbb{Z}^2$ and let $\Delta t^h(u)$ be an interpolation interval, whose use will be made clear momentarily. We assume that $p^h(x, y|u)$ and $\Delta t^h(u)$

satisfy the following "local consistency" condition: for each $x \in \mathscr{D}^h$,

$$\sum_y (y - x) p^h(x, y|u) = u\,\Delta t^h(u), \tag{2.10}$$

$$\begin{aligned}\sum_y \left[(y - x) - u\,\Delta t^h(u)\right]\left[(y - x) - u\,\Delta t^h(u)\right]' p^h(x, y|u) \\ = o\left(\|u\|\Delta t^h(u)\right).\end{aligned} \tag{2.11}$$

In the proof of convergence for the numerical procedure we will define a continuous time interpolation of the related discrete time Markov chain, and the interpolation intervals used will be $\Delta t^h(u)$. With such a scaling, (2.10) states that the average increment per unit of continuous time obtained when control u is applied is u, while (2.11) implies that the variance about this mean tends to zero as $h \to 0$. Hence the controlled Markov process is a natural replacement for the deterministic controlled dynamics $\dot{\phi} = u$, given that the state space of the approximating process should be $h\mathbb{Z}^2$. Note that with deterministic dynamics [i.e., $p^h(x, y(x, u)|u) = 1$, for some $y(x, u) \in \mathscr{D}^h$], it is, in general, impossible to satisfy (2.10) for all $u \in \mathbb{R}^2$ because of the discrete nature of $\mathscr{D}^h$. We also need to assume that $\Delta t^h(u) \to 0$ for each $u \in \mathbb{R}^2$, and that

$$\lim_{h \to 0} \sup\left\{\|y - x\|: p^h(x, y|u) > 0,\ x \in \mathscr{D}^h,\ y \in \mathscr{D}^h,\ u \in \mathbb{R}^2\right\} = 0.$$

These last two conditions are easy to satisfy.

EXAMPLE 2.2. Let $t^+ = t \vee 0$ and $t^- = -(t \wedge 0)$. An obvious choice for the transition function is

$$p^h(x, y|u) = \begin{cases} u_i^\pm / \Sigma_j |u_j|, & \text{if } y = x \pm h e_i, \\ 0, & \text{otherwise,} \end{cases}$$

and $\Delta t^h(u) = h(\Sigma_j |u_j|)^{-1}$. This definition actually makes sense only when $u \neq 0$, and we take care of the omitted case by setting

$$p^h(x, y|0) = \begin{cases} 1, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases}$$

The sequence $\Delta t^h(0) > 0$ is arbitrary [for the purposes of satisfying (2.10) and (2.11)] and for simplicity we can take $\Delta t^h(0) = h$.

It seems likely that other possibilities, such as a less regular grid, may be important. For example, it may be that the grid spacings in the coordinate directions differ. Such variations are easily accommodated. However, to simplify the presentation, we will use the transition probabilities of Example 2.2 for the rest of the paper.

We next consider algorithms for approximating $V(\cdot)$, and hence $z(\cdot)$. The key tool we will use to produce recursive algorithms is the dynamic programming equation for stochastic optimal control. This equation will be valid if we restrict ourselves to controls under which the controlled process has a Markov property. The appropriate definitions are as follows.

Admissible control schemes. A sequence $\{u_i^h,\ i \in \mathbb{Z}^+\}$ is an admissible control scheme for the controlled chain $\{\xi_i^h,\ i \in \mathbb{Z}^+\}$ if it is of the feedback form $u_i^h = U^h(\xi_i^h, i)$, where U^h is a mapping from $h\mathbb{Z}^2 \times \mathbb{Z}^+$ to $\mathbb{R}^2$. Let $(\Omega, \mathscr{F}, P)$ be the underlying probability space on which the controlled Markov chain is defined and suppose that the controlled transition probabilities $p^h(x,y|u)$ are to be used. In general, we could allow this space to depend on h, the control scheme, etc. To simplify the notation this dependence will not be explicitly indicated. Then under an admissible control scheme the following Markov property holds:

$$P\left\{\xi_{i+1}^h = y \middle| \left(u_j^h, \xi_j^h\right), j = 0, 1, \dots, i\right\} = P\left\{\xi_{i+1}^h = y \middle| \left(u_i^h, \xi_i^h\right)\right\} = p^h\left(\xi_i^h, y \middle| u_i^h\right).$$

Admissible stopping times. A random time M^h with values in $\mathbb{Z}^+$ is said to be an admissible stopping time if there is a $\{0,1\}$-valued feedback control $w_i^h = W^h(\xi_i^h,\ i)$ such that $M^h = \min\{i\colon w_i^h = 1\}$. Note that the definition implies M^h is finite with probability 1.

Because the "target set" $\mathscr{M}$ for the deterministic optimal control problem associated with $V(x)$ can possibly contain isolated points that may not be included in $\mathscr{D}^h$, we really need to introduce a "discretized target set" $\mathscr{M}^h \subset \mathscr{D}^h$, and redefine $g(\cdot)$ in the obvious way. We would need that $\mathscr{M}^h \to \mathscr{M}$ in the Hausdorff metric [i.e., $d(x, \mathscr{M}) \le \varepsilon^h$ for all $x \in \mathscr{M}^h$, $d(x, \mathscr{M}^h) \le \varepsilon^h$ for all $x \in \mathscr{M}$ and $\varepsilon^h \to 0$]. To simplify the notation we will just assume $\mathscr{M} \subset \mathscr{D}^h$.

The approximation $V^h(x)$. We now define the approximation $V^h(x)$. For a given admissible control scheme $\{u_i^h,\ i \in \mathbb{Z}^+\}$ let $\{\xi_i^h,\ i \in \mathbb{Z}^+\}$ denote the associated controlled process. Define $N^h = \inf\{i\colon \xi_i^h \notin \mathscr{D}$ or $\xi_i^h \in \mathscr{M}\}$. We then set

$$V^h(x) = \inf E_x\left[\sum_{i=0}^{(N^h \wedge M^h)-1} L\left(\xi_i^h, u_i^h\right) \Delta t^h\left(u_i^h\right) + g\left(\xi_{(N^h \wedge M^h)}^h\right)\right], \tag{2.12}$$

where the infimum is over all admissible control sequences and stopping times and E_x denotes expectation conditioned on $\xi_0^h = x$. It is shown in Theorem 2.3 that $V^h(x) \to V(x)$ for $x \in \mathscr{G}$, where $\mathscr{G}$ is any set that satisfies Assumption 2.1. This is at least intuitively plausible, given the characteristics of the controlled chain $\{\xi_i^h,\ i \in \mathbb{Z}^+\}$ described after (2.10) and (2.11), and the resemblance of (2.12) to (2.9).

Suppose that instead of minimizing over a potentially unbounded time horizon (as here), we consider the problem

$$V_n^h(x) = \inf E_x\left[\sum_{i=0}^{(N^h \wedge M^h \wedge n)-1} L\left(\xi_i^h, u_i^h\right) \Delta t^h\left(u_i^h\right) + g\left(\xi_{(N^h \wedge M^h \wedge n)}^h\right)\right], \tag{2.13}$$

where the infimum is over the same controls and stopping times. Then $V_n^h(x)$ is clearly nonincreasing in n and $V_n^h(x) \downarrow V^h(x)$ as $n \to \infty$. By the dynamic programming principle of optimality [14], $V_n^h(x)$ and $V_{n+1}^h(x)$ are related by

$$V_{n+1}^h(x) = \min\left[\inf_{u\in\mathbb{R}^2}\left(L(x,u)\,\Delta t^h(u) + \sum_y p^h(x,y|u)V_n^h(y)\right), g(x)\right] \tag{2.14}$$

for $x \in \mathscr{D}^h \setminus \mathscr{M}$, while $V_n^h(x) = g(x)$ for all $x \notin \mathscr{D}^h$, $x \in \mathscr{M}$ and all $n \in \mathbb{Z}^+$.

This recursive equation (as well as a number of variations) will serve as the basis for the computation of $V^h(x)$. Starting with the initial condition $V_0^h(x) = g(x)$, we iterate using (2.14). If $V_0^h(x) \geq g(x)$, then the iteration (2.14) extends to $x \in \mathscr{M}$, that is, we can use (2.14) to define the algorithm for all $x \in \mathscr{D}^h$. It can be shown in the vertical light case [$\gamma = (0,0,1)$] that (2.14) converges in no more than $O(h^{-2})$ steps. Typically, for vertical or oblique light, convergence is effectively achieved after $O(h^{-1})$ steps. Further remarks on this point appear in Section 4.

In numerical analysis parlance, (2.14) corresponds to what would be called a Jacobi iteration, since the calculation of each $V_{n+1}^h(x)$, $x \in \mathscr{D}^h$, is based only on the values $V_n^h(x)$, $x \in \mathscr{D}^h$. An alternative is to always use the most recently updated values $V_{n+1}^h(x)$, if they are available, rather than $V_n^h(x)$. Such an algorithm will be referred to as a Gauss–Seidel type algorithm. Let $<$ denote an ordering of the states in $\mathscr{D}^h$. Then (2.14) is replaced by

$$V_{n+1}^h(x) = \min\left[\inf_{u\in\mathbb{R}^2}\left(L(x,u)\,\Delta t^h(u) + \sum_{y<x} p^h(x,y|u)V_{n+1}^h(y) + \sum_{y\geq x} p^h(x,y|u)V_n^h(y)\right), g(x)\right]. \tag{2.15}$$

It is usually preferable to use a different ordering for each iteration. One can show that the Gauss–Seidel procedure is never worse than the Jacobi. (In fact this is easily shown by adapting the argument used to prove part 1 of Proposition 3.2.) We refer the reader to [14] for further discussion and a description of the Markov chain interpretation of the Gauss–Seidel procedure.

It is straightforward to check that $V^h(x)$ is a fixed point of (2.15) if and only if it is a fixed point of (2.14). We prove in Proposition 3.2 of the next section that both procedures in fact produce the same approximation $V^h(x)$ if initialized with the same value $V_0^h(x)$ (recall that, in general, the fixed points may not be unique). Thus the approximation $V^h(x)$ referred to in the convergence theorem, which we now state, is well defined. The proof of the convergence theorem is deferred to Section 5.

THEOREM 2.3. *Let $\mathscr{G}$ be any domain that satisfies Assumption* 2.1, *and let B be an upper bound for $f(\cdot)$ on $\mathscr{G}$. Define $L(\cdot,\cdot)$ by* (2.5), *$g(\cdot)$ by* (2.8) *and $V(x)$ by* (2.9). *For $h > 0$ let $V^h(\cdot)$ be the approximation to $V(\cdot)$ defined in this subsection. Then $V^h(x) \to V(x)$ as $h \to 0$ for all $x \in \mathscr{G}$.*

REMARK. By combining the proof given in Section 5 with an argument by contradiction, one can show the convergence is in fact uniform on $\mathscr{G}$.

3. A more attractive computational scheme. In the last section we introduced an optimal control representation for the height function in the shape from the shading problem (Theorem 2.1). An approximation scheme was derived that was based on this representation, and a convergence result for this approximation was stated in Theorem 2.3. As was noted in the heuristic proof of Theorem 2.1 that was given immediately before the statement of the theorem, various ways of rewriting the image irradiance equation (1.2) give rise to different approximation schemes. In this section we will derive a second computational scheme. In this new scheme the function that replaces the function $L(x, \beta)$ defined in (2.5) will be quadratic in β. Because of this, the iterative algorithm $V_{n+1}^h = F^h(V_n^h, I^h)$ that is obtained by following essentially the same procedure as in Section 2.3 will be easier to evaluate, implement and iterate. As before, the quantities V_n^h will be monotonically nondecreasing, and our approximation to f will be defined to be the limit $V^h = \lim_{n\to\infty} V_n^h$. Although the iterative schemes used to produce the approximation V^h differ in this section and the previous section, the approximations themselves are the same. This is the content of Proposition 3.1. Thus the convergence asserted in Theorem 2.3 carries over to the approximations of this section.

In contrast with the last section, the equation (2.2) can also be rewritten as

$$H(x, f_x(x)) = 0, \tag{3.1}$$

where

$$\begin{aligned} H(x, \alpha) = \tfrac{1}{2}\big[& I^2(x)\alpha_1^2 + v(x)\alpha_2^2 \\ & +2(1 - I^2(x))\gamma_2\alpha_2 - (1 - I^2(x))\big] \end{aligned} \tag{3.2}$$

and $v(x) = I^2(x) - \gamma_2^2$. [The last two equations are not necessarily equivalent to (2.2), since their derivation involves taking a square. Thus additional zeros may appear in (3.1) that do not appear in (2.3). As we shall see, this problem is easy to resolve.]

We now proceed as in Section 2.3 to produce an approximation scheme. We first consider the subset of the image plane where $v(x) \geq 0$. Note that the interior of this set includes $\mathscr{S}$. Let B be an upper bound for $\{f(x),\ x \in \mathscr{D}\}$, and define the terminal cost

$$g(x) = \begin{cases} f(x), & \text{for } x \in \mathscr{M}, \\ B, & \text{for } x \notin \mathscr{M}. \end{cases} \tag{3.3}$$

We also define the running cost $L(x, \beta)$ via equation (2.4) as the Legendre transform of $H(x, \alpha)$ in α:

$$L(x, \beta) = \frac{1}{2}\frac{|\beta_1|^2}{I^2(x)} + \frac{1}{2}\frac{|\beta_2 + (1 - I^2(x))\gamma_2|^2}{v(x)} + \frac{1}{2}(1 - I^2(x)). \tag{3.4}$$

(By convention we set $c/0 = \infty$ if $c > 0$ and $c/0 = 0$ if $c = 0$.) For the case $v(x) \geq 0$ we have $L(x, \beta) = 0$ if and only if $x \in \mathscr{S}$ and $\beta = 0$. The optimal control problem that is appropriate for the region where $v(x) \geq 0$ is the same as that of Section 2.3, save that this new running cost is used. (If desired, the possibility of zeros appearing in denominators in the running cost can be eliminated if one is willing to modify the dynamics to accommodate the change. Such a modification would in fact give the control problem we have previously used for oblique light [22]. However, the algorithms derived will be exactly the same. The control problem chosen here allows a development that more closely parallels that of the previous section.)

On the other hand, for regions where $v(x) < 0$ we observe a negative coefficient in front of α_2^2 in (3.2). This means that in order to obtain an analogous representation in such a region we must work with a differential game. Specifically, the control $u = (u_1, u_2)$ splits into two opposing controls, one of which (u_1) seeks to minimize the cost, while the other (u_2) seeks to maximize. Since the role of the current optimization problem is only to suggest a scheme, our description of this game will not be precise.

For such regions, the relationship between L and H is now an extension of the Legendre transform in (2.4):

$$\begin{aligned}
\inf_{u_2} \sup_{u_1} \Bigg[& -\alpha_1 u_1 - \alpha_2 u_2 - \frac{1}{2} \frac{|u_1|^2}{I^2(x)} \\
& - \frac{1}{2} \frac{\left| u_2 + (1 - I^2(x)) \gamma_2 \right|^2}{v(x)} - \frac{1}{2}(1 - I^2(x)) \Bigg] \\
&= \frac{1}{2}\left[I^2(x) \alpha_1^2 + v(x) \alpha_2^2 + 2(1 - I^2(x)) \gamma_2 \alpha_2 - (1 - I^2(x)) \right] \\
&\equiv H(x, \alpha).
\end{aligned}$$

Note that for $v(x) < 0$ only one of the zeros of $H(x, \cdot)$ for a given α_1 is actually a valid solution of the image irradiance equation (2.1). The other zero corresponds to a surface normal pointing away from the camera, which is impossible if $f(x)$ is a function. This spurious zero is due to taking a square in the course of rewriting the image irradiance equation in the form (3.2). The valid zero is characterized by its being achieved at a value u_2 such that

$$u_2 \gamma_2 < 0 \tag{3.5}$$

(see the proof of Proposition 3.1). Moreover, if $H(x, \alpha) = 0$, a simple computation shows that the supremum and infimum are achieved at u obeying

$$\frac{u_1^2}{I^2(x)} + \frac{u_2^2}{v(x)} < 0.$$

Together, these facts imply the angular constraint on the control

$$\frac{\gamma_2 u_2}{\|u\|} < -\left(\gamma_2^2 - I(x)^2\right)^{1/2}. \tag{3.6}$$

This inequality suggests that the differential game should be defined with a restriction on the control to pick out the valid zero of $H(x,\cdot)$. The restriction we will impose is (3.5), since this is simple to implement in our numerical scheme, although the stronger constraint on the maximizer $\operatorname{sign}(\gamma_2)u_2 < -|u_1|(-v(x))^{1/2}/I(x)$ could be used instead. These restrictions may be unnecessary to achieve a valid differential game representation, but result in more efficient algorithms, and are needed to prove the equivalence of the fixed points for the two algorithms. For further discussion, see the proof of Proposition 3.1.

As in the previous section the relationship between H and L suggests a numerical scheme. Let $p^h(x, y|u)$ and $\Delta t^h(u)$ be as in Example 2.2. We can then define an analogue of (2.12), in which the control problem becomes a discrete time discrete state stochastic game in the region where $v(x) < 0$. Only the minimizer will be allowed to stop the game and pay the stopping cost. A formal application of the principle of dynamic programming suggests the following algorithm: $V_0^h(x) = g(x)$ and

$$\begin{aligned}
V_{n+1}^h(x) &= \min\left[\inf_{u\in\mathbb{R}^2}\left(L(x,u)\,\Delta t^h(u) + \sum_y p^h(x,y|u)V_n^h(y)\right), g(x)\right], && \text{if } v(x) > 0,\\
V_{n+1}^h(x) &= \min\left[\sup_{u_2:\,u_2\gamma_2<0}\ \inf_{u_1\in\mathbb{R}}\left(L(x,u)\,\Delta t^h(u) + \sum_y p^h(x,y|u)V_n^h(y)\right), g(x)\right], && \text{if } v(x) \le 0,
\end{aligned} \tag{3.7}$$

together with boundary conditions as in (2.14).

EXAMPLE 2.2 (Continued). If we insert $p^h(x, y|u)$ and $\Delta t^h(u)$ from Example 2.2 into (3.7), we obtain the following recursive formulas. Fix $x \in \mathscr{D}^h$. Define $m = (1 - I^2(x))/v(x)$,

$$v_2 = \begin{cases} h^{-1}\min\{V_n^h(x + h(0,1)) + m\gamma_2, \\ \qquad V_n^h(x - h(0,1)) - m\gamma_2\}, & \text{if } v(x) > 0,\\ h^{-1}v(x - \operatorname{sign}(\gamma_2)h(0,1)) - m|\gamma_2|, & \text{if } v(x) < 0,\end{cases}$$

$$v_1 = h^{-1}\min\{V_n^h(x + h(1,0)), V_n^h(x - h(1,0))\}, \qquad A = v_2 - v_1,$$

and

$$S = \left(\frac{I(x)|v(x)|^{1/2}}{I^2(x) + v(x)}\right)\left|m\gamma_3^2\frac{(I^2(x) + v(x))}{v(x)} - A^2\right|^{1/2}.$$

For $v(x) > 0$, let

$$\hat{V} = \begin{cases} hv_1 + |m|^{1/2}\gamma_3, & \text{if } A > |m|^{1/2}\gamma_3, \\ hS + \dfrac{I^2(x)v_1 + v(x)v_2}{I^2(x) + v(x)}, & \text{if } -|m|^{1/2}\gamma_3\left(\dfrac{I(x)}{v^{1/2}(x)}\right) \le A \le |m|^{1/2}\gamma_3, \\ hv_2 + |m|^{1/2}\gamma_3\left(\dfrac{I(x)}{v^{1/2}(x)}\right), & \text{if } A < -|m|^{1/2}\gamma_3\left(\dfrac{I(x)}{v^{1/2}(x)}\right). \end{cases}$$

For $v(x) < 0$, we let

$$\hat{V} = \begin{cases} hv_2 - |m|^{1/2}\gamma_3\left(\dfrac{I(x)}{|v(x)|^{1/2}}\right), & \text{if } A < |m|^{1/2}\gamma_3\left(\dfrac{I(x)}{|v(x)|^{1/2}}\right), \\ hS + \dfrac{I^2(x)v_1 + v(x)v_2}{I^2(x) + v(x)}, & \text{otherwise.} \end{cases}$$

The recursions defined above are continuous in $v(x)$, so that the recursions for $v(x) = 0,\ -I(x)^2$ are just given by the appropriate limits. We then set

$$V_{n+1}^h(x) = \min\left[\hat{V}, g(x)\right].$$

For $x \notin \mathscr{D}^h$ we set $V_{n+1}^h(x) = g(x) = B$.

If the constraint $u_2\gamma_2 < 0$ were not imposed for $v(x) < 0$, then the definition of v_2 would be altered to

$$v_2 = h^{-1}\max\{V_n^h(x + h(0,1)) + m\gamma_2, V_n^h(x - h(0,1)) - m\gamma_2\}$$

for $v(x) < 0$, and the recursion would otherwise be unchanged.

As in Section 2, we define $V^h(x) = \lim_{n\to\infty} V_n^h(x)$. The existence of such a limit is proved in Proposition 3.2, and the convergence of $V^h(x)$ to $f(x)$ as $h \to 0$ follows from Proposition 3.1 and Theorem 2.3. Without the foregoing constraint on the maximizer for $v(x) < 0$, the algorithm would probably converge to the same V^h in most cases. However, computational experience suggests the convergence may be slow over those regions of $\mathscr{D}^h$ in which the maximizer, for at least some number of iterations, can prevent the entry of the controlled path into the domain where there is only a minimizer [i.e., $v(x) \ge 0$]. By this delay, the maximizer attempts to win a large positive terminal cost by not allowing the controlled path to terminate in the target set $\mathscr{M}$.

The preceding algorithm can be implemented in a Gauss–Seidel version just as for the algorithm of Section 2.3. This is the algorithm actually used in the computations that will be presented in the next section. In the special case where the camera and light directions are the same [$\gamma = (0,0,1)$], the equations take a simpler form, which we include here for convenience.

EXAMPLE 2.2 (Continued). Assume that $\gamma = (0,0,1)$. For $x \in \mathscr{D}^h$, let v_1 and v_2 be the smallest values from the sets

$$\{V_n^h(x + h(1,0)), V_n^h(x - h(1,0))\}$$

and

$$\{V_n^h(x + h(0,1)), V_n^h(x - h(0,1))\},$$

respectively. Define $m = (1/I^2(x)) - 1$. If $0 \le h^2 m < (v_1 - v_2)^2$, then we use the recursion

$$V_{n+1}^h(x) = \min\left[g(x), (v_1 \wedge v_2) + hm^{1/2}\right].$$

If $h^2 m \ge (v_1 - v_2)^2$, then we use

$$V_{n+1}^h(x) = \min\left[g(x), \tfrac{1}{2}\left[\left(2h^2 m - (v_1 - v_2)^2\right)^{1/2} + (v_1 + v_2)\right]\right].$$

Of central importance is the following result, which relates the approximations obtained from the two control problems. The proof is given in Section 5.

PROPOSITION 3.1. *Consider the transition probability $p^h(x, y|u)$ and interpolation interval $\Delta t^h(u)$ of Example* 2.2, *as well as the recursive equation* (2.14) *for the control problem used in Section* 2.3 *and* (3.7) *for the control problem of this section. Then $w(x)$ is a fixed point of* (3.7) *if and only if it is a fixed point for* (2.14).

In the proposition just given we have asserted that the fixed points of Jacobi-type algorithms obtained from the two control representations are all actually the same. It has also been noted that the fixed points of the Gauss–Seidel type procedures [e.g., (2.15)] (under any ordering of the states) and those of the Jacobi procedures [e.g., (2.14)] are also identical for either of the control problems. One of the more subtle points in the analysis of these algorithms has to do with the fact that the iterations we have described may have multiple fixed points. The existence of this nonuniqueness is tied to the fact that the running costs $L^{(i)}(x, u)$ are not bounded from below away from zero (which is itself due to the presence of singular points in the region over which the reconstruction is to be done). However, the nonuniqueness also turns out to be quite illuminating, since, as we will see, its proper resolution leads naturally to algorithms that are quite efficient. The following result will be essential in relating the various algorithms and dealing with the nonuniqueness. The proof is given in Section 5. Recall that B is an upper bound for $f(\cdot)$ on $\mathscr{G}$, and that $g(\cdot)$ is defined in (3.3).

PROPOSITION 3.2. *Consider any of the recursive algorithms derived in either this section or Section* 2.3. *Let an initial condition V_0^h be given and* *define the sequence* $\{V_i^h, i \in \mathbb{N}\}$ *according to either the Jacobi iteration* [*e.g.*, (2.14)] *or the Gauss–Seidel iteration* [*e.g.*, (2.15)], *where for the Gauss–Seidel procedure we allow a possibly different ordering to be used for each iteration. Assume that* $V_0^h(x) \geq g(x)$ *for all* $x \in \mathscr{D}^h$. *Then the following conclusions hold.*

1. *For each* $x \in \mathscr{D}^h$, $V_i^h(x)$ *is nonincreasing in* i *and bounded from below. Define* $V^h(x) = \lim_{i \to \infty} V_i^h(x)$. *Then the function* $V^h(\cdot)$ *is a fixed point of* (2.14) [*or* (3.7) *if appropriate*].
2. *The function* $V^h(\cdot)$ *can be uniquely characterized as the largest fixed point of* (2.14) [*or* (3.7) *if appropriate*] *that satisfies* $V^h(x) \leq V_0^h(x)$ *for all* $x \in \mathscr{D}^h$. *Thus, whenever the initial condition is sufficiently large, the limits for all the various procedures* (*control problem of Section* 2.3 *or Section* 3, *Jacobi or Gauss–Seidel*) *are identical.*

REMARK. In the actual use of the algorithm, $g(x)$ is always taken as the initial condition, and thus the assumption $V_0^h(x) \geq g(x)$ in the proposition is automatically satisfied.

4. Experiments and a description of the global algorithm. In this section we present some experimental results obtained with the algorithm described in Section 3. In all of the discussion so far, we have required that the surface "height" $z(\cdot)$ be known at all points $x \in \mathscr{M}$, which are a subset of the singular points in $\mathscr{S}$. This is equivalent to knowing the value of $f(\cdot)$ at all of its local minima in $\mathscr{S}$. (We remind the reader that the definition of local minimum given in Section 2 is not the standard definition.) In Section 4.2, this restriction is relaxed. An approach is outlined in which the surface can be reconstructed with no boundary data, that is, using only the intensity data $I(x)$ in addition to Assumption 2.1. To do this, we must impose some additional conditions, the most important of which is that $f(\cdot)$ is C^2. A fuller discussion of this approach is deferred to future work. Throughout this section we continue to use the transition probabilities of Example 2.1.

In the discussion that follows we use the term "boundary data" to refer to data that are specified on subsets of the set $\mathscr{S}$ of singular points. Of course we never specify data on boundaries of sets such as $\mathscr{D}$ or $\mathscr{S}$. Hopefully, this use of the term "boundary" will not cause confusion.

4.1. *Boundary data given on* $\mathscr{M}$. Figure 1 displays a 32 by 32 parabolic surface that is assumed to be imaged from above. The image has one singular point, corresponding to the local minimum of the height $z(\cdot)$. Assuming vertical light, the image intensity was first computed using the forward/backward discretization of the derivative implicit in the algorithm of Example 2.1 (see also the proof of Proposition 3.1). Thus we are defining the data to correspond to the discretized (rather than continuous) surface. With this choice, the original surface is a fixed point of the algorithm and should be reconstructed exactly. Our purpose in doing this is to assess the convergence

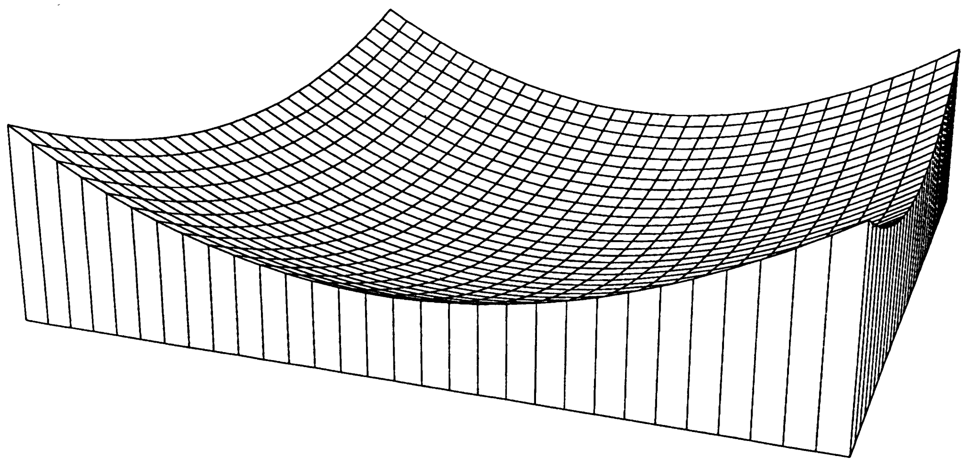

FIG. 1.

time of the algorithm. Using Jacobi updates, the algorithm converged to the correct solution to within, on average, one part in 10^7 after 63 iterations. In general, the convergence time is expected to be on the order of the maximum length of an optimal path, that is, a path of steepest descent. For the controlled transition probabilities that we use, the Markov chain ξ_i^h jumps one lattice site per discrete time step. Thus, when the number of iterations becomes greater than the maximum optimal path length, all points x should have "learned" their heights from the singular points. For the given surface, the maximal path length is on the order of 32, since paths starting at the image corners must zigzag to the singular point at the center of the image.

Convergence using Gauss–Seidel updating was faster: it was obtained after just four iterations. Gauss–Seidel was performed using a row major pass over the image, changing the direction of the pass after each iteration. In this case, the information from the singular point is able to propagate outward to a portion of the boundary in just one iteration, since the ordering of the states corresponds to the steepest ascent direction throughout one quadrant of the image.

If we use for data the values that correspond to the true derivatives of the (continuous) surface at the grid points, the average and maximal errors were 0.8 and 1.6 (the latter obtained at the image boundary), compared with a range for the surface height of 25.

For comparison, Figures 2 and 3 display the result of applying our implementation of a more traditional algorithm [10] to a similar surface. The intensity was computed using discrete forward derivatives such that the given surface is a fixed point of the algorithm. Even after 3072 iterations, the algorithm has not converged to the correct fixed point solution. We have also implemented other algorithms such as those of [15] and [32], and applied them to this surface with similar results.

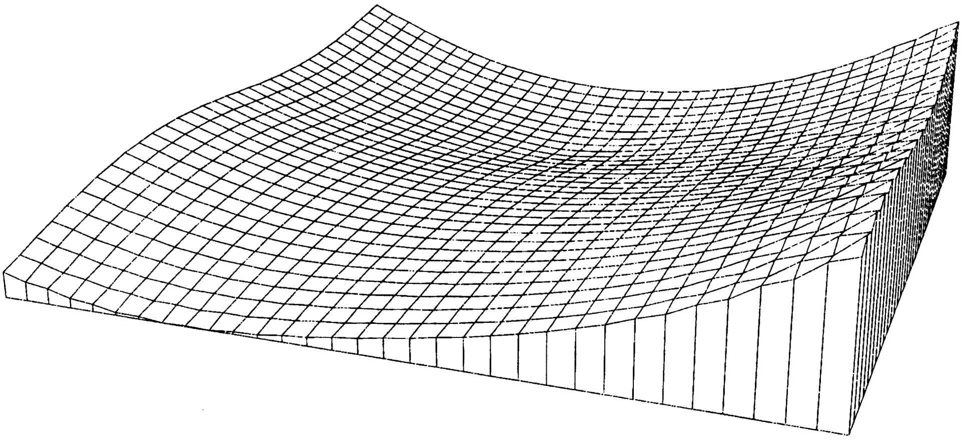

FIG. 2.

Figure 4 shows a more complicated 128 by 128 surface. Under vertical light, the intensity was first computed as for Figure 1 so as to make this surface a fixed point of the algorithm. At the local minimum points $\mathscr{M}$, the initial estimate $V_0^h(x)$ and the terminal cost $g(x)$ were set to the known height values $z(x)$, as discussed in Section 2. The algorithm converged to a perfect reconstruction of the original surface in 100 iterations. As expected, the convergence time is on the order of the longest optimal path. Using Gauss–Seidel, convergence was achieved in 10 iterations. When the intensity was derived analytically, the algorithm again converged in 10 iterations using Gauss–Seidel, with an average error of 1.7 compared to a surface

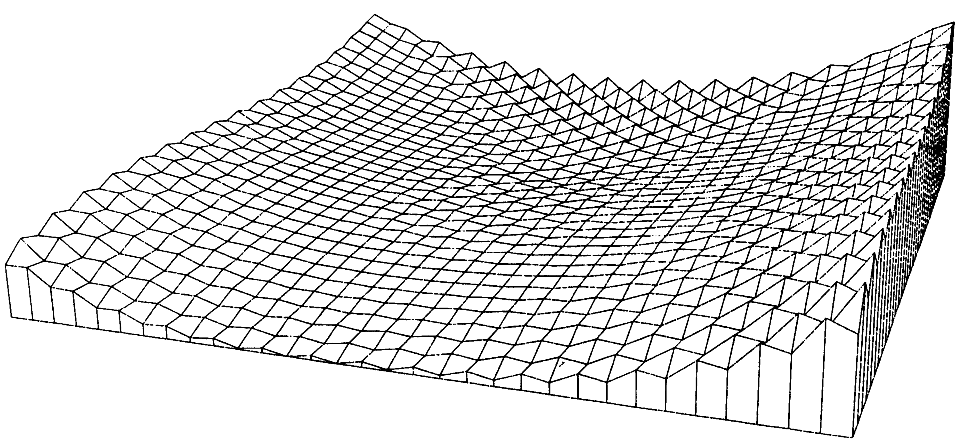

FIG. 3.

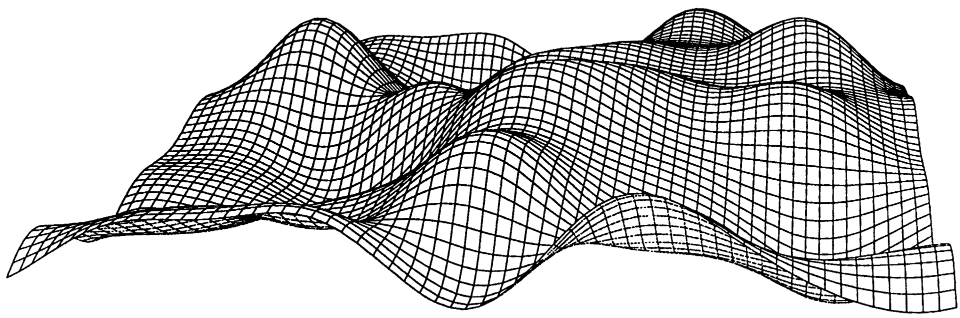

FIG. 4.

range of 51 (Figure 5). Because the surface does not obey the condition in Assumption 2.1 that it be decreasing in from the boundary, the reconstruction is incorrect (as it should be) in some regions near the boundary of $\mathscr{D}$, though it is good in the interior. This is clear in Figure 6, which displays the difference between the reconstruction and the original surface. This surface was also reconstructed assuming oblique light at an angle of 17.5° to the vertical. For an intensity derived so that the surface is a fixed point, convergence to within one part in 10^{-7} was obtained using the Jacobi scheme in 120 iterations. Using Gauss–Seidel, convergence was obtained in 11 iterations. Reconstruction for the analytically derived intensity function was obtained in 14 iterations, with an average error of 2.2. As previously, the reconstruction was good in the interior but incorrect along one boundary (Figure 7). Note that the domains which satisfy Assumption 2.1 will, in general, vary with the direction of the incoming light.

Finally, our algorithm has been applied to the real 200×200 image shown in Figure 8. The data for this problem were kindly provided by Leclerc and Bobick [15]. The light is from above at $(0, 0.488, 0.873)$. This image is a photograph of a mannequin. The particular object was chosen since

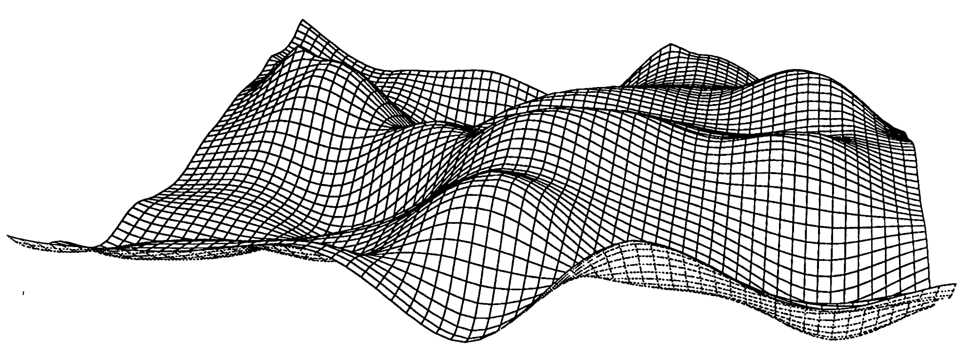

FIG. 5.

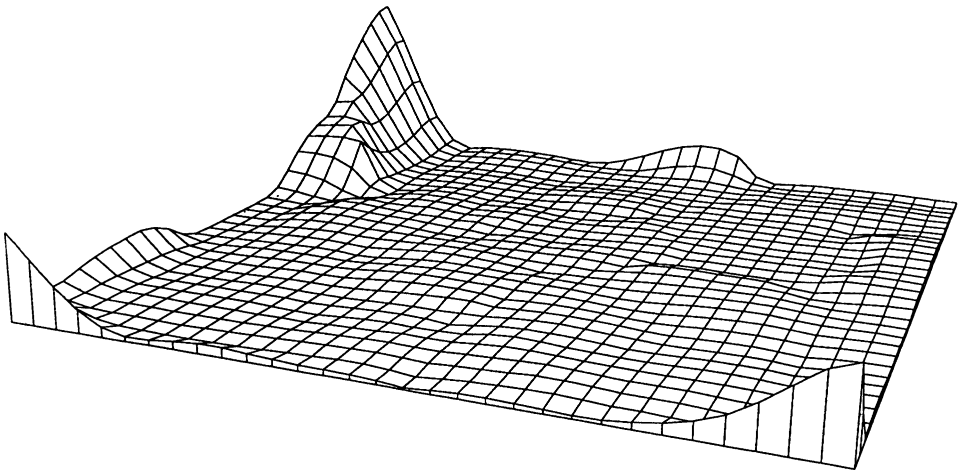

FIG. 6.

its reflectance properties seem to be reasonably in line with the assumed Lambertian model. For this example, one of course specifies the height at the local maxima, rather than minima. For the reconstruction, boundary conditions were specified only at the point that corresponds to the nose of the mannequin, though $\mathscr{M}$ actually contains several other points which correspond to shallow local maxima. Figure 9 shows the reconstruction obtained using Gauss–Seidel after six iterations, illuminated from the same direction as the original. Convergence essentially has been achieved over the face. Since a fixed point of the algorithm must correspond to the original intensity

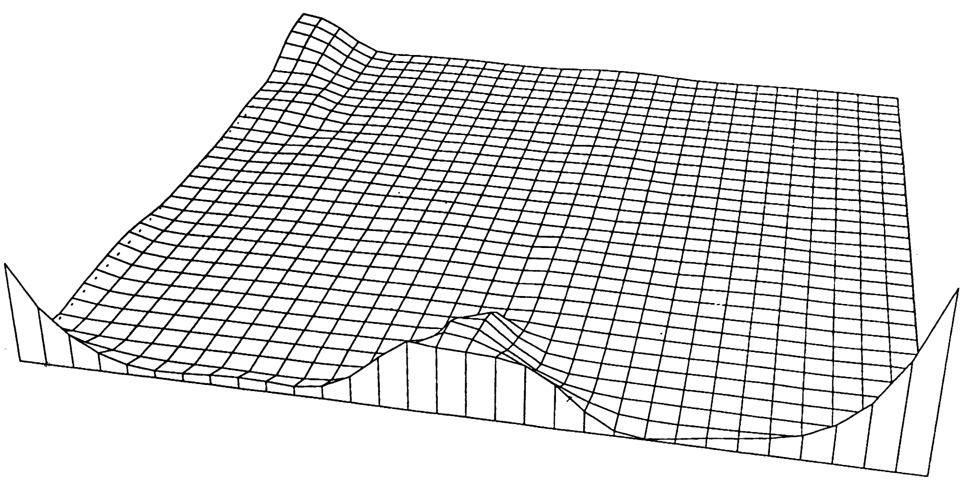

FIG. 7.

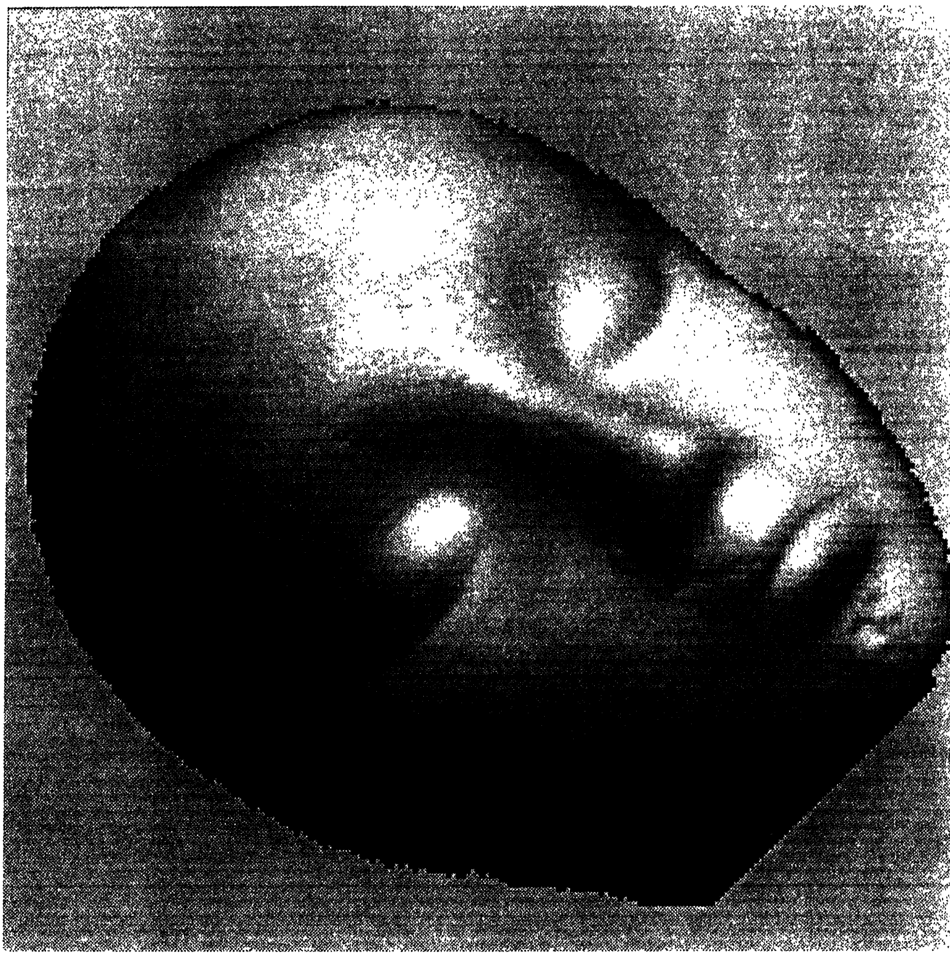

FIG. 8.

data (modulo the effects of any neglected local maxima), this image essentially reproduces Figure 8. However, errors due to incorrect modeling of the surface as Lambertian and the neglect of the shallow local maxima show up as spurious lines in Figure 9. Their occurrence suggests that the truly Lambertian surface corresponding to the intensity data in Figure 8, and with the prescribed boundary conditions, is not everywhere C^1, but has at least line discontinuities in f_x. Figure 10 shows the reconstruction illuminated from below. Since this image is not guaranteed to correspond to the original intensity, it better displays the errors due to incorrect modeling, neglect of local maxima and discretization effects (i.e., noninfinitesimal h). Finally, Figure 11 shows the surface reconstruction. this reconstruction took about 9 seconds of CPU time on a DEC 5000 workstation. Standard variational algorithms typically require thousands of iterations [10].

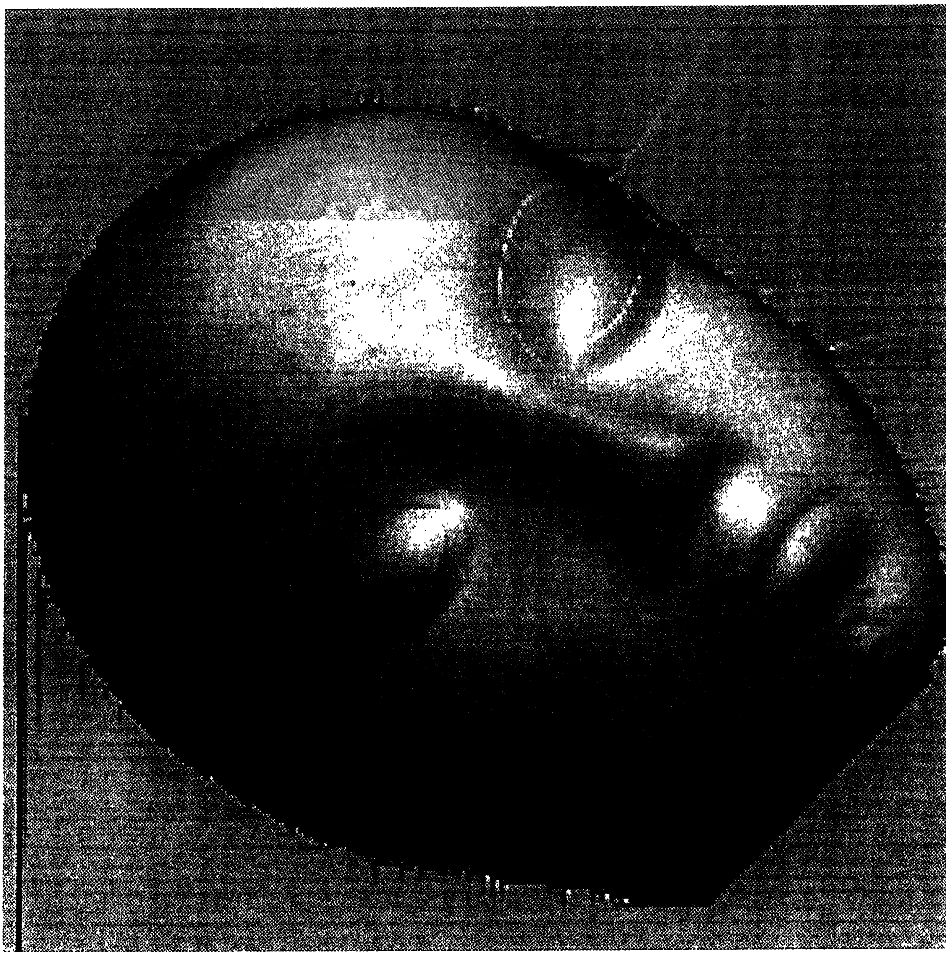

FIG. 9.

For comparison, Figures 12 and 13 display the reconstruction obtained by Leclerc and Bobick [15] using a more standard variational method, developed for the purpose of including stereo information. Stereo information was used as an initial condition for this reconstruction.

4.2. *Reconstruction without boundary data.* We present a preliminary description of our approach and some experimental results that employ it. For simplicity, our description will be for the case of vertical light, although our method is also valid in the more general case of oblique light. Our experimental results are presented for oblique light.

Reconstruction without boundary data is possible only if some additional assumptions are placed on the surface, the most important of which is that $z(\cdot)$ is C^2. In addition, we assume that the extremal points of $z(\cdot)$ are isolated

FIG. 10.

and that the surface has nonzero curvature at these points. (These assumptions are not necessarily respected in our experiments.) These two conditions are implied by the following assumption on the intensity data $I(\cdot)$:

ASSUMPTION 4.1. The set $\mathscr{S}$ consists of isolated singular points. At these points, the matrix of second derivatives of $I(\cdot)$ has nonzero, unequal eigenvalues.

In addition, we assume as before that $I(\cdot)$ is defined over a region $\mathscr{D}$ and that we are considering the reconstruction in a subset $\mathscr{G}$ that satisfies Assumption 2.1. Finally, we will assume for simplicity that the set $\mathscr{G}$ is simply connected. The uniqueness of $z(\cdot)$ given $I(\cdot)$ under these assumptions was investigated in [19] and [20].

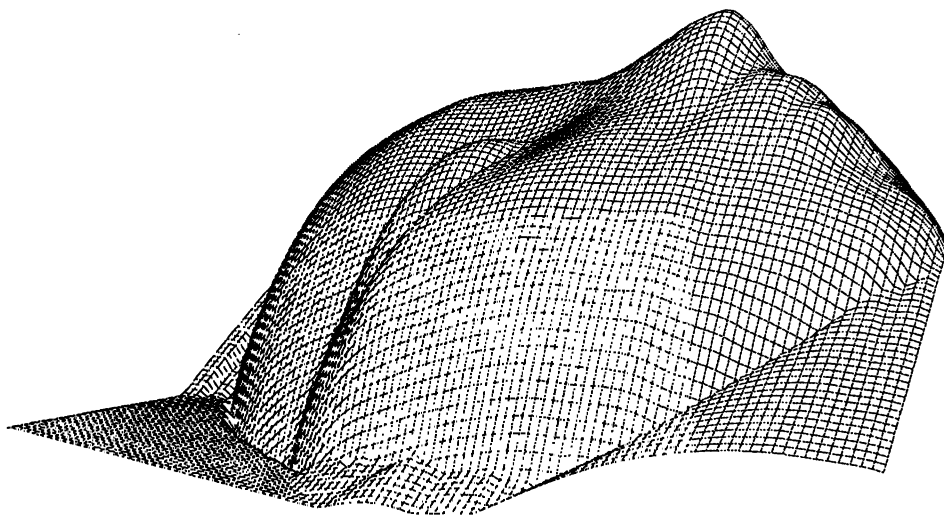

FIG. 11.

Suppose that a point x_0 is known to be an isolated, local minimum of the height $z(\cdot)$ (the discussion is similar if x_0 is a local maximum). We define the domain of attraction of x_0 to be the set A_{x_0} that consists of all points $y \in \mathscr{G}$ such that the projected steepest descent curve passing through y converges to x_0. These curves are uniquely determined since $z(\cdot)$ is C^2. Suppose that we set $g(x_0) = z(x_0)$ and $g(x) = B$ for all $x \neq x_0$, where B is an upper bound for $z(\cdot)$ on $\mathscr{G}$. The algorithm with g as its initial data will accurately reconstruct the surface in the set A_{x_0} (although the reconstruction will generally be poor elsewhere). This follows by considering an increasing sequence of domains $\mathscr{G}_i \subset A_{x_0}$ with $\mathscr{G}_i \uparrow A_{x_0}$ as $i \to \infty$ and each $\mathscr{G}_i$ satisfying Assumption 2.1, and by applying the convergence and representation theorems on each of these domains.

Typically, there will be other singular points in ∂A_{x_0}. By continuity, the heights of these points will be correctly approximated by the algorithm. From Assumption 4.1, these points are either local maxima or saddle points of $z(\cdot)$. Suppose it is possible to identify one of them (say x_1) as a local maximum. The algorithm can then be applied again with the approximate height provided as initial datum at the new singular point x_1 (as well as any other points $x_2, \dots, x_n$ on ∂A_x that are identified as local maxima), which extends the computed approximation $V^h(\cdot)$ over the region $\bigcup_{i=1}^n B_{x_i} \cup A_{x_0}$, where B_{x_i} is the analogous domain of attraction for the local maximum point x_i. Again by continuity, the heights of any local minima on the boundary of this region will be correctly approximated, and the whole process can be repeated. It is argued in [19] and [20] that every singular point in $\mathscr{G}$ is connected to every other singular point by a collection of steepest descent curves. Thus, iterating the foregoing process should eventually identify all local minima and max-

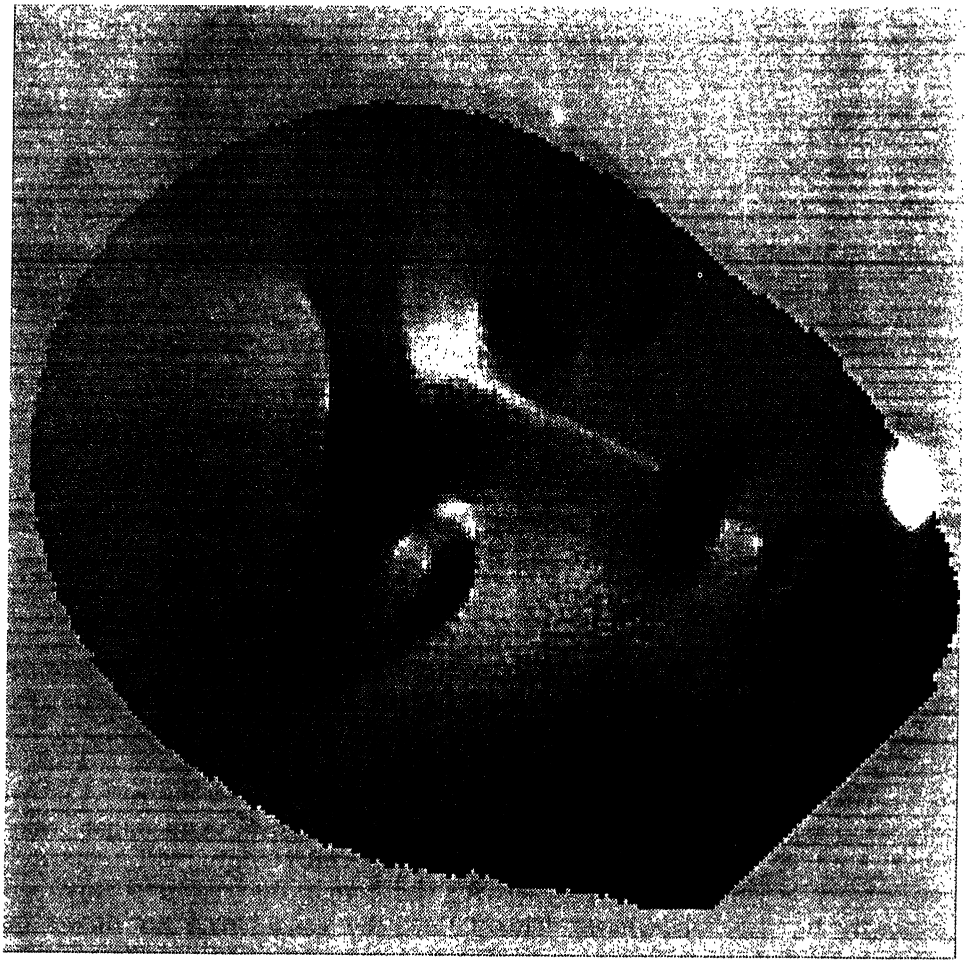

FIG. 12.

ima, and lead to a computed approximation to $z(\cdot)$ over the entire domain $\mathscr{G}$.

For the preceding strategy to work, all that we need is a method for identifying as such the local maxima or minima on ∂A_{x_0}. This is nontrivial, since for the computed $V^h(\cdot)$ that is based on just one singular point, all other singular points will be reconstructed as though they were inflection points.

We now sketch such a strategy. There are a number of possible variations, and it is not clear that the approach we now outline is the best in any sense. Let $x \in \mathscr{M}$, and assume $A_{x_0} \subset \mathscr{G}$ and $\partial \mathscr{G} \cap \partial A_{x_0} = \emptyset$. From Assumption 4.1, $\mathscr{S} \cap \partial A_{x_0}$ consists of a finite set of points. It can be shown that it contains at least two, and at least one local maximum. Since $z(\cdot)$ is C^2, the steepest ascent paths are uniquely determined and fill out A_{x_0}. It is easy to show that

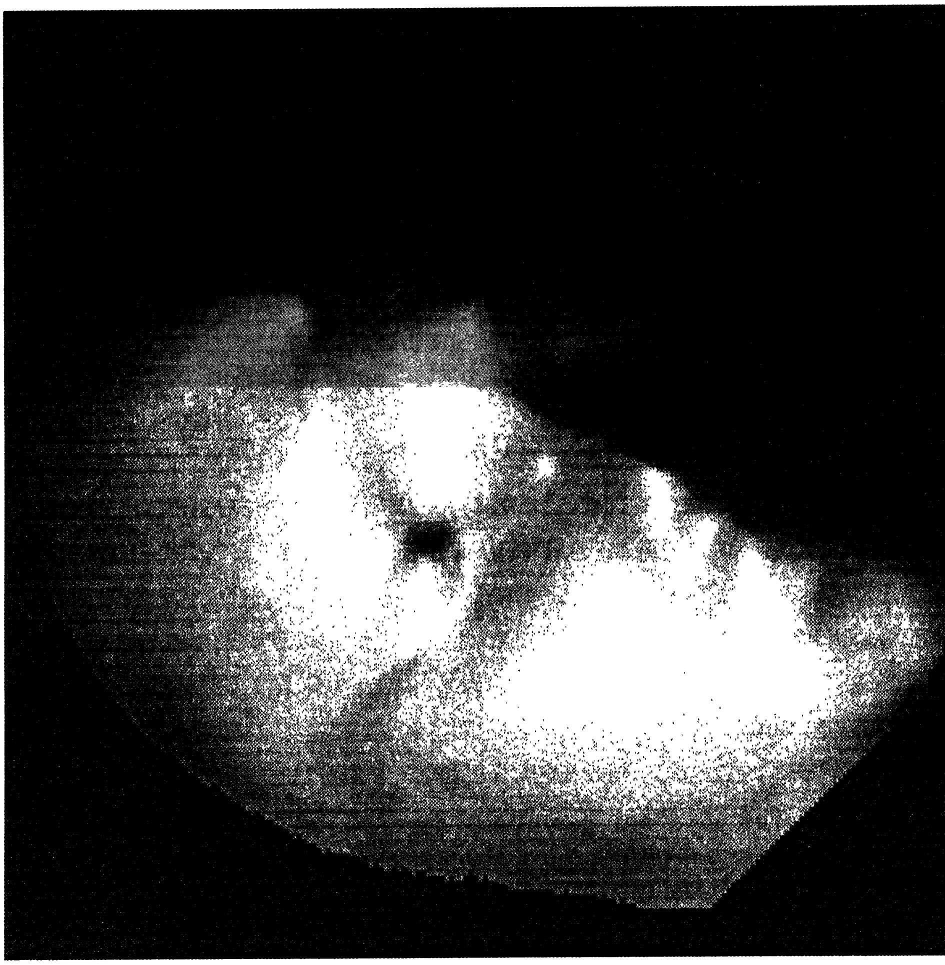

FIG. 13.

all but a finite number converge to local maximum points in $\mathscr{S} \cap \partial A_{x_0}$. The finite number of exceptions converge to saddle points in $\mathscr{S} \cap \partial A_{x_0}$, exactly one to each saddle point.

The following discussion is purely formal, since our purpose here is simply to suggest a numerical scheme. According to the discussion in the previous paragraph, if we start a particle off from x_0 in a random direction selected according to the uniform distribution and let it proceed along a steepest ascent path, then it has probability 1 of ending up at a local maximum point. Suppose we could compute the probability distribution $\mu(dy)$ on $\partial\mathscr{G}$ that describes the distribution of the particle in the limit as $t \to \infty$. If x_i, $i = 1, 2, \ldots, n$, denote the local maxima in ∂A_{x_0}, then $\mu(\{\mu_1, \ldots, x_n\}) = 1$, and obviously $\mu(\partial A_{x_0} \setminus \{x_1, \ldots, x_n\}) = 0$. In particular, for at least one $i \in$

$\{1, \ldots, n\}$ we must have $\mu(\{x_i\}) > 0$. Thus, the distribution μ allows the identification of at least one local maximum point in ∂A_{x_0}.

A scheme for computing an approximation to μ on $\mathscr{G}^h$ can be developed that is similar in spirit to our reconstruction algorithms. However, it is much simpler. Let $V^h(\cdot)$ be the reconstruction obtained by the algorithm using only the value $z(x_0)$ as the initial boundary datum. Let ξ_i^h be a Markov chain defined on $\mathscr{G}^h$ such that $\xi_0^h = x_0$. The transition probabilities are defined so that the discrete time evolution of ξ_i^h approximates via $V^h(\cdot)$ a steepest ascent curve on $z(\cdot)$. A simple choice for these probabilities is the following. At x_0, let $p^h(x_0, x_0 \pm h(1,0)) = p^h(x_0, x_0 \pm h(0,1)) = 1/4$, giving an equal probability of emerging from x_0 in any of the compass directions. We can also allow diagonal transitions to give a better approximation to the uniform distribution. Now consider $y \neq x_0$ and define

$$s_1 = \operatorname{sign}\left[V^h(y + h(1,0)) - V^h(y - h(1,0))\right],$$
$$s_2 = \operatorname{sign}\left[V^h(y + h(0,1)) - V^h(y - h(0,1))\right].$$

Also, let v_1 and v_2 be the largest values from the sets

$$\{V^h(y + h(1,0)), V^h(y - h(1,0))\}$$

and

$$\{V^h(y + h(0,1)), V^h(y - h(0,1))\},$$

respectively, and

$$d_{y,i} \equiv \max\left(0, h^{-1}\left(v_i - V^h(y)\right)\right),$$

for $i = 1, 2$. The $d_{y,i}$ are essentially forward or backward derivatives, depending on which direction gives the steepest ascent. For $d_{y,1} = d_{y,2} = 0$, define $p^h(y, y) = 1$. Otherwise, define

$$p^h(y, y + s_1 h(1,0)) = \frac{d_{y,1}}{d_{y,1} + d_{y,2}},$$
$$p^h(y, y + s_2 h(0,1)) = \frac{d_{y,2}}{d_{y,1} + d_{y,2}},$$

with all other probabilities zero.

Let $\{\xi_i^h, i \in \mathbb{N}\}$ be the Markov chain that is defined by the construction outlined previously. For $y \in \mathscr{D}^h$ set

$$P_n^h(y) \equiv P\{\xi_n^h = y\}.$$

The Markov property gives the recursion

$$P_{n+1}^h(y) = \sum_x p^h(x, y) P_n^h(x). \tag{4.1}$$

We would like to use $\lim_{n \to \infty} P_n^h(y)$ as the approximation to μ. However, because the reconstruction $V^h(\cdot)$ is computed with $g(y) < B$ only at $y = x_0$, the infimal expected cost $V^h(y)$ for $y \in \mathscr{G} \setminus A_{x_0}$ will approximately satisfy

$V^h(y) \geq z(y)$ and also $V^h(y) \geq \inf_{y' \in \partial A_{x_0}} V^h(y')$. Since the evolution of the chain approximates steepest ascent on $V^h(\cdot)$ and because of the discretization, it is likely that ξ_n^h will evolve outside a neighborhood of the set A_{x_0} and that the support of $P_n^h(y)$ as defined will eventually extend beyond A_{x_0}. A direct use of this distribution therefore gives a poor approximation to μ as $n \to \infty$, and may result in spurious concentrations of probability at singular points in $\mathscr{S} \setminus \bar{A}_{x_0}$, leading to the false identification of these points as local maxima.

We avoid this problem by modifying the transition probabilities. Since the steepest ascent paths converge on the local maxima points in ∂A_{x_0}, ξ_i^h should pass near a point in the set $\{x_1, \ldots, x_n\}$ with probability that approaches 1 as $h \to 0$. Let $m > 0$ be such that $m < (1/2)\min\{\|y - y'\|: y, y' \in \mathscr{S}\}$. We modify the probabilities so that the chain is stopped as soon as it enters the set $\{y': \|y' - y\| < m,\ y \in \mathscr{S} \setminus x_0\}$, that is, all points $x \in \mathscr{D}^h \cap \{y': \|y' - y\| < m,\ y \in \mathscr{S} \setminus x_0\}$ are taken to be absorbing, so that $p^h(x, x) = 1$. With this modification, the strength of $P_n^h(y)$ at singular points outside ∂A_{x_0} should be sharply reduced. Let $P^h(y) = \lim_{n \to \infty} P_n^h(y)$. This limit should be well defined at all points and is obviously well defined because of monotonicity for $x \in \mathscr{D}^h \cap \{y': \|y' - y\| < m,\ y \in \mathscr{S} \setminus x_0\}$. Since the structure of the recursion (4.1) is similar to that of the reconstruction algorithm itself, the number of iterations required for convergence to P^h is expected to be similar to the number of iterations used in the convergence to V^h. Once an approximation to $P^h(y)$ has been computed, those singular points near which this approximation has a large value can be identified as local maxima.

In our actual experiments, a variation of this procedure was used. Rather than an approximation to the probability distribution μ, we consider the sum

$$Q_n^h(y) \equiv \sum_{i=0}^{n} P_i^h(y).$$

The Markov chain ξ_i^h is the same as before, except that at y such that $d_{y,1} = d_{y,2} = 0$, we define the transition probability to be $p^h(y, \mathscr{T}) = 1$, where $\mathscr{T}$ is a "fictitious" absorbing point adjoined to $\mathscr{D}^h$. We also define $p^h(y, \mathscr{T}) = 1$ for $y \notin \mathscr{D}^h$. At points $y \in \mathscr{D}^h \setminus \{x_0\}$, $Q_n^h(y)$ satisfies the same recursion (4.1) as before, while $Q_n^h(x_0) = 1$ for all n. The advantage of working with the summed distribution is that it is monotonically nondecreasing in n for all y. Define $Q^h(y) = \lim_{n \to \infty} Q_n^h(y)$. As before, this limit should exist at all points in $\mathscr{D}^h$ because of the new definition of the chain with an absorbing state and since, apart from the case $d_{y,1} = d_{y,2} = 0$, transitions are uphill with respect to $V^h(\cdot)$. Due to the monotonicity, $Q^h(y)$ can be calculated using a Gauss–Seidel iteration similar to that of the reconstruction algorithm, where the ordering of the states is changed after each iteration. As before, the number of iterations necessary for convergence is expected to be the same as for the original reconstruction of $V^h(\cdot)$. Also, it is again necessary to modify the transition probabilities so as to reduce the probability of the process exiting a neighborhood of the set A_{x_0}. In this approach, local maxima in $z(\cdot)$

are identified by looking for a singular point near a local maximum of $\overline{Q}^h(y)$. More precisely, parameters $\theta_1 > 0$ and $\theta_2 > 0$ are chosen and we identify a singular point as a local maximum if there is a local maximum of $Q^h(y)$ with value greater than θ_1 within θ_2 of the singular point.

The algorithm as outlined still requires that at least one singular point be identified as a local maximum or minimum point. Since the height $z(\cdot)$ is ambiguous up to the addition of an overall constant, the height of this point is not needed to start the procedure. However, for any singular point x_0, it is often possible to determine a priori whether it is a local maximum, minimum or saddle point, as we now discuss.

Let $I'(\cdot)$ be an arbitrary function in some region around x_0 such that $I'(x_0) = 1$ and $I'(y) \in (0,1)$, $y \neq x_0$. Suppose $g(x_0) = z(x_0)$ and $g(x) = B$ for all $x \neq x_0$, where B is an upper bound for $z(\cdot)$ on $\mathscr{G}$. The corresponding height function $V(x)$, given by the control representation of (2.9), will in general not be C^1. In the typical case, $V_x(\cdot)$ will have "line" discontinuities, and the $V^h(\cdot)$ reconstructed by our algorithm will approximately reproduce these discontinuities. For $I(\cdot)$, our experiments show in general that if $V^h(\cdot)$ is computed using a singular point x_0 under the incorrect assumption that it is a local minimum (or maximum), then the discrete "derivatives," for example, $\Delta V^h_{x_1}(y) \equiv V^h(y + h(0,1)) - V^h(y)$, will display abrupt changes that can be interpreted as "discontinuities." These can be easily detected. We have used the occurrence of these "discontinuities" both to determine an initial local minimum point and to check the assignments of singular points as local minima, maxima or saddle points, as determined by our iterative procedure.

We have applied this procedure to the surface of Figure 4, under the oblique lighting used previously. In order to reduce the effects of discretization, the surface was first scaled by a factor 0.5, so that the height range was approximately 25 units compared to a range for x_1 and x_2 of 128 units. Using no boundary data other than $I(\cdot)$, the surface shown in Figure 14 was reconstructed by this procedure. The reconstruction took four cycles: that is, a surface was first reconstructed using a single local minimum, then again

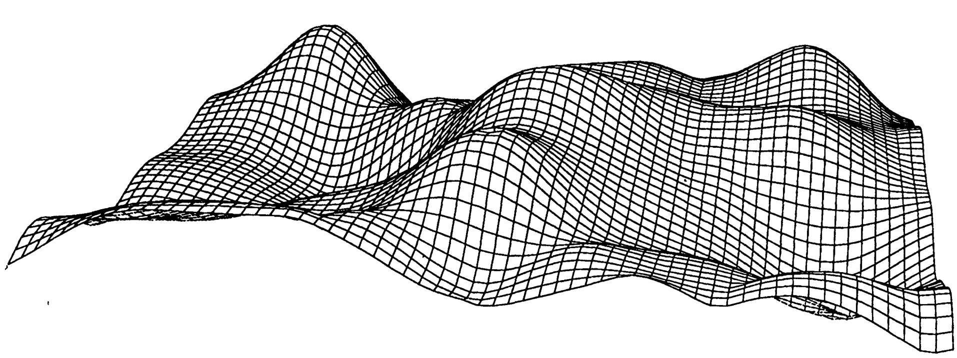

FIG. 14.

using the recovered local maxima, again using local minima and the result of reconstructing a final time using the final recovered local maxima is displayed in the figure. The reconstruction took less than 30 seconds of CPU time on a DEC 5000 workstation.

Figure 15 illustrates the reconstruction error, the magnitude of the difference between the original surface height and that of the reconstruction. The reconstruction is good except near the edges of the image. This is due in part to the fact that, as in the previous subsection, the surface does not satisfy Assumption 2.1 for the entire set $\mathscr{D}$ (or its analog for reconstruction based on local maxima). The average reconstruction error in the interior of the image with $20 \leq x_2 \leq 105$ is 0.5 units, or about 2% of the height range. Figures 16 and 17 show the surface and its reconstruction, respectively, over this region, illustrating the accuracy of the reconstruction there.

A second surface (illuminated as before) and its reconstruction are displayed in Figures 18 and 19, respectively. The reconstruction error is shown in Figure 20. Only three cycles, again starting from a local minimum singular point, were enough to give this reconstruction. The surface also was reconstructed starting from a different local minimum singular point, with comparably good results. The average height error in the interior of the image is 1 unit, in comparison to the overall height range for this surface of 44 units. As before, the accuracy of reconstruction is on the order of 2%.

Figure 21 depicts the summed probability distribution Q^h generated from Figure 18 after one cycle, together with the singular points in the image of this surface (shown as dark isolated clusters). Larger magnitudes for Q^h are indicated by increasing darkness. The distribution Q^h achieves its maximum at the central local minimum singular point used to generate this distribution. The figure clearly shows the dominant evolution of the chain toward

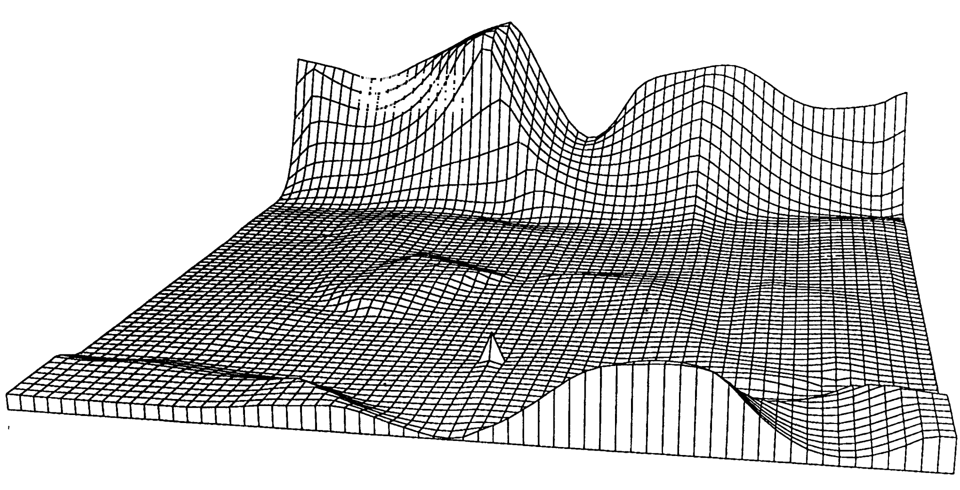

FIG. 15.

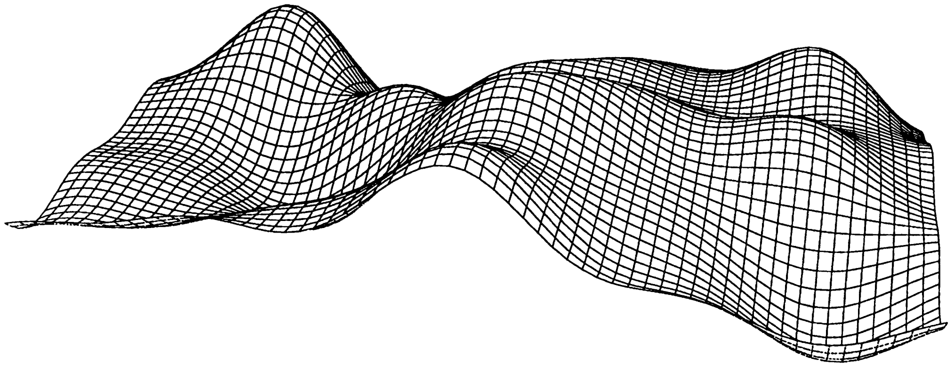

FIG. 16.

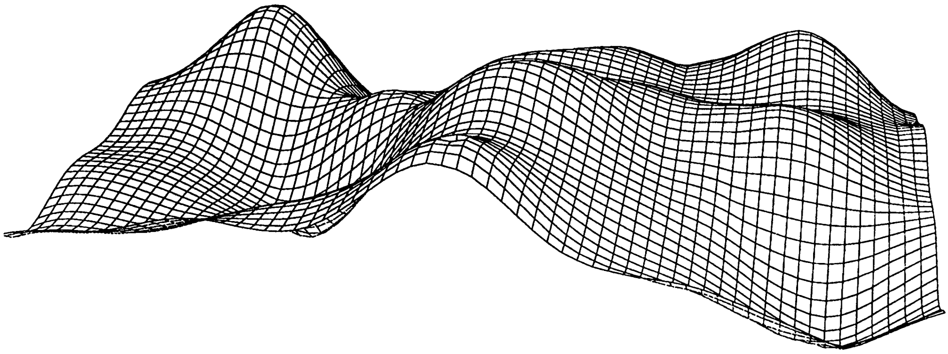

FIG. 17.

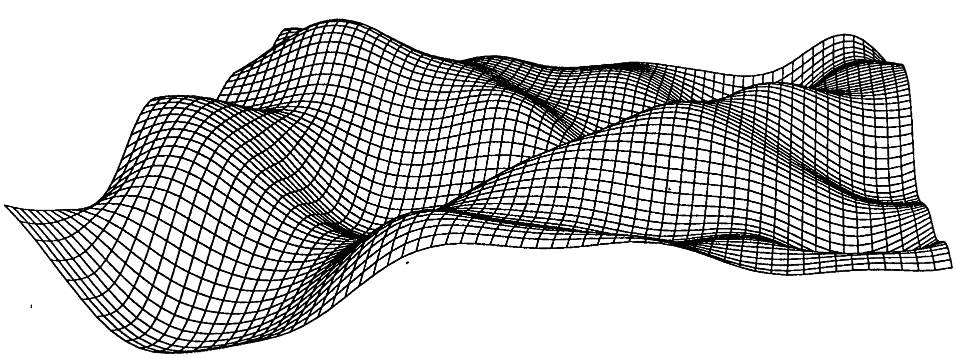

FIG. 18.

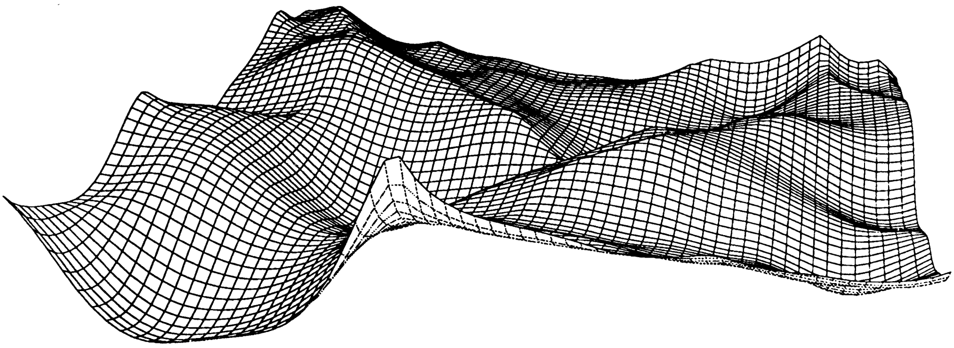

FIG. 19.

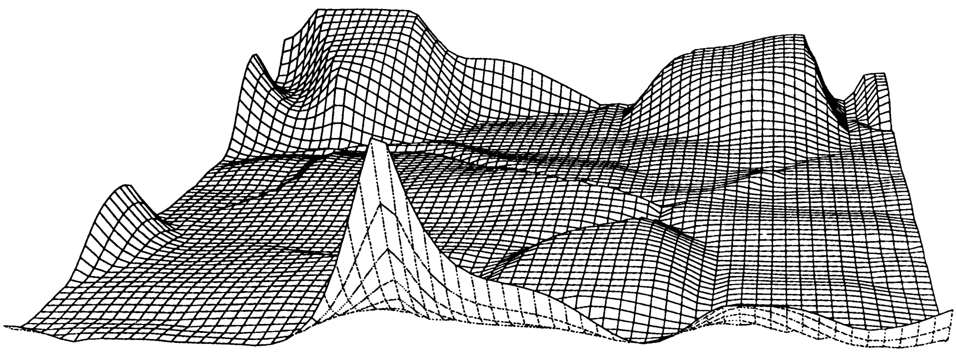

FIG. 20.

four other singular points, which in fact correspond to local maxima of the height function. There are local maxima of Q^h near each of the four singular points, as indicated by the shaded patches near these points. A fifth point has a shaded patch nearby, but is not identified as a maximum since it does not satisfy the threshold rule.

We have also studied the effect of adding noise to the image of Figure 18. The noise added had a uniform distribution on the interval $[-0.1, 0.1]$, and was independent for different lattice points. Since the maximum image intensity is only $I = 1$, this is a large noise of $\pm 10\%$. The reconstruction based on the image with added noise is shown in Figure 22. The surface shown was generated using the computed local minima, and required three cycles. Although there are large errors in some parts of the image, the reconstruction is still good over much of the image. The error in the height is displayed in Figure 23, where saturated white represents a height error of 3. The error is less than 3 units over most of the image. In the region of the image with $127 > x_{1,2} > 40$, the mean height error is just 1.6. This represents a surprising immunity to the large image noise.

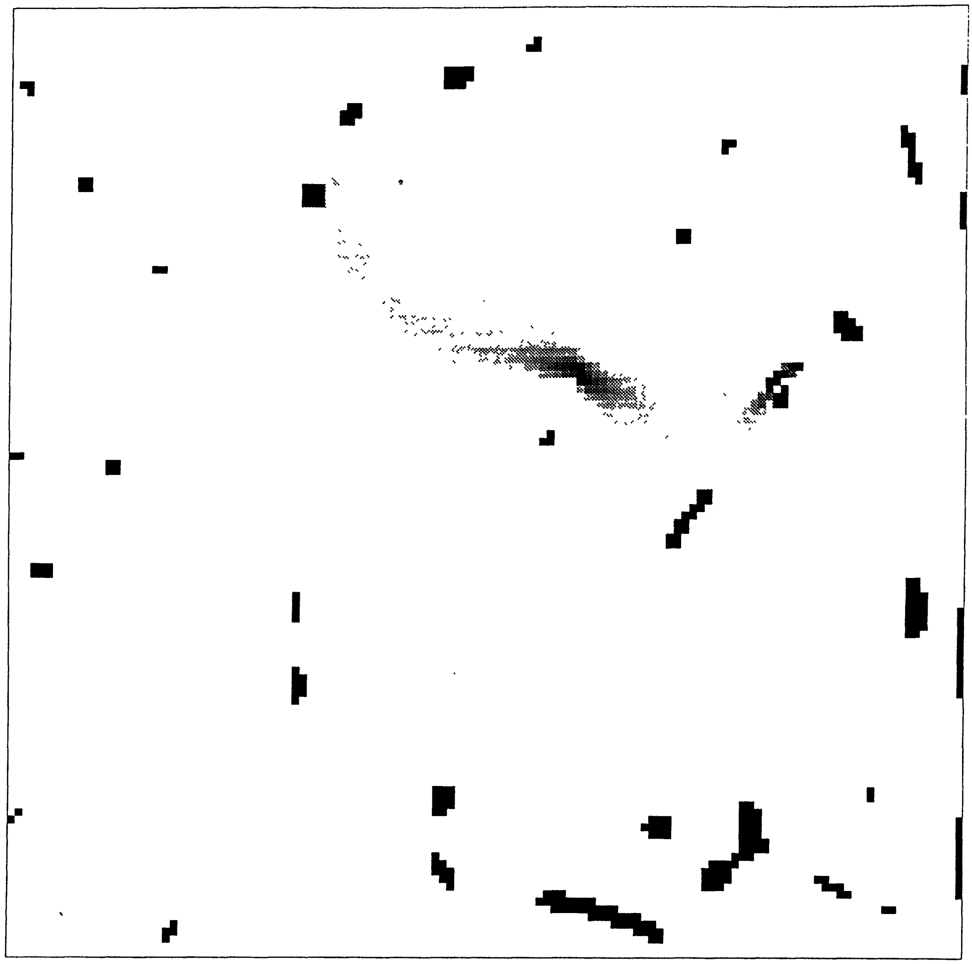

FIG. 21.

In Figure 24 the error is shown for a different reconstruction from the same noisy image with the same scale as before. In this case, the surface was generated from the local maxima after just two cycles. As expected, near the boundary of the image, the region of accurate reconstruction for the maxima-based method is complementary to that of the minima-based method. Since the image boundary does not respect Assumption 2.1 (for either method), the maximum-based method does better at those points near the boundary where the steepest descent direction is outward, while the minima-based method does better where this direction is inward. Together, the two methods give reconstruction with error less than 3 units over most of the image.

5. Proofs of the theorems and propositions. In this section we give the proofs of the main theoretical results stated in this paper, including the

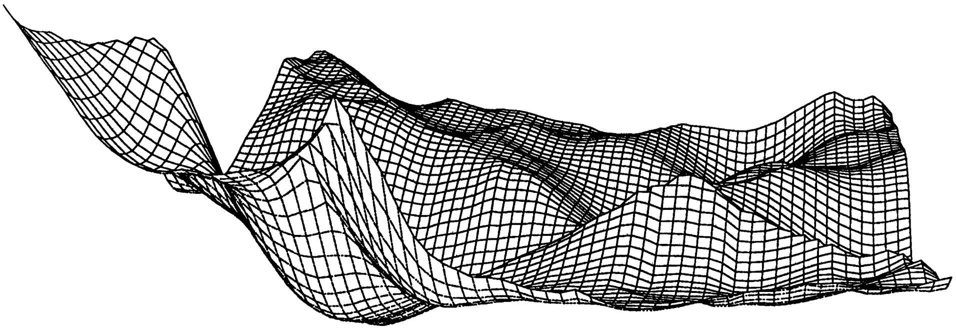

FIG. 22.

representation and convergence theorems, Theorems 2.1 and 2.3. We begin with the representation theorem, which states that in certain regions the minimal cost of the control problem specified in Section 2 equals the unknown function f.

PROOF OF THEOREM 2.1. We first show that $V(x) \geq f(x)$. Let $u(\cdot)$ be any admissible control and define

$$\phi(t) = x + \int_0^t u(s)\,ds, \qquad \tau = \inf\{t: \phi(t) \in \partial\mathscr{D} \cap \mathscr{M}\}. \tag{5.1}$$

Since L is the Legendre transform of H and since $H(x, f_x(x)) = 0$ for $x \in \mathscr{D}$,

$$0 \geq -\langle f_x(x), \beta\rangle - L(x, \beta)$$

for all $\beta \in \mathbb{R}^2$, and in particular

$$-\langle f_x(\phi(t)), u(t)\rangle \leq L(\phi(t), u(t))$$

for $t \in [0, \rho \wedge \tau]$. (Recall that ρ is the controlled stopping time.) This implies that

$$\begin{aligned} -f(\phi(\rho \wedge \tau)) + f(x) &= -\int_0^{\rho\wedge\tau} \langle f_x(\phi(t)), u(t)\rangle\,dt \\ &\leq \int_0^{\rho\wedge\tau} L(\phi(t), u(t))\,dt, \end{aligned}$$

and thus for any admissible control and any $\rho \in [0, \infty)$,

$$\int_0^{\rho\wedge\tau} L(\phi(t), u(t))\,dt + f(\phi(\rho \wedge \tau)) \geq f(x).$$

Since $g(\phi(\rho \wedge \tau)) \geq f(\phi(\rho \wedge \tau))$, we obtain $V(x) \geq f(x)$.

Next we show $V(x) \leq f(x)$. In order to do so we will verify that for each $\varepsilon > 0$ there exists a control $u(\cdot)$ such that for ϕ and τ defined by (5.1) we have $\tau < \infty$, and

$$\int_0^{\tau} L(\phi(t), u(t))\,dt + g(\phi(\tau)) \leq f(x) + \varepsilon. \tag{5.2}$$

FIG. 23.

Recall that $\mathscr{U}(x) = \{(u_1, u_2): |u_1|^2 + |u_2 + \gamma_2|^2 \leq 1\}$ and $L(x,0) = 0$ for $x \in \mathscr{S}$. Let $\mathscr{S}_C$ be a maximal smoothly connected component of $\mathscr{S}$. We first show that any two points near $\mathscr{S}_C$ can be connected by a piecewise continuous control with low cost. For any two points x, y in $\mathscr{S}_C$, there exists a uniformly bounded control $u(t)$ and a time $t^* < \infty$ such that if $\phi(t) = x + \int_0^t u(s)\,ds$, then $\phi(t) \in \mathscr{S}_C$ for $t \in [0, t^*]$ and $y = \phi(t^*)$. Define a new control $u_\lambda(t) \equiv \lambda u(t\lambda)$, where $\lambda > 0$, and let $\phi_\lambda(t) = \phi(t\lambda)$ be the corresponding path. Since

$$\frac{L(x,u)}{\|u\|} \to 0 \quad \text{as } \|u\| \to 0$$

FIG. 24.

for x such that $I(x) = 1$, we can choose λ such that

$$\int_0^{t^*/\lambda} L(\phi_\lambda(t), u_\lambda(t))\, dt = \int_0^{t^*} \frac{L(\phi(t), \lambda u(t))}{\lambda}\, dt \leq \varepsilon/3.$$

Since $L(\cdot, \cdot)$ is continuous on an open neighborhood of $\mathscr{S}_C \times \{0\}$, we can also assume u_λ is continuous. Further, since $|\gamma_2| < 1$, $U(x)$ contains an open neighborhood of the origin for all $x \in \mathscr{S}_C$. Hence there exists $a > 0$ such that for any component $\mathscr{S}_C$ as above and x such that $d(x, \mathscr{S}_C) \leq a$, we have the following. Let y be the point in $\mathscr{S}_C$ closest to x. Then there exists a time $t_a \in [0, \infty)$, a constant control $u(t) = (y - x)/t_a$ and corresponding path $\phi(t) = x + \int_0^t u(s)\, ds$, such that $\phi(t_a) = y$ and $\int_0^{t_a} L(\phi(t), u(t))\, dt \leq \varepsilon/3$. Finally, this shows that for any $\mathscr{S}_C$, and x, y such that $d(x, \mathscr{S}_C) \leq a$ and $d(y, \mathscr{S}_C) \leq a$, there exists a piecewise continuous control $\tilde{u}_{xy}(t)$ and time $\sigma_{xy} \in [0,\infty)$ such that for the corresponding path $\phi_{xy}(t)$ we have

$$\phi_{xy}(0) = x, \qquad \phi_{xy}(\sigma_{xy}) = y$$

and

$$\int_0^{\sigma_{xy}} L(\phi_{xy}(t), \tilde{u}_{xy}(t))\, dt \le \varepsilon.$$

Since f is constant on $\mathscr{S}_C$, we can assume (by choosing $a > 0$ smaller if need be) that $|f(x) - f(y)| \le \varepsilon$.

We now construct the control that satisfies (5.2). If $x \in \mathscr{S}_C$ and $\mathscr{S}_C \subset \mathscr{M}$, then we simply take $\tau = 0$ and are done. There are then two remaining cases: (1) x is contained in some $\mathscr{S}_C$ with $\mathscr{S}_C \cap \mathscr{M} = \varnothing$ or (2) $x \notin \mathscr{S}$. If case (1) holds, then $\mathscr{S}_C$ is either a set of local maxima or saddle points, which implies the existence of a point y such that $f(y) < f(x)$ and $d(y, \mathscr{S}_C) \le a$. Since Assumption 2.1 implies $\mathscr{S} \subset \mathscr{G}^0$, we can assume that $y \in \mathscr{G}$. In this case we will set $u(t) = \tilde{u}_{xy}(t)$ for $t \in [0, \sigma_{xy})$.

Next consider the definition of the control for $t \ge \sigma_{xy}$. For $c > 0$ let $b = \inf\{L(x,u)\colon x \in \mathscr{G},\ d(x,\mathscr{S}) > c,\ u \in \mathbb{R}^2\}$. The continuity of $I(\cdot)$ and the fact that $I(x) < 1$ for $x \notin \mathscr{S}$ implies $b > 0$. Consider any solution (there may be more than one) to

$$\dot{\phi}(t) = \bar{u}(\phi(t)), \qquad \phi(0) = y. \tag{5.3}$$

According to (2.7), for any t such that $\phi(t) \in \mathscr{G} \setminus \mathscr{S}$ and $d(\phi(t), \mathscr{S}) > c$,

$$\begin{aligned} \frac{d}{dt} f(\phi(t)) &= \langle f_x(\phi(t)), \bar{u}(\phi(t)) \rangle \\ &= -L(\phi(t), \bar{u}(\phi(t))) \\ &\le -b. \end{aligned} \tag{5.4}$$

Assumption 2.1 implies $\phi(t)$ cannot exit $\mathscr{G}$. Thus, since $f(x)$ is bounded on $\mathscr{G}$, (5.4) implies that $\phi(t)$ must enter the set $\{x\colon d(x,\mathscr{S}) \le c\}$ in finite time, for any $c > 0$. If $\phi(t) \in \mathscr{S}$ for some $t < \infty$ we define $\eta_y = \inf\{t\colon \phi(t) \in \mathscr{S}\}$ and $w = \phi(\eta_y)$. Otherwise, let t_i be any sequence tending to ∞ as $i \to \infty$. Since $\mathscr{G}$ is compact, we can extract a subsequence (again labeled by i) such that $\phi(t_i) \to v$ for some $v \in \mathscr{S}$. Let i^* be large enough that $\|\phi(t_{i^*}) - v\| \le a$. Since $f(\phi(t_i)) \downarrow f(v)$, we have $f(\phi(t_{i^*})) > f(v)$. For this case we define $\eta_y = t_{i^*}$ and $w = \phi(\eta_y)$.

Integrating (5.4) gives

$$f(y) - f(w) = \int_0^{\eta_y} L(\phi(t), \bar{u}(\phi(t)))\, dt.$$

We then define the control $u(t)$ to be used for $t \in [\sigma_{xy}, \sigma_{xy} + \eta_y)$ to be $\bar{u}(\phi(t - \sigma_{xy}))$.

We now consider the point w. We first examine the case in which the solution to (5.3) does not enter $\mathscr{S}$ in finite time. Since $\|w - v\| \le a$, $\tilde{u}_{wv}(t)$ gives a control such that the application of this control moves $\phi(\cdot)$ from w to

v with accumulated running cost less than or equal to ε. We define $u(t) = \tilde{u}_{wv}(t - (\eta_y + \sigma_{xy}))$ for $t \in [\sigma_{xy} + \eta_y, \sigma_{xy} + \eta_y + \sigma_{wv})$. If the solution to (5.3) reached $\mathscr{S}$ in finite time, we define $w = v$ and $\sigma_{wv} = 0$. Let $\sigma = \sigma_{xy} + \eta_y + \sigma_{wv}$.

Let us summarize the results of this construction. Given any point $x \in \mathscr{S}$ that is not a local minimum, we have constructed a piecewise continuous control $u(\cdot)$ and $\sigma < \infty$ such that if $\phi(t) = x + \int_0^\sigma u(s)\,ds$, then

$$\begin{aligned} f(x) - f(\phi(\sigma)) &= f(x) - f(y) + f(y) - f(w) + f(w) - f(v) \\ &\geq \int_{\sigma_{xy}}^{\sigma_{xy}+\eta_y} L(\phi(t), u(t))\,dt \\ &\geq -2\varepsilon + \int_0^\sigma L(\phi(t), u(t))\,dt. \end{aligned}$$

We have also shown that $f(x) > f(v) = f(\phi(\sigma))$, $\phi(\sigma) \in \mathscr{S}$. Thus, either the component $\mathscr{S}_C$ containing $\phi(\sigma)$ satisfies $\mathscr{S}_C \cap \mathscr{M} \neq \varnothing$ and we are done, or we are back into case 1 and can repeat the procedure. Let K be the number of disjoint compact connected sets that comprise $\mathscr{S}$. Then the strict inequality $f(x) > f(\phi(\sigma))$ and the fact that $f(\cdot)$ is constant on each $\mathscr{S}_C$ imply the procedure can be repeated no more than K times before reaching some $\mathscr{S}_C$ contained in $\mathscr{M}$. If case 2 holds, we can use the same procedure, save that the very first step is omitted. Thus, in general, we have exhibited a control $u(\cdot)$ such that

$$\int_0^\tau L(\phi(t), u(t))\,dt + g(\phi(\tau)) \leq f(x) + (2K+1)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the theorem is proved. □

The following lemma is used in the proof of Theorem 2.3.

LEMMA 5.1. *Let Assumption* 2.1 *hold, and let* $L(x, \beta)$ *and* $\mathscr{U}(x)$ *be defined as in Section* 2. *Assume that* $I \in (0,1]$ *on* $\mathscr{G}$, *let* u *be the piecewise continuous control constructed in the proof of Theorem* 2.1 *that gives the inequality in* (5.2) *and let* ϕ *and* τ *be as defined as in* (5.1). *Then the following conclusions hold.*

1. *The function* $L(x, \cdot)$ *satisfies the following property: Given* $B < \infty$, *there exists* $\delta > 0$ *such that* $\|\nabla_u L(x,u)\| \leq B$ *implies* $d(u, \mathscr{U}(x)^c) \geq \delta$, *where the superscript c denotes complement.*
2. *The path* $\phi(t)$ *remains in* $\mathscr{G}$ *for all* $t \in [0, \tau]$ *and there exists* $\delta > 0$ *such that* $d(u(t), \mathscr{U}(\phi(t))^c) \geq \delta$ *for all* $t \in [0, \tau]$.

The assumption $I(x) > 0$ implies that $\mathscr{U}(x)$ has a nonempty interior and is clearly needed for part 2 of the lemma. Although we must also assume this condition in the convergence theorem (Theorem 2.3), by working with a sequence of domains (see the remark after the statement of Theorem 2.1), we can extend the convergence theorem to cover the case $I(x) \in [0,1]$.

PROOF. If part 1 were not true, then there would exist a sequence $u_i \to u \in \partial\mathscr{U}(x)$ for which $\|\nabla_u L(x, u_i)\| \le B$ (i.e., $L(x, \cdot)$ would not be *essentially smooth* in the terminology of Rockafellar [26]). This can be contradicted either via a direct calculation or the general result given in Theorem 26.3 of [26].

Since $f_x(\cdot)$ is bounded on $\mathscr{G}$, (2.6), (2.7) and part 1 of the lemma imply that for each $x \in \mathscr{G}$ there exists $\delta > 0$ such that $d(\bar{u}(x), \mathscr{U}(x)^c) \ge \delta$. Since $\bar{u}(x)$ and $\mathscr{U}(x)$ are continuous on $\mathscr{G}$, we can assume that δ is independent of x for $x \in \mathscr{G}$.

It is also straightforward to show that we can take $\delta > 0$ small enough that $d(\tilde{u}_{xy}(t), \mathscr{U}(\phi(t))^c) \ge \delta$ for all $t \in [0, \sigma_{xy}]$ whenever both $d(x, \mathscr{S}_C)$ and $d(y, \mathscr{S}_C)$ are sufficiently small, where $\tilde{u}_{xy}$ and σ_{xy} are as defined in the proof of Theorem 2.1 and ϕ is the associated controlled path. This implies part 2 of the lemma. □

We next give the proof of the convergence theorem.

PROOF OF THEOREM 2.3. The basis for the proof of the theorem will be the representation for $V^h(x)$ as the minimal cost for a stochastic control problem, as given in Section 2. Thus we have

$$V^h(x) = \inf E_x\left[\sum_{i=0}^{(N^h \wedge M^h)-1} L(\xi_i^h, u_i^h)\,\Delta t^h(u_i^h) + g\left(\xi^h_{(N^h \wedge M^h)}\right)\right], \tag{5.5}$$

where the infimum is over all admissible control sequences and controlled stopping times M^h, and N^h is the first time of exit from $\mathscr{D}$ or entrance into $\mathscr{M}$. The transition probabilities and interpolation times are those of Example 2.2, and $L(x, u)$ is the running cost used in Section 2 and given in (2.5). General references for terminology, basic results from weak convergence theory and the properties of martingales that are used are references 5 and 14. Let $\{y^n,\ n \in \mathbb{N}\}$ and y be random variables that take values in some metric space S. We follow the standard practice of saying that the random variables y^n converge weakly to y (denoted $y^n \Rightarrow y$) if the measures induced by y^n on S converge weakly to the measure induced by y on S.

In order to study $V^h(x)$, we must first put the representation into a form suitable to taking limits. Let ξ_i^h and u_i^h be the state of the controlled chain and the control applied at time i, respectively. Define $t_n^h = \sum_0^{n-1} \Delta t^h(u_i^h)$. Thus t_n^h is the interpolated time up until discrete time n. Define $\tau^h = t^h_{N^h}$ and $\rho^h = t^h_{M^h}$, and also the piecewise constant interpolations

$$\xi^h(t) = \xi_n^h, \qquad u^h(t) = u_n^h \quad \text{for } t \in \left[t_n^h, t_{n+1}^h\right). \tag{5.6}$$

We consider the processes $\xi^h(\cdot)$ as taking values in $D([0,\infty): \mathbb{R}^2)$, the metric space of $\mathbb{R}^2$-valued functions that are continuous from the right and have limits from the left. [Our interest in the control actually stops when $i = N^h \wedge M^h$. However, to simplify the notation we may assume the control is defined for all $i \in \mathbb{N}$. Specifically, its value will be defined as $(0, -\gamma_2)$ when

$i \geq N^h \wedge M^h$, so that the running costs per unit time after $N^h \wedge M^h$ are automatically bounded. For points $x \notin \mathscr{D}^h$ we can let $L(x, \beta)$ be $L(y, \beta)$, where y is any point in $\mathscr{S}$]. In order to take limits of the sequences of controls it will be convenient to use an alternative representation for the control processes. This alternative representation will allow the use of weak convergence methods and also provide a topology on the space of controls. For all $t \geq 0$ and $h > 0$ define

$$(5.7) \qquad m_t^h(d\alpha) = \delta_{u^h(t)}(d\alpha) \quad \text{and} \quad m^h(A \times B) = \int_B m_t^h(A)\, dt,$$

where $\delta_u(d\alpha)$ is the probability measure that puts a unit mass at u. The m^h are random and take values in the space of measures on $\mathbb{R}^2 \times [0, \infty)$. We consider this space as endowed with the following topology: A sequence l_n of measures on $\mathbb{R}^2 \times [0, \infty)$ converges to l if

$$\int_{\mathbb{R}^2 \times [0,\infty)} s(u,t) l_n(du \times dt) \to \int_{\mathbb{R}^2 \times [0,\infty)} s(u,t) l(du \times dt)$$

for every $s \in C(\mathbb{R}^2 \times [0, \infty))$ with compact support. The random measures $\{m^h,\ h > 0\}$ will be called tight if the corresponding restrictions of the measures to $\mathbb{R}^2 \times [0, T]$ are tight in the usual sense for each $T < \infty$. The representation m^h for the control process $u^h(\cdot)$ is known as the *relaxed control representation.* A measure l on $\mathbb{R}^2 \times [0, \infty)$ will be called a relaxed control process if $l(\mathbb{R}^2 \times [0, t]) = t$ for all $t \in [0, \infty)$. For such a measure, it follows from the existence of regular conditional probability measures that $l(\cdot)$ has a derivative in the following sense: For each $t \in [0, \infty)$ there exists a probability measure $l_t(\cdot)$ on $\mathbb{R}^2$ such that $l(A \times B) = \int_B l_t(A)\, dt$ for all Borel sets $B \subset [0, \infty)$ and $A \subset \mathbb{R}^2$ (see [5], page 502).

With these definitions, we can write

$$(5.8) \quad V^h(x) = \inf E_x\left[\int_0^{(\rho^h \wedge \tau^h)} L(\xi^h(t), u) m^h(du \times dt) + g(\xi^h(\rho^h \wedge \tau^h))\right].$$

In the next lemma we derive some properties of the controlled processes under an arbitrary admissible control with bounded running cost. In the lemma's statement, the expectation operator actually depends on the admissible control strategy that is used. In order to simplify the notation, this dependence is not denoted explicitly. The function L in the statement of the lemma is of course the same as that used in (5.8) and is the running cost defined in Section 2.

LEMMA 5.2. *Consider any sequence of initial conditions* $\xi_0^h \in \mathscr{D}$ *and admissible controls for which*

$$(5.9) \qquad \limsup_{h \to 0} E_{\xi_0^h} \int_0^T L(\xi^h(t), u) m^h(du \times dt) < \infty$$

for all $T < \infty$. *Then the random measures* $\{m^h(\cdot),\ h > 0\}$ *are tight. Suppose that a subsequence (again indexed by* h*) is given such that* $m^h(\cdot)$ *converges* *weakly to a limit $m(\cdot)$ and that ξ_0^h converges to x_0. Then $m(\cdot)$ is a relaxed control process (w.p.1), and hence can be written $m(du \times dt) = m_t(du)\,dt$. Furthermore, the sequence $\{(\xi^h, m^h),\ h > 0\}$ converges weakly to a limit (x, m) that satisfies*

$$x(t) - x_0 = \int_0^t \int_{\mathbb{R}^2} u m(du \times ds) = \int_0^t \int_{\mathbb{R}^2} u m_s(du)\, ds \tag{5.10}$$

for all $t \in [0, \infty)$, w.p.1.

PROOF. If (5.9) holds, then the fact that $L(x, u) = \infty$ whenever $\|u\| > 2$ implies for all $h > 0$ that the restrictions of $m^h(\cdot)$ to $\mathbb{R}^2 \times [0, T]$ are supported on the compact set $\{u: \|u\| \le 2\} \times [0, T]$ (w.p.1). Hence the tightness of $\{m^h,\ h > 0\}$ is automatic. Assume that m^h converges weakly to m. Since $m^h(\mathbb{R}^2 \times [0, t]) = t$ for all ω and $t \in [0, \infty)$, $m(\mathbb{R}^2 \times [0, t]) = t$ for $t \in [0, \infty)$ (w.p.1). Thus $m(\cdot)$ is a relaxed control process (w.p.1). Define the random processes

$$x^h(t) = \int_0^t \int_{\mathbb{R}^2} u m^h(du \times ds) + \xi_0^h \quad \text{and} \quad x(t) = \int_0^t \int_{\mathbb{R}^2} u m(du \times ds) + x_0.$$

We view these processes as taking values in $C([0, \infty); \mathbb{R}^2)$, the space of continuous functions from $[0, \infty)$ to $\mathbb{R}^2$. This space is equipped with the metric inherited as a subset of $D([0, \infty); \mathbb{R}^2)$, under which convergence is equivalent to uniform convergence on each compact subset of $[0, \infty)$. Then the fact that the second marginal of m is Lebesgue measure together with the uniform (in t, h and ω) boundedness of the supports of the m_t^h implies $x^h \Rightarrow x$.

We next prove the weak convergence $\xi^h \Rightarrow x$. If we use the fact that $x^h \Rightarrow x$, then to prove $\xi^h \Rightarrow x$ it suffices to show for each $T < \infty$ that

$$\sup_{t \in [0, T]} \| \xi^h(t) - x^h(t) \| \to 0 \tag{5.11}$$

is probability as $h \to 0$. Let $N^h(T) = \inf\{n: t_n^h \ge T\}$. By (2.10),

$$E\left[\xi_{i+1}^h - \xi_i^h - u_i^h \,\Delta t^h(u_i^h) \middle| (\xi_j^h, u_j^h), j \le i \right] = 0.$$

This implies that

$$\xi_i^h - \xi_0^h - \int_0^{t_i^h} \int_{\mathbb{R}^2} u m^h(du \times ds) \tag{5.12}$$

is a martingale in i. Using (2.11), we compute that

$$E_{\xi_0^h}\left[\xi_{N^h(T)}^h - \xi_0^h - \int_0^{t_{N^h(T)}^h} \int_{\mathbb{R}^2} u m^h(du \times ds) \right]^2 \to 0$$

as $h \to 0$. Hence by the submartingale inequality ([5], Chapter 2),

$$\sup_{0 \le i \le N^h(T)} \left\| \xi_i^h - \xi_0^h - \int_0^{t_i^h} \int_{\mathbb{R}^2} u m^h(du \times ds) \right\| \to 0$$

in probability as $h \to 0$. This clearly implies (5.11), and completes the proof of the lemma. □

PROOF OF THE UPPER BOUND. We first prove $\limsup_{h \to 0} V^h(x) \le V(x)$. We begin by describing a "controllability" property of the chain $\{\xi_i^h, i \in \mathbb{N}\}$. An elementary calculation shows $L(x,u) \le 2$ for all $x \in \mathscr{D}$ and $u \in \mathscr{U}(x)$. Let y be any point in $\mathscr{M}$. The running cost at such a point is $\gamma_3^2 - \gamma_2 u_2 - \gamma_3(1 - |u_1|^2 - |u_2 + \gamma_2|^2)^{1/2}$ for all u such that $|u_1|^2 + |u_2 + \gamma_2|^2 \le 1$, and ∞ otherwise. Since $|\gamma_2| < 1$, there exists $c > 0$ such that $L(y,u) \le 2$ for all u satisfying $\|u\| \le c$. Owing to the continuity of $I(\cdot)$ and the form of $L(x,u)$, we can in fact assume $L(x,u) \le 2$ whenever $\|x - y\| \le c$ and $\|u\| \le c$.

Fix x such that $\|x - y\| \le c$ and define $j_i = |x_i - y_i|/h$, $i = 1, 2$. Suppose the control $u = ([-\text{sign}(x_1 - y_1)]c, 0)$ is applied for exactly j_1 steps, and that after this the control $(0, [-\text{sign}(x_2 - y_2)c])$ is applied for j_2 steps. It is easy to see that this control is admissible. The "deterministic" form of the transition probabilities under these controls guarantees that the controlled chain arrives at y at discrete time $j = j_1 + j_2$. Since for all steps $\Delta t^h(u) = h/c$, the running cost accumulated during this time will be bounded above by

$$(5.13) \quad 2\left[\frac{1}{h}(|x_1 - y_1| + |x_2 - y_2|)\right]\left[\frac{h}{c}\right] = \frac{2}{c}(|x_1 - y_1| + |x_2 - y_2|).$$

We now give the proof of the upper bound. Let $\mathscr{G}$ be any set that satisfies Assumption 2.1. According to Lemma 5.1, for each $x \in \mathscr{G}$ and $\varepsilon > 0$ there exists $\delta > 0$, $\tau < \infty$ and piecewise continuous $u\colon [0,\tau] \to \mathbb{R}^2$ such that if $x(t) = x + \int_0^t u(s)\,ds$, then:

1. $\int_0^\tau L(x(t), u(t))\,dt + g(x(\tau)) \le V(x) + \varepsilon$.
2. $x(\tau) \in \mathscr{M}$, $x(t) \notin \mathscr{M}$ for all $t \in [0,\tau)$.
3. $d(u(t), \mathscr{U}^c(x(t))) \ge \delta$ for all $t \in [0,\tau]$.

Since $\mathscr{U}(x)$ is continuous in x, we can assume $\delta > 0$ has been chosen small enough that

$$(5.14) \quad \begin{aligned} &d(u(t), \mathscr{U}^c(x)) \ge \delta \\ &\text{for all } x \text{ such that } |x - x(t)| \le \delta \text{ and a.e. } t \in [0,\tau]. \end{aligned}$$

Since $\mathscr{G} \subset \mathscr{D}$ is compact and $\mathscr{D}$ is open, we can also assume $d(x, \mathscr{D}^c) \ge \delta$ for all $x \in \mathscr{G}$.

Let $y = x(\tau) \in \mathscr{M}$. Fix $c > 0$ so that the "controllability" property and the bound (5.13) on the running cost will hold. We now define an admissible control scheme for the Markov chain in terms of $u(\cdot)$. To apply $u(\cdot)$ to the chain $\{\xi_i^h, i < \infty\}$ we recursively define the control applied at discrete time i by $u_i^h = u(t_i^h)$ and $t_{i+1}^h = t_i^h + \Delta t^h(u_i^h)$. This defines a control until i such that $t_{i+1}^h \ge \tau$. Let $\{\xi_i^h, i < \infty\}$ be the chain that starts at x and uses this control.

Define

$$S^h = \inf\{i\colon t_i^h \ge \tau \text{ or } \xi_i^h \in \mathscr{M} \text{ or } \xi_i^h \notin \mathscr{D} \text{ or } \|\xi_i^h - x(t_i^h)\| \ge \delta\},$$

and let $\sigma^h = t^h_{S^h}$. By construction the restrictions of the measures $m^h(du \times ds)$ to $\mathbb{R}^2 \times [0, \tau]$ converge weakly to the measure $m(du \times ds) = \delta_{u(s)}(du)\, ds$. Hence by Lemma 5.2, we have $\sup_{0 \le t \le \sigma^h} \|\xi^h(t) - x(t)\| \to 0$ in probability, and for each $\theta > 0$, $P_x\{\|\xi^h_{S^h} - x(\tau)\| \ge \theta\} \to 0$ as $h \to 0$.

Assume $\theta \in (0, c)$. If $\|\xi^h_{S^h} - x(\tau)\| \ge \theta$, then we stop the process at discrete time S^h and pay the stopping cost $g(\xi^h_{S^h}) \le B$. On the set where $\|\xi^h_{S^h} - x(\tau)\| < \theta$, we extend the definition of the control sequence for discrete times larger than S^h according to the discussion at the beginning of the proof. The control applied after S^h will drive ξ^h_i to y in fewer than $2\theta/h$ steps with a running cost that is bounded above by $4\theta/c$.

The total cost for the control scheme and stopping time defined in this way is bounded above by

$$\begin{aligned} E_x \int_0^{\sigma^h} \int_{\mathbb{R}^2} L(\xi^h(s), u) m^h(du \times ds) + 4\theta/c \\ + P_x\{\|\xi^h_{S^h} - x(\tau)\| \ge \theta\} B \\ + P_x\{\|\xi^h_{S^h} - x(\tau)\| < \theta\} \sup\{g(x) : \|x - y\| \le \theta,\ x \in \mathscr{M}\}. \end{aligned}$$

According to the Skorokhod representation theorem ([5], Theorem 3.1.8), we can assume that the convergence $(\xi^h(\cdot), m^h(\cdot)) \to (x(\cdot), m(\cdot))$ is w.p.1 for the purposes of evaluating the limits of the expectations above, and that x and m satisfy (5.10). Owing to the definition of S^h and (5.14), we can apply the dominated convergence theorem to obtain

$$\begin{aligned} \limsup_{h \to 0} V^h(x) &\le \int_0^{\tau} \int_{\mathbb{R}^2} L(x(s), u) m(du \times ds) \\ &\quad + \sup\{g(z) : \|z - y\| \le \theta,\ z \in \mathscr{M}\} + 4\theta/c \\ &= \int_0^{\tau} L(x(s), u(s))\, ds + \sup\{g(z) : \|z - y\| \le \theta,\ z \in \mathscr{M}\} \\ &\quad + 4\theta/c. \end{aligned}$$

Since $\varepsilon > 0$ and $\theta \in (0, c)$ are arbitrary and $y = x(\tau)$,

$$\limsup_{h \to 0} V^h(x) \le V(x).$$

Proof of the lower bound. We now prove $\liminf_{h \to 0} V^h(x) \ge V(x)$. Fix $x \in \mathscr{G} - \mathscr{M}$ and $\varepsilon > 0$. Owing to the definition of $V^h(x)$, there is a controlled Markov chain $\{\xi^h_i,\ i < \infty\}$ with admissible control sequence $\{u^h_i,\ i < \infty\}$ that satisfies $\xi^h_0 = x$, and a stopping time M^h such that

$$V^h(x) \ge E_x \sum_{j=0}^{(N^h \wedge M^h) - 1} L(\xi^h_j, u^h_j)\, \Delta t^h(u^h_j) + E_x g(\xi^h_{N^h \wedge M^h}) - \varepsilon, \tag{5.15}$$

where N^h is the time of first exit from $\mathscr{D}$ or entrance into the set $\mathscr{M}$. Let $\xi^h(\cdot)$ and $u^h(\cdot)$ be the continuous parameter interpolations of $\{\xi^h_i,\ i < \infty\}$ and $\{u^h_i,\ i < \infty\}$ as defined by (5.6), and define $\rho^h = t^h_{M^h}$, $\tau^h = t^h_{N^h}$. We can then rewrite (5.15) as

$$V^h(x) \geq E_x \int_0^{\rho_h \wedge \tau_h} \int_{\mathbb{R}^2} L(\xi^h(s), u) m^h(du \times ds) + E_x g(\xi^h(\rho_h \wedge \tau_h)) - \varepsilon, \tag{5.16}$$

where $m^h(\cdot)$ is the relaxed control representation of the ordinary control $u^h(\cdot)$.

Let $\mathscr{S}_q$, $q = 1, \dots, Q$, be disjoint compact connected sets such that $\mathscr{S} = \bigcup_{q=1}^{Q} \mathscr{S}_q$. The existence of such a decomposition has been assumed in the statement of Theorem 2.3. Now $V(x)$ is constant on each $\mathscr{S}_q$, so there exists $\theta > 0$ such that

$$x \in \mathscr{S}_q, \quad y \in N_\theta(\mathscr{S}_q) \Rightarrow |V(x) - V(y)| \leq \varepsilon \tag{5.17}$$

and such that the sets $N_\theta(\mathscr{S}_q)$ are separated by a distance greater than θ for distinct q. Because the reflected light intensity $I(\cdot)$ is continuous on the closure of $\mathscr{D}$, there is $c > 0$ such that

$$L(x, u) \geq c \quad \text{for all } u \in \mathbb{R}^2,\ x \in \mathscr{D} - \bigcup_{q=1}^{Q} N_{\theta/2}(\mathscr{S}_q). \tag{5.18}$$

For simplicity, we will consider the proof of the lower bound for the case when the initial condition satisfies $x \in N_{\theta/2}(\mathscr{S}_{q^*})$ for some q^*. The general case follows easily using the same arguments. We define a sequence of stopping times by

$$\tau_0^h = 0,$$

$$\sigma_j^h = \inf\left\{ t \geq \tau_j^h : \xi^h(t) \notin \bigcup_{q=1}^{Q} N_\theta(\mathscr{S}_q) \right\},$$

$$\tau_j^h = \inf\left\{ t \geq \sigma_{j-1}^h : \xi^h(t) \in \bigcup_{q=1}^{Q} N_{\theta/2}(\mathscr{S}_q) \text{ or } \xi^h(t) \notin \mathscr{D} \right\}.$$

Consider the processes

$$\Xi^h(\cdot) = (\xi_0^h(\cdot), \xi_1^h(\cdot), \dots), \qquad M^h(\cdot) = (m_0^h(\cdot), m_1^h(\cdot), \dots),$$

where $\xi_j^h(\cdot) = \xi^h(\cdot + \sigma_j^h)$ and where $m_j^h(\cdot)$ is the relaxed control representation of the ordinary control $u^h(\cdot + \sigma_j^h)$. We consider $(\Xi^h(\cdot), M^h(\cdot))$ as taking values in the product space endowed with the usual product space topology, that is, a sequence converges if and only if each finite subset of components converges.

Consider any subsequence along which $V^h(x)$ converges to a point in $(-\infty, \infty]$. Then it is enough to prove that the limit of this subsequence is no less than $V(x)$. We can eliminate the case where the limit is ∞, since in this case the lower bound is automatic. For the rest of the proof we shall assume that we are working with a subsequence (again labeled by h) along which

$V^h(x)$ converges to a bounded limit. Thus the sequence $V^h(x)$ will be uniformly bounded from above.

Lemma 5.2 shows that given any subsequence of $\{(\Xi^h(\cdot), M^h(\cdot)), h > 0\}$, we can extract a further subsequence that converges weakly, and that any limit point

$$(X(\cdot), M(\cdot)) = ((x_0(\cdot), x_1(\cdot), \dots), (m_0(\cdot), m_1(\cdot), \dots))$$

of such a convergent subsequence satisfies

$$x_j(t) - x_j(0) = \int_0^t \int_{\mathbb{R}^2} u m_j(du \times ds),$$

where each $m_j(\cdot)$ is a relaxed control process. In addition, the definition of the stopping times $\{\sigma_j^h\}$ guarantees that $x_0(0) \in \partial N_\theta(S_{q^*})$, and that for all $j > 0$ either $x_j(0) \in \partial N_\theta(\mathscr{S}_q)$ for some q or $x_j(0) \notin \mathscr{D}$.

Let $J^h = \min\{j: \tau_j^h \geq \tau^h\}$, where τ^h has been defined to be the interpolated time at which $\xi^h(\cdot)$ first left $\mathscr{D}$ or entered $\mathscr{M}$. By construction, if $\xi^h(\tau^h) \in \mathscr{S}_q$ for some q (i.e., ξ^h enters $\mathscr{M}$ before it leaves $\mathscr{D}$), then $\xi^h(\tau^h_{J^h-1}) \in N_{\theta/2}(\mathscr{S}_q)$ for the same q. It follows from $\lim_{h\to 0} V^h(x) < \infty$ and the uniform bound from below given in (5.18) that

$$\limsup_{h \to 0} E_x \sum_{0 \leq j < J^h} \left(\tau_{j+1}^h - \sigma_j^h\right) < \infty. \tag{5.19}$$

Define $s_j^h = \tau_{j+1}^h - \sigma_j^h$ and $S^h = (s_0^h, s_1^h, \dots)$. It also follows from (5.18) that there exists $\bar{c} > 0$ such that for all q_1 and q_2,

$$\inf\left\{\int_0^T L(\phi, \dot{\phi})\, ds \colon \phi(0) \in \partial N_\theta(\mathscr{S}_{q_1}),\ \phi(T) \in N_{\theta/2}(\mathscr{S}_{q_2}),\ \phi(t) \in \bar{\mathscr{D}},\ t \in [0,T],\ T > 0\right\} \geq \bar{c} \tag{5.20}$$

and $\tilde{c} > 0$ that is independent of θ for all small $\theta > 0$ such that for all $q_1 \neq q_2$,

$$\inf\left\{\int_0^T L(\phi, \dot{\phi})\, ds \colon \phi(0) \in \partial N_\theta(\mathscr{S}_{q_1}),\ \phi(T) \in N_{\theta/2}(\mathscr{S}_{q_2}),\ \phi(t) \in \bar{\mathscr{D}},\ t \in [0,T],\ T > 0\right\} \geq \tilde{c}. \tag{5.21}$$

We now prove the lower bound $\lim_{h\to 0} V^h(x) \geq V(x)$. Extract a subsequence along which

$$(\xi^h(\cdot), m^h(\cdot), \rho^h, \tau^h, \Xi^h(\cdot), M^h(\cdot), J^h, S^h)$$

converges weakly to a limit

$$(x(\cdot), m(\cdot), \rho, \tau, \Xi(\cdot), M(\cdot), J, S).$$

Note that the bounds (5.19) and (5.20) imply that J and the s_j, $j \in \{0, \dots, J-1\}$, are finite w.p.1. The ρ^h and τ^h are regarded as taking values in the compactified space $[0, \infty]$ in order to guarantee tightness, and hence τ and ρ may take the value ∞. We assume via the Skorokhod representation [5] that the convergence is w.p.1, and consider any ω for which there is convergence. If $\rho < \tau$, that is, if we choose to stop before entering $\mathscr{M}$ or leaving $\mathscr{D}$, we have

$$\liminf_{h \to 0} g\left(\xi^h (\rho^h \wedge \tau^h) \right) = B \geq V(x).$$

Next assume $\tau \leq \rho$. The nonnegativity and lower semicontinuity of $L(x, u)$ in (x, u) then imply

$$\begin{aligned} &\liminf_{h \to 0} \int_0^{\rho^h \wedge \tau^h} \int_{\mathbb{R}^2} L(\xi^h(s), u) m^h(du \times ds) \\ (5.22) \qquad &\geq \liminf_{h \to 0} \sum_{0 \leq j < J^h} \int_{\sigma_j^h}^{\tau_{j+1}^h} \int_{\mathbb{R}^2} L(\xi^h(s), u) m^h(du \times ds) \\ &\geq \sum_{0 \leq j < J} \int_0^{s_j} \int_{\mathbb{R}^2} L\left(x_j(s), u\right) m_j(du \times ds). \end{aligned}$$

Let j_k, $k = 1, \dots, K$, index those values of $j \in \{0, \dots, J-1\}$ such that $x_j(0) \in N_\theta(\mathscr{S}_q)$ and $x_j(s_j) \notin N_{\theta/2}(\mathscr{S}_q)$, that is, the j_k label those trajectories that actually leave the neighborhood of one of the $\mathscr{S}_q$ and either enter the neighborhood of some $\mathscr{S}_{q'}$, $q' \neq q$, or else end up at $\partial\mathscr{D}$. Assume that $j_{k_1} < j_{k_2}$ whenever $k_1 < k_2$.

For any $j \in \{0, \dots, J-1\}$ and $s \in [0, s_j]$ define $u_j(s) = \int_{\mathbb{R}^2} u m_{j,s}(du)$, where $m_{j,s}(\cdot)$ is the derivative of $m_j(\cdot)$. Then $x_j(t) - x_j(0) = \int_0^t u_j(s)\, ds$ for $t \in [0, s_j]$ and by Jensen's inequality,

$$\int_0^{s_j} \int_{\mathbb{R}^2} L\left(x_j(s), u\right) m_j(du \times ds) \geq \int_0^{s_j} L\left(x_j(s), u_j(s)\right) ds.$$

From the definition of $V(\cdot)$ and an elementary dynamic programming argument,

$$V\left(x_j(0)\right) \leq \int_0^{s_j} L\left(x_j(s), u_j(s)\right) ds + V\left(x_j(s_j)\right).$$

Assembling these inequalities gives that for each $j \in \{0, \dots, J-1\}$,

$$(5.23) \qquad \int_0^{s_j} \int_{\mathbb{R}^2} L\left(x_j(s), u\right) m_j(du \times ds) \geq V\left(x_j(0)\right) - V\left(x_j(s_j)\right).$$

According to the definitions of the τ_i^h, σ_i^h and the indices j_k, if $x_{j_k}(s_{j_k}) \in \partial N_{\theta/2}(\mathscr{S}_q)$, then $x_{j_{k+1}}(0) \in N_\theta(\mathscr{S}_q)$ for all $k \in \{1, \dots, K-1\}$. Thus $|V(x_{j_k}(s_{j_k})) - V(x_{j_{k+1}}(0))| \leq 2\varepsilon$ for all such k. Together with the fact that

$x_{j_1}(0) \in N_\theta(\mathscr{S}_{q^*})$ (recall that $x \in N_{\theta/2}(\mathscr{S}_{q^*})$, the last sentence together with (5.22) and (5.23) implies

$$\begin{aligned}
&\liminf_{h \to 0} \int_0^{\rho^h \wedge \tau^h} \int_{\mathbb{R}^2} L(\xi^h(s), u) m^h(du \times ds) \\
&\geq \sum_{k \in \{1, \dots, K\}} \int_0^{s_{j_k}} \int_{\mathbb{R}^2} L(x_{j_k}(s), u) m_{j_k}(du \times ds) \\
&\geq \sum_{k \in \{1, \dots, K\}} \left[V(x_{j_k}(0)) - V(x_{j_k}(s_{j_k})) \right] \\
&\geq V(x) - V(x_{j_K}(s_{j_K})) \\
&\quad + \sum_{k \in \{2, \dots, K\}} \left[V(x_{j_k}(0)) - V(x_{j_{k-1}}(s_{j_{k-1}})) \right] - \varepsilon \\
&\geq V(x) - V(x_{j_K}(s_{j_K})) - (2K - 1)\varepsilon
\end{aligned} \tag{5.24}$$

w.p.1.

Next consider $\liminf_{h \to 0} g(\xi^h(\rho^h \wedge \tau^h))$. As previously noted, $\liminf_{h \to 0} g(\xi^h(\rho^h \wedge \tau^h)) = B$ if $\rho < \tau$. If $\tau \leq \rho$, there are two possibilities. Recall that ρ^h is the controlled stopping time and that τ^h is the first time the process enters $\mathscr{M}$ or leaves $\mathscr{D}$. If $\rho^h < \tau^h$ or $\rho^h \geq \tau^h$ and $\xi^h(\tau^h) \notin \mathscr{M}$, then $g(\xi^h(\rho^h \wedge \tau^h)) = B$. If $\rho^h \geq \tau^h$ and $\xi^h(\tau^h) \in \mathscr{S}_q \subset \mathscr{M}$, then as observed previously $\xi^h(\tau^h_{J^h-1}) \in N_{\theta/2}(\mathscr{S}_q)$ and therefore $V(\xi^h(\tau^h_{J^h-1})) \leq g(\xi^h(\rho^h \wedge \tau^h)) + \varepsilon$. Now j_K is the last index for which there is a transition between different $N_\theta(\mathscr{S}_q)$. Thus, in general,

$$\liminf_{h \to 0} g(\xi^h(\rho^h \wedge \tau^h)) \geq V(x_{j_K}(s_{j_K})) - \varepsilon. \tag{5.25}$$

Combining (5.24) and (5.25) gives

$$\lim_{h \to 0} V^h(x) \geq V(x) - 2\varepsilon E_x K.$$

Now by (5.21) we also have

$$\lim_{h \to 0} V^h(x) \geq \tilde{c} E_x K,$$

where $\tilde{c} > 0$ is independent of $\varepsilon > 0$. Thus $E_x K$ has a bound that is independent of $\varepsilon > 0$. Sending $\varepsilon \to 0$ gives the desired lower bound $\lim_{h \to 0} V^h(x) \geq V(x)$. □

REMARK. Dupuis would like to acknowledge an error in the proof of the lower bound as it appears in [14], Chapter 13. That proof considered convergence in the special case of vertical light only. In the proof, the distinction between the paths which move between neighborhoods of different $\mathscr{S}_q$ and those that do not was omitted, making the assertion of a uniform upper bound on J incorrect. The correct assertion is the uniform upper bound on K, as noted previously.

The next two proofs are of Propositions 3.1 and 3.2, respectively. The first proposition shows that if the initial conditions are the same, then so are the approximations to V that are produced when either of the algorithms from Sections 2 or 3 is used. The second proposition provides a characterization of the approximation in terms of the initial condition, and proves that the iterates of any of the algorithms are monotonic.

PROOF OF PROPOSITION 3.1. We will distinguish the functions that were defined in Sections 3 and 2 by using the superscripts (1) and (2), respectively. It is sufficient to consider only the case $\gamma_2 \le 0$. For convenience we recall the various functions of interest. We have

$$L^{(1)}(x,\beta) = \frac{1}{2}\frac{|\beta_1|^2}{I^2(x)} + \frac{1}{2}\frac{|\beta_2 + (1 - I^2(x))\gamma_2|^2}{v(x)} + \frac{1}{2}(1 - I^2(x)), \tag{5.26}$$

$$L^{(2)}(x,\beta) = \begin{cases} \gamma_3^2 - \gamma_2\beta_2 - \gamma_3\left(I^2(x) - |\beta_1|^2 - |\beta_2 + \gamma_2|^2\right)^{1/2}, & \text{if } |\beta_1|^2 + |\beta_2 + \gamma_2|^2 \le I^2(x), \\ \infty, & \text{if } |\beta_1|^2 + |\beta_2 + \gamma_2|^2 > I^2(x), \end{cases} \tag{5.27}$$

$$\begin{aligned} H^{(1)}(x,\alpha) &= \sup_{\beta \in \mathbb{R}^2}\left[-\langle \alpha, \beta\rangle - L^{(1)}(x,\beta)\right] \\ &= \frac{1}{2}\big[I^2(x)\alpha_1^2 + v(x)\alpha_2^2 \\ &\qquad + 2(1 - I^2(x))\gamma_2\alpha_2 - (1 - I^2(x))\big] \\ &\qquad\qquad \text{if } v(x) = I^2(x) - \gamma_2^2 \ge 0, \end{aligned} \tag{5.28}$$

$$\begin{aligned} H^{(1)}(x,\alpha) &= \inf_{\beta_2}\sup_{\beta_1}\left[-\langle \alpha, \beta\rangle - L^{(1)}(x,\beta)\right] \\ &= \frac{1}{2}\big[I^2(x)\alpha_1^2 + v(x)\alpha_2^2 \\ &\qquad + 2(1 - I^2(x))\gamma_2\alpha_2 - (1 - I^2(x))\big] \\ &\qquad\qquad \text{if } v(x) < 0, \end{aligned} \tag{5.29}$$

and finally

$$\begin{aligned} H^{(2)}(x,\alpha) &= \sup_{\beta \in \mathbb{R}^2}\left[-\langle \alpha, \beta\rangle - L^{(2)}(x,\beta)\right] \\ &= I(x)\left(1 + \|\alpha\|^2 - 2\alpha_2\gamma_2\right)^{1/2} + \alpha_2\gamma_2 - 1. \end{aligned} \tag{5.30}$$

We also recall that the functions $H^{(1)}$ and $H^{(2)}$ arise as two ways of rewriting the image irradiance equation (2.1). The key to establishing the equivalence of the fixed points will be to relate the discrete equations that characterize a fixed point back to these functions. For points x where $v(x) > 0$, it is easy to see that the convexity of both $H^{(1)}$ and $H^{(2)}$ in α implies

$$(5.31)\qquad H^{(1)}(x,\alpha)\begin{Bmatrix} > \\ = \\ < \end{Bmatrix}0 \quad\Leftrightarrow\quad H^{(2)}(x,\alpha)\begin{Bmatrix} > \\ = \\ < \end{Bmatrix}0.$$

For the points where $v(x) \le 0$, the set $\{\alpha\colon H^{(1)}(x,\alpha) = 0\}$ takes the form of a hyperbola with two branches: one that opens in the positive α_2 direction and one that opens in the negative α_2 direction. For the case $\gamma_2 \le 0$ considered here, the image irradiance equation must be satisfied with $f_x(x)$ taking a value in the branch of the hyperbola that opens in the positive α_2 direction. This is because a value in the branch that opens in the negative α_2 direction corresponds to a surface normal that points away from the camera direction, which is impossible if $f(\cdot)$ is a function. Even more to the point is that the positive branch coincides precisely with the zeros of $H^{(2)}(x,\cdot)$.

A consequence of the convex duality formula used in the demonstration of (5.40) is that

$$(5.32)\qquad \begin{aligned} \tilde{H}^{(1)}(x,\alpha) &\equiv \inf_{\beta_2\ge 0}\sup_{\beta_1}\left[-\langle\alpha,\beta\rangle - L^{(1)}(x,\beta)\right] \\ &= \sup_{\alpha_2^*\ge 0} H^{(1)}(x,\alpha+(0,\alpha_2^*)). \end{aligned}$$

The function $\tilde{H}^{(1)}(x,\alpha)$ is automatically nonincreasing in α_2 for each fixed α_1. Note that the zeros of the function $\tilde{H}^{(1)}(x,\alpha)$ correspond exactly to the branch of the hyperbola that opens in the positive α_2 direction. It follows that

$$(5.33)\qquad \tilde{H}^{(1)}(x,\alpha)\begin{Bmatrix} > \\ = \\ < \end{Bmatrix}0 \quad\Leftrightarrow\quad H^{(2)}(x,\alpha)\begin{Bmatrix} > \\ = \\ < \end{Bmatrix}0.$$

(It would indeed be possible to construct the algorithm of Section 3 directly from the equation $\tilde{H}^{(1)}(x, f_x(x)) = 0$, which by (5.33) is equivalent to the image irradiance equation.)

We now consider the equation for a fixed point of (3.7) with running cost $L^{(1)}(x,\beta)$, and that for a fixed point of (2.14) with running cost $L^{(2)}(x,\beta)$. By the statement "(3.7) [or (2.14)] holds with $w(\cdot)$ at x," we mean that (3.7) [or (2.14)] holds at x with V_n^h and V_{n+1}^h replaced by w. It will be convenient to introduce some new notation. Define $Q_1 = \{x\colon\ x_1 \ge 0,\ x_2 \ge 0\}$, $Q_2 = \{x\colon\ x_1 \le 0,\ x_2 \ge 0\}$, $Q_3 = \{x\colon\ x_1 \le 0,\ x_2 \le 0\}$ and $Q_4 = \{x\colon\ x_1 \ge 0,\ x_2 \le 0\}$. We set $w_1(x) = (w(x+h(1,0)) - w(x), w(x+h(0,1)) - w(x))$ and $w_2(x) = (w(x+h(-1,0)) - w(x), w(x+h(0,1)) - w(x))$, and define $w_3(x)$ and $w_4(x)$ analogously.

First assume $w(x) < g(x)$. If we insert the transition probabilities and interpolation interval of Example 2.2, (2.14) implies

$$(5.34)\qquad \left(\min_{i=1,2,3,4}\left[\inf_{u\in Q_i\setminus\{0\}}\frac{L^{(2)}(x,u)+\langle u, w_i(x)\rangle}{|u_1|+|u_2|}\right]\right)\wedge\left(L^{(2)}(x,0)\right) = 0.$$

Recall that $L^{(2)}(x, \beta) \geq 0$ and that $L^{(2)}(x, \beta) = 0$ if and only if $(x, u) \in \mathscr{S} \times \{0\}$. Thus for $x \in \mathscr{S}$ we obtain

$$\min_{i=1,2,3,4}\left[\inf_{u \in Q_i \setminus \{0\}} \frac{L^{(2)}(x,u) + \langle u, w_i(x)\rangle}{|u_1| + |u_2|}\right] \geq 0, \tag{5.35}$$

while for $x \notin \mathscr{S}$ we have

$$\min_{i=1,2,3,4}\left[\inf_{u \in Q_i \setminus \{0\}} \frac{L^{(2)}(x,u) + \langle u, w_i(x)\rangle}{|u_1| + |u_2|}\right] = 0. \tag{5.36}$$

Next consider the case $w(x) = g(x)$. For this case we deduce that (5.35) holds for all x.

We next simplify the inequalities by showing that we can eliminate the denominator. Consider any $x \in \mathscr{S}$. Then for each $i = 1, 2, 3, 4$,

$$\begin{aligned} &\inf_{u \in Q_i \setminus \{0\}} \frac{L^{(2)}(x,u) + \langle u, w_i(x)\rangle}{|u_1| + |u_2|} \geq 0 \\ &\Leftrightarrow \inf_{u \in Q_i \setminus \{0\}} \left[L^{(2)}(x,u) + \langle u, w_i(x)\rangle\right] \geq 0. \end{aligned} \tag{5.37}$$

Now consider any point $x \notin \mathscr{S}$. Owing to the strictly positive lower bound on $L^{(2)}(x, \cdot)$ for such points, for each $i = 1, 2, 3, 4$,

$$\begin{aligned} &\inf_{u \in Q_i \setminus \{0\}} \frac{L^{(2)}(x,u) + \langle u, w_i(x)\rangle}{|u_1| + |u_2|} \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0 \\ &\Leftrightarrow \inf_{u \in Q_i \setminus \{0\}} \left[L^{(2)}(x,u) + \langle u, w_i(x)\rangle\right] \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0. \end{aligned} \tag{5.38}$$

It will be demonstrated below that (5.31), (5.33) and a convex duality formula imply

$$\begin{aligned} &\inf_{u \in Q_i \setminus \{0\}} \left[L^{(2)}(x,u) + \langle u, w_i(x)\rangle\right] \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0 \\ &\Leftrightarrow \inf_{u \in Q_i \setminus \{0\}} \left[L^{(1)}(x,u) + \langle u, w_i(x)\rangle\right] \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0 \end{aligned} \tag{5.39}$$

for $i = 1, 2, 3, 4$ when $v(x) > 0$ and

$$\begin{aligned} &\inf_{u \in Q_i \setminus \{0\}} \left[L^{(2)}(x,u) + \langle u, w_i(x)\rangle\right] \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0 \\ &\Leftrightarrow \sup_{u_2 > 0} \inf_{(-1)^{i+1}u_1 \geq 0} \left[L^{(1)}(x,u) + \langle u, w_i(x)\rangle\right] \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0 \end{aligned} \tag{5.40}$$

for $i = 1, 2$ when $v(x) \leq 0$. Postponing temporarily the proofs of (5.39) and (5.40), we now complete the proof of the equivalence of the fixed points. It will be convenient to separate the cases $\mathscr{A}_1 = \{x\colon x \in \mathscr{S}\}$, $\mathscr{A}_2 = \{x\colon x \notin \mathscr{S}, w(x) < g(x), v(x) > 0\}$, $\mathscr{A}_3 = \{x\colon x \notin \mathscr{S}, w(x) < g(x), v(x) \leq 0\}$, $\mathscr{A}_4 = \{x\colon x \notin \mathscr{S}, w(x) = g(x), v(x) > 0\}$ and $\mathscr{A}_5 = \{x\colon x \notin \mathscr{S}, w(x) = g(x), v(x) \leq 0\}$.

Assume (3.7) holds with $w(\cdot)$ at x. Note that (5.37) holds for $L^{(1)}$ as well as $L^{(2)}$. For $x \in \mathscr{A}_1$ it is always true that (5.35) holds, and by (5.37) with $L^{(2)}$ replaced by $L^{(1)}$ and (5.39), equation (5.35) holds with $L^{(2)}$ replaced by $L^{(1)}$. Since $L^{(2)}(x,0) = L^{(1)}(x,0) = 0$, (3.7) is satisfied for $x \in \mathscr{A}_1$. Next consider $x \in \mathscr{A}_2$. In this case (5.36) holds. Since $L^{(1)}(x,\cdot)$ has a strictly positive lower bound for $x \in \mathscr{A}_2$, (5.38) holds with $L^{(2)}$ replaced by $L^{(1)}$. If we now use (5.38) with $L^{(2)}$ replaced by $L^{(1)}$ and (5.39) we find that (5.36) holds with $L^{(2)}$ replaced by $L^{(1)}$. Since $L^{(1)}(x,0) > 0$ for $x \in \mathscr{A}_2$, (3.7) holds for $x \in \mathscr{A}_2$.

Next consider $x \in \mathscr{A}_3$. The analogue of (5.37) that is appropriate for $L^{(1)}$ is

$$\text{(5.41)} \quad \begin{aligned} &\sup_{u_2>0} \inf_{(-1)^{i+1}u_1 \geq 0} \frac{L^{(1)}(x,u) + \langle u, w_i(x)\rangle}{|u_1| + |u_2|} \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0 \\ &\Leftrightarrow \sup_{u_2>0} \inf_{(-1)^{i+1}u_1 \geq 0} \left[L^{(1)}(x,u) + \langle u, w_i(x)\rangle \right] \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0. \end{aligned}$$

for $i = 1,2$. This equation is most easily verified by showing that strict positivity (respectively, strict negativity) of the left-hand side implies strict positivity (respectively, strict negativity) of the right-hand side, and conversely. For $x \in \mathscr{A}_3$ we have (5.36) for $i = 1,2$, which by (5.40) and (5.41) implies

$$\text{(5.42)} \quad \sup_{u_2>0} \inf_{(-1)^{i+1}u_1 \geq 0} \frac{L^{(1)}(x,u) + \langle u, w_i(x)\rangle}{|u_1| + |u_2|} = 0$$

for $i = 1,2$. Thus (3.7) holds for $x \in \mathscr{A}_3$.

The proofs for $x \in \mathscr{A}_4$ and $x \in \mathscr{A}_5$ are very similar and are omitted. Thus we have proved that any fixed point of (2.14) is also a fixed point of (3.7). The proof of the reverse implication is essentially the same, and for the sake of brevity we give details only for the case $x \in \mathscr{A}_3$. For such points, (3.7) implies (5.42) for $i = 1,2$. Tracing back through (5.40) and (5.41) we see that (5.36) holds for $i = 1,2$. Since for $x \in \mathscr{A}_3$, $L^{(2)}(x,u) = \infty$ for $u \in Q_3 \cup Q_4$, this implies (2.14) at the point x.

Thus all that remains are the proofs of (5.39) and (5.40). Consider first (5.39) and the case $i = 1$. All other cases of i may be treated similarly. Owing to continuity of the bracketed quantities below,

$$\text{(5.43)} \quad \begin{aligned} &\inf_{u \in Q_1 \setminus \{0\}} \left[L^{(j)}(x,u) + \langle u, w_i(x)\rangle \right] \\ &\qquad = \inf_{u \in Q_1} \left[L^{(j)}(x,u) + \langle u, w_i(x)\rangle \right] \end{aligned}$$

for $j = 1,2$. Thus we can consider the infima over Q_1. Define

$$I_{Q_1}(\beta) = \begin{cases} 0, & \beta \in Q_1, \\ +\infty, & \beta \notin Q_1. \end{cases}$$

We can then write

$$\inf_{u \in Q_1} \left[L^{(j)}(x,u) + \langle u, w_i(x)\rangle \right] = \inf_{u \in \mathbb{R}^2} \left[\left(L^{(j)}(x,u) + I_{Q_1}(u) \right) + \langle u, w_i(x)\rangle \right]$$

for $j = 1, 2$. Define the Legendre transforms

$$L^{(j),*}(x, \alpha) = \inf_{u \in \mathbb{R}^2} \left[L^{(j)}(x, u) + \langle u, \alpha \rangle \right],$$
$$I^*_{Q_1}(\alpha) = \inf_{u \in \mathbb{R}^2} \left[I_{Q_1}(u) + \langle u, \alpha \rangle \right].$$

Note that the Legendre transform considered here differs by a sign convention from the one used previously. The relationship between the two transforms is simply

$$H^{(j)}(x, \alpha) = -L^{(j),*}(x, \alpha) \tag{5.44}$$

for $j = 1, 2$. One can easily calculate

$$I^*_{Q_1}(\alpha) = \begin{cases} -\infty, & \alpha_1 < 0 \text{ or } \alpha_2 < 0, \\ 0, & \text{else.} \end{cases}$$

For x such that $v(x) > 0$, the domains of finiteness of $L^{(j)}(x, \cdot)$ and $I_{Q_1}(\cdot)$ have nonempty intersection for $j = 1, 2$. We can therefore apply the convex duality formula for the Legendre transform of a sum ([26], Theorem 16.4) to obtain

$$\begin{aligned} \inf_{u \in Q_1} \left[L^{(j)}(x, u) + \langle u, \alpha \rangle \right] &= \sup_{\alpha^*} \left[L^{(j),*}(x, \alpha - \alpha^*) + I^*_{Q_1}(\alpha^*) \right] \\ &= \sup_{\substack{\alpha_1^* \geq 0 \\ \alpha_2^* \geq 0}} L^{(j),*}(x, \alpha - \alpha^*). \end{aligned}$$

The last equation, (5.31), (5.43) and (5.44) then imply (5.39).

The proof of (5.40) is very similar. We consider only $i = 1$, since $i = 2$ is treated in the same way. An application of the same convex duality formula gives

$$\begin{aligned} \inf_{u \in Q_1 \setminus \{0\}} \left[L^{(2)}(x, u) + \langle u, \alpha \rangle \right] &= \sup_{\substack{\alpha_1^* \geq 0 \\ \alpha_2^* \geq 0}} - H^{(2)}(x, \alpha - \alpha^*) \\ &= - \inf_{\alpha_1^* \geq 0} H^{(2)}(x, (\alpha_1 - \alpha_1^*, \alpha_2)), \end{aligned}$$

where the last equality follows from the fact that $H^{(2)}(x, \alpha)$ is nonincreasing in α_2 for each fixed α_1. A second application gives

$$\begin{aligned} &\sup_{u_2 > 0} \inf_{(-1)^{i+1} u_1 \geq 0} \left[L^{(1)}(x, u) + \langle u, \alpha \rangle \right] \\ &\quad = - \sup_{\alpha_2^* \geq 0} \inf_{\alpha_1^* \geq 0} H^{(1)}(x, (\alpha_1 - \alpha_1^*, \alpha_2 + \alpha_2^*)) \\ &\quad = - \inf_{\alpha_1^* \geq 0} \tilde{H}^{(1)}(x, (\alpha_1 - \alpha_1^*, \alpha_2)). \end{aligned}$$

Thus (5.40) follows from (5.33), which completes the proof. □

PROOF OF PROPOSITION 3.2. For each fixed $x \in \mathscr{D}^h$ and $i \in \mathbb{N}$, any of the Jacobi and Gauss–Seidel iterations we have defined may be written in one of the following forms:

$$(5.45)\quad V_{i+1}^h(x) = \min\left[\inf_u\left(c^h(x,u) + \sum_y p^h(x,y|u)w(y)\right), g(x)\right]$$

or

$$(5.46)\quad V_{i+1}^h(x) = \min\left[\sup_{u_2}\inf_{u_1}\left(c^h(x,u) + \sum_y p^h(x,y|u)w(y)\right), g(x)\right].$$

Here $c^h(x,u)$ denotes the running cost and $w(\cdot)$ is a function that depends on the particular type of iteration used as well as (in the Gauss–Seidel case) the ordering of the states. Note that for both (5.45) and (5.46) the right-hand sides are monotonically nondecreasing in $w(\cdot)$ if we use the partial ordering of real valued functions on $\mathscr{D}^h$ defined by $w_1(\cdot) \le w_2(\cdot)$ whenever $w_1(x) \le w_2(x)$ for all $x \in \mathscr{D}^h$.

First consider the Jacobi iteration. The monotonicity property just described implies that $V_{i+1}^h \le V_i^h$ whenever $V_i^h \le V_{i-1}^h$. Since the initial condition satisfies $V_0^h \ge g$ and since $V_1^h \le g$, we conclude $V_{i+1}^h \le V_i^h$ for all $i \in \mathbb{N}$ by induction.

Next consider the Gauss–Seidel procedure, with a possibly different ordering for each iteration. Regardless of the ordering used on the first iteration, the fact that $V_0^h \ge g$ implies $V_1^h \le V_0^h$. We will again complete the proof via an induction argument. Suppose that $V_i^h \le V_{i-1}^h$. Let $<_i$ denote the ordering that is used on the ith iteration. Let $x_1, x_2, \dots$ denote the states of $\mathscr{D}^h$, ordered according to $<_{i+1}$. The values $V_1^h(y)$, $y \in \mathscr{D}^h$, are used to define $V_{i+1}^h(x_1)$, while the values $V_i^h(y)$, $y <_i x_1$, and $V_{i-1}^h(y)$, $x_1 <_i y$, were used to define $V_i^h(x_1)$. Since $V_i^h \le V_{i-1}^h$, $V_{i+1}^h(x_1) \le V_i^h(x_1)$. We now proceed by induction according to $<_{i+1}$. Fix j and assume $V_{i+1}^h(x_k) \le V_i^h(x_k)$ for $k < j$. The values $V_{i+1}^h(y)$, $y <_{i+1} x_j$, and $V_i^h(y)$, $x_j <_{i+1} y$, are used to define $V_{i+1}^h(x_j)$, while the values $V_i^h(y)$, $y <_i x_j$, and $V_{i-1}^h(y)$, $x_j <_i y$, were used to define $V_i^h(x_j)$. In all cases the values used to define $V_{i+1}^h(x_j)$ are no larger than those used to define $V_i^h(x_j)$. Thus $V_{i+1}^h(x_j) \le V_i^h(x_j)$. By induction on $<_{i+1}$ we conclude $V_{i+1}^h \le V_i^h$, and by induction on the usual ordering on $\mathbb{N}$ we obtain the monotonicity described in part 1 of the proposition.

Since the running costs are nonnegative for the control problem of Section 2, the proved monotonicity establishes the existence of $V^h(x) = \lim_{i\to\infty} V_i^h(x)$ for both the Jacobi and Gauss–Seidel procedures. This is also the case for the control problem of Section 3 with vertical light or for $v(x) \ge 0$ for oblique light. When $v(x) < 0$ for that control problem, we first consider the case $w(y) = 0$ in (5.46). A simple calculation shows that

$$\sup_{u_2\gamma_2<0}\inf_{u_1}\left(L^{(1)}(x,u)\right) > 0$$

and therefore,

$$\sup_{u_2\gamma_2<0} \inf_{u_1} \left(L^{(1)}(x,u)\,\Delta t^h(u) \right) \geq 0.$$

Since the probabilities are nonnegative and $\sum_y p^h(x, y|u) = 1$, this implies

$$\sup_{u_2\gamma_2<0} \inf_{u_1} \left(L^{(1)}(x,u)\,\Delta t^h + \sum_y p^h(x,y|u) w(y) \right) \geq w_{\min},$$

where $w_{\min} \equiv \min_y w(y)$. Thus for the control problem of Section 3 and when $v(x) < 0$, V_n^h is bounded from below by $\min_x V_0^h(x)$. This gives part 1 of the proposition.

We next turn to part 2. Let $\overline{V}^h$ be any fixed point of (2.14) or (3.7) that satisfies $\overline{V}^h(x) \leq V_0^h(x)$ for all $x \in \mathscr{D}^h$. An argument very similar to the one used to prove part 1 shows that

$$\overline{V}^h(x) \leq V_i^h(x) \quad \Rightarrow \quad \overline{V}^h(x) \leq V_{i+1}^h(x).$$

Therefore, by induction, $\overline{V}^h(x) \leq V^h(x)$ for all $x \in \mathscr{D}^h$. □

Acknowledgment. We would like to thank an anonymous referee for many helpful suggestions regarding the organization of this paper.

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