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RATES OF CONVERGENCE OF MEANS FOR DISTANCE-MINIMIZING SUBADDITIVE EUCLIDEAN FUNCTIONALS[1]

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Functionals L on finite subsets A of $\mathbb{R}^d$ are considered for which the value is the minimum total edge length among a class of graphs with vertex set equal to, or in some cases containing, A. Examples include minimal spanning trees, the traveling salesman problem, minimal matching and Steiner trees. Beardwood, Halton and Hammersley, and later Steele, have shown essentially that for $\{X_1, \dots, X_n\}$ a uniform i.i.d. sample from $[0,1]^d$, $EL(\{X_1, \dots, X_n\})/n^{(d-1)/d}$ converges to a finite constant. Here we bound the rate of this convergence, proving a conjecture of Beardwood, Halton and Hammersley.

1. Introduction. In [8], Steele introduced subadditive Euclidean functionals, a class of real-valued functions on finite subsets of $\mathbb{R}^d$. In several examples of interest, the functional $L(\{x_1, \dots, x_n\})$ is the minimal total edge length for a class of graphs with vertex set equal to, or in some examples containing, $\{x_1, \dots, x_n\}$. Examples include Steiner trees and the traveling salesman problem (TSP). In addition, minimal matching, triangulation in the plane and minimal spanning trees (MST's) are not quite subadditive Euclidean functionals, but can be handled by similar methodology ([9]–[11]).

One main result about such functionals (see Steele [8], [9], [11] and references therein) is that there is a constant $\beta = \beta(L, d)$ such that for $X_1, \dots, X_n$ i.i.d. uniform in $[0,1]^d$,

$$\lim_n L(\{X_1, \dots, X_n\})/n^{(d-1)/d} = \beta \quad \text{a.s.} \tag{1.1}$$

From Steele's methods, it is easily seen that also

$$\lim_n EL(\{X_1, \dots, X_n\})/n^{(d-1)/d} = \beta. \tag{1.2}$$

The question we examine here is, how fast is this latter convergence? In [1], Beardwood, Halton and Hammersley conjectured for TSP, and implicitly also for MST and Steiner trees, that at least in the Poissonized version of the problem,

$$\left| EL(\{X_1, \dots, X_{N_n}\}) - \beta n^{(d-1)/d} \right| = O(n^{(d-2)/d}), \qquad d \geq 2,$$

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when N_n is Poisson(n). We will show that in fact, for minimal matching, TSP, Steiner trees and MST, one has for some positive $C_i(L, d)$,

$$\beta n^{(d-1)/d} - C_1 n^{(d-2)/(2(d-1))} \le EL(\{X_1, \dots, X_n\}) \le \beta n^{(d-1)/d} + C_2 n^{(d-2)/d}. \tag{1.3}$$

Note in particular that for dimension $d = 2$,

$$|EL(\{X_1, \dots, X_n\}) - \beta n^{1/2}| \quad \text{stays bounded as } n \to \infty.$$

For TSP with $d = 2$, Rhee [6] showed that for some $K_1, K_2 > 0$,

$$\beta n^{1/2} + K_1 \le EL(\{X_1, \dots, X_{N_n}\}) \le \beta n^{1/2} + K_2.$$

Rhee and Talagrand [7] showed that for TSP and Steiner trees with $d = 2$, for some constant K,

$$P\left[|L(\{X_1, \dots, X_n\}) - EL(\{X_1, \dots, X_n\})| > t\right] \le K \exp(-t^2/K), \qquad t > 0.$$

Thus this same result is valid when centering at $\beta n^{1/2}$ instead of at the expectation.

In order to obtain (1.3), we will have to restrict our class of functionals somewhat more than did Steele in [8]. Our main purpose is to include minimal matching, TSP, Steiner trees and MST. These are examples of distance-minimizing functionals, defined as follows: Let

$$\mathscr{E}_n := \{\{i, j\} : 1 \le i < j \le n\}.$$

We call L *distance-minimizing* if it satisfies:

(A1) For each n there is a set $\mathscr{G}_n \subset 2^{\mathscr{E}_n}$ such that either

(i) $$L(\{x_1, \dots, x_n\}) = \min_{G \in \mathscr{G}_n} \sum_{\{i, j\} \in G} |x_i - x_j|$$

or

(ii) $$L(\{x_1, \dots, x_n\}) = \inf_{m \ge n} \min_{x_{n+1}, \dots, x_m \in \mathbb{R}^d} \min_{G \in \mathscr{G}_m} \sum_{\{i, j\} \in G} |x_i - x_j|,$$

with the infimum achieved for every choice of $\{x_1, \dots, x_n\}$.

The latter case includes, for example, Steiner trees. Here $2^{\mathscr{E}_n}$ denotes the collection of all subsets of $\mathscr{E}_n$. We call the additional vertices $x_{n+1}, \dots, x_m$ *Steiner points*. It is clear that the minimum is achieved for fixed m and/or n. A subset $\mathscr{G}_n$ of $2^{\mathscr{E}_n}$ is *permutation-invariant* if for every permutation σ of $\{1, \dots, n\}$ and $G \in \mathscr{G}_n$,

$$\{\{\sigma(i), \sigma(j)\} : \{i, j\} \in G\} \in \mathscr{G}_n.$$

Note that since L is a function of the set $\{x_1, \dots, x_n\}$, rather than of the sequence $(x_1, \dots, x_n)$, $\mathscr{G}_n$ must be permutation-invariant. Given the set $\{x_1, \dots, x_n\}$, some Steiner points $x_{n+1}, \dots, x_m$ if in case (ii) of (A1), and

$G \in \mathscr{G}_m$, we call

$$\tilde{G} := \{[x_i, x_j] : \{i, j\} \in G\}$$

an *allowable graph* on $\{x_1, \dots, x_n\}$. Here $[x, y]$ denotes $\{tx + (1-t)y: 0 \le t \le 1\}$. Thus the set of allowable graphs cannot, in an obvious sense, depend on the locations or the labeling of the points $x_1, \dots, x_n$. Therefore, minimal triangulation length (see [10]), for example, is excluded.

Our second assumption is:

(A2) The functional L is *scale-bounded for all dimensions*; that is, for each $k \ge 1$ there exists K_k such that if $\{x_1, \dots, x_n\} \subset [0,1]^k \times \{0\}^{d-k}$ in $\mathbb{R}^d$, then $L(\{x_1, \dots, x_n\}) \le K_k n^{(k-1)/k}$.

This is closely related to (A6) of [8].

The lower bound in (1.3) is easy and is essentially implicit in Steele's work [8]. Lower bounds are closely related to the following: Suppose we have disjoint finite subsets A and B of $\mathbb{R}^d$, and corresponding allowable graphs $\tilde{G}_A$ and $\tilde{G}_B$. What minimal total length of edges must be added to tie $\tilde{G}_A$ and $\tilde{G}_B$ together to create an allowable graph on $A \cup B$? This minimum length is an upper bound for the quantity $L(A \cup B) - L(A) - L(B)$. We will use the following assumption:

(A3) The functional L is *simply adjoining*; that is, there is a constant $C_3(L, d)$ such that for A, B, $\tilde{G}_A$ and $\tilde{G}_B$ as above, at most C_3 edges must be added to $\tilde{G}_A \cup \tilde{G}_B$ to create an allowable graph on $A \cup B$.

For minimal matching, TSP, Steiner trees and MST, at most one edge must be added, and its length is bounded by diam$(A \cup B)$. This is what underlies the fact that these functionals are subadditive (cf. (A5) in [8]), which, in turn, is roughly what gives rise to the lower bound in (1.3).

Upper bounds, by contrast, are related to superadditivity. Suppose we have a finite subset $A = \{x_1, \dots, x_n\}$ of a rectangle in $\mathbb{R}^d$, and an allowable graph $\tilde{G}$ on A. We then split the rectangle into two adjacent rectangles R and S using a hyperplane H. If all the broken edges, that is, those passing through H, are deleted, what minimum total length of edges must be added on average to the remaining two graphs to "patch" them, that is, to create a separate allowable graph in each of R and S? This minimum length is an upper bound for the quantity $E[L(A) + L(B) - L(A \cup B)]$ and is related to the expected total length of the broken edges and/or to the expected number of broken-edge endpoints. In the non-Steiner case (i) of (A1), this expected number can be bounded by n, and for the Steiner tree by $2n$. Using Steele's arguments (see, e.g., (2.2) and Lemma 4.2 in [8]), this leads to a weaker upper bound than in (1.3):

$$EL(\{X_1, \dots, X_n\}) \le \beta n^{(d-1)/d} + C_4 n^{(d-2)/(d-1)} \tag{1.4}$$

for the Steiner tree, minimal matching and TSP. One of the major tasks in improving (1.4) to (1.3), then, is to get a better estimate on the expected total number of endpoints in H and/or total length of the broken edges.

Before formulating all this more precisely, let us consider this patching procedure in some examples. Let $l(e)$ denote the length of an edge e and $l(E) := \sum_{e \in E} l(e)$ for E a set of edges. Given a graph $\tilde{G}$ and a subset U of its vertices, let

$$\tilde{G}|_U := \{[x, y] \in \tilde{G}: x, y \in U\},$$

$$\partial_{\tilde{G}} U := \{[x, y] \in \tilde{G}: x \in U,\ y \notin U\}.$$

Consider first minimal matching in dimension $d = 2$. The allowable graphs are those that are maximal subject to having degree at most 1 for each vertex. Thus vertices are paired off, with one extra unattached vertex if n is odd. Suppose we have a finite subset $A = \{x_1, \dots, x_n\}$ of some rectangle in $\mathbb{R}^2$, an allowable graph $\tilde{G}$ representing a matching on A and a line H which splits the rectangle into two subrectangles R and S; see Figure 1(a). For each edge e which crosses H, that is, $e \in \partial_{\tilde{G}}(A \cap R)$, let z_e denote the point where e meets H and let x_e denote the endpoint of e in R.

One can create a matching on $A \cap R$ by the following four-step procedure. First replace each edge e crossing H with $[x_e, z_e]$. Second, pair off all (or all but one) of the points z_e in H and add in the corresponding edges, with the pairing done so as to minimize the total length of the added edges; see Figure 1(b). Third, if z_e and $z_{e'}$ are paired in this way, replace the three segments $[x_e, z_e]$, $[z_e, z_{e'}]$ and $[z_{e'}, x_{e'}]$ with $[x_e, x_{e'}]$; see Figure 1(c). Fourth, if $|A|$ and $|\partial_{\tilde{G}}(A \cap R)|$ are both odd, and the originally unmatched vertex in A is in R, one must add an extra edge, which we will call a *leftovers edge*, of length at most diam(A) to connect the originally unmatched vertex in A to the unmatched vertex in $\partial_{\tilde{G}}(A \cap R)$. Let F_R denote the set of edges added to

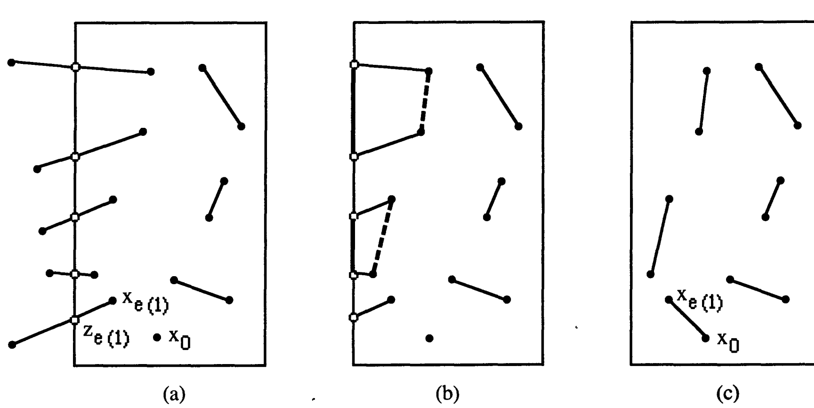

FIG. 1. (a) *Rectangle R, edges meeting R and points z_e (circles)*; (b) *matching in H (heavy lines) and corresponding edges of form $[x_e, x_{e'}]$ (dashed lines)*; (c) *final matching of $A \cap R$, with leftovers edge $[x_{e(1)}, x_0]$*.

$\tilde{G}|_{(A\cap R)}$ by this four-step procedure, that is, the edges of form $[x_e, x_{e'}]$ together with the leftovers edge, if any. Here $|F|$ denotes the cardinality of a finite set F. We have

$$|x_e - x_{e'}| \le |x_e - z_e| + |z_e - z_{e'}| + |z_{e'} - x_{e'}|. \tag{1.5}$$

Summing this inequality over all edges $[z_e, z_{e'}]$ and adding the leftovers edge, we obtain

$$\begin{aligned} l(F_R) \le\ & l(\{[x_e, z_e]: e \in \partial_{\tilde{G}}(A \cap R)\}) \\ & + L(\{z_e: e \in \partial_{\tilde{G}}(A \cap R)\}) + \operatorname{diam}(A). \end{aligned} \tag{1.6}$$

If we start with the graph $\tilde{G} \cap R$ produced by the first of the four steps, the second through fourth steps reduce its length by $l(\{[x_e, z_e]: e \in \partial_{\tilde{G}}(A \cap R)\})$, since the edges $\{[x_e, z_e]: e \in \partial_{\tilde{G}}(A \cap R)\}$ are removed, but also increase it by $l(F_R)$, for a net increase of at most $L(\{z_e: e \in \partial_{\tilde{G}}(A \cap R)\}) + \operatorname{diam}(A)$. The latter sum is bounded, uniformly in $n = |A|$, by twice the length of $H \cap (R \cup S)$. This and the analogous fact for S imply that

$$\begin{aligned} & L(A \cap R) + L(A \cap S) - L(A \cap (R \cup S)) \\ & \text{is bounded uniformly in } n \text{ and in } A \text{ with } |A| = n, \end{aligned} \tag{1.7}$$

a strong form of superadditivity which leads readily to the upper bound in (1.3), using the arguments in [8].

Our four-step patching procedure—truncate edges crossing H, add in the edges of an allowable graph on the set of crossing points z_e, replace each added edge $[z_e, z_{e'}]$ with the corresponding edge $[x_e, x_{e'}]$, then add/delete a small number of additional edges—can be applied to other functionals as well. For TSP, the allowable graphs are self-avoiding loops which visit every vertex. In dimension $d = 2$, the graph $\tilde{G} \cap R$ consists of one or more disjoint connected paths; roughly, the edges $[z_e, z_{e'}]$ are used to connect these paths into a single path. The differences from minimal matching are roughly the following; see Example 3.1 for a more complete description.

1. In the second step, one first constructs a self-avoiding loop path γ of minimal length visiting all sites z_e in H, then constructs a loop path α in $\gamma \cup (\tilde{G} \cap R)$ which traverses each component of $\tilde{G} \cap R$ exactly once and each edge of γ at most twice. The edges $[z_e, z_{e'}]$ which get added in our second step are those which are in α, with two edges from z_e to $z_{e'}$ added if $[z_e, z_{e'}]$ is traversed twice by α.
2. As a consequence, the net increase in length in steps two and three is at most $2L(\{z_e: e \in \partial_{\tilde{G}}(A \cap R)\})$, which is bounded uniformly in $n = |A|$ by $2l(H \cap (R \cup S))$.
3. In the third step, a single edge of the form $[x_e, x_{e'}]$ may replace several edges of the form $[z_e, z_{e'}]$.
4. In the fourth step there is no leftovers edge. Thus (1.7) is valid for TSP as well. This entire procedure for TSP, including the existence of the path α, is described in the proof of Lemma 2 of [1]; see also [5].

Our four-step procedure is also valid in dimensions $d \geq 3$ for both minimal matching and TSP, but a significant complication arises: the length $L(\{z_e: e \in \partial_{\tilde{G}}(A \cap R)\})$ of the allowable graph in the hyperplane is no longer bounded uniformly in A and/or n. In fact, this length depends very much on the number of sites z_e in H, that is, on the number of edges which cross H. From (A2) we know that

$$L(\{z_e: e \in \partial_{\tilde{G}}(A \cap R)\}) \leq K_{d-1}\left|\{z_e: e \in \partial_{\tilde{g}}(A \cap R)\}\right|^{(d-2)/(d-1)}.$$

If A is a "typical" uniform random sample of n points from $R \cup S$, then $|\{z_e: e \in \partial_{\tilde{G}}(A \cap R)\}|$ is of order $n^{(d-1)/d}$. Thus we would hope to show that, roughly, in place of (1.7),

$$E[L(A \cap R) + L(A \cap S) - L(A \cap (R \cup S))] = O(n^{(d-2)/d}), \tag{1.8}$$

which would lead to the upper bound in (1.3), but we must overcome the problem of "atypical" samples A in which too many edges cross H.

For MST (see Figure 2), an additional difficulty arises in all dimensions. In the second step, we add the edges of an MST T of the set of all vertices z_e in H. In the third step, for each edge $[z_e, z_{e'}]$ in T we replace the three edges $[x_e, z_e]$, $[z_e, z_{e'}]$ and $[z_{e'}, x_{e'}]$ with $[x_e, x_{e'}]$. When the inequalities (1.5) are summed, the number of appearances of a given term $|x_e - z_e|$ in the sum is the degree $\deg_T(z_e)$ of z_e in T. This degree is bounded by some constant D_d in dimension d, as observed by Steele [10]. In the fourth step, there is no leftovers edge, but one may need to delete some edges $[x_e, x_{e'}]$ to make the resulting graph a tree. Hence instead of (1.6) we obtain

$$\begin{aligned} l(F_R) &\leq \sum_{e \in \partial_{\tilde{G}}(A \cap R)} \deg_T(z_e)|x_e - z_e| + L(\{z_e: e \in \partial_{\tilde{G}}(A \cap R)\}) \\ &\leq D_{d-1} l(\{[x_e, z_e]: e \in \partial_{\tilde{G}}(A \cap R)\}) + L(\{z_e: e \in \partial_{\tilde{G}}(A \cap R)\}). \end{aligned} \tag{1.9}$$

Unlike TSP and minimal matching, the first term on the right side of (1.9) is not cancelled out by the removal of the edges $\{[x_e, z_e]: e \in \partial_{\tilde{G}}(A \cap R)\}$. Hence to bound the net increase in length in steps two and three, we must be concerned with not only the number of edges crossing H, but also with the total length of these edges. As there are typically of order $n^{(d-1)/d}$ such edges when A is a uniform random sample from $R \cup S$, and the typical edge has length of order $n^{-1/d}$, this total length should typically be of order $n^{(d-2)/d}$, leading us to hope that (1.8) is again valid. Again the difficulty is "atypical" samples A in which edges crossing H are too numerous or too long.

We will at times need to split finite sets A not only using a hyperplane H, but also more generally into $A \cap R$ and $A \cap R^c$ for some subrectangle R of a rectangle containing the whole set A. Further, when it exists, the set of Steiner points in the allowable graph on A needs to be split as well. The natural split of Steiner points is according to location in R versus R^c, but we will need a different split for (2.20). Therefore, we formulate our assumption on L in terms of general splits of A and of the Steiner points, as follows.

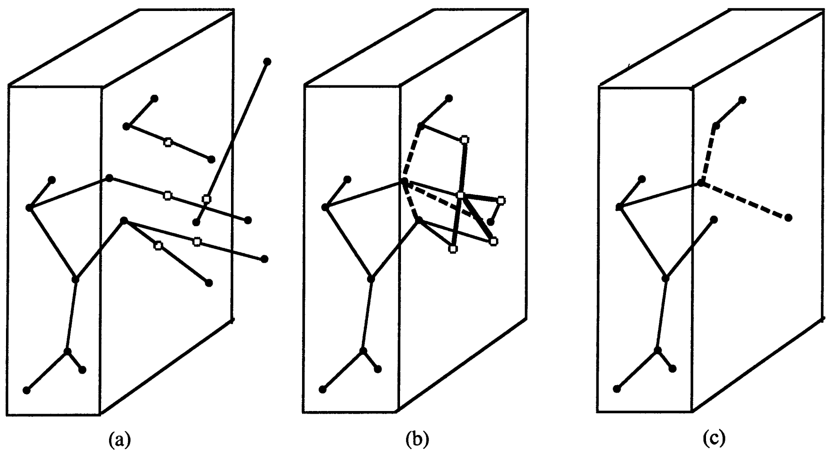

FIG. 2. (a) *Rectangle R, edges meeting R and points* z_e *(circles)*; (b) *spanning tree in H (heavy lines) and corresponding edges of form* $[x_e, x_{e'}]$ *(dashed lines)*; (c) *final tree spanning* $A \cap R$.

(A4) L is *patchable*, that is, for some constants $C_i(L, d)$, given finite sets $B \subset A$, given an allowable graph $\tilde{G}$ on A with vertex set denoted V, given a split $V = V' \cup V''$ into disjoint subsets with $V' \supset B$ and $V'' \supset A \setminus B$ and given a distinct point $z_e \in e$ for each edge $e \in \partial_{\tilde{G}} V'$, there exists a set F_B of edges [each with both endpoints in B in the non-Steiner case (i) of (A1)] such that:

(i) $\{[x, y] \in \tilde{G}: x, y \in V'\} \cup F_B$ is an allowable graph on B;

(ii)
$$l(F_B) \le C_5 l(\{[x, z_e]: e = [x, y] \in \partial_{\tilde{G}} V', x \in V'\}) + C_6 L(\{z_e: e \in \partial_{\tilde{G}} V'\}) + C_7 \operatorname{diam}(B).$$

The points of $V' \setminus B$ and of $V'' \setminus (A \setminus B)$ are, of course, the Steiner points. If the constant C_5 in (A4) is 1, we say L is *efficiently patchable*. In this case, which we have shown includes minimal matching and TSP, the first term on the right side of (A4)(ii) is cancelled out by the reduction in total length when the edges of $\partial_{\tilde{G}} V'$ are deleted.

REMARK 1.1. By taking limits, one sees that if (A4) is true as stated, when it is true without the requirement that the z_e's be distinct for distinct e, provided that, in constructing allowable graphs, coinciding z_e's are interpreted as distinct vertices in the same spatial location. We will tacitly use this interpretation henceforth. This is equivalent to extending the domain of L in (A1) in the obvious way to multisets, that is, sets in which each element has a multiplicity. In (A4)(ii), $\{z_e: e \in \partial_{\tilde{G}} V'\}$ must be interpreted as a multiset.

The main tool in the proof of Lemmas 4.2 and 4.3 of [8], and the analogous result for Steiner trees, is essentially the observation that TSP, minimal matching and Steiner trees are efficiently patchable. As we have mentioned, for TSP this fact is implicit in Lemma 2 of [1]. We have shown that MST is patchable, but with $C_5 > 1$. Thus for MST we will (roughly) have to bound the total length of the edges which cross the boundary of a rectangle.

The problem of "atypical" samples in which edges crossing a dividing hyperplane are too numerous or too long will be attacked roughly as follows. The arguments in [8] establishing (1.1) involve subdividing a cube of side t, containing a random set of vertices, into a grid of m^d cubes of side t/m. We will instead use a grid of only $(m-1)^d$ cubes of side t/m. This grid can be moved around while remaining entirely inside the cube of side t. Averaging over the locations of this grid eliminates this atypical-samples problem, thanks to some elementary integral geometry; see (2.7) and (2.11). For large m, the price paid for reducing m is minimal.

As was done for some results in [1] and [8], we will first prove something like (1.3) for a Poisson process rather than a uniform sample. Let $\Pi(R)$ denote the set of sites in $R \subset \mathbb{R}^d$ of a Poisson process of unit intensity, and $\Pi_r(R)$ similarly for a Poisson process in $[0,1]^d$ with intensity r. Note that

$$(1.10) \qquad L(\Pi_r(R)) =_d r^{-1/d} L\big(\Pi(r^{1/d}R)\big) \quad \text{for } R \subset [0,1]^d,$$

where $=_d$ denotes equality in distribution. Let

$$\psi_L(n) := EL(\{X_1, \dots, X_n\})$$

and let N_n be a Poisson(n) r.v. independent of $\{X_1, \dots, X_n\}$, so that

$$(1.11) \qquad EL\big(\Pi_n([0,1]^d)\big) = E\psi_L(N_n), \qquad n \geq 1.$$

The sequence of values $\{EL(\Pi_n([0,1]^d)),\ n \geq 1\}$ for the Poisson process can thus be thought of as a smoothed version of the original sequence $\{EL(\{X_1, \dots, X_n\}),\ n \geq 1\}$. We will obtain an analog of (1.3) for the Poisson process:

$$(1.12) \quad \beta r^{(d-1)/d} - C_8 \leq EL\big(\Pi_r([0,1]^d)\big) \leq \beta r^{(d-1)/d} + C_9 r^{(d-2)/d}.$$

To obtain (1.3) from this requires "desmoothing." For the result (1.2) without error terms, such desmoothing was done, for example, for MST by Steele in [11] using Abelian and Tauberian theorems. Here we will rely more on brute force, making use of the properties (A1)–(A4) of the functionals of interest, along with the following assumption.

(A5) L has *bounded graph degree*, that is, there exists $C_{10}(L, d)$ such that every vertex of every allowable graph which achieves the minimum in (A1) has degree at most C_{10}.

These will be shown to imply

$$(1.13) \quad \Big|EL(\{X_1, \dots, X_n\}) - EL\big(\Pi_n([0,1]^d)\big)\Big| = O\big(n^{(d-2)/(2(d-1))}\big).$$

In (A5), one can change "every allowable graph" to "some allowable graph" if either L is efficiently patchable or (1.17) remains true when the graphs are restricted to those satisfying the degree bound.

We can now formulate our main results. Let $\mathscr{H}(\{x_1, \dots, x_n\}; L)$ denote the set of allowable graphs $\tilde{G}$ for which the corresponding G achieves the minimum in (A1).

THEOREM 1.1. *Suppose the functional L on finite subsets of $\mathbb{R}^d$ $(d \geq 2)$ is distance-minimizing and simply adjoining. Then for some $C_8(L, d)$,*

$$\beta = \beta(L, d) = \lim_{r \to \infty} EL\big(\Pi_r([0,1]^d)\big)/r^{(d-1)/d} \quad \textit{exists} \tag{1.14}$$

and

$$\beta r^{(d-1)/d} - C_8 \leq EL\big(\Pi_r([0,1]^d)\big). \tag{1.15}$$

If also L is scale-bounded for all dimensions, and either

$$L \textit{ is efficiently patchable} \tag{1.16}$$

or L is patchable and for $\{X_1, \dots, X_n\}$ uniform in $[0,1]^d$,

$$E\left(\inf_{\tilde{G} \in \mathscr{H}(\{X_1, \dots, X_n\}; L)} \sum_{e \in \tilde{G}} l(e)^2 \right) = O(n^{(d-2)/d}), \tag{1.17}$$

then for some $C_9(L, d)$,

$$EL\big(\Pi_r([0,1]^d)\big) \leq \beta r^{(d-1)/d} + C_9 r^{(d-2)/d}. \tag{1.18}$$

When Steiner points are allowed, (1.17) is typically valid, as the sum of squares can be made arbitrarily small by adding vertices to break up edges into smaller edges, provided the resulting graph is also allowable.

Theorem 1.1 and (1.13) immediately yield the following theorem.

THEOREM 1.2. *Suppose the functional L on finite subsets of $\mathbb{R}^d$ $(d \geq 2)$ is distance-minimizing, is simply adjoining, is scale-bounded for all dimensions, has bounded graph degree and either is efficiently patchable or is patchable and satisfies (1.17). Then for $X_1, \dots, X_n$ uniform in $[0,1]^d$, for some positive $C_i(L, d)$,*

$$\begin{aligned} \beta n^{(d-1)/d} - C_1 n^{(d-2)/(2(d-1))} &\leq EL(\{X_1, \dots, X_n\}) \\ &\leq \beta n^{(d-1)/d} + C_2 n^{(d-2)/d}. \end{aligned} \tag{1.19}$$

2. Proofs. Following [8], for fixed m we divide the unit cube into m^d cubes of side $1/m$, and label these $Q_1, \dots, Q_{m^d}$. We also define

$$\varphi(u) := EL\big(\Pi([0,u]^d)\big)$$

and

$$H_i(t) := \{y \in \mathbb{R}^d : y_i = t\}.$$

Throughout our proofs, $c_1, c_2, \dots$ denote constants which depend only on L and d, and $C_1, C_2, \dots$ are the constants specified in the introduction. "Rectangle" or "cube" will mean a product of left-closed, right-open intervals. Note that, if R is a rectangle of maximum side length s with faces $S_1, \dots, S_{2d}$ and A is a finite subset of ∂R, then by (A2) and (A3),

$$(2.1)\qquad \begin{aligned} L(A) &\le \sum_{i=1}^{2d} sK_{d-1}|A \cap S_i|^{(d-2)/(d-1)} + (2d-1)C_3 d^{1/2} s \\ &\le c_1 |A|^{(d-2)/(d-1)} s. \end{aligned}$$

Lemma 2.1. *Let R be a cube in $\mathbb{R}^d$ of side t, consisting of 2^{kd} cubes $Q_1, \dots, Q_{2^{kd}}$ of side $t/2^k$. Let A be a finite subset of $\mathbb{R}^d$. Suppose L satisfies the hypotheses of Theorem 1.1 and $\tilde{G}$ is an allowable graph on A. Let*

$$F := \{e \in \tilde{G} \colon e \text{ crosses } \partial Q_i \text{ for some } Q_i\}.$$

Then there exist disjoint $F_1, \dots, F_{2^{kd}}$ and positive constants $\delta_1, \dots, \delta_{2^{kd}}$ such that $F = \bigcup F_i$,

$$(2.2)\qquad \sum_{i=1}^{2^{kd}} \delta_i^{d-1} \le 2^{d+k}$$

and

$$(2.3)\qquad \begin{aligned} &\sum_{i=1}^{2^{kd}} L(A \cap Q_i) \\ &\quad \le L(A) + (C_5 - 1) l(F) + c_2 \sum_{i=1}^{2^{kd}} |F_i|^{(d-2)/(d-1)} t\delta_i + c_3 2^{(d-1)k} t. \end{aligned}$$

It is important that $d-1$, and not d, appear in the exponent in the last term of the latter inequality. That is, we do not allow an error term as large as $c_3 t$ per cube Q_i, as would follow from (A4) if we removed the Q_i's one at a time from R, patching what was left after each removal.

Proof of Lemma 2.1. Let us first consider $A \subset R$ and $k = 1$. We first split R into two halves R' and R'' with a hyperplane H and let F_1 be the set of edges broken by the split, that is, those which have one endpoint in each half. (For specificity let us say that the splitting hyperplane is part of the upper half.) For $e \in F_1$ let z_e be the point where e intersects H. Let

$$F_1' := \{[x, z_e] \colon e = [x, y] \in F_1,\ x \in R'\},$$
$$F_1'' := \{[z_e, y] \colon e = [x, y] \in F_1,\ y \in R''\},$$

let V be the set of vertices of G and let $V' := V \cap R'$, $V'' := V \cap R''$. By (A4), (A2) and (2.1),

$$\begin{aligned} &L(A \cap R') + L(A \cap R'') \\ &\quad \le L(A) + (C_5 - 1)(l(F_1') + l(F_1'')) + 2C_6 L(\{z_e \colon e \in F_1\}) + c_4 t \\ &\quad \le L(A) + (C_5 - 1) l(F_1) + c_5 |F_1|^{(d-2)/(d-1)} t + c_4 t. \end{aligned}$$

Repeating this entire splitting procedure on $A \cap R'$ and on $A \cap R''$, and continuing through $2^d - 1$ splits, we obtain

$$\sum_{i=1}^{2^d} L(A \cap Q_i) \le L(A) + (C_5 - 1) \sum_{i=1}^{2^d - 1} l(F_i) + c_5 \sum_{i=1}^{2^d - 1} |F_i|^{(d-2)/(d-1)} t\delta_i + (2^d - 1)c_4 t,$$

where F_i is the set of edges broken by the ith split and $\delta_i = 1$ for all $i \le 2^d - 1$.

For $k = 2$ this entire procedure can be repeated on each of the 2^d cubes of side $t/2$, with additional iterations for general k. This yields

$$\begin{aligned} \sum_{i=1}^{2^{kd}} L(A \cap Q_i) &\le L(A) + (C_5 - 1) \sum_{i=1}^{2^{kd}-1} l(F_i) \\ &\quad + c_5 \sum_{i=1}^{2^{kd}-1} |F_i|^{(d-2)/(d-1)} t\delta_i + c_4 \sum_{i=1}^{2^{kd}-1} t\delta_i, \end{aligned} \tag{2.4}$$

where $t\delta_i$ is the maximum side length of the rectangle which is divided by the ith split. For each $0 \le j \le k - 1$, there are $(2^d - 1)2^{jd}$ values of $i \le 2^{kd} - 1$ for which $\delta_i = 2^{-j}$. Therefore

$$\sum_{i=1}^{2^{kd}-1} \delta_i = 2^{(d-1)k} - 1 \quad \text{and} \quad \sum_{i=1}^{2^{kd}-1} \delta_i^{d-1} = 2^d(2^k - 1),$$

and the result follows for $A \subset R$.

If $A \not\subset R$, one additional split preceding the above procedure is needed, separating R from R^c. Letting $F_{2^{kd}}$ denote the set of edges with exactly one endpoint in R, for $e \in F_{2^{kd}}$ let z_e be the point where e exits R. Let $\delta_{2^{kd}} := 1$. Using (A4) and (2.1),

$$L(A \cap R) \le L(A) + (C_5 - 1) l(F_{2^{kd}}) + c_6 |F_{2^{kd}}|^{(d-2)/(d-1)} t + c_4 t.$$

This, with (2.4) applied to $A \cap R$, gives (2.3). □

Proof of Theorem 1.1. It follows from the fact that L is simply adjoining that for $\varepsilon, t > 0$,

$$\begin{aligned} L\big(\Pi([0, t + \varepsilon]^d)\big) &\le L\big(\Pi([0, t]^d)\big) \\ &\quad + c_7(t + \varepsilon)\big|\Pi\big([0, t + \varepsilon]^d \setminus [0, t]^d\big)\big|, \end{aligned} \tag{2.5}$$

so that, taking expectations,

$$\varphi(t + \varepsilon) \le \varphi(t) + c_8 t^d \varepsilon \quad \text{for } 0 \le \varepsilon \le t. \tag{2.6}$$

Steele's proof in (2.1) of [8], with our (2.6) replacing Steele's monotonicity assumption, yields the existence of

$$\beta := \lim_{t\to\infty} \varphi(t)/t^d.$$

Because of (1.10), this is equivalent to (1.14).

Suppose that $m = 2^k$ for some $k \geq 0$. We claim that for some c_9,

$$L\big(\Pi([0,t]^d)\big) \leq \sum_{i=1}^{m^d} L(\Pi(tQ_i)) + c_9 m^{d-1}t.$$

We proceed by induction on k, assuming validity for $m = 2^{k-1}$; $k = 1$ is clear. Dividing $[0,t]^d$ into $(m/2)^d$ cubes of side $2t/m$, then dividing each of these into 2^d cubes of side t/m, we obtain from the simply adjoining property and the induction hypothesis:

$$\begin{aligned} L\big(\Pi([0,t]^d)\big) &\leq \sum_{i=1}^{m^d} L(\Pi(tQ_i)) + (m/2)^d 2^d C_3 d^{1/2} 2t/m + c_9 (m/2)^{d-1} t \\ &\leq \sum_{i=1}^{m^d} L(\Pi(tQ_i)) + \big(2C_3 d^{1/2} + c_9/2^{d-1}\big) m^{d-1} t \\ &\leq \sum_{i=1}^{m^d} L(\Pi(tQ_i)) + c_9 m^{d-1} t, \end{aligned}$$

provided that we let $c_9 := 4C_3 d^{1/2}$. After taking expectations, for $u = t/m$ this gives (2.2) of [8]:

$$\varphi(mu)/(mu)^d \leq \varphi(u)/u^d + c_9 u^{-(d-1)}.$$

By (1.10), letting $m \to \infty$ and setting $u = r^{1/d}$ yields (1.15).

Suppos' now that L is patchable and scale-bounded for all dimensions. We may ass me the Q_i are numbered in such a way that

$$\bigcup_{i=1}^{(m-1)^d} Q_i = [0,(m-1)/m]^d.$$

Let $\tilde{G}$ be an allowable graph on $\Pi([0,t]^d)$ which minimizes L. The key observation is that

$$\begin{aligned} &\int_0^t \big|\{e \in \tilde{G}: e \text{ crosses } H_i(s)\}\big|\, ds \\ (2.7)\qquad &= \sum_{e\in\tilde{G}} \int_0^t 1_{[e \text{ crosses } H_i(s)]}\, ds \leq l(\tilde{G}) \quad \text{for each } i. \end{aligned}$$

Fix k, let $m := 2^k + 1$ and suppose $u \in \mathbb{R}^d$ with $0 \le u_i \le 1/m$ for all $i \le d$. Note $u + Q_i \subset [0,1]^d$ for all $i \le (m-1)^d$. By Lemma 2.1 and Hölder's inequality, for some decomposition of $F := \{e \in \tilde{G}\colon e \text{ crosses } \partial(t(u+Q_i)) \text{ for some } i \le (m-1)^d\}$ into disjoint $F_1 \cup \cdots \cup F_{(m-1)^d}$ and some δ_i satisfying (2.2),

$$\begin{aligned}
&\sum_{i=1}^{(m-1)^d} L(\Pi(t(u+Q_i))) \\
&\quad \le L\big(\Pi([0,t]^d)\big) + (C_5 - 1)l(F) \\
&\qquad + c_2 \sum_{i=1}^{(m-1)^d} |F_i|^{(d-2)/(d-1)} t\delta_i + c_3 m^{d-1} t \\
&\quad \le L\big(\Pi([0,t]^d)\big) \\
&\qquad + (C_5 - 1)\sum_{i=1}^{d}\sum_{j=0}^{m-1} l\Big(\big\{e \in \tilde{G}\colon e \text{ crosses } H_i(t(u_i + j/m))\big\}\Big) \\
&\qquad + c_2 t\left(\sum_{i=1}^{(m-1)^d} \delta_i^{d-1}\right)^{1/(d-1)} |F|^{(d-2)/(d-1)} + c_3 m^{d-1} t.
\end{aligned} \tag{2.8}$$

From (2.2), the third term on the right side of (2.8) is bounded by

$$\begin{aligned}
&c_{10} t m^{1/(d-1)} \left(\sum_{i=1}^{d}\sum_{j=0}^{m-1} \Big|\big\{e \in \tilde{G}\colon e \text{ crosses } H_i(t(u_i + j/m))\big\}\Big|\right)^{(d-2)/(d-1)} \\
&\quad \le c_{10} t m^{1/(d-1)} \sum_{i=1}^{d} \left(\sum_{j=0}^{m-1} \Big|\big\{e \in \tilde{G}\colon e \text{ crosses } H_i(t(u_i + j/m))\big\}\Big|\right)^{(d-2)/(d-1)}
\end{aligned}$$

Taking expected values in (2.8) therefore yields

$$\begin{aligned}
&(m-1)^d \varphi(t/m) \\
&\quad \le \varphi(t) + (C_5 - 1)\sum_{i=1}^{d}\sum_{j=0}^{m-1} El\Big(\big\{e \in \tilde{G}\colon e \text{ crosses } H_i(t(u_i + j/m))\big\}\Big) \\
&\qquad + c_{10} t m^{1/(d-1)} \\
&\qquad \times \sum_{i=1}^{d} E\left(\sum_{j=0}^{m-1} \Big|\big\{e \in \tilde{G}\colon e \text{ crosses } H_i(t(u_i + j/m))\big\}\Big|\right)^{(d-2)/(d-1)} \\
&\qquad + c_3 m^{d-1} t.
\end{aligned} \tag{2.9}$$

We now average this inequality over $u_i \in [0, 1/m]$ for each $i \le d$. The average of the ith summand in the third term on the right side of (2.9) is, using (2.7),

$$
\begin{aligned}
m\int_0^{1/m} & E\left(\sum_{j=0}^{m-1}\left|\left\{e\in\tilde{G}: e \text{ crosses } H_i(t(u_i+j/m))\right\}\right|\right)^{(d-2)/(d-1)} du_i \\
&\le \left(mE\int_0^{1/m}\sum_{j=0}^{m-1}\left|\left\{e\in\tilde{G}:\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. e \text{ crosses } H_i(t(u_i+j/m))\right\}\right| du_i\right)^{(d-2)/(d-1)} \qquad (2.10)\\
&\le \left(mE\int_0^1\left|\left\{e\in\tilde{G}: e \text{ crosses } H_i(tu)\right\}\right| du\right)^{(d-2)/(d-1)} \\
&\le \left(mt^{-1}El(\tilde{G})\right)^{(d-2)/(d-1)} \\
&\le c_{11} m^{(d-2)/(d-1)} t^{d-2}.
\end{aligned}
$$

Suppose now that L is efficiently patchable, so that the second term on the right side of (2.9) is 0. Then (2.9) and (2.10) give

$$(m-1)^d\varphi(t/m) \le \varphi(t) + c_{12}mt^{d-1} + c_3 m^{d-1}t.$$

Letting $s = t/m$, this becomes

$$\frac{(m-1)^d}{m^d}\frac{\varphi(s)}{s^d} \le \frac{\varphi(ms)}{(ms)^d} + \frac{c_{12}}{s} + \frac{c_3}{s^{d-1}}.$$

Letting k, or equivalently m, approach ∞, setting $s = r^{1/d}$ and applying (1.10) yields (1.18).

Alternatively, suppose that (1.17) holds. Observe that

$$
\begin{aligned}
\int_0^t l(\{e\in G: e \text{ crosses } H_i(s)\})\, ds &= \sum_{e\in\tilde{G}} l(e)\int_0^t 1_{[e \text{ crosses } H_i(s)]}\, ds \qquad (2.11)\\
&\le \sum_{e\in\tilde{G}} l(e)^2.
\end{aligned}
$$

It is easy to see that (1.17) implies its analog for the Poisson process:

$$E\left(\inf_{\tilde{G}\in\mathscr{H}(\Pi_r([0,1]^d;\, \acute{L})}\sum_{e\in\tilde{G}} l(e)^2\right) = O(r^{(d-2)/d}). \qquad (2.12)$$

We may assume G is always chosen so that the sum in (1.17) is within 1 of its infimum over all allowable graphs. The average of the ith summand in the second term on the right side of (2.9) over $u_i \in [0, 1/m]$ is therefore, using (2.11),

$$\begin{aligned} &mE\int_0^{1/m}\sum_{j=0}^{m-1} l\big(\{e \in \tilde{G}: e \text{ crosses } H_i(t(u_i + j/m))\}\big)\, du_i \\ &\quad = mE\int_0^1 l\big(\{e \in \tilde{G}: e \text{ crosses } H_i(tu)\}\big)\, du \\ &\quad \le mt^{-1}E\Bigg(\sum_{e\in\tilde{G}} l(e)^2\Bigg). \end{aligned}$$

Substituting this, (2.10) and (2.12) into (2.9), we obtain

$$(m-1)^d\varphi(t/m) \le \varphi(t) + c_{13}mt^{-1}E\Bigg(\sum_{e\in\tilde{G}} l(e)^2\Bigg) + c_{12}mt^{d-1} + c_3m^{d-1}t.$$

Setting $s = t/m$ and using (1.17) rescaled by ms, this becomes

$$(m-1)^d\varphi(s) \le \varphi(ms) + c_{14}m^2s(ms)^{d-2} + c_{12}m^ds^{d-1} + c_3m^ds,$$

and (1.18) follows as in the efficiently patchable case. □

Here is the monotonicity result needed for desmoothing the sequence $\{E\psi_L(N_n)\}$.

LEMMA 2.2. *Under the hypotheses of Theorem* 1.2,

$$|\psi_L(n \pm k) - \psi_L(n)| \le \begin{cases} c_{15}k^{(d-2)/(d-1)}, & \text{for } 1 \le k \le n^{(d-1)/d}, \\ c_{16}kn^{-1/d}, & \text{for } n^{(d-1)/d} < k \le n/2. \end{cases}$$

PROOF. Let us first show that

$$\psi_L(n+k) \le \psi_L(n) + c_{15}k^{(d-2)/(d-1)} \quad \text{for } 1 \le k \le n^{(d-1)/d}. \tag{2.13}$$

Fix such n and k and let $\{X_1, \dots, X_{n+k}\}$ be an i.i.d. uniform sample from $[0,1]^d$. Since L is a function only of the set $\{X_1, \dots, X_{n+k}\}$, we may assume these points are relabeled by increasing dth coordinate, that is,

$$(X_1)_d < \cdots < (X_{n+k})_d.$$

Define $\tau \in [0, 1]$ by

$$1 - \tau = (X_{n+1})_d.$$

By (A3),

$$\begin{aligned} &EL(\{X_1, \dots, X_{n+k}\}) \\ &\quad \le EL(\{X_1, \dots, X_n\}) \\ &\qquad + EL(\{X_{n+1}\}) + EL(\{X_{n+2}, \dots, X_{n+k}\}) + 2C_3d^{1/2}. \end{aligned} \tag{2.14}$$

Now given τ, $\{X_1, \dots, X_n\}$ is an i.i.d. uniform sample from $[0,1]^{d-1} \times [0, 1 - \tau]$, as is $\{X_{n+2}, \dots, X_{n+k}\}$ from $[0,1]^{d-1} \times [1 - \tau, 1]$. Multiplying each

$(X_i)_d$, $i \le n$, by $(1-\tau)^{-1}$ gives an i.i.d. uniform sample from $[0,1]^d$. It follows that

$$EL(\{X_1,\dots,X_n\}) \le \psi_L(n). \tag{2.15}$$

By (A1),

$$L(\{X_{n+1}\}) = 0. \tag{2.16}$$

Let Y_i denote the projection of X_i onto $[0,1]^{d-1} \times \{1\}$, for $n+2 \le i \le n+k$. Let $\tilde{G}_Y$ be an allowable graph on $\{Y_{n+2},\dots,Y_{n+k}\}$ with vertex degree bounded by C_{10} and let $\tilde{G}_X$ be the allowable graph on $\{X_{n+2},\dots,X_{n+k}\}$ obtained by replacing each Y_i with X_i. [In case (ii) of (A1), any Steiner points are not moved.] By (A5),

$$l(\tilde{G}_X) \le l(\tilde{G}_Y) + C_{10}k\tau. \tag{2.17}$$

It follows using (A2), $E\tau \le k/n$ and $k \le n^{(d-1)/d}$ that

$$EL(\{X_{n+2},\dots,X_{n+k}\}) \le K_{d-1}k^{(d-2)/(d-1)} + C_{10}kE\tau \le c_{17}k^{(d-2)/(d-1)}.$$

This and (2.14)–(2.16) prove (2.13).

Next suppose $n^{(d-1)/d} < k \le n$. Define $\gamma > 1$ by $k = \gamma n^{(d-1)/d}$. Iterating (2.13) $[\gamma]+1$ times gives

$$\begin{aligned} \psi_L(n+k) &\le \psi_L(n) + c_{15}([\gamma]+1)(n+k)^{(d-2)/d} \\ &\le \psi_L(n) + c_{18}kn^{-1/d}, \qquad n^{(d-1)/d} < k \le n/2. \end{aligned} \tag{2.18}$$

Next we will show

$$\psi_L(n-k) \le \psi_L(n) + c_{10}k^{(d-2)/(d-1)} \quad \text{for } 1 \le k \le n^{(d-1)/d}. \tag{2.19}$$

Let $A := \{X_1,\dots,X_n\}$ be an i.i.d. uniform sample from $[0,1]^d$, relabeled as above so that $(X_1)_d < \cdots < (X_n)_d$. Define $(X_0)_d := 0$ and $(X_{n+1})_d := 1$. Let $\tilde{G}$ be an allowable graph on A which achieves the minimum in (A1) and let V be the set of vertices of $\tilde{G}$. Fix $i \le n-k+1$ and let $B_i := A \setminus \{X_i,\dots,X_{i+k-1}\}$ and $\tau_i := (X_{i+k})_d - (X_i)_d$. Let E_j be the set of edges which have X_j as one endpoint and let E_j^* be the set of edges in E_j which have the other endpoint outside $\{X_i,\dots,X_{i+k-1}\}$. Let $z_e := X_j$ if $e \in E_j^*$ for some $i \le j \le i+k-1$. (These z_e are not necessarily distinct—see Remark 1.1.) By (A4), with $V' := V \setminus \{X_i,\dots,X_{i+k-1}\}$ and $V'' := \{X_i,\dots,X_{i+k-1}\}$,

$$L(B_i) \le L(A) + (C_5 - 1)\sum_{j=i}^{i+k-1} l(E_j^*) + C_6 L(W_i) + C_7, \tag{2.20}$$

where $W_i \subset \{X_i,\dots,X_{i+k-1}\}$. [Actually W_i may be a multiset—see Remark 1.1—but the multiplicities of the elements are bounded, by (A5).] Analogously to (2.17),

$$L(W_i) \le c_{19}k^{(d-2)/(d-1)} + C_{10}k\tau_i. \tag{2.21}$$

Let $X_j^* := X_j - \tau_i \xi_d$ for $i+k \le j \le n$, where ξ_d is the dth coordinate vector, and let $B_i^* := \{X_1,\dots,X_{i-1},X_{i+k}^*,\dots,X_n^*\}$. Then given τ_i, because of exchangeability of uniform spacings, B_i^* is an i.i.d. uniform sample from $[0,1]^{d-1} \times [0, 1-\tau_i]$. Further,

$$L(B_i^*) \le L(B_i). \tag{2.22}$$

Multiplying the dth coordinates of the points of B_i^* by $(1-\tau_i)^{-1}$ gives a uniform sample from $[0,1]^d$ and increases edge lengths in any allowable graph on B_i^* by a factor of at most $(1-\tau_i)^{-1}$. Hence

$$\psi_L(n-k) \le (1-\tau_i)^{-1} E(L(B_i^*)|\tau_i) \quad \text{a.s.} \tag{2.23}$$

If $\tau_i \le 1/2$, then

$$(1-\tau_i)^{-1} \le 1 + 2\tau_i.$$

Hence averaging (2.23) over the event $[\tau_i \le 1/2]$ gives

$$\psi_L(n-k) \le E[(1+2\tau_i)E(L(B_i^*)|\tau_i)]/P[\tau_i \le 1/2]. \tag{2.24}$$

Since $1 + 2\tau_i$ is an increasing and $E(L(B_i^*)|\tau_i)$ is a decreasing function of τ_i, since $E\tau_i = k/(n+1)$ and since $P[\tau_i \le 1/2]$ is exponentially small in n, this and (2.20)–(2.22) imply

$$\begin{aligned} \psi_L(n-k) &\le (1-\exp(-c_{20}n))^{-1} E(1+2\tau_i)E(L(B_i^*)) \\ &\le (1-\exp(-c_{20}n))^{-1}(1+2k/n) \\ &\quad \times \left[\psi_L(n) + (C_5 - 1)E\left(\sum_{j=i}^{i+k-1} l(E_j) \right) + c_{21}k^{(d-2)/(d-1)} \right. \\ &\qquad \left. + c_{22}k^2/n + C_7 \right]. \end{aligned} \tag{2.25}$$

For $n-k+1 \le i \le n$, a similar argument with $\{X_i, \dots, X_n, X_1, \dots, X_{i+k-1-n}\}$ in place of $\{X_i, \dots, X_{i+k-1}\}$, with $\tau_i := 1 - (X_i)_d + (X_{i+k-n-1})_d$ and with W_i split into two parts yields (2.25) for these i as well, provided we evaluate the summation index j mod n when $j > n$.

Since $k \le n^{(d-1)/d}$, we have

$$\psi_L(n)k/n \le K_d k n^{-1/d} \le K_d k^{(d-2)/(d-1)} \quad \text{and} \quad k^2/n \le k^{(d-2)/(d-1)}.$$

Because of this and

$$E\left(\sum_{i=1}^{n} \sum_{j=i}^{i+k-1} l(E_j) \right) \le 2kEl(\tilde{G}) = 2k\psi_L(n),$$

averaging (2.25) over $1 \le i \le n$ shows that

$$\begin{aligned} \psi_L(n-k) &\le (1-\exp(-c_{20}n))^{-1}(1+2k/n) \\ &\quad \times [\psi_L(n)(1 + c_{23}k/n) + c_{24}k^{(d-2)/(d-1)}] \\ &\le \psi_L(n) + c_{25}k^{(d-2)/(d-1)}, \end{aligned} \tag{2.26}$$

as desired.

Analogously to (2.18), it follows from (2.19) that

$$\psi_L(n-k) \le \psi_L(n) + c_{16} k n^{-1/d}, \qquad n^{(d-1)/d} < k \le n/2. \tag{2.27}$$

The other inequalities implicit in the lemma statement, specifically

$$\psi_L(n \pm k) \ge \begin{cases} \psi_L(n) - c_{15} k^{(d-2)/(d-1)}, & \text{for } 1 \le k \le n^{(d-1)/d}, \\ \psi_L(n) - c_{16} k n^{-1/d}, & \text{for } n^{(d-1)/d} < k \le n/2, \end{cases}$$

are consequences of (2.13), (2.18), (2.19) and (2.27). □

As mentioned in the introduction, Theorem 1.2 is an immediate consequence of Theorem 1.1 and (i) of the following result.

LEMMA 2.3. (i) *Under the hypotheses of Theorem* 1.2,

$$|\psi_L(n) - E\psi_L(N_n)| = O(n^{(d-2)/(2(d-1))}).$$

(ii) *If also*

$$|\psi_L(n+1) - \psi_L(n)| = O(n^{-1/d}), \tag{2.28}$$

then

$$|\psi_L(n) - E\psi_L(N_n)| = O(n^{(d-2)/(2d)}).$$

(iii) *If also* ψ_L *is monotone increasing* [*but not assuming* (2.28)], *then for some* $C_{11} > 0$,

$$\psi_L(n) \ge \beta n^{(d-1)/d} - C_{11} n^{(d-2)/(2d)} (\log n)^{1/2}.$$

Note that (2.28) is satisfied if for some $C_{12} > 0$ and all $\{x_1, \dots, x_{n+1}\}$,

$$|L(\{x_1, \dots, x_{n+1}\}) - L(\{x_1, \dots, x_n\})| \le C_{12} \min_{i \le n} |x_i - x_{n+1}|. \tag{2.29}$$

PROOF OF LEMMA 2.3. (i) From Lemma 2.2 and (A2),

$$\begin{aligned} |\psi_L(n) - E\psi_L(N_n)| \le\ & c_{15} E|N_n - n|^{(d-2)/(d-1)} + c_{16} n^{-1/d} E|N_n - n| \\ & + K_d n^{(d-1)/d} P[N_n < n/2] \\ & + K_d E\big(N_n^{(d-1)/d} 1_{[N_n > 3n/2]}\big). \end{aligned} \tag{2.30}$$

Since $(N_n - n)/n^{1/2}$ has exponential tails, the last two terms approach 0. Since $E|N_n - n| = O(n^{1/2})$, part (i) follows from Hölder's inequality.

(ii) In place of Lemma 2.2, under (2.28) we have

$$|\psi_L(n \pm k) - \psi_L(n)| \le c_{26} k n^{-1/d} \quad \text{for } 1 \le k \le n/2.$$

This yields (2.30) without the first term on the right side, which implies (ii).

(iii) Fix n and let $j := [2n^{1/2}(\log n)^{1/2}]$, where $[\cdot]$ denotes the integer part. Then

$$\begin{aligned} E\psi_L(N_{n-j}) &\le \psi_L(n) + E\big(\psi_L(N_{n-j}) - \psi_L(n)\big) 1_{[N_{n-j} > n]} \\ &\le \psi_L(n) + c_{15} E(N_{n-j} - n)^{(d-2)/(d-1)} 1_{[N_{n-j} > n]} \\ &\quad + c_{16} n^{-1/d} E(N_{n-j} - n) 1_{[N_{n-j} > n]}. \end{aligned} \tag{2.31}$$

It is easily checked that since N_{n-j} is Poisson$(n-j)$,

$$\begin{aligned} E(N_{n-j}-n)1_{[N_{n-j}>n]} &\le E(N_{n-j}-(n-j))1_{[N_{n-j}>n]} \\ &= (n-j)P[N_{n-j}=n] \\ &\le (n-j)P[N_{n-j}=n-j]\exp(-j^2/4n) \\ &\le c_{27}n^{-1/2} \end{aligned}$$

so that by (2.31),

$$E\psi_L(N_{n-j}) \le \psi_L(n) + o(1). \tag{2.32}$$

Further, by Theorem 1.1,

$$\begin{aligned} E\psi_L(N_{n-j}) &\ge \beta(n-j)^{(d-1)/d} - C_8 \\ &\ge \beta n^{(d-1)/d} - c_{28}n^{-1/d}j - C_8 \\ &\ge \beta n^{(d-1)/d} - c_{29}n^{(d-2)/2d}(\log n)^{1/2}, \end{aligned}$$

which with (2.32) proves (iii). □

3. Examples.

EXAMPLE 3.1 (Traveling salesman problem). Here the allowable graphs are self-avoiding loops which visit every vertex. Properties (A1)(i), (A3) and (A5) are clear. Property (A2) can be found in [2], [3] or [12]. Property (A4) is essentially the content of Lemma 2 of [1]. Specifically, given a closed loop $\tilde{G}$ through all of a finite set A of vertices and $B \subset A$, choose a point z_e in each edge $e \in \partial_{\tilde{G}} B$ which connects a point of B to a point of $A \setminus B$. We now cut the path at each z_e and discard the portion with vertices in $A \setminus B$. What remains is a collection of paths, which we call *chords*, each starting at some z_e and ending at another. Let P be a self-avoiding closed loop through these endpoints $\{z_e: e \in \partial_{\tilde{G}} B\}$. The proof of Lemma 2 of [1] shows that there exists a closed path Γ which traverses each chord exactly once and each edge of P at most twice. The path Γ will contain segments of the form $x \to z_e \to \cdots \to z_{e'} \to y$, where $x \in B$ is an endpoint of e, $y \in B$ is an endpoint of e' and all the unnamed middle vertices are from $\{z_e: e \in \partial_{\tilde{G}} B\}$. Each such segment can be replaced with the single edge $[x, y]$. The result is a self-avoiding closed loop with vertex set B. Letting F_B be the set of all such $[x, y]$ gives (A4) with $C_5 = 1$ and $C_6 = 2$.

It is easily checked that (2.29) holds, so from Theorem 1.1 and Lemma 2.3,

$$\beta n^{(d-1)/d} - c_{30}n^{(d-2)/(2d)} \le EL(\{X_1, \dots, X_n\}) \le \beta n^{(d-1)/d} + C_2 n^{(d-2)/d}.$$

EXAMPLE 3.2 (Minimal spanning trees). Here $L(A)$ is the minimal total edge length among all trees with vertex set A. As observed by Steele [10], in any optimal tree, no two edges emanating from the same vertex make an angle of less than 60°, so that (A5) holds. Properties (A1)(i) and (A3) are clear, and (A2) follows from the same result for TSP.

Suppose that we have a minimal spanning tree $\tilde{G}$ with vertex set A, a subset $B \subset A$ and a point z_e in each edge $e \in \partial_{\tilde{G}} B$. Let $\partial_{v,\tilde{G}} B := \{x \in B: [x, y] \in \tilde{G}$ for some $y \in A \setminus B\}$. Let P be a minimal spanning tree of $\{z_e: e \in \partial_{\tilde{G}} B\}$. For each edge $[z_e, z_{e'}]$ in P, there are points $x_e, x_{e'}$ of $\partial_{v,\tilde{G}} B$ which are endpoints of e and e', respectively. Let F_B^* be the set of all such edges $[x_e, x_{e'}]$. Then F_B^* is a graph which spans $\partial_{v,\tilde{G}} B$. Since the vertices of P have degree at most C_{10}, we have

$$l(F_B^*) \le C_{10}(\{[x, z_e]: e = [x, y] \in \partial_{\tilde{G}} B,\ x \in B\}) + L(\{z_e: e \in \partial_{\tilde{G}} B\}).$$

By deleting some edges from F_B^*, we can obtain a set F_B of edges such that $\{[x, y] \in \tilde{G}: x, y \in B\} \cup F_B$ is a tree which spans B, and (A4) follows.

We have not established efficient patchability, but it is not needed because (1.17) follows from Lemma 2.2 and the remark following in [11].

Clearly, adding a point x_{n+1} to $\{x_1, \dots, x_n\}$ adds at most $\min_{i \le n} |x_i - x_{n+1}|$ to the value of L. If x_{n+1} is removed from $\{x_1, \dots, x_{n+1}\}$, the vertices formerly connected to x_{n+1} can be connected instead to an x_i which minimizes $|x_i - x_{n+1}|$. Thus, using (A5), (2.29) holds, so by Lemma 2.3 and Theorem 1.1,

$$\beta n^{(d-1)/d} - c_{31} n^{(d-2)/(2d)} \le EL(\{X_1, \dots, X_n\}) \le \beta n^{(d-1)/d} + C_2 n^{(d-2)/d}.$$

EXAMPLE 3.3 (Steiner trees). Here L is again the length of the minimal spanning tree, but with Steiner points allowed. It is readily checked (see [4]) that optimal trees have at most $n - 1$ Steiner points, so that the infimum in (A1) is always achieved. Property (A3) is clear, and (A2) and (A5) follow from the same result for minimal spanning trees.

In the notation of Example 3.2, for Steiner trees we can let $F_B := P \cup \{[x, z_e]: e = [x, y] \in \partial_{\tilde{G}} V',\ x \in V'\}$ to establish (A4) with $C_5 = C_6 = 1$. As in Example 3.2, (2.29) holds, so Lemma 2.3 and Theorem 1.1 give

$$\begin{aligned} \beta n^{(d-1)/d} - c_{32} n^{(d-2)/(2d)} &\le EL(\{X_1, \dots, X_n\}) \\ &\le \beta n^{(d-1)/d} + C_2 n^{(d-2)/d}. \end{aligned} \tag{3.1}$$

EXAMPLE 3.4 (Rectilinear Steiner trees). These are Steiner trees in which all edges must be parallel to the coordinate axes. This is equivalent to replacing the Euclidean norm in (A1) with the L^1 norm. The proofs of all our results remain unchanged for the L^1 norm, since it is equivalent to the Euclidean norm. Thus (3.1) is valid for rectilinear Steiner trees as well.

EXAMPLE 3.5 (Minimal matching). Minimal matching was described, and efficient patchability established, in the Introduction. Properties (A1), (A3) and (A5) are obvious, and (A2) follows from the same result for TSP. Thus by Theorem 1.2,

$$\beta n^{(d-1)/d} - c_{33} n^{(d-2)/(2(d-1))} \le EL(\{X_1, \dots, X_n\}) \le \beta n^{(d-1)/d} + C_3 n^{(d-2)/d}.$$

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