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# SUPEREXTREMAL PROCESSES, MAX-STABILITY AND DYNAMIC CONTINUOUS CHOICE

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A general framework in an ordinal utility setting for the analysis of dynamic choice from a continuum of alternatives $E$ is proposed. The model is based on the theory of random utility maximization in continuous time. We work with *superextremal processes* $\mathbf{Y} = \{\mathbf{Y}_t,\ t \in (0,\infty)\}$, where $\mathbf{Y}_t = \{Y_t(\tau),\ \tau \in E\}$ is a random element of the space of upper semicontinuous functions on a compact metric space $E$. Here $Y_t(\tau)$ represents the utility at time $t$ for alternative $\tau \in E$. The choice process $\mathbf{M} = \{M_t,\ t \in (0,\infty)\}$, is studied, where $M_t$ is the set of utility maximizing alternatives at time $t$, that is, $M_t$ is the set of $\tau \in E$ at which the sample paths of $\mathbf{Y}_t$ on $E$ achieve their maximum. Independence properties of $\mathbf{Y}$ and $\mathbf{M}$ are developed, and general conditions for $\mathbf{M}$ to have the Markov property are described. An example of such conditions is that $\mathbf{Y}$ have max-stable marginals.

**1. Introduction.** This paper presents a general class of probabilistic models for the analysis of the *dynamic* choice behavior of individuals from sets of alternatives which may be arbitrarily large. The choice behavior of individuals at any time $E$ is postulated to conform to the theory of random utility maximization [cf. McFadden (1981)]. The set of alternatives $E$ is a compact metric space. The preferences of an individual for alternatives in $E$, at any given time $t$, are captured by a real-valued random function $\mathbf{Y}_t = \{Y_t(\tau), \tau \in E\}$ called the *random utility function*. In keeping with utility maximization, individuals select alternatives in $E$ which achieve the maximum value of $\mathbf{Y}_t$. The randomness is assumed because the analyst does not actually observe all of the factors determining choice.

Choice models from sets with finitely many alternatives have employed max-stability for quite some time [cf. McFadden (1981)] and it is natural to investigate the prospects of continuous choice modeling under the auspices of max-stability in infinite dimensions. The origins of the continuous choice problem stem from the need to analyze data when the sets of alternatives are arbitrarily large [see Cosslett (1988), Dagsvik (1988) and Resnick and Roy (1991a)]. In transportation research, Ben-Akiva and Watanada (1981) and Ben-Akiva, Litinas and Tsunokawa (1984) developed the continuous Logit

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model for approximating choices over large regional areas. Subsequently, McFadden (1989) considered continuous choice models for problems with large choice sets in the context of location choice modeling. Rust (1991) surveyed discrete choice modeling and discussed applications of max-stability to the problem of making a single choice from an infinite set. Pakes (1991) provided a survey of models where the set of alternatives is either discrete or continuous. Both Pakes and Rust discussed dynamic models in their respective papers, but their emphasis was primarily on static models.

There are several streams of research in the economics literature on continuous choice modeling. These models have usually addressed slightly different classes of problems than the ones which are the focus of this paper. For instance, there has been extensive research on dynamic asset pricing models where the choice space represents investments and is continuous [Pakes (1991), Duffie (1988, 1992)]. These models usually have dynamic programming foundations, and are often based on "Euler" equation techniques [cf. Lucas (1978)]. Martingales are modern tools for the analysis of consumption-investment decisions [Duffie (1992)]. These models are based on the von-Neumann–Morgenstern theory of decision-making [see Kreps (1988)], where the individuals make decisions on the basis of maximizing expected utility, and the utility functions are cardinal, that is, invariant to affine transformations. In the setup of our paper, the utility functions are invariant to monotone transformations, that is, ordinal. The randomness in these ordinal utility functions is introduced to account for the fact that one cannot observe all of the factors determining an individual's preferences. Individual decisions are determined by maximizing their random utilities at a choice occasion.

One approach to generating random utility models in economics from an underlying utility maximization problem is where one derives indirect utility functions to which random error terms are added [Haneman (1984)]. Another approach in mathematical psychology is where random utility models are derived from axioms on the choice probabilities for alternatives and distributions of the random errors are deduced from the axioms [Luce (1959); McFadden (1973)]. Cohen (1980) developed extensions of this approach to the case of continuous choice, and Dagsvik (1990) generalized this approach to derive a max-stable process model. Finally, the "social surplus function" approach of McFadden (1981) also gave rise to random utility models, and this setup was utilized in describing max-stable process models in Resnick and Roy (1991a).

Max-stable process models for the analysis of the static continuous choice problem were proposed by Cosslett (1988) and Dagsvik (1990). Cosslett defined the utility functions as max-moving averages [cf. Balkema and de Haan (1988)] with continuous sample paths on a closed interval of the real line, which represented the choice set. Dagsvik (1990) specified conditions on the choice probabilities which led to max-stable random utility processes. Resnick and Roy (1991a) rigorously discussed static continuous choice modeling on compact metric choice sets and gave general characterization theorems which provided guidelines for the construction of continuous choice models within the framework of max-stability. A major factor in favor of employing max-stable processes for modeling the continuous choice problem is that the resulting formulae for the choice probabilities are in closed form, making them amenable to future econometric analysis.

One way to extend static continuous choice models to a dynamic framework is to embed the static model (such as the max-stable process model mentioned previously) within a dynamic programming formulation [Rust (1988); Resnick and Roy (1991a), Section 6], but the key drawback of this approach is that tractability considerations usually force the simplifying assumption that utilities for alternatives at different points in time are independent [see Rust (1991) and Pakes (1991) for more on this, though in a discrete choice setup]. Therefore, we propose another approach. One interpretation of our framework is as a model of dynamic choice under conditions of perfect foresight available to the individual making the choices. There is no uncertainty from the individual's frame of reference. The uncertainty in the model arises from the underlying premise that the analyst does not observe all the relevant factors (or processes) which go into the making of an individual's decision [cf. Rust (1991)] and, thus, in economics language, we have a model in reduced form.

We present a model for time-varying choice from a continuous set that generalizes the finite-dimensional analysis of Dagsvik (1988) and Resnick and Roy (1990), which is based on multivariate extremal processes. We replace the multivariate extremal process by an infinite-dimensional counterpart called the *superextremal process* [Resnick and Roy (1991b)] $\mathbf{Y} = \{\mathbf{Y}_t, t > 0\}$, which models the dynamic evolution of utilities. For any $t > 0$, $\mathbf{Y}_t$ is a random element of the space of nonnegative upper semicontinuous functions on $E$, where $E$ represents the choice set. The quantity $Y_t(\tau)$ represents the random utility for alternatives $\tau \in E$ at time $t$. We define the arg max or choice process $\mathbf{M} = \{M_t, t > 0\}$ [Resnick and Roy (1991b)]:

$$M_t = \left\{\tau \in E: Y_t(\tau) = \bigvee_{s \in E} Y_t(s)\right\};$$

$\mathbf{M}$ has state-space $\mathscr{F}(E)$, the space of closed subsets of $E$. In this paper, we have kept the choice set $E$ time-invariant and nonrandom, which allows for a relatively clear exposition of the main ideas behind the model. Extensions with a random process describing the evolution of the choice set, which delivers similar properties as the model in this paper, are possible but require rather stringent conditions on the process describing the temporal evolution of the choice set. Time-varying nonrandom choice sets could be handled fairly easily as in Resnick and Roy (1990).

The distribution of a superextremal process $\mathbf{Y}$ is characterized by its sup-Lévy measure $\mu$ (see Section 2). When this measure admits a particular decomposition [see (23) and (24)], then $\mathbf{M}$ is Markov. In particular, we shall show that if for each $t > 0$, $\mathbf{Y}_t$ is a max-stable process, then $\mathbf{M}$ is Markov. In the latter case, the choice and transition probabilities of $\mathbf{M}$ are available in closed form and are natural generalizations of their finite-dimensional counterparts [see Resnick and Roy (1990)].

In Section 4, we describe de Haan's spectral function construction of max-stable processes and show that this approach is equivalent to our construction in function space in Sections 2 and 3. The spectral function approach lends itself somewhat naturally to specific parametric forms and an example is provided with a continuous Logit model being deduced from dynamic utility maximization. The Appendix describes the appropriate topologies and deals with some measurability issues.

**2. The superextremal process and max-stability.** Suppose $(\Omega, \mathscr{A}, \mathbf{P})$ is a complete space and $E$ is a compact metric space with countable dense subset $D_E$ and metric $d$. Let $\mathscr{F}(E)$ be the closed subsets of $E$; because $E$ is compact, this is the same as $\mathscr{K}(E)$, the compact subsets of $E$. Let $\mathscr{B}(E)$ denote the Borel $\sigma$-algebra on $E$. Let $US(E)$ be the space of upper semicontinuous (USC) functions from $E \to (0, \infty]$ and with the sup-vague topology [cf. Vervaat (1988) and Appendix A.2). Let $\mathscr{B}(US(E))$ denote the usual Borel $\sigma$-algebra on $(US(E)$, that is, the $\sigma$-algebra generated by open sets. Let $US_0(E) = US(E) - \{\mathbf{0}\}$, that is, $US(E)$ punctured by removal of the function identically zero on $E$. If $(\Omega, \mathscr{A}, \mathbf{P})$ is a complete probability space, we say that the map $\xi\colon \Omega \to US(E)$ is a random usc function if it is a random element of $(US(E), \mathscr{B}(US(E)))$. This means $\xi^{-1}(\mathscr{B}(US(E))) \subseteq \mathscr{A}$. Henceforth, for any measurable $B \subseteq E$ and $f \in US(E)$ we use the notation

$$f^{\vee}(B) := \bigvee_{\tau \in B} f(\tau).$$

We begin with the definition of a superextremal process given in Resnick and Roy (1991b). Let

$$N = \sum_{k \geq 1} \varepsilon_{(t_k, \eta_k)}$$

be a Poisson random measure (PRM) on $(0, \infty) \times US_0(E)$ with mean measure $\mu$ such that $\mu$ is Radon (finite on compact sets) on $(0, \infty) \times US_0(E)$. We assume for all $t > 0$,

$$\mu\big((0,t] \times \{f \in US(E) \colon f^{\vee}(K) = \infty\}\big) = 0 \qquad \forall\, K \in \mathscr{K}(E), \tag{1}$$

$$\mu\big((0,t] \times US_0(E)\big) = \infty, \tag{2}$$

$$\mu(\{t\} \times \cdot) = 0. \tag{3}$$

Also, for notational convenience we will often write $\mu((0,t] \times \cdot\,)$ as $\mu_t(\cdot)$.

The PRM $N$ is time-homogeneous if there exists a Radon measure $\nu$ on $US_0(E)$, such that for $A \in \mathscr{B}(US_0(E))$,

$$\mu((0,t] \times A) =: t\nu(A)$$

and $\nu$ satisfies the analogues of (1) and (2).

The *superextremal process* $\mathbf{Y} = \{\mathbf{Y}_t,\ t > 0\}$ is defined by

$$\mathbf{Y}_t := \bigvee_{t_k \leq t} \eta_k \tag{4}$$

[cf. Resnick and Roy (1991b)]. Often $\mu$ is called the *sup-Lévy* or *exponent* measure of $\mathbf{Y}$. Also define

$$\mathbf{Y}_{st} = \bigvee_{s < t_k \le t} \eta_k .$$

The random variable $\mathbf{Y}_t$ represents individual's random utility for alternatives in $E$ at time $t$, and the process $\mathbf{Y}$ describes the evolution of the individual's utility. The superextremal process $\mathbf{Y} = \{\mathbf{Y}_t,\ t > 0\}$ is Markov, stochastically continuous, has a version in $D((0,\infty), US(E))$ and $\mathbf{Y}$ is $\mathscr{B}((0,\infty)) \times \mathscr{A}/\mathscr{B}(US(E))$ measurable. Furthermore, $\mathbf{Y}$ has a version (also called $\mathbf{Y}$) such that for each fixed $t > 0$, $\mathbf{Y}_t$ is a random element of $US(E)$ and is $\mathscr{B}(E) \times \mathscr{A}/\mathscr{B}((0,\infty))$ measurable. For any $B \in \mathscr{B}(E)$, $Y_t^{\vee}(B)$ is a random variable. The process $\mathbf{Y}$ is Markov with state space $US_0(E)$, and its transition probabilities are determined by ($0 < s < t$, $h \in US_0(E)$, $K_i \in \mathscr{K}(E)$, $x_i \ge 0$, $i = 1, \dots, m$):

$$\mathbf{P}\big[\mathbf{Y}_t^{\vee}(K_i) \le x_i,\ i = 1,\dots,m \,|\, \mathbf{Y}_s = h\big] = \begin{cases} 0, & h^{\vee}(K_i) > x_i \text{ for some } i \in \{1,\dots,m\}, \\ \exp\Big(-\mu\big((s,t] \times \{f: f^{\vee}(K_i) \le x_i,\ i = 1,\dots,m\}^c\big)\Big), & \text{otherwise.} \end{cases}$$

Also note for $B \in \mathscr{B}(E)$, the process $Y^{\vee}(B) = \{Y_t^{\vee}(B),\ t > 0\}$ is a classical univariate extremal process. See Resnick and Roy, (1991b) for details.

In this paper, we show that somewhat richer properties are inherited by the superextremal processes $\mathbf{Y}$, whose sup-Levy measures admit a particular decomposition [defined in (23) and (24)]. Max-stable processes are an important example of processes that belong to this class and are introduced next. Henceforth we assume that we have a fixed version of $\mathbf{Y}$ which is a random element of $D((0,\infty), US(E))$.

2.1. *Max-stability.* We say that the superextremal process $\mathbf{Y}$ has max-stable components if for every $t > 0$ and any $\theta > 0$ and $A \in \mathscr{B}(US(E))$,

$$\theta\mu_t(\theta A) = \mu_t(A) \tag{5}$$

[cf. de Haan (1984) and Gine, Hahn and Vatan (1990)]. Note that

$$\begin{aligned}\{f \in US_0(E): f^{\vee}(K) > \theta x\} &= \{\theta f \in US_0(E): f^{\vee}(K) > x\} \\ &= \theta\{f \in US_0(E): f^{\vee}(K) > x\}\end{aligned}$$

and from this it is easy to see that for each fixed $t$, $\{Y_t(\tau),\ \tau \in E\}$ is a max-stable process [de Haan (1984)]. Also, $\mu_t$ is the analogue to McFadden's (1981) "social surplus function" [cf. Resnick and Roy (1991a)].

By (1), the sup-Levy measure $\mu$ places no mass on $(0,\infty) \times \{f \in US(E): f^{\vee}(E) = \infty\}$ and hence for any $t$,

$$\mathbf{P}[\mathbf{Y}_t \in US(E) \setminus US_b(E)] = 0,$$

where $US_b(E) \subset US(E)$ are bounded functions in $US(E)$. On $US_b(E)$ we define $f^{\vee}(E) := \|f\|$. From now on we use $US_0(E)$ to denote $US_b(E) - \{\mathbf{0}\}$.

Define the unit ball in $US(E)$ as

$$\aleph_{US} = \{f \in US(E): \|f\| = f^{\vee}(E) = 1\}, \tag{6}$$

which is compact in the sup-vague topology. Construct a measure $\sigma$ on $\mathscr{B}((0,\infty)) \times \mathscr{B}(\aleph_{US})$ as follows: For $A \in \mathscr{B}(\aleph_{US})$ and $t > 0$,

$$\sigma((0,t] \times A) =: \sigma_t(A) = \mu_t\left(\left\{f \in US_0(E): \frac{f}{\|f\|} \in A, \|f\| > 1\right\}\right). \tag{7}$$

Also define for $0 < s < t$,

$$\sigma_{st}(\cdot) = \mu_{st}\left(\left\{g: \|g\| > 1, \frac{g}{\|g\|} \in \cdot\right\}\right).$$

Define a generalized polar coordinate transformation

$$R: (0,\infty) \times US_0(E) \to (0,\infty) \times (0,\infty) \times \aleph_{US}$$

by

$$R(t,f) := \left(t, \|f\|, \frac{f}{\|f\|}\right). \tag{8}$$

From the definition of max-stability, for $r > 0$,

$$\begin{aligned}
&\mu_t\left(\left\{f \in US_0(E): \|f\| > r, \|f\|^{-1}f \in A\right\}\right)\\
&= r^{-1}\mu_t\left(\left\{r^{-1}f \in US_0(E): \|f\| > r, \|f\|^{-1}f \in A\right\}\right)\\
&= r^{-1}\mu_t\left(\left\{r^{-1}f \in US_0(E): \|r^{-1}f\| > 1, \|r^{-1}f\|^{-1}r^{-1}f \in A\right\}\right)\\
&= r^{-1}\mu_t\left(\left\{g \in US_0(E): \|g\| > 1, \|g\|^{-1}g \in A\right\}\right)\\
&= r^{-1}\sigma_t(A).
\end{aligned}$$

Then we have

$$\mu \circ R^{-1}(dt, dr, dg) = r^{-2}\, dr\, \sigma(dt, dg) \tag{9}$$

and for fixed $t > 0$, the finite-dimensional distributions of $\mathbf{Y}_t$ are specified as follows. For $\{K_i\}_{i=1}^n \in \mathscr{K}(E)$ and $x_i > 0$, $i = 1,\dots,n$,

$$\begin{aligned}
&-\log \mathbf{P}\left[\bigcap_{i=1}^{n} \{Y_t^{\vee}(K_i) \le x_i\}\right]\\
&= -\mu_t\left(\left\{f \in US(E): f^{\vee}(K_i) \le x_i, i = 1,\dots,n\right\}^c\right)\\
&= \int_{\{(r,g) \in (0,\infty) \times \aleph_{US}:\, rg^{\vee}(K_i) \le x_i,\, i=1,\dots,n\}} r^{-2}\, dr\, \sigma_t(dg)\\
&= \int_{g \in \aleph_{US}} \left(\int_{\{r:\, r \le \wedge_{i=1}^{n}(x_i/g^{\vee}(K_i))\}} r^{-2}\, dr\right) \sigma_t(dg)\\
&= \int_{g \in \aleph_{US}} \left(\bigvee_{i=1}^{n} \frac{g^{\vee}(K_i)}{x_i}\right) \sigma_t(dg).
\end{aligned} \tag{10}$$

If $\mathbf{Y}$ is time-homogeneous, that is, $\mu_t = t\nu$, then $\sigma_t = t\sigma$, where $\sigma$ is a measure on $\mathscr{B}(\aleph_{US})$, constructed analogously to (7).

The $US(E)$-valued process $\mathbf{Y}$ is Markov [Resnick and Roy (1991b)], and when for each $t > 0$, $\mathbf{Y}_t$ is max-stable, its transition probabilities take on the tractable form determined by ($0 < s < t$, $h \in US_0(E)$, $K_i \in \mathscr{F}(E)$, $x_i \geq 0$, $i = 1, \dots, m$):

$$\mathbf{P}\left[\bigcap_{i=1}^{m} \{Y_t^{\vee}(K_i) \leq x_i\} | \mathbf{Y}_s = h\right]$$
$$= \begin{cases} 0, & \text{if } h^{\vee}(K_i) > x_i \text{ for some } i \in \{1, \dots, m\}, \\ \exp\left(-\int_{\aleph_{US}} \left(\bigvee_{i=1}^{m} \frac{g^{\vee}(K_i)}{x_i}\right) d\sigma_{st}(g)\right), & \text{otherwise.} \end{cases}$$

If $\mathbf{Y}$ is time-homogeneous, then the transition probabilities simplify further. Since $\sigma_t = t\sigma$, we have for $0 < s < t$, $h \in US_0(E)$, $K_i \in \mathscr{F}(E)$, $x_i \geq 0$, $i = 1, \dots, m$, that

$$\mathbf{P}\left[\bigcap_{i=1}^{m} \{Y_t^{\vee}(K_i) \leq x_i\} | \mathbf{Y}_s = h\right]$$
$$= \begin{cases} 0, & \text{if } h^{\vee}(K_i) > x_i \text{ for some } i \in \{1, \dots, m\}, \\ \exp\left(-(t-s)\int_{\aleph_{US}} \left(\bigvee_{i=1}^{m} \frac{g^{\vee}(K_i)}{x_i}\right) d\sigma(g)\right), & \text{otherwise.} \end{cases}$$

Now that we have collected the basic properties of the random utility process $\mathbf{Y}$ and defined max-stable superextremal utilities, we proceed toward developing the properties of the corresponding choice process $\mathbf{M}$.

**3. The choice process.** In this section, we have a fixed version of the superextremal utility process $\mathbf{Y} = \{\mathbf{Y}_t, t > 0\}$ defined in (4), which is a random element of $D((0,\infty), US(E))$, representing the evolution of the alternatives in $E$. Recall $\mathscr{F} = \mathscr{F}(E)$ is the class of closed subsets of $E$, and since $E$ is compact, $\mathscr{F}(E) = \mathscr{K}(E)$. The class $\mathscr{F}(E)$ is given the *vague* topology (cf. Appendix A.3) and $\mathscr{B}(\mathscr{F}(E))$ denotes the Borel $\sigma$-algebra generated by the open subsets of $\mathscr{F}(E)$. A random element of $(\mathscr{F}(E), \mathscr{B}(\mathscr{F}(E)))$ is a *random (closed) set* [cf. Castaing and Valadier (1977) and Vervaat (1988)].

The *arg max functional* $A_{\vee}$ on $US(E)$ is defined as

$$A_{\vee}(f) := \{\tau \in E: f(\tau) = f^{\vee}(E)\} = \{\tau \in E: f(\tau) \geq f^{\vee}(E)\} \\ = f^{-1}[f^{\vee}(E), \infty]. \tag{11}$$

Since $f \in US(E)$, $A_{\vee}(f)$ is closed [Resnick and Roy (1991a)]. Furthermore the arg max functional $A_{\vee}$: $US(E) \to \mathscr{F}(E)$ is upper continuous and $\mathscr{B}(US(E))/\mathscr{B}(\mathscr{F}(E))$ measurable [cf. Resnick and Roy (1991b) and Appendix A.3].

For a superextremal utility process $\mathbf{Y} = \{\mathbf{Y}_t,\ t > 0\}$, the *choice process* $\mathbf{M} = \{M_t,\ t > 0\}$ is defined by

$$M_t := A_\vee(\mathbf{Y}_t). \tag{12}$$

Therefore, $M_t$ represents the collection of utility maximizing alternatives at time $t$. Some noteworthy properties of $\mathbf{M}$ [Resnick and Roy (1991b)] are that for each $t > 0$, $M_t$ is a random element of $\mathscr{F}(E)$, $M$ is a.s. right upper continuous in $\mathscr{F}(E)$ and $\mathbf{M}$ is $\mathscr{B}((0,\infty)) \times \mathscr{A}/\mathscr{B}(\mathscr{F}(E))$ measurable.

For most applications, it is standard (and convenient) to assume that utility $\mathbf{Y}_t(\cdot)$ is maximized by a single alternative in the choice set of each $t > 0$. This is not true, in general, and we specify conditions in the following text which ensure this [Resnick and Roy (1991b)].

Define $US(E)_{\mathrm{SING}}$ to be the functions in $US_0(E)$ which achieve their maxima at a unique point in $E$, that is,

$$\begin{aligned} US(E)_{\mathrm{SING}} &:= \bigcup_{\tau\in E} \{f \in US_0(E) \colon f^\vee(E) = f(\tau) > f(\tau'), \forall\, \tau' \in E - \{\tau\}\} \\ &= \bigcup_{\tau\in E} \{f \in US_0(E) \colon A_\vee(f) = \{\tau\}\} \in \mathscr{B}(US(E)). \end{aligned}$$

Then from Resnick and Roy (1991b), we know that (i) for any fixed $t > 0$, the set

$$\mathrm{SING}_{\mathscr{F}(t)}(E) = \{\omega \colon M_t(\omega) \text{ is singleton}\} \in \mathscr{A}$$

and

$$\mathrm{SING}_{\mathscr{F}(E)} = \bigcap_{t>0} \{\omega \colon M_t(\omega) \text{ is singleton}\} \in \mathscr{A},$$

and (ii) if $\mathbf{P}[\mathrm{SING}_{\mathscr{F}(E)}] = 1$, then $\mathbf{M}$ is a.s. right continuous in $\mathscr{F}(E)$, and stochastically continuous. Furthermore, if the exponent measure $\mu$ of the superextremal process $\mathbf{Y} = \{\mathbf{Y}_t,\ t > 0\}$ satisfies

$$\mu\big((0,t] \times \{f \in US(E) \colon f^\vee(E) \in \cdot\}\big) \tag{13}$$

is atomless for every $t > 0$, then:

1. $\mathbf{P}[\mathrm{SING}_{\mathscr{F}(E)}] = 1$ if and only if
$$\mu\big((0,\infty) \times [US(E)_{\mathrm{SING}}]^c\big) = 0,$$
2. For any $t > 0$, $\mathbf{M}_t$ is $\mathbf{P}$-a.s. singleton, that is, $\mathbf{P}[\mathrm{SING}_{\mathscr{F}(E)}(t)] = 1$ if and only if
$$\mu\big((0,t] \times [US(E)_{\mathrm{SING}}]^c\big) = 0.$$

Finally, when the sup-Lévy measure satisfies the atomless condition in (13), then the joint evolution of the process $\{(\mathbf{M}, \mathbf{Y}^\vee(E))\} = \{(M_t, \mathbf{Y}_t^\vee(E)),\ t > 0\}$ is Markov.

3.1. *The Markov property of the choice process.* Now we study the dynamic properties of the choice process $\mathbf{M} = \{M_t,\ t > 0\}$.

We define some auxiliary processes [cf. Resnick and Roy (1991a)] needed for the calculation to follow. First, define the following sets: For any $K \in \mathscr{F}(E)$, $K^{(>)} = \{f \in US(E): A_{\vee}(f) \subseteq K\}$, $K^{(<)} = \{f \in US(E): A_{\vee}(f) \cap K = \varnothing\}$ and $K^{(=)} = [K^{(>)} \cup K^{(<)}]^c$, all of which are measurable (Appendix A.1). Also define $K^{(\geq)} = K^{(>)} \cup K^{(=)}$ and $K^{(\leq)} = K^{(<)} \cup K^{(=)}$.

Next, for any $K \in \mathscr{F}(E)$, define three auxiliary Poisson processes

$$N_{K^{(>)}} = \sum_j \varepsilon_{(t_j, \eta_j)} \mathbf{1}_{\{\eta_j \in K^{(>)}\}},$$

$$N_{K^{(<)}} = \sum_j \varepsilon_{(t_j, \eta_j)} \mathbf{1}_{\{\eta_j \in K^{(<)}\}},$$

$$N_{K^{(=)}} = \sum_j \varepsilon_{(t_j, \eta_j)} \mathbf{1}_{\{\eta_j \in K^{(=)}\}},$$

which are mutually independent as a consequence of the complete randomness of $N$. Then, for all $K \in \mathscr{F}(E)$ and any $t > 0$, define the random variables

$$X_t(K^{(>)}) = \bigvee_{t_k \leq t} \eta_k^{\vee}(E) \mathbf{1}_{[\eta_k \in K^{(>)}]}, \tag{14}$$

$$X_t(K^{(<)}) = \bigvee_{t_k \leq t} \eta_k^{\vee}(E) \mathbf{1}_{[\eta_k \in K^{(<)}]}, \tag{15}$$

$$X_t(K^{(=)}) = \bigvee_{t_k \leq t} \eta_k^{\vee}(E) \mathbf{1}_{[\eta_k \in K^{(=)}]}. \tag{16}$$

Again, the complete randomness of the PRM $N$ implies that $X_t(K^{(>)})$, $X_t(K^{(<)})$ and $X_t(K^{(=)})$ are independent random variables. It is also convenient to define the random variables

$$X_t(K^{(\geq)}) = X_t(K^{(>)}) \vee X_t(K^{(=)}),$$

$$X_t(K^{(\leq)}) = X_t(K^{(<)}) \vee X_t(K^{(=)}).$$

We assume throughout the rest of the paper that the sup-Levy measure $\mu$ of the process $\mathbf{Y}$ satisfies the atomless condition defined in (13). It is easy to see that (13) is satisfied by max-stable superextremal processes.

We begin by reproducing a lemma from Resnick and Roy [(1990), Lemma 3.2], which is needed in some of the proofs to follow.

LEMMA 3.1. *Suppose $X_1$ and $X_2$ are nonnegative independent random variables with distributions $F_1$ and $F_2$, respectively. For $0 < c < 1$,*

$$\mathbf{P}[x \geq X_1 \vee X_2, X_1 \geq X_2] = \mathbf{P}[X_1 \geq X_2]\mathbf{P}[X_1 \vee X_2 \leq x]$$

*if and only if*

$$F_2 = F_1^{c^{-1}(1-c)},$$

*where $c = \mathbf{P}[X_1 \geq X_2]$.*

Recall $\mathbf{Y}$ is a superextremal process and $\mathbf{M}$ is the corresponding choice process defined in (12). We write $[g^{\vee}(E) > x]$ for $\{g \in US(E): g^{\vee}(E) > x\}$.

THEOREM 3.1. *For any* $t > 0$, $Y_t^{\vee}(E)$ *and* $M_t$ *are independent iff for any* $K \in \mathscr{K}(E)$, $x > 0$,

$$\mu_t\left(K^{(>)} \cap \left[g^{\vee}(E) > x\right]\right) = \mathbf{P}[M_t \subseteq K]\, \mu_t\left(\left[g^{\vee}(E) > x\right]\right) \tag{17}$$

*or, equivalently,*

$$\mu_t\left(K^{(\geq)} \cap \left[g^{\vee}(E) > x\right]\right) = \mathbf{P}[M_t \cap K \neq \varnothing]\, \mu_t\left(\left[g^{\vee}(E) > x\right]\right). \tag{18}$$

PROOF. Suppose $Y_t^{\vee}(E) = X_t(K^{(\geq)}) \vee X_t(K^{(<)})$ and $M_t$ are independent. Since

$$[M_t \cap K \neq \varnothing] = \left[X_t(K^{(\geq)}) \geq X_t(K^{(<)})\right],$$

we have

$$\mathbf{P}\left[x \geq Y_t^{\vee}(E), M_t \cap K \neq \varnothing\right] = \mathbf{P}\left[x \geq Y_t^{\vee}(E)\right]\mathbf{P}[M_t \cap K \neq \varnothing]$$

or, equivalently,

$$\begin{aligned}\mathbf{P}&\left[x \geq X_t(K^{(\geq)}) \vee X_t(K^{(<)}), X_t(K^{(\geq)}) \geq X_t(K^{(<)})\right] \\ &= \mathbf{P}\left[x \geq X_t(K^{(\geq)}) \vee X_t(K^{(<)})\right]\mathbf{P}\left[X_t(K^{(\geq)}) \geq X_t(K^{(<)})\right].\end{aligned}$$

By Lemma 3.1 we get

$$\frac{-\log \mathbf{P}\left[X_t(K^{(\geq)}) \leq x\right]}{-\log \mathbf{P}\left[X_t(K^{(<)}) \leq x\right]} = \frac{\mathbf{P}[M_t \cap K = \varnothing]}{\mathbf{P}[M_t \cap K \neq \varnothing]}.$$

Since

$$\begin{aligned}-\log \mathbf{P}\left[X_t(K^{(<)}) \leq x\right] &= \mu_t\left(K^{(<)} \cap \left[g^{\vee}(E) > x\right]\right) \\ &= \mu_t\left(\left[g^{\vee}(E) > x\right]\right) - \mu_t\left(K^{(\geq)} \cap \left[g^{\vee}(E) > x\right]\right),\end{aligned}$$

the result follows. The converse can be verified directly. □

When $\mathbf{Y}_t$ is max-stable, then $\mu_t$ satisfies (17) and (18), and since the resulting formulae for the choice probability are of interest in their own right, we collect them in the following corollary.

COROLLARY 3.1. *If* $\mathbf{Y}$ *is a superextremal process with max-stable components, then for each* $t > 0$, $Y_t^{\vee}(E)$ *and* $M_t$ *are independent;*

(i) *the containment functional is*

$$\mathbf{P}[M_t \subseteq K] = \frac{\sigma_t\left(K^{(>)} \cap \aleph_{US}\right)}{\sigma_t(\aleph_{US})} \tag{19}$$

*and is called the choice probability;*

(ii) *the hitting functional is*

$$\mathbf{P}[M_t \cap K \neq \varnothing] = \frac{\sigma_t\left(K^{(\geq)} \cap \aleph_{US}\right)}{\sigma_t(\aleph_{US})}. \tag{20}$$

*Additionally, if* $\mathbf{Y}$ *is time-homogeneous, then recall* $\sigma_t(\cdot) = t\sigma(\cdot)$, *where* $\sigma$ *is a measure on* $\mathscr{B}(\aleph_{US})$. *In this case,*

(iii) *the choice probability is*

$$\mathbf{P}[M_t \subseteq K] = \frac{\sigma(K^{(>)} \cap \aleph_{US})}{\sigma(\aleph_{US})} =: \pi_K^C; \tag{21}$$

(iv) *the hitting functional is*

$$\mathbf{P}[M_t \cap K \neq \phi] = \frac{\sigma(K^{(\geq)} \cap \aleph_{US})}{\sigma(\aleph_{US})} =: \pi_K^H, \tag{22}$$

*where both do not depend on* $t$.

Proof. The formulae for the choice probability and hitting functional follow from the definitions in (5) and (6) and some direct calculations. We compute the containment functional (i) for illustrative purposes:

$$\begin{aligned}\mathbf{P}[M_t \subseteq K] &= \mathbf{P}\big[X_t(K^{(>)}) > X_t(K^{(\leq)})\big]\\ &= \int_{(0,\infty)} \exp\big(-x^{-1}\sigma_t\big(K^{(\leq)} \cap \aleph_{US}\big)\big)\, d\big[\exp\big(-x^{-1}\sigma_t\big(K^{(>)} \cap \aleph_{US}\big)\big)\big]\\ &= \frac{\sigma_t\big(K^{(>)} \cap \aleph_{US}\big)}{\sigma_t(\aleph_{US})}.\end{aligned}$$

The independence property follows since $\mu_t$ satisfies (17) and (18). We check that $\mu_t$ satisfies (17). For $x > 0$,

$$\begin{aligned}\mu_t\big(K^{(>)} \cap \big[g^{\vee}(E) > x\big]\big) &= x^{-1}\mu_t\big(K^{(>)} \cap \big[g^{\vee}(E) > 1\big]\big)\\ &= x^{-1}\sigma_t\big(K^{(>)} \cap \aleph_{US}\big)\\ &= \frac{\sigma_t\big(K^{(>)} \cap \aleph_{US}\big)}{\sigma_t(\aleph_{US})} x^{-1}\sigma_t(\aleph_{US})\\ &= \mathbf{P}[M_t \subseteq K]\, \mu_t\big(\big[g^{\vee}(E) > x\big]\big). \qquad \square\end{aligned}$$

The next result develops a critical independence property that is used in showing that the choice process $\mathbf{M}$ is Markov.

Corollary 3.2. *Suppose the measure* $\mu_t$ *satisfies the following equivalent conditions: For any* $0 < s < t$, $x > 0$, $K \in \mathscr{F}(E)$,

$$\mu_{st}\big(K^{(>)} \cap \big[f^{\vee}(E) > x\big]\big) = c_{K^{(>)}}(s,t]\, \mu_{st}\big(\big[f^{\vee}(E) > x\big]\big) \tag{23}$$

*or, equivalently,*

$$\mu_{st}\big(K^{(\geq)} \cap \big[f^{\vee}(E) > x\big]\big) = c_{K^{(\geq)}}(s,t]\, \mu_{st}\big(\big[f^{\vee}(E) > x\big]\big), \tag{24}$$

*where* $c_{K^{(\geq)}}(s,t]$ *and* $c_{K^{(>)}}(s,t]$ *are constants independent of* $x$. *Then for any fixed* $t > 0$, $Y_t^{\vee}(E)$ *is independent of* $\{M_u,\ u \leq t\}$.

*When* $\mathbf{Y}$ *is a superextremal process with max-stable components, the sup-Levy measure satisfies* (23) *and* (24), *since*

$$\mu_{st}\big(K^{(>)} \cap [f^{\vee}(E) > x]\big) = \frac{\sigma_{st}\big(K^{(>)} \cap \aleph_{US}\big)}{\sigma_{st}(\aleph_{US})}\mu_{st}\big([f^{\vee}(E) > x]\big)$$
$$= c_{K^{(>)}}(s,t]\,\mu_{st}\big([f^{\vee}(E) > x]\big)$$

*and*

$$\mu_{st}\big(K^{(\geq)} \cap [f^{\vee}(E) > x]\big) = \frac{\sigma_{st}\big(K^{(\geq)} \cap \aleph_{US}\big)}{\sigma_{st}(\aleph_{US})}\mu_{st}\big([f^{\vee}(E) > x]\big)$$
$$= c_{K^{(\geq)}}(s,t]\,\mu_{st}\big([f^{\vee}(E) > x]\big).$$

PROOF. We prove the result by induction, as in Resnick and Roy (1990). For proving independence it is enough to show for $x > 0$, $0 < t_1 < \cdots < t_n$, $K_i \in \mathscr{K}(E)$, $i = 1, \dots, n$,

$$\mathbf{P}\big[Y_{t_n}^{\vee}(E) \leq x,\, M_{t_i} \cap K_i \neq \varnothing,\, i = 1,\dots,n\big]$$
$$= \mathbf{P}\big[Y_{t_n}^{\vee}(E) \leq x\big]\mathbf{P}\big[M_{t_i} \cap K_i \neq \varnothing,\, i = 1,\dots,n\big].$$

From Theorem 3.1, we know that $Y_{t_1}^{\vee}(E)$ is independent of $M_{t_1}$. For notational convenience, set $Y_t^{\vee}(E) = Z_t$ for any $t > 0$.

For $0 < t_1 < \cdots < t_n$, assume as the induction hypothesis

$$Z_{t_{n-1}} \text{ is independent of } \{M_{t_1}, \dots, M_{t_{n-1}}\}.$$

Suppose $K_i \in \mathscr{K}(E)$, $i = 1, \dots, n$, and $y > 0$. Then

$$\mathbf{P}\big[Z_{t_n} \leq y,\, M_{t_i} \cap K_i \neq \varnothing,\, i = 1,\dots,n\big]$$
$$= \mathbf{E}\mathbf{P}\big[Z_{t_n} \leq y,\, M_{t_i} \cap K_i \neq \varnothing,$$
$$i = 1,\dots,n \,|\, (Z_{t_l}, M_{t_l}),\, l = 1,\dots,n-1\big]$$
$$= \mathbf{E}\mathbf{1}_{[\cap_{l=1}^{n}\{M_{t_l} \cap K_l \neq \phi\}]}$$
$$\times \mathbf{P}\big[Z_{t_n} \leq y,\, M_{t_n} \cap K_n \neq \varnothing \,|\, (Z_{t_l}, M_{t_l}),\, l = 1,\dots,n-1\big]$$

and, since $\{(Z_t, M_t),\, t > 0\}$ are jointly Markov (see the remarks at the end of Section 3), the preceding expression becomes

$$= \mathbf{E}\mathbf{1}_{[\cap_{l=1}^{n-1}\{M_{t_l} \cap K_l \neq \varnothing\}]}\mathbf{P}\big[Z_{t_n} \leq y,\, M_{t_n} \cap K_n \neq \phi \,|\, Z_{t_{n-1}}, M_{t_{n-1}}\big] \tag{25}$$

$$= \mathbf{E}\mathbf{1}_{[\cap_{l=1}^{n-1}\{M_{t_l} \cap K_l \neq \varnothing\}]}\,g\big(Z_{t_{n-1}}, M_{t_{n-1}}\big), \tag{26}$$

where

$$g\big(Z_{t_{n-1}}, M_{t_{n-1}}\big) = \mathbf{P}\big[Z_{t_n} \leq y,\, M_{t_n} \cap K_n \neq \varnothing \,|\, Z_{t_{n-1}}, M_{t_{n-1}}\big].$$

Now integrate using the joint distribution of $Z_{t_{n-1}}$ and $(M_{t_1}, \dots, M_{t_{n-1}})$ and (26) becomes

$$\begin{aligned}
&\int_{\{(F_1,\dots,F_{n-1}):\, F_l \cap K_l \neq \varnothing,\, l=1,\dots,n-1\}} \\
&\quad \cdots \int_{x \le y} g(x, F_{n-1}) \mathbf{P}\big[Z_{t_{n-1}} \in dx,\, M_{t_i} \in dF_i,\, i = 1,\dots,n-1\big] \\
&\quad = \int_{\{(F_1,\dots,F_{n-1}):\, F_l \cap K_l \neq \varnothing,\, l=1,\dots,n-1\}} \\
&\qquad \cdots \int_{x \le y} g(x, F_{n-1}) \mathbf{P}\big[Z_{t_{n-1}} \in dx\big] \mathbf{P}\big[M_{t_i} \in dF_i,\, i = 1,\dots,n-1\big],
\end{aligned} \tag{27}$$

where the last expression is deduced by invoking the induction hypothesis. Now write

$$A = \mathbf{P}\big[y \ge Z_{t_{n-1},t_n} > Z_{t_{n-1}},\, M_{t_n} \cap K_n \neq \varnothing | Z_{t_{n-1}} = x,\, M_{t_{n-1}} = F_{n-1}\big],$$

$$B = \mathbf{P}\big[y \ge Z_{t_{n-1}} \ge Z_{t_{n-1},t_n},\, M_{t_n} \cap K_n \neq \varnothing | Z_{t_{n-1}} = x,\, M_{t_{n-1}} = F_{n-1}\big]$$

so that (27) becomes

$$\begin{aligned}
&\int_{\{(F_1,\dots,F_{n-1}):\, F_l \cap K_l \neq \varnothing,\, l=1,\dots,n-1\}} \\
&\quad \cdots \int_{x \le y} (A + B) \mathbf{P}\big[Z_{t_{n-1}} \in dx\big] \mathbf{P}\big[M_{t_i} \in dF_i,\, i = 1,\dots,n-1\big].
\end{aligned} \tag{28}$$

The term involving $A$ in (28) we have

$$\begin{aligned}
&\int_{\{(F_1,\dots,F_{n-1}):\, F_l \cap K_l \neq \varnothing,\, l=1,\dots,n-1\}} \\
&\quad \cdots \int_{x \le y} \mathbf{P}\big[x < Z_{t_{n-1},t_n} \le y,\, M_{t_{n-1},t_n} \cap K_n \neq \phi\big] \mathbf{P}\big[Z_{t_{n-1}} \in dx\big] \\
&\qquad \times \mathbf{P}\big[M_{t_i} \in dF_i,\, i = 1,\dots,n-1\big],
\end{aligned} \tag{29}$$

where $M_{t_{n-1},t_n} = \{\tau \in E\colon\ Y_{t_{n-1},t_n}(\tau) = Y^{\vee}_{t_{n-1},t_n}(E)\}$ is independent of $(Z_{t_{n-1}}, M_{t_{n-1}})$. Now apply Theorem 3.1 to see that $M_{t_{n-1},t_n}$ and $Z_{t_{n-1},t_n}$ are independent. Hence (29) becomes

$$\begin{aligned}
&= \int_{\{(F_1,\dots,F_{n-1}):\, F_l \cap K_l \neq \varnothing,\, l=1,\dots,n-1\}} \\
&\quad \cdots \int_{x \le y} \mathbf{P}\big[x < Z_{t_{n-1},t_n} \le y\big] \mathbf{P}\big[M_{t_{n-1},t_n} \cap K_n \neq \varnothing\big] \mathbf{P}\big[Z_{t_{n-1}} \in dx\big] \\
&\qquad \times \mathbf{P}\big[M_{t_i} \in dF_i,\, i = 1,\dots,n-1\big] \\
&= \mathbf{P}\big[Z_{t_{n-1}} < Z_{t_{n-1},t_n} \le y\big] \mathbf{P}\big[M_{t_{n-1},t_n} \cap K_n \neq \varnothing\big] \\
&\quad \times \mathbf{P}\big[M_{t_i} \cap K_i \neq \varnothing,\, i = 1,\dots,n-1\big].
\end{aligned}$$

Applying Lemma 3.1, this becomes

$$(30)\qquad \begin{aligned} &= \mathbf{P}\left[Z_{t_n} \le y\right]\mathbf{P}\left[Z_{t_{n-1}} < Z_{t_{n-1},t_n}\right]\mathbf{P}\left[M_{t_{n-1},t_n} \cap K_n \ne \varnothing\right] \\ &\quad \times \mathbf{P}\left[\bigcap_{i=1}^{n-1}\{M_{t_i} \cap K_i \ne \varnothing\}\right]. \end{aligned}$$

Now consider the term containing $B$ in (28). This is the case where $Y^{\vee}_{t_{n-1},t_n}(E) \le Y^{\vee}_{t_{n-1}}(E)$, and in our current notation this is $Z_{t_{n-1},t_n} \le Z_{t_{n-1}}$. Since $\mu_t(\{f > x\mathbf{1}\}) = x^{-1}\mu_t(\{f > \mathbf{1}\})$ is continuous in $x$, we have $M_{t_{n-1}} = M_{t_n}$. Then from (28) we have for the $B$ term:

$$\begin{aligned} &\int_{\{(F_1,\dots,F_{n-1}):\, F_l\cap K_l \ne \varnothing,\, l=1,\dots,n-1\}} \\ &\quad \cdots \int_{x\le y} \mathbf{P}\left[Z_{t_{n-1},t_n} \le x,\, M_{t_n} \cap K_n \ne \varnothing | Z_{t_{n-1}} = x,\, M_{t_{n-1}} = F_{n-1}\right] \\ &\qquad\qquad \times \mathbf{P}\left[Z_{t_{n-1}} \in dx\right]\mathbf{P}\left[M_{t_i} \in dF_i,\, i = 1,\dots,n-1\right] \\ &= \int_{\{(F_1,\dots,F_{n-1}):\, F_l\cap K_l \ne \varnothing,\, l=1,\dots,n-1\}} \\ &\quad \cdots \int_{x\le y} \mathbf{1}_{[F_{n-1}\cap K_n \ne \varnothing]}\mathbf{P}\left[Z_{t_{n-1},t_n} \le x\right]\mathbf{P}\left[Z_{t_{n-1}} \in dx\right] \\ &\qquad\qquad \times \mathbf{P}\left[M_{t_i} \in dF_i,\, i = 1,\dots,n-1\right]. \end{aligned}$$

Another application of Lemma 3.1 yields

$$(31)\qquad \begin{aligned} &\mathbf{P}\left[Z_{t_n} \le y\right]\mathbf{P}\left[Z_{t_{n-1}} \ge Z_{t_{n-1},t_n}\right] \\ &\quad \times \mathbf{P}\left[\bigcap_{i=1}^{n-1}\{M_{t_i} \cap K_i \ne \varnothing\},\, M_{t_{n-1}} \cap K_n \ne \varnothing\right]. \end{aligned}$$

Thus from (30) and (31),

$$\begin{aligned} &\mathbf{P}\left[Z_{t_n} \le y,\, M_{t_i} \cap K_i \ne \phi,\, i = 1,\dots,n\right] \\ &\quad = \mathbf{P}\left[Z_{t_n} \le y\right]\mathbf{P}\left[Z_{t_{n-1}} < Z_{t_{n-1},t_n}\right]\mathbf{P}\left[M_{t_{n-1},t_n} \cap K_n \ne \varnothing\right] \\ &\qquad \times \mathbf{P}\left[\bigcap_{i=1}^{n-1}\{M_{t_i} \cap K_i \ne \varnothing\}\right] \\ &\qquad + \mathbf{P}\left[Z_{t_n} \le y\right]\mathbf{P}\left[Z_{t_{n-1}} \ge Z_{t_{n-1},t_n}\right] \\ &\qquad \times \mathbf{P}\left[\bigcap_{i=1}^{n-1}\{M_{t_i} \cap K_i \ne \varnothing\},\, M_{t_{n-1}} \cap K_n \ne \varnothing\right] \\ &\quad = \psi(y)\phi(K_1,\dots,K_n), \end{aligned}$$

which implies the desired independence. □

In the following theorem, the Markov property for the choice process $\mathbf{M}$ is established and the formulae for the transition probabilities are given.

THEOREM 3.2. (i) *If* (23) *or* (*and*) (24) *hold, then the choice process* $\mathbf{M} = \{M_t, t > 0\}$ *is Markov with state space* $\mathscr{F}(E)$. *For* $0 < s < t$, $K, F \in \mathscr{F}(E)$, $K \cap F = \varnothing$,

$$\mathbf{P}[M_t \subseteq K | M_s = F] = \mathbf{P}[X_{st}(K^{(>)}) > X_{st}(K^{(\leq)})] \times \mathbf{P}[Y_{st}^{\vee}(E) > Y_s^{\vee}(E)]. \tag{32}$$

(ii) *If* $\mathbf{Y}$ *is a superextremal process with max-stable components, then* (23) *and* (24) *hold, so* $\mathbf{M}$ *is Markov. For* $0 < s < t$, $K, F \in \mathscr{F}(E)$, *the transition probabilities are determined* (a) *in terms of the choice probability by*

$$\mathbf{P}[M_t \subseteq K | M_s = F] = \mathbf{P}[M_t \subseteq K] - \frac{\sigma_s(\aleph)}{\sigma_t(\aleph)}\mathbf{P}[M_s \subseteq K] + \mathbf{1}_{[K \cap F \neq \varnothing]}\frac{\sigma_s(\aleph)}{\sigma_t(\aleph)}; \tag{33}$$

(b) *in terms of the hitting functional are determined by*

$$\mathbf{P}[M_t \cap K \neq \varnothing | M_s = F] = \mathbf{P}[M_s \cap K \neq \varnothing] - \frac{\sigma_s(\aleph)}{\sigma_t(\aleph)}\mathbf{P}[M_s \cap K \neq \varnothing] + \mathbf{1}_{[K \cap F \neq \varnothing]}\frac{\sigma_s(\aleph)}{\sigma_t(\aleph)}. \tag{34}$$

(iii) *If* $\mathbf{Y}$ *is time-homogeneous, superextremal process with max-stable components, then* $\mathbf{M}$ *is Markov and for* $0 < s < t$, $F, K \in \mathscr{F}(E)$, *the transition probabilities are* (a) *in terms of the containment functional*:

$$\mathbf{P}[M_t \subseteq K | M_s = F] = \left(1 - \frac{s}{t}\right)\pi_K^C + \mathbf{1}_{[K \cap F \neq \varnothing]}\frac{s}{t}; \tag{35}$$

(b) *in terms of the hitting function*:

$$\mathbf{P}[M_t \cap K \neq \varnothing | M_s = F] = \left(1 - \frac{s}{t}\right)\pi_K^H + \mathbf{1}_{[K \cap F \neq \varnothing]}\frac{s}{t}. \tag{36}$$

*In the time-homogeneous case, the deterministically time-changed* ($t \mapsto e^t$) *choice process* $\mathbf{M}_e = \{M_{e^t}, t > 0\}$ *is a* $\mathscr{F}(E)$-*valued, time-homogenous Markov process whose stationary transition probabilities are* (a′) *in terms of the containment functional*:

$$\mathbf{P}[M_{e^t} \subseteq K | M_{e^s} = F] = (1 - e^{-(t-s)})\pi_K^C + \mathbf{1}_{[K \cap F \neq \varnothing]}e^{-(t-s)}; \tag{37}$$

(b′) *in terms of the hitting functional*:

$$\mathbf{P}[M_{e^t} \cap K \neq \varnothing | M_{e^s} = F] = (1 - e^{-(t-s)})\pi_K^H + \mathbf{1}_{[K \cap F \neq \varnothing]}e^{-(t-s)}, \tag{38}$$

*and* $\{M_{e^t}, t > 0\}$ *is a stationary process.*

PROOF. For $0 < t_1 < \cdots < t_n$ and $K_i \in \mathscr{K}(E)$, $i = 1, \dots, n$,

$$\begin{aligned}
\mathbf{P}\big[M_{t_n} \subseteq K_n | M_{t_i} = K_i, i = 1, \dots, n-1\big] \\
&= \mathbf{P}\big[M_{t_n} \subseteq K_n, Y^{\vee}_{t_{n-1}, t_n}(E) > Y^{\vee}_{t_{n-1}}(E) | M_{t_i} = K_i, i = 1, \dots, n-1\big] \\
&\quad + \mathbf{P}\big[M_{t_{n-1}, t_n} \subseteq K_n, Y^{\vee}_{t_{n-1}, t_n}(E) \le Y^{\vee}_{t_{n-1}}(E) | M_{t_i} = K_i, i = 1, \dots, n-1\big] \\
&= \mathbf{P}\big[M_{t_{n-1}, t_n} \subset K_n, Y^{\vee}_{t_{n-1}, t_n}(E) > Y^{\vee}_{t_{n-1}}(E) | M_{t_i} = K_i, i = 1, \dots, n-1\big] \\
&\quad + \mathbf{1}_{[K_n \cap K_{n-1} \neq \varnothing]} \mathbf{P}\big[M_{t_{n-1}} \subset K_n, \\
&\qquad Y^{\vee}_{t_{n-1}, t_n}(E) \le Y^{\vee}_{t_{n-1}}(E) | M_{t_i} = K_i, i = 1, \dots, n-1\big]
\end{aligned}$$

and by Corollary 3.2, $(M_{t_{n-1}, t_n}, Y_{t_{n-1}, t_n}, Y_{t_{n-1}})$ is independent of $(M_{t_1}, \dots, M_{t_{n-1}})$ so the preceding equation is

$$\begin{aligned}
&= \mathbf{P}\big[M_{n-1, t_n} \subset K_n, Y^{\vee}_{t_{n-1}, t_n}(E) > Y^{\vee}_{t_{n-1}}(E)\big] \\
&\quad + \mathbf{1}_{[K_n \cap K_{n-1} \neq \varnothing]} \mathbf{P}\big[Y^{\vee}_{t_{n-1}, t_n}(E) \le Y^{\vee}_{t_{n-1}}(E)\big].
\end{aligned}$$

This proves the Markov property. The rest of the formulae for the transition probabilities are obtained by straightforward calculation. The stationarity of $\mathbf{M}$ for time-homogeneous $\mathbf{Y}$ with max-stable marginals is readily checked. □

REMARK 3.1. A direct calculation shows that if $0 < s < t$,

$$\operatorname{Corr}\left(\frac{1}{Y_s^{\vee}(E)}, \frac{1}{Y_t^{\vee}(E)}\right) = \frac{\sigma_s(\aleph_{US})}{\sigma_t(\aleph_{US})}$$

[cf. Resnick and Roy (1990) for the calculation in finite dimensions]. Therefore, the transition probabilities may be rewritten by replacing $(\sigma_s(\aleph_{US}))/(\sigma_t(\aleph_{US}))$ by the correlation. For instance, (33) becomes

$$\begin{aligned}
\mathbf{P}[M_t \subseteq K | M_s = F] &= \left(1 - \operatorname{Corr}\left(\frac{1}{Y_s^{\vee}(E)}, \frac{1}{Y_t^{\vee}(E)}\right)\right)\pi_K^C \\
&\quad + \mathbf{1}_{[K \cap F \neq \varnothing]} \operatorname{Corr}\left(\frac{1}{Y_s^{\vee}(E)}, \frac{1}{Y_t^{\vee}(E)}\right).
\end{aligned}$$

The following independence property is the infinite-dimensional counterpart of Proposition 4.1 of Resnick and Roy (1990). The proof uses techniques from there and Corollary 3.2 of the present paper, and is omitted.

PROPOSITION 3.1. *If* $\mathbf{Y}$ *is a time-homogeneous, superextremal process and the sup-Levy measure* $\mu$ *satisfies* (23) *or* (24), *then* $M_t$ *is independent of* $\{Y_u^{\vee}(E), u \le t\}$, *for any* $t > 0$.

One can construct a counterexample, as in Resnick and Roy [(1990), Section 4], to show that time homogeneity is necessary for the result in Proposition 3.1.

**4. The spectral representation of max-stable superextremal processes.** In this section we discuss another method for constructing max-stable superextremal processes using spectral functions [cf. de Haan (1984)]. For $US(E)$-valued superextremal processes with max-stable components, this method is *equivalent* to the approach to the subject in Section 3, which utilized the canonical representation of sup-infinitely divisible processes on $US_0(E)$ of Norberg (1986). The methods in this section are important because when modelling choice, it is more convenient to pick a family of upper semicontinuous functions than to select a measure on the function space $US_0(E)$, which is required by the construction in Section 3 [see also Rust (1991) for a related discussion]. On the other hand, in Section 3 we have general characterization theorems, which apply to all $US(E)$-valued superextremal processes, and are obviously in force when max-stability is present.

For what follows, $\lambda$ denotes Lebesgue measure.

THEOREM 4.1. *Let* $\mathbf{Y}$ *be a superextremal process with max-stable components with sup-Levy measure* $\mu$ *and accompanying measure* $\sigma$ *on* $\mathscr{B}((0,\infty)) \times \mathscr{B}(\aleph_{US})$. *Construct a PRM* $N = \sum_k \varepsilon_{(u_k, v_k, \Gamma_k)}$ *with points in* $(0,\infty)^3$ *and mean measure* $\lambda^3$. *There exists a measurable function* $\mathbf{f} := (f_1, f_2)\colon (0,\infty)^2 \to (0,\infty) \times \aleph_{US}$ *such that*

$$\sigma = \lambda^2 \circ \mathbf{f}^{-1}.$$

*The process* $\tilde{\mathbf{Y}} = \{\tilde{\mathbf{Y}}_t,\ t > 0\}$ *defined by*

$$\tilde{\mathbf{Y}}_t := \bigvee_{f_1(u_k, v_k) \le t} \frac{f_2(u_k, v_k)}{\Gamma_k} \tag{39}$$

*is a superextremal process with max-stable components and* $\mathbf{Y} =_d \tilde{\mathbf{Y}}$.

PROOF. See Theorem 3.2 of Resnick and Roy (1991b). We need to show

$$\sum_k \varepsilon_{(f_1(u_k, v_k), (f_2(u_k, v_k))/\Gamma_k)}$$

is PRM ($\mu$) or equivalently, because $\|f_2\| = 1$, we need to show

$$\sum_k \varepsilon_{(f_1(u_k, v_k),\ \Gamma_k^{-1},\ f_2(u_k, v_k))}$$

is PRM with mean measure of $[0, t] \times (r_0, \infty) \times \Lambda$ [where $\Lambda \in \mathscr{B}(\aleph_{US(E)})$, $r_0 > 0$] equal to $r_0^{-1}\sigma_t(\Lambda)$. This follows in a straightforward way from the choice of $\mathbf{f}$ to satisfy $\sigma = \lambda^2 \circ \mathbf{f}^{-1}$. $\square$

Theorem 4.1 suggests the following method of constructing a superextremal process with max-stable components. Find two functions $f_1\colon (0,\infty)^2 \to (0,\infty)$ and $f_2\colon (0,\infty)^2 \to \aleph_{US}$. Often, in fact, the range of $f_2$ will be $C(E)$, the space of continuous functions on $E$. Define $\sigma = \lambda^2 \circ \mathbf{f}^{-1}$. Construct a PRM ($\sigma$) and call it

$$\sum_k \varepsilon_{(t_k, r_k, \mathbf{a}_k)},$$

where $t_k \in (0, \infty)$, $r_k \in (0, \infty)$ and $\mathbf{a}_k \in \aleph_{US}$. Then

$$\sum_k \varepsilon_{(t_k, r_k \mathbf{a}_k)}$$

is also PRM. Call its mean measure $\mu$. Assuming (1), (2) and (3) are satisfied,

$$\mathbf{Y}_t = \bigvee_{t_k \le t} r_k \mathbf{a}_k$$

is the required superextremal process with max-stable components.

A time-homogeneous, superextremal process with max-stable components readily can be generated by the following variant of the previous construction. Suppose

$$\sum_k \varepsilon_{(t_k, u_k, \Gamma_k)}$$

is homogenous PRM on $(0, \infty) \times (0, 1) \times (0, \infty)$ and let $f\colon (0, 1) \mapsto US(E)$ satisfy for every $K \in \mathscr{K}(E)$,

$$\int_0^1 (f(u))^{\vee}(K)\, du < \infty.$$

Then

$$\sum \varepsilon_{(t_k, \Gamma_k^{-1} f(u_k))}$$

is PRM and if

$$\mathbf{Y}_t = \bigvee_{t_k \le t} \frac{f(u_k)}{\Gamma_k},$$

then $\{\mathbf{Y}_t,\ t > 0\}$ is a superextremal process with max-stable components.

Observe that for $K_j \in \mathscr{K}(E)$, $x_j > 0$, $j = 1, \dots, J$, we have for any $t$ fixed,

$$\begin{aligned}
P\left[\bigcap_{j=1}^{J} \left[Y_t^{\vee}(K_j) \le x_j\right]\right] &= P\left[\bigcap_{j=1}^{J} \left\{\bigvee_{t_k \le t} \frac{(f(u_k))^{\vee}(K_j)}{\Gamma_k} \le x_j\right\}\right] \\
&= P\left[\bigvee_{t_k \le t} \bigvee_{j=1}^{J} \frac{(f(u_k))^{\vee}(K_j)}{x_j \Gamma_k} \le 1\right] \\
&= \exp\left(-t \int_{\{(u, w):\ \bigvee_{j=1}^{J}((f(u))^{\vee}(K_j))/x_j w \le 1\}} dw\, du\right) \\
&= \exp\left(-t \int_{u=0}^{1} \bigvee_{j=1}^{J} \frac{(f(u))^{\vee}(K_j)}{x_j}\, du\right). \qquad \square
\end{aligned}$$

EXAMPLE 4.1 (The dynamic continuous Logit model). Let the dynamic random utility process be defined as follows: The choice set is $E = [0, 1]$. For $\tau \in E$, $u \in U = [0, 1]$, define the spectral functions

$$f(\tau, u) = V(\tau) - |\tau - u|.$$

An interpretation of this functional form is that $U$ corresponds to an individual's "ideal" choices and the set $E$ represents those which are offered to the individual for selection. Then $|\tau - u|$ represents an individual's disutility for the alternative $\tau$ when $u$ is the ideal.

Now suppose that the functions $f(\tau, u)$ are maximized at unique $\tau \in E$; such that $\tau = u$, for each fixed $u \in U$. This means that the functions $V(\tau)$ satisfy $V(u) > V(\tau) - |\tau - u|$. Examples of such functions are those which are Lipschitz of finite order. In particular, consider

$$f(\tau, u) = \exp(a\tau^2 - |\tau - u|) \tag{40}$$

for $a > 0$.

Now the underlying dynamic random utility process is a superextremal process with max-stable components, defined by

$$\left\{\mathbf{Y}_t := \bigvee_{t_k \le t} \frac{f(u_k)}{\Gamma_k}, t > 0\right\},$$

where for $E$ fixed,

$$\mathbf{Y}_t(\tau) = \bigvee_{t_k \le t} \frac{f(\tau, u_k)}{\Gamma_k},$$

any $\tau \in E$. Also suppose $\mathbf{Y}$ is time homogeneous, that is, $\mathbf{Y}$ has sup-Levy measure $dt \times du \times dw$ over $\mathscr{B}(0, \infty) \times \mathscr{B}(U) \times \mathscr{B}(0, \infty)$. Then we know the corresponding choice process $\mathbf{M}$ is Markov (cf. Theorem 3.2), the transition probabilities [cf. (35)] are for $0 < s < t$, $K_1, K_2 \in \mathscr{F}(E)$,

$$\mathbf{P}[M_t \subseteq K_2 | M_s = K_1] = (1 - s/t)\mathbf{P}[M_t \subseteq K_2] + 1_{\{K_1 \cap K_2 \neq \varnothing\}} s/t.$$

First recall that these spectral functions in (40) have unique maxima in $\tau \in E$ for fixed $u \in U$. Recall our characterizations in Section 3, where we note for unique maxima in utilities (i.e., singleton $M_t$) the sup-Levy measure must be concentrated on $US(E)_{\mathrm{SING}}$ or, equivalently in the language of Section 4, the spectral functions must have unique maxima [as described in Corollary 4.2 in Resnick and Roy (1990)]. This implies $K_2^{(>)} = K_2$. Now applying the formulae (21) for the choice probability and Resnick and Roy [(1990), Theorem 4.1], the choice probability is

$$\mathbf{P}[M_t \subseteq K_2] = \frac{\int_{K_2} f^{\vee}(T, u)\, du}{\int_{[0,1]} f^{\vee}(T, u)\, du} = \frac{\int_{K_2} e^{V_u}\, du}{\int_{[0,1]} e^{V_u}\, du} = \frac{\int_{K_2} e^{au^2}\, du}{\int_{[0,1]} e^{au^2}\, du}.$$

Hence, the transition probability (35) simplifies to

$$\mathbf{P}[M_t \subseteq K_2 | M_s = K_1] = \left(1 - \frac{s}{t}\right) \frac{\int_{K_2} e^{au^2}\, du}{\int_{[0,1]} e^{au^2}\, du} + 1_{\{K_1 \cap K_2 \neq \varnothing\}} s/t.$$

## APPENDIX

**A.1. Measurability.** Let $E$ be a compact, metric space. The arg max functional $A_\vee$ is $\mathscr{B}(US(E))/\mathscr{B}(\mathscr{F}(E))$ measurable. Hence, for any $K \in \mathscr{K}(E)$, $K^{(>)} = \{f \in US(E)\colon A_\vee(f) \subseteq K\} \in \mathscr{B}(US(E))$, $K^{(<)} = \{f \in US(E)\colon A_\vee(f) \cap K = \varnothing\} \in \mathscr{B}(US(E))$, and $K^{(=)} = [K^{(>)} \cup K^{(<)}]^c \in \mathscr{B}(US(E))$.

**A.2. The space of upper semicontinuous functions $US(E)$.** Let $E$ be a compact, metric space with countable dense subset $D_E$ and metric $d$; $\mathscr{B}(E)$ denotes the Borel $\sigma$-algebra on $E$; $US(E)$ is the space of upper semicontinuous functions from $E \mapsto [0, \infty]$, endowed with the sup-vague topology [cf. Vervaat (1988)]. The *sup-vague* topology has basis sets of the form $\{f \in US(E)\colon \bigvee_{t \in K} f(E) < x\}$, $\{f \in US(E)\colon \bigvee_{t \in G} f(E) > x\}$, where $K \in \mathscr{K}(E)$, the compact subsets of $E$, and $G \in \mathscr{G}(E)$, the open subsets of $E$. Then $\mathscr{B}(US(E))$ denotes the usual Borel $\sigma$-algebra on $US(E)$, that is, the $\sigma$-algebra generated by open sets; $US(E)$ is compact, separable and metrizable [cf. Dolecki, Salinetti and Wets (1983) and Norberg (1986)].

**A.3. The space of closed sets $\mathscr{F}(E)$.** Denote $\mathscr{F} = \mathscr{F}(E)$ by the class of closed subsets of $E$, which is given the *Fell* or *hit-miss* or *vague* topology by declaring the following collection as subbasis sets of the topology: (i) $\{F \in \mathscr{F}(E)\colon F \cap K = \varnothing\}$; (ii) $\{F \in \mathscr{F}(E)\colon F \cap G \neq \varnothing\}$, for $K \in \mathscr{K}(E)$, $G \in \mathscr{G}(E)$. Since $E$ is compact, this topology coincides with the topology generated by the Hausdorff metric on $\mathscr{F}(E)$ [cf. Vervaat (1988)]. $\mathscr{F}(E)$ is a compact, metric space in the vague topology.

The *upper topology* [cf. Castaing and Valadier (1977) and Vervaat (1988)] on $\mathscr{F}(E)$ is generated by taking the collection of sets in (i) as subbasis. Note we always have $F_n \to_u E$, so limits are not unique and the upper topology is not Hausdorff.

Similarly, the *lower topology* on $\mathscr{F}(E)$ is generated by taking the collection in (ii) as subbasis. Therefore a function $H$ is vaguely continuous iff $H$ is upper and lower continuous.

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