

## AN ANALYSIS OF THE MODIFIED OFFERED-LOAD APPROXIMATION FOR THE NONSTATIONARY ERLANG LOSS MODEL

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A fundamental problem that led to the development of queueing theory is the probabilistic modelling of the number of busy lines in telephone trunk groups. Based on the behavior of real telephone systems, a natural model to use would be the  $M_t/G/s/0$  queue, which has  $s$  servers, no extra waiting space and a nonhomogeneous Poisson arrival process ( $M_t$ ). Unfortunately, so far queueing theory has provided an exact analysis for only the  $M/G/s/0$  queue in steady state, which yields the Erlang blocking formula, and the  $M_t/G/\infty$  queue, which treats nonstationary arrivals at the expense of having infinitely many servers. However, these results can be synthesized to create a modified offered-load (MOL) approximation for the  $M_t/G/s/0$  queue: the distribution of the number of busy servers in the  $M_t/G/s/0$  queue at time  $t$  is approximated by the steady-state distribution of the stationary  $M/G/s/0$  queue with an offered load (arrival rate times mean service time) equal to the mean number of busy servers in the  $M_t/G/\infty$  queue at time  $t$ . In addition to being a simple effective approximation scheme, the MOL approximation makes all of the exact results for infinite server queues relevant to the analysis of nonstationary loss systems. In this paper, we provide a rigorous mathematical basis for the MOL approximation. We find an expression for the difference between the  $M_t/G/s/0$  queue length distribution and its MOL approximation. From this expression we extract bounds on the error and deduce when one distribution stochastically dominates the other.

**1. Introduction.** The probabilistic modelling of the number of busy lines in telephone trunk groups is one of the fundamental problems that led to the development of queueing theory. It was first formulated as an  $M/M/s/0$  queue by Erlang [7]. He gave an exact solution for the steady-state distribution, which gave rise to the well known *Erlang blocking formula*. This formula states that if  $Q_s(t)$  is the random queue length at time  $t$  for the  $M/M/s/0$  system (queueing here means “waiting” for service completion), then

$$(1.1) \quad \lim_{t \rightarrow \infty} P(Q_s(t) = s) = \beta_s \left( \frac{\lambda}{\mu} \right) \equiv \frac{(\lambda/\mu)^s}{s!} \bigg/ \sum_{k=0}^s \frac{(\lambda/\mu)^k}{k!},$$

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where  $\lambda$  is the Poisson arrival rate,  $1/\mu \equiv E[S]$  is the mean of the exponentially distributed random service time  $S$  and  $s$  equals the total number of servers (trunk lines). Since Poisson arrivals see time averages,  $\beta_s(\lambda/\mu)$  is also the long-run proportion of arrivals that are lost.

The Erlang blocking formula also applies to the  $M/G/s/0$  queue with a general service-time distribution, having the same Poisson arrival rate and mean service time. This *insensitivity property* means that the assumption of exponential service is superfluous, which expands the model's range of applicability. Moreover, limit theorems for general point processes show that modelling the arrival process as Poisson is not too restrictive; for example, see [2], page 281.

In fact, the most restrictive assumption in the  $M/M/s/0$  model is having a constant arrival rate. Significant steps were made to solve this problem starting in the 1930s; see [16], [11] and [17]. They found the exact solution for the time-dependent distribution in the  $M_t/G/\infty$  model. This infinite server queue captures the effect of a time-varying mean arrival rate and general service times, but at the expense of letting the total number of servers be infinite. If  $Q_\infty(t)$  equals the queue length at time  $t$  in the  $M_t/G/\infty$  model and  $Q_\infty(t_0) = 0$  for some  $t_0 < t$ , then

$$(1.2) \quad P(Q_\infty(t) = k) = \exp(-m_\infty(t)) \frac{m_\infty(t)^k}{k!}$$

for all nonnegative integers  $k$ , where

$$(1.3) \quad m_\infty(t) = E \left[ \int_{t-S}^t \lambda(\tau) d\tau \right],$$

with  $\lambda(t) = 0$  for all  $t < t_0$ . A simple direct approximation for the blocking probability  $P(Q_s(t) = s)$  in the  $M_t/G/s/0$  model is the tail probability  $P(Q_\infty(t) \geq s)$ .

These exact solutions to the  $M/G/s/0$  and  $M_t/G/\infty$  models led to a better technique for approximating the time-dependent queue length distribution in the  $M_t/G/s/0$  model. It is called the *modified offered-load approximation* (MOL); see [8]. Since the Erlang blocking formula is a function of  $\lambda/\mu$  and  $\lambda/\mu$  is the mean queue length in the steady-state stationary  $M/G/\infty$  queue, we should obtain a reasonable approximation for the time-dependent blocking probability in the  $M_t/G/s/0$  queue if we substitute  $m_\infty(t)$  for  $\lambda/\mu$  in the Erlang blocking formula. Thus the MOL approximation is

$$(1.4) \quad P(Q_s(t) = s) \approx \beta_s(m_\infty(t)) = P(Q_\infty(t) = s | Q_\infty(t) \leq s),$$

where  $\beta$  is given by (1.1) and  $m_\infty(t)$  is given by (1.3).

From (1.4), we see that MOL enables us to apply the exact results for the  $M_t/G/\infty$  model to the analysis of the  $M_t/G/s/0$  model. For example, we applied the MOL approximation to help understand the impact of the service-time distribution in an  $M_t/G/s/0$  queue in [3]. The MOL approximation was also a major motivation for the papers by Eick, Massey and Whitt [5, 6] on the  $M_t/G/\infty$  model. Moreover, since a solution exists for the transient

distribution of the  $M_t/G/\infty$  queue (see [1] and [15]), we can apply the MOL approximation to the  $M_t/G_t/s/0$  queue as well.

The goal of this paper is to create a mathematical theory supporting this heuristic approximation. In Section 4, we do so by constructing a formal solution to the error between the exact probability solution and the MOL approximation for the case of time-dependent phase-type service. From this main result, we derive simple, computable error bounds for MOL. For the  $M_t/M/s/0$  queue, we will show that

$$(1.5) \quad \sup_{0 \leq \tau \leq t} |P(Q_s(\tau) = s) - \beta_s(m_\infty(\tau))| \leq 2 \int_0^t \beta_s(m_\infty(\tau))(1 - \beta_s(m_\infty(\tau))) \left| \frac{dm_\infty}{d\tau}(\tau) \right| d\tau,$$

where we assume that the distribution of  $Q_s(0)$  is the steady-state  $M/M/s/0$  distribution with parameter  $m_\infty(0)$ , which is a family of distributions that includes the point masses at 0 and  $s$ ; see (4.6). For the more general  $M_t/G/s/0$  system, we will also show that

$$(1.6) \quad \sup_{0 \leq \tau \leq t} |P(Q_s(\tau) = s) - \beta_s(m_\infty(\tau))| \leq 2 \cdot \int_0^t \beta_s(m_\infty(\tau)) \left| \frac{dm_\infty}{d\tau}(\tau) \right| d\tau;$$

see (4.7). These error bounds imply that the MOL approximation is asymptotically correct as either the derivative of  $m_\infty(t)$  or the tail probability  $P(Q_\infty(t) \geq s)$  in the  $M_t/G_t/\infty$  model approaches 0. In turn, these limits for the  $M_t/G/\infty$  model hold as the derivative of  $\lambda(t)$  approaches 0 and as  $s \rightarrow \infty$ . More generally, these bounds support the intuition that MOL should perform better when the arrival rate  $\lambda(t)$  changes more slowly and when the blocking probability is lower.

We obtain alternative bounds for the  $M_t/M/s/0$  system in Section 5 by using the  $l_1$ -norm on cumulative distribution functions instead of the  $l_1$ -norm on probability mass functions. For example, with the same initial conditions, if  $\mu = 1$  and  $\lambda$  is bounded with a bounded derivative  $\lambda'$  on  $[0, \infty)$ , then

$$(1.7) \quad \sup_{t \geq 0} |E[Q_s(t)] - m_\infty(t)(1 - \beta_s(m_\infty(t)))| \leq |\lambda'|_\infty s \beta_s(|\lambda|_\infty),$$

where  $|f|_\infty = \sup_{x \geq 0} |f(x)|$  for all bounded functions  $f$  on  $[0, \infty)$ . Note that (1.7) is uniform over all time.

In Section 6, we investigate in detail the special case of an  $M/M/s/0$  model which experiences a change of parameters at time 0. Hence we are describing the transient behavior going from one stationary regime to another. Here we exploit the fact that the generator after time 0 is not time-dependent.

**2. The  $M_t/PH_t/s/0$  queue.** We define the  $M_t/PH_t/s/0$  queueing system as follows. It has  $s$  independent servers, each with a common time-dependent phase-type service, and an arrival process that is nonhomogeneous

Poisson. The class of phase-type service-time distributions is quite general, because phase-type distributions are dense in the space of all distributions. This assumption enables us to construct an extended finite state space such that the queue length process is Markovian in continuous time. Let  $C$  equal the finite set of service phases (which we assume does not change with time). To obtain a general state description that makes our system Markovian, we count the number of customers in each phase of service. We define  $S_C$  to be the corresponding state space, allowing arbitrary numbers of customers. The states in  $S_C$  can be denoted by  $\mathbf{k}$ , where every  $\mathbf{k} \in S_C$  is written as the formal sum

$$(2.1) \quad \mathbf{k} = \sum_{\alpha \in C} k_\alpha \mathbf{e}_\alpha,$$

such that  $\mathbf{e}_\alpha$  is an independent basis vector, corresponding to the service phase  $\alpha$ , and each  $k_\alpha$  is a nonnegative integer, representing the number of customers in service phase  $\alpha$ . The set  $S_C$  is the state space for the case of  $s = \infty$ . In algebraic terms,  $S_C$  is referred to as the *free Abelian semigroup* generated by the set  $C$ , in contrast to the free non-Abelian subgroup structure used in [13] for the state space of a multiclass single server queue. Finally, if we denote the *length* of  $\mathbf{k}$  as  $|\mathbf{k}|$ , which equals  $\sum_{\alpha \in C} k_\alpha$ , then the state space for our queueing model  $M_t/PH_t/s/0$  will be  $S_C(s)$ , where

$$(2.2) \quad S_C(s) = \{\mathbf{k} \mid \mathbf{k} \in S_C \text{ and } |\mathbf{k}| \leq s\}.$$

Now let  $\{\mathbf{Q}_s(t) \mid t \geq 0\}$  be the Markovian queue length process with state space  $S_C(s)$ . Its infinitesimal generator will be constructed from the following parameters:

- $\lambda_\alpha(t)$  = the external arrival rate at time  $t$  for a customer that initiates service in phase  $\alpha$ ,
- $\mu_\alpha(t)$  = the service rate at time  $t$  for phase  $\alpha$ ,
- $p_{\alpha\beta}(t)$  = the probability that phase  $\beta$  service is initiated at time  $t$ , given that phase  $\alpha$  service has just terminated,
- $q_\alpha(t)$  = the probability that the entire service has terminated at time  $t$ , given that phase  $\alpha$  service has just terminated.

If  $p(\mathbf{k}, t) \equiv P(\mathbf{Q}_s(t) = \mathbf{k})$ , then for  $|\mathbf{k}| < s$ ,  $\mathbf{Q}_s(t)$  has the following set of forward equations:

$$\begin{aligned} \frac{d}{dt} p(\mathbf{k}, t) = & \sum_{\alpha \in C} \left[ \lambda_\alpha(t) \text{sgn}(k_\alpha) p(\mathbf{k} - \mathbf{e}_\alpha, t) \right. \\ & + \mu_\alpha(t) (k_\alpha + 1) q_\alpha(t) p(\mathbf{k} + \mathbf{e}_\alpha, t) \\ & + \sum_{\beta \in C} \mu_\beta(t) (k_\beta + 1) p_{\beta\alpha}(t) \text{sgn}(k_\alpha) p(\mathbf{k} - \mathbf{e}_\alpha + \mathbf{e}_\beta, t) \\ & \left. - (\lambda_\alpha(t) + \mu_\alpha(t) k_\alpha) p(\mathbf{k}, t) \right], \end{aligned}$$

where  $\text{sgn}(k) = 0$  if  $k = 0$  and  $= 1$  if  $k > 0$ . When  $|\mathbf{k}| = s$ , we have

$$\begin{aligned} \frac{d}{dt}p(\mathbf{k}, t) = \sum_{\alpha \in C} \left[ \lambda_{\alpha}(t)\text{sgn}(k_{\alpha})p(\mathbf{k} - \mathbf{e}_{\alpha}, t) \right. \\ \left. + \sum_{\beta \in C} \mu_{\beta}(t)(k_{\beta} + 1)p_{\beta\alpha}(t)\text{sgn}(k_{\alpha})p(\mathbf{k} - \mathbf{e}_{\alpha} + \mathbf{e}_{\beta}, t) \right. \\ \left. - \mu_{\alpha}(t)k_{\alpha}p(\mathbf{k}, t) \right]. \end{aligned}$$

Letting  $l(S_C(s))$  be the vector space of real-valued functions on  $S_C(s)$ , we can encode these equations as

$$(2.3) \quad \frac{d}{dt}\mathbf{p}(t) = \mathbf{p}(t)\mathbf{A}(t),$$

where

$$(2.4) \quad \mathbf{p}(t) = \sum_{\mathbf{k} \in S_C(s)} P(\mathbf{Q}_s(t) = \mathbf{k})\mathbf{e}_{\mathbf{k}}$$

and  $\mathbf{A}(t)$  is the corresponding infinitesimal generator; that is, a linear operator on  $l(S_C(s))$  composed of the arrival and service rates for the queueing process. The  $\mathbf{e}_{\mathbf{k}}$ 's are the unit basis vectors for  $l(S_C(s))$ , where each  $\mathbf{e}_{\mathbf{k}}$  corresponds to the indicator function for the singleton set  $\{\mathbf{k}\}$ . In general,  $\mathbf{p}(t)$  is a *probability vector*, since it is a vector encoding of the probability distribution given by  $p(\mathbf{k}, t)$ . We will use the terms probability vector and probability distribution interchangeably. Formally, we can solve for  $\mathbf{p}(t)$  and get

$$(2.5) \quad \mathbf{p}(t) = \mathbf{p}(0)\mathbf{E}_{\mathbf{A}}(t),$$

where  $\mathbf{E}_{\mathbf{A}}(t)$  is the *time-ordered exponential* of the family of generators  $\{\mathbf{A}(\tau) | 0 \leq \tau \leq t\}$ . When  $\mathbf{A}$  is a constant operator, then the corresponding time-ordered exponential is just  $\exp(t\mathbf{A})$ . In general, it is the unique operator solution to the equation

$$(2.6) \quad \frac{d}{dt}\mathbf{E}_{\mathbf{A}}(t) = \mathbf{E}_{\mathbf{A}}(t)\mathbf{A}(t),$$

where  $\mathbf{E}_{\mathbf{A}}(0) = \mathbf{I}$ , the identity operator. For all  $\tau, 0 \leq \tau \leq t$ , it will also be useful to define

$$(2.7) \quad \mathbf{E}_{\mathbf{A}}(\tau, t) \equiv \mathbf{E}_{\mathbf{A}}(\tau)^{-1}\mathbf{E}_{\mathbf{A}}(t).$$

A thorough treatment of the issues of existence, uniqueness and construction of time-ordered exponentials can be found in [4].

**3. The  $M_t / PH_t / \infty$  queue.** Our approximate analysis of the  $M_t / PH_t / s / 0$  queue employs the exact solution for its infinite-server counterpart, the  $M_t / PH_t / \infty$  queue. Let  $\{\mathbf{Q}_{\infty}(t) | t \geq 0\}$  be the  $M_t / PH_t / \infty$  queue length process. Its marginal probabilities,  $q(\mathbf{k}, t) \equiv P(\mathbf{Q}_{\infty}(t) = \mathbf{k})$  for all  $\mathbf{k} \in S_C$ , will

then solve the following set of forward equations:

$$\begin{aligned} \frac{d}{dt}q(\mathbf{k}, t) = \sum_{\alpha \in C} & \left[ \lambda_{\alpha}(t) \operatorname{sgn}(k_{\alpha}) q(\mathbf{k} - \mathbf{e}_{\alpha}, t) \right. \\ & + \mu_{\alpha}(t)(k_{\alpha} + 1) q_{\alpha}(t) q(\mathbf{k} + \mathbf{e}_{\alpha}, t) \\ & + \sum_{\beta \in C} \mu_{\beta}(t)(k_{\beta} + 1) p_{\beta\alpha}(t) \operatorname{sgn}(k_{\alpha}) q(\mathbf{k} - \mathbf{e}_{\alpha} + \mathbf{e}_{\beta}, t) \\ & \left. - (\lambda_{\alpha}(t) + \mu_{\alpha}(t)k_{\alpha}) q(\mathbf{k}, t) \right]. \end{aligned}$$

Now for any  $\mathbf{x}$  in  $l(C)$ , the vector space of real-valued functions on  $C$ , and any state  $\mathbf{k} \in S_C$ , define the following useful operations:

$$(3.1) \quad \mathbf{x}^{\mathbf{k}} \equiv \prod_{\alpha \in C} x_{\alpha}^{k_{\alpha}}, \quad \mathbf{k}! \equiv \prod_{\alpha \in C} k_{\alpha}!, \quad |\mathbf{x}| \equiv \sum_{\alpha \in C} |x_{\alpha}|,$$

where  $x_{\alpha} = \mathbf{x}(\alpha)$ . We will also represent  $\mathbf{x}$  by the formal sum  $\sum_{\alpha \in C} x_{\alpha} \mathbf{e}_{\alpha}$ . Hence  $|\mathbf{x}|$  is the  $l_1$ -norm applied to  $\mathbf{x}$ . In this notation, the multinomial theorem is transformed into

$$(3.2) \quad \sum_{|\mathbf{k}|=s} \frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!} = \frac{|\mathbf{x}|^s}{s!}.$$

Theorem 8.2 of [15] gives the exact solution for the  $M_t/PH_t/\infty$  queue, with appropriate initial distributions, as

$$(3.3) \quad q(\mathbf{k}, t) = \frac{e^{-m_{\infty}(t)} \mathbf{m}_{\infty}(t)^{\mathbf{k}}}{\mathbf{k}!},$$

where  $\mathbf{m}_{\infty}(t) = \sum_{\alpha \in C} m_{\infty}^{\alpha}(t) \mathbf{e}_{\alpha}$  and  $m_{\infty}(t) = |\mathbf{m}_{\infty}(t)| = \sum_{\alpha \in C} m_{\infty}^{\alpha}(t)$ , such that the  $m_{\infty}^{\alpha}(t)$ 's solve the set of differential equations

$$(3.4) \quad \frac{d}{dt} m_{\infty}^{\alpha}(t) = \lambda_{\alpha}(t) + \sum_{\beta \in C} \mu_{\beta}(t) m_{\infty}^{\beta}(t) p_{\beta\alpha}(t) - \mu_{\alpha}(t) m_{\infty}^{\alpha}(t)$$

for all  $\alpha \in C$ , with arbitrary  $\mathbf{m}_{\infty}(0)$ . The solution (3.3) is valid provided that the initial distribution  $p(\mathbf{k}, 0)$  is also of the same form depending on the initial mean vector  $\mathbf{m}_{\infty}(0)$ .

**4. The fundamental identity and bounds for MOL.** The MOL approximation is defined to be  $p^*(\mathbf{k}, t)$  for  $S_C$ , where

$$(4.1) \quad \begin{aligned} P(\mathbf{Q}_s(t) = \mathbf{k}) \approx p^*(\mathbf{k}, t) & \equiv \frac{\mathbf{m}_{\infty}(t)^{\mathbf{k}}}{\mathbf{k}!} \bigg/ \sum_{j=0}^s \frac{m_{\infty}(t)^j}{j!} \\ & = P(\mathbf{Q}_{\infty}(t) = \mathbf{k} \mid |\mathbf{Q}_{\infty}(t)| \leq s), \end{aligned}$$

where the components of the vector  $\mathbf{m}_{\infty}(t) = \sum_{\alpha \in C} m_{\infty}^{\alpha}(t) \mathbf{e}_{\alpha}$  solve the differential equations given by (3.4), with arbitrary initial vector  $\mathbf{m}_{\infty}(0)$ . We now present our main result, which we prove in Section 7.

**THEOREM 4.1.** *Let  $\{\mathbf{Q}_s(t) \mid t \geq 0\}$  be the Markovian queueing process for  $M_t/PH_t/s/0$  with the family of infinitesimal generators  $\{\mathbf{A}(t) \mid t \geq 0\}$ . Let  $\mathbf{p}(t)$  be the probability vector for the distribution of  $\mathbf{Q}_s(t)$ , with an initial distribution  $\mathbf{p}(0) = \mathbf{p}^*(0)$ , which is of the form (3.3) for arbitrary  $\mathbf{m}_\infty(0)$ . Let  $\mathbf{p}^*(t)$  be the probability vector for the modified offered-load approximation. Then*

$$(4.2) \quad \mathbf{p}^*(t) - \mathbf{p}(t) = \sum_{\|\mathbf{k}\|=s} \int_0^t p^*(\mathbf{k}, \tau) (\mathbf{p}^*(\tau) - \mathbf{e}_\mathbf{k}) \mathbf{E}_\mathbf{A}(\tau, t) dm_\infty(\tau),$$

where  $\mathbf{E}_\mathbf{A}(\tau, t)$  is given by (2.7), the signed measure  $dm_\infty(\tau)$  is formally the derivative of  $m_\infty$  times  $d\tau$  and

$$(4.3) \quad dm_\infty(\tau) = \left( \sum_{\alpha \in C} \lambda_\alpha(\tau) - \mu_\alpha(\tau) m_\infty^\alpha(\tau) q_\alpha(\tau) \right) d\tau.$$

We now apply Theorem 4.1 to obtain bounds and inequalities. First, we obtain bounds by simply bounding the time-ordered exponential  $\mathbf{E}_\mathbf{A}(\tau, t)$  in (4.2) by 1. It may be possible to obtain more refined relations by more carefully examining the time-ordered exponential, as we illustrate by example in Section 6. Recall that  $\|\mathbf{x}\|$  is the  $l_1$ -norm, defined in (3.1). Let  $|dm_\infty(\tau)|$  be the measure

$$(4.4) \quad |dm_\infty(\tau)| \equiv \left| \frac{dm_\infty}{d\tau}(\tau) \right| d\tau.$$

**COROLLARY 4.2.** *In the setting of Theorem 4.2, we have the following bounds for the error due to the modified offered-load approximation:*

$$(4.5) \quad \begin{aligned} & \sup_{0 \leq \tau \leq t} |\mathbf{p}^*(\tau) - \mathbf{p}(\tau)| \\ & \leq 2 \sum_{\|\mathbf{k}\|=s} \int_0^t p^*(\mathbf{k}, \tau) (1 - p^*(\mathbf{k}, \tau)) |dm_\infty(\tau)| \end{aligned}$$

$$(4.6) \quad \leq 2 \int_0^t \beta_s(m_\infty(\tau)) \left( 1 - \frac{\beta_s(m_\infty(\tau))}{\binom{s + |C| - 1}{s}} \right) |dm_\infty(\tau)|$$

$$(4.7) \quad \leq 2 \int_0^t \beta_s(m_\infty(\tau)) |dm_\infty(\tau)|,$$

where  $\beta_s(m_\infty(t))$  is given in (1.4), that is,

$$(4.8) \quad \beta_s(m_\infty(t)) = \sum_{\|\mathbf{k}\|=s} p^*(\mathbf{k}, t).$$

**PROOF.** The first bound follows from Theorem 4.1, the identity

$$(4.9) \quad \|\mathbf{p}^*(t) - \mathbf{e}_\mathbf{k}\| = 2(1 - p^*(\mathbf{k}, t))$$

for all  $\mathbf{k} \in S_C$  and the fact that  $\|\mathbf{E}_\mathbf{A}(t)\| = 1$ , where  $\|\cdot\|$  is an operator norm induced by the  $l_1$ -norm on row vectors.

For the second inequality, we observe that  $x(1 - x)$  is a concave function of  $x$ , and  $\binom{s + |C| - 1}{s}$  equals the number of states  $\mathbf{k}$  with  $|\mathbf{k}| = s$ . Now apply Jensen's inequality to the first bound.  $\square$

If  $Q_s(0)$  has a distribution that is *not* of the form (4.1), then we can construct a process  $\tilde{Q}_s$  that has the same infinitesimal generator, but an initial distribution of the proper form. We then have

$$(4.10) \quad \sup_{0 \leq \tau \leq t} |\mathbf{p}^*(\tau) - \mathbf{p}(\tau)| \leq |\mathbf{p}(0) - \tilde{\mathbf{p}}(0)| + \sup_{0 \leq \tau \leq t} |\mathbf{p}^*(\tau) - \tilde{\mathbf{p}}(\tau)|,$$

where  $\tilde{\mathbf{p}}$  is the probability vector for  $\tilde{Q}_s$ , and now Corollary 4.2 applies.

**5. MOL bounds for the  $M_t/M_t/s/0$  queue.** Now we restrict ourselves to one class or  $|C| = 1$ , which gives us the  $M_t/M_t/s/0$  queue. It follows that  $S_C(s) = \{0, 1, \dots, s\}$ , which is a totally ordered set. Moreover, (4.2) simplifies to

$$(5.1) \quad \mathbf{p}^*(t) - \mathbf{p}(t) = \int_0^t \beta_s(m_\infty(\tau))(\mathbf{p}^*(\tau) - \mathbf{e}_s) \mathbf{E}_A(\tau, t) dm_\infty(\tau).$$

The next proposition establishes a stochastic comparison between the  $M_t/M_t/s/0$  queue and its MOL approximation. (All proofs appear at the end of the section.) We say that a probability vector  $\mathbf{p}_1$  is *stochastically dominated* by  $\mathbf{p}_2$ , and write  $\mathbf{p}_1 \leq_{st} \mathbf{p}_2$ , if

$$(5.2) \quad \sum_{j=k}^s p_1(j) \leq \sum_{j=k}^s p_2(j) \quad \text{for all } k = 0, 1, \dots, s.$$

In terms of operators and componentwise ordering of vectors,  $\mathbf{p}_1 \leq_{st} \mathbf{p}_2$  is equivalent to  $\mathbf{p}_1 \mathbf{K} \leq \mathbf{p}_2 \mathbf{K}$ , where  $\mathbf{K} = (\mathbf{I} - \mathbf{L})^{-1}$  with  $\mathbf{L}$  equalling the left shift operator on row vectors, or

$$(5.3) \quad \mathbf{K} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

**THEOREM 5.1.** *For the  $M_t/M_t/s/0$  system, if  $m_\infty(0) \leq \lambda(0)/\mu(0)$  and  $\lambda/\mu$  is an increasing function on  $[0, t]$ , then the modified offered-load distribution is stochastically dominated by the exact distribution for  $Q_s$  on  $[0, t]$ , or*

$$(5.4) \quad \mathbf{p}^*(\tau) \leq_{st} \mathbf{p}(\tau) \quad \text{for all } \tau \in [0, t].$$

*In particular,  $\beta_s(m_\infty)$  underestimates the actual blocking probability on  $[0, t]$ . Conversely, if  $m_\infty(0) \geq \lambda(0)/\mu(0)$  and  $\lambda/\mu$  is an decreasing function on  $[0, t]$ , then the exact distribution for  $Q_s(t)$  is stochastically dominated by the modified offered-load distribution at time  $t$  and  $\beta_s(m_\infty)$  overestimates the actual blocking probability on  $[0, t]$ .*



In order to obtain better bounds on the error of MOL in the blocking probability, we focus on the  $l_1$ -norm of cumulative distribution functions (cdf's) or, equivalently complementary cdf's, instead of probability mass functions. For any vector  $\mathbf{x}$  on  $\{0, 1, \dots, s\}$  we define  $\|\mathbf{x}\|_{\mathbf{K}} = \|\mathbf{x}\mathbf{K}\|$ , that is, the  $l_1$ -norm applied to tail sums. Recall that  $E[X] = \sum_{k=0}^{\infty} P(X > k)$  if  $X$  is a nonnegative integer-valued random variable. Thus, if  $\mathbf{p}_1$  and  $\mathbf{p}_2$  are two probability vectors corresponding to  $\{0, 1, \dots, s\}$ -valued random variables  $X_1$  and  $X_2$ , we have

$$(5.5) \quad \max(|P(X_1 = s) - P(X_2 = s)|, |E[X_1] - E[X_2]|) \leq \|\mathbf{p}_1 - \mathbf{p}_2\|_{\mathbf{K}}.$$

**THEOREM 5.2.** *Let  $\mathbf{p}(0) = \mathbf{p}^*(0)$ . For all  $t \geq 0$ , we have*

$$(5.6) \quad \begin{aligned} \|\mathbf{p}^*(t) - \mathbf{p}(t)\|_{\mathbf{K}} &\leq \int_0^t \beta_s(m_{\infty}(\tau))(s - m_{\infty}(\tau)(1 - \beta_s(m_{\infty}(\tau)))) \\ &\times \exp\left(-\int_{\tau}^t \mu(r) dr\right) dm_{\infty}(\tau). \end{aligned}$$

We now apply Theorem 5.2 to obtain bounds that hold for all time. Note that  $s\beta_s(x) \rightarrow 0$  as  $s \rightarrow \infty$ .

**COROLLARY 5.3.** *In the setting of Theorem 5.2, if  $\lambda$  is a bounded function on  $[0, \infty)$  and  $\mu$  is a constant function, then*

$$(5.7) \quad \begin{aligned} \sup_{t \geq 0} \|\mathbf{p}^*(t) - \mathbf{p}(t)\|_{\mathbf{K}} &\leq \frac{s}{\mu} \max(2|\lambda|_{\infty}, |\lambda|_{\infty} + \mu m_{\infty}(0)) \\ &\times \beta_s\left(\max\left(\frac{|\lambda|_{\infty}}{\mu}, m_{\infty}(0)\right)\right). \end{aligned}$$

If, in addition,  $\lambda$  is differentiable and its derivative  $\lambda'$  is bounded on  $[0, \infty)$ , then

$$(5.8) \quad \begin{aligned} \sup_{t \geq 0} \|\mathbf{p}^*(t) - \mathbf{p}(t)\|_{\mathbf{K}} &\leq \frac{s}{\mu} \max\left(|\lambda(0) - \mu m_{\infty}(0)|, \frac{|\lambda'|_{\infty}}{\mu}\right) \\ &\times \beta_s\left(\max\left(\frac{|\lambda|_{\infty}}{\mu}, m_{\infty}(0)\right)\right). \end{aligned}$$

We remark that Corollary 5.3 is not good for the blocking probabilities, because we can use stochastic comparisons to deduce directly with proper initial conditions the sharper bound  $P(Q_s(t) = s) \leq \beta_s(|\lambda|_{\infty}/\mu)$ , for example, by Theorem 10 of [18]. However, Corollary 5.3 yields useful bounds for the mean, as stated in (1.7).

**PROOF OF THEOREM 5.1.** The basis vector  $\mathbf{e}_s$  is a probability vector for the point mass distribution of being in state  $s$ , which is the maximum probability distribution, with respect to stochastic dominance, on  $\{0, 1, \dots, s\}$ . It follows

that the probability vector  $\mathbf{p}^*(t)$  is always stochastically dominated by  $\mathbf{e}_s$ . Now  $\mathbf{A}(t)$  for fixed  $t$  is the generator for a birth-death process, which is stochastically monotone. Using Theorem 7.5 of [14], it follows that the probability vector  $\mathbf{p}^*(\tau)\mathbf{E}_A(\tau, t)$  is always stochastically dominated by  $\mathbf{e}_s\mathbf{E}_A(\tau, t)$  for all  $0 \leq \tau \leq t$ . After combining this result with (4.2), we will be done once we show that the derivative of  $m_\infty$  is nonnegative (or nonpositive) on  $[0, t]$ . This will follow from the lemma below.  $\square$

LEMMA 5.4. *If  $m_\infty(0) \leq \lambda(0)/\mu(0)$  and  $\lambda/\mu$  is a right-continuous increasing function on  $[0, t]$ , then  $m_\infty$  is increasing on  $[0, t]$  also. Conversely, if  $m_\infty(0) \geq \lambda(0)/\mu(0)$  and  $\lambda/\mu$  is a right-continuous function on  $[0, t]$ , then  $m_\infty$  is decreasing on  $[0, t]$ .*

PROOF. Since  $|C| = 1$ , (3.4) becomes

$$(5.9) \quad \frac{d}{dt}m_\infty(t) = \lambda(t) - \mu(t)m_\infty(t)$$

and so

$$(5.10) \quad m_\infty(t) = m_\infty(0)\exp\left(-\int_0^t \mu(\tau) d\tau\right) + \int_0^t \lambda(\tau)\exp\left(-\int_\tau^t \mu(v) dv\right) d\tau.$$

Now let  $\rho \equiv \lambda/\mu$ . Since by hypothesis  $\rho$  is right-continuous and of bounded variation, we can apply the integration by parts formula (see [2], page 104) and get

$$(5.11) \quad \begin{aligned} \frac{1}{\mu(t)} \frac{d}{dt}m_\infty(t) &= (\rho(0) - m_\infty(0))\exp\left(-\int_0^t \mu(\tau) d\tau\right) \\ &\quad + \int_0^t \exp\left(-\int_\tau^t \mu(v) dv\right) d\rho(\tau). \end{aligned}$$

We now observe that the hypothesis gives precisely the conditions that makes the two summands above nonnegative or nonpositive on  $[0, t]$ .  $\square$

PROOF OF THEOREM 5.2 AND COROLLARY 5.3. Since  $\mu$  is constant, we can write (5.10) as

$$(5.12) \quad m_\infty(t) = m_\infty(0)e^{-\mu t} + \int_0^t \lambda(t - \tau)e^{-\mu\tau} d\tau$$

$$(5.13) \quad = \int_t^\infty \mu m_\infty(0)e^{-\mu\tau} d\tau + \int_0^t \lambda(t - \tau)e^{-\mu\tau} d\tau$$

$$(5.14) \quad \leq \int_0^\infty \max(|\lambda|_\infty, \mu m_\infty(0))e^{-\mu\tau} d\tau$$

$$(5.15) \quad \leq \max\left(\frac{|\lambda|_\infty}{\mu}, m_\infty(0)\right).$$

Combining this with (5.9) gives us

$$(5.16) \quad \left| \frac{dm_\infty}{dt} \right|_\infty \leq |\lambda|_\infty + \mu |m_\infty|_\infty \leq |\lambda|_\infty + \max(|\lambda|_\infty, \mu m_\infty(0)) \leq \max(2|\lambda|_\infty, |\lambda|_\infty + \mu m_\infty(0)).$$

When  $\lambda'$  exists and is bounded, we have by (5.11),

$$(5.17) \quad \left| \frac{dm_\infty}{dt} \right|_\infty = \left| (\lambda(0) - \mu m_\infty(0))e^{-\mu t} + \int_0^t e^{-\mu \tau} \lambda'(t - \tau) d\tau \right|$$

$$(5.18) \quad \leq |\lambda(0) - \mu m_\infty(0)|e^{-\mu t} + \frac{|\lambda'|_\infty}{\mu} (1 - e^{-\mu t})$$

$$(5.19) \quad \leq \max\left(|\lambda(0) - \mu m_\infty(0)|, \frac{|\lambda'|_\infty}{\mu}\right).$$

Finally, we apply the following lemma.

LEMMA 5.5. *If  $\mathbf{E}_A(t)$  is the transition probability matrix for an  $M_t/M_t/s/0$  queue at time  $t$ , then for any two probability vectors  $\mathbf{p}$  and  $\mathbf{q}$  we have*

$$(5.20) \quad |(\mathbf{p} - \mathbf{q})\mathbf{E}_A(t)\mathbf{K}| \leq |(\mathbf{p} - \mathbf{q})\mathbf{K}| \exp\left(-\int_0^t \mu(\tau) d\tau\right).$$

PROOF. If  $\mathbf{A}$  is the generator for an  $M/M/s/0$  queue, then it has the form

$$(5.21) \quad \mathbf{A} = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots & 0 & 0 \\ \mu & -(\lambda + \mu) & \lambda & \cdots & 0 & 0 \\ 0 & 2\mu & -(\lambda + 2\mu) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -(\lambda + (s - 1)\mu) & \lambda \\ 0 & 0 & 0 & \cdots & s\mu & -s\mu \end{bmatrix}.$$

Using right and left shift operators, we have

$$(5.22) \quad \mathbf{A} = \lambda\mathbf{R} + \mu\mathbf{\Delta L} - \lambda\mathbf{RL} - \mu\mathbf{\Delta} = (\lambda\mathbf{R} - \mu\mathbf{\Delta})(\mathbf{I} - \mathbf{L}).$$

Since  $\mathbf{K} = (\mathbf{I} - \mathbf{L})^{-1}$ , then

$$(5.23) \quad \mathbf{K}^{-1}\mathbf{AK} = (\mathbf{I} - \mathbf{L})(\lambda\mathbf{R} - \mu\mathbf{\Delta}) = \lambda\mathbf{R} + \mu\mathbf{L\Delta} - \lambda\mathbf{LR} - \mu\mathbf{\Delta}.$$

Translating back into matrix form, we get

$$(5.24) \quad \mathbf{K}^{-1}\mathbf{AK} = \begin{bmatrix} 0 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & -(\lambda + \mu) & \lambda & \cdots & 0 & 0 \\ 0 & \mu & -(\lambda + 2\mu) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -(\lambda + (s - 1)\mu) & \lambda \\ 0 & 0 & 0 & \cdots & (s - 1)\mu & -(\lambda + s\mu) \end{bmatrix}.$$

Let  $\mathbf{A}^*$  be the lower right-hand  $s \times s$  submatrix of  $\mathbf{K}^{-1}\mathbf{A}\mathbf{K}$ , namely,

$$(5.25) \quad \mathbf{A}^* = \begin{bmatrix} -(\lambda + \mu) & \lambda & \cdots & 0 & 0 \\ \mu & -(\lambda + 2\mu) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -(\lambda + (s - 1)\mu) & \lambda \\ 0 & 0 & \cdots & (s - 1)\mu & -(\lambda + s\mu) \end{bmatrix}.$$

The off-diagonal terms of  $\mathbf{A}^*$  are nonnegative, which makes  $\exp(t\mathbf{A}^*)$  a nonnegative matrix. Moreover, the row sums of  $\mathbf{A}^*$  are all nonnegative and so  $\exp(t\mathbf{A}^*)$  is substochastic, meaning its  $l_1$  operator norm is less than or equal to 1. Since  $\mathbf{A}^* + \mu\mathbf{I}$  has these same properties, we get

$$(5.26) \quad |\exp(t\mathbf{A}^*)| \leq \exp(-\mu t).$$

Now observe that if  $\mathbf{p}$  and  $\mathbf{q}$  are probability vectors, then the first (or zeroth) entry of the row vector  $(\mathbf{p} - \mathbf{q})\mathbf{K}$  is 0. Hence  $\mathbf{A}$  acts on  $(\mathbf{p} - \mathbf{q})\mathbf{K}$  the same way that  $\mathbf{A}^*$  acts on the nonzero, right-handed,  $s$ -dimensional subvector of  $(\mathbf{p} - \mathbf{q})\mathbf{K}$ . Taking norms, we get

$$(5.27) \quad \begin{aligned} |(\mathbf{p} - \mathbf{q})\exp(t\mathbf{A})\mathbf{K}| &\leq |(\mathbf{p} - \mathbf{q})\mathbf{K}\exp(t\mathbf{K}^{-1}\mathbf{A}\mathbf{K})| \\ &\leq |(\mathbf{p} - \mathbf{q})\mathbf{K}|\exp(t\mathbf{A}^*)| \\ &\leq |(\mathbf{p} - \mathbf{q})\mathbf{K}|\exp(-\mu t). \end{aligned}$$

Now we consider the transition matrix  $\prod_{i=1}^n \exp(t_i \mathbf{A}_i)$ , where  $\mathbf{A}_i$  is an  $M/M/s/0$  generator for each  $i$ . By (5.27) and induction,

$$(5.28) \quad \left| (\mathbf{p} - \mathbf{q}) \prod_{i=1}^n \exp(t_i \mathbf{A}_i) \mathbf{K} \right| \leq |(\mathbf{p} - \mathbf{q})\mathbf{K}| \exp\left(-\sum_{i=1}^n \mu_i t_i\right).$$

However, we can approximate  $\mathbf{E}_A(t)$  arbitrarily closely by  $\prod_{i=1}^n \exp(t_i \mathbf{A}_i)$ . This allows us to deduce (5.20).  $\square$

**6. Example: Changing  $M/M/s/0$  rates in midstream.** Suppose we consider the case of  $\lambda(t) = \lambda_+$  and  $\mu(t) = \mu_+$  for all  $t \geq 0$  and  $\mathbf{p}(0) = \mathbf{p}^*(0)$ , where  $m_\infty(0) = \lambda_-/\mu_-$ . The time-dependent behavior of  $Q_s$  is that of a stationary  $M/M/s/0$  queue with rates  $\lambda_-$  and  $\mu_-$  for all time  $t < 0$  that suddenly switches to rates  $\lambda_+$  and  $\mu_+$  for all times  $t \geq 0$ . We now want to compute an upper bound for the error between the transient distribution of  $Q_s$  and its MOL approximation. Now, in addition to Theorem 4.1, we exploit the fact that  $\mathbf{E}_A(t) = \exp(\mathbf{A}t)$ , where  $\mathbf{A}$  is the infinitesimal generator of the  $M/M/s/0$  queue with parameters  $\lambda_+$  and  $\mu_+$ . In this case, (5.1) becomes

$$(6.1) \quad \mathbf{p}^*(t) - \mathbf{p}(t) = \int_0^t \beta_s(m_\infty(\tau))(\mathbf{p}^*(\tau) - \mathbf{e}_s)\exp((t - \tau)\mathbf{A}) dm_\infty(\tau).$$

If we let  $\rho_+ \equiv \lambda_+/\mu_+$  and  $\rho_- \equiv \lambda_-/\mu_-$ , then

$$(6.2) \quad m_\infty(\tau) = \rho_- \exp(-\mu_+ \tau) + \rho_+ (1 - \exp(-\mu_+ \tau))$$

and

$$(6.3) \quad \frac{d}{d\tau} m_\infty(\tau) = \mu_+ (\rho_+ - \rho_-) \exp(-\mu_+ \tau).$$

Since  $Q_s$  is reversible (see [10], page 32), the generator  $\mathbf{A}$  is diagonally similar to a symmetric matrix. By the spectral decomposition theorem, we have

$$(6.4) \quad \exp(t\mathbf{A}) = \mathbf{1}^T \boldsymbol{\pi} + \sum_{j=1}^s \exp(-\sigma_j t) \boldsymbol{\Delta}(\sqrt{\boldsymbol{\pi}})^{-1} \mathbf{x}_j^T \mathbf{x}_j \boldsymbol{\Delta}(\sqrt{\boldsymbol{\pi}}),$$

where  $\boldsymbol{\pi}$  is the steady-state probability vector for  $\mathbf{A}$  such that  $\boldsymbol{\pi}\mathbf{A} = \mathbf{0}$ ,  $\sqrt{\boldsymbol{\pi}}$  is the positive vector whose components are the square roots of the components of  $\boldsymbol{\pi}$  and  $\boldsymbol{\Delta}(\sqrt{\boldsymbol{\pi}})$  is the corresponding diagonal matrix. The negatives of the  $s + 1$  real numbers  $0 < \sigma_1 < \dots < \sigma_s$  are the eigenvalues for  $\mathbf{A}$ . Finally,  $\{\sqrt{\boldsymbol{\pi}}, \mathbf{x}_1, \dots, \mathbf{x}_s\}$  is the corresponding set of orthonormal eigenvectors for  $\boldsymbol{\Delta}(\sqrt{\boldsymbol{\pi}})\mathbf{A}\boldsymbol{\Delta}(\sqrt{\boldsymbol{\pi}})^{-1}$ . The eigenvalues and eigenvectors for this model are readily obtained, as shown in [12] and [9]. In this case the orthogonal polynomials are the Poisson-Charlier polynomials; also see [8].

Since  $\mathbf{p}^*(\tau)$  and  $\mathbf{e}_s$  are probability vectors, we have

$$(6.5) \quad (\mathbf{p}^*(\tau) - \mathbf{e}_s)\mathbf{1}^T = 0.$$

Using the Cauchy-Schwarz inequality, we have

$$(6.6) \quad |\mathbf{x}_j \boldsymbol{\Delta}(\sqrt{\boldsymbol{\pi}})| \leq 1 \quad \text{and} \quad |\mathbf{x}_j \boldsymbol{\Delta}(\sqrt{\boldsymbol{\pi}})^{-1}| \leq \sqrt{\sum_{k=0}^s \frac{1}{\pi_k}}.$$

Hence, we get

$$(6.7) \quad \begin{aligned} & |(\mathbf{p}^*(\tau) - \mathbf{e}_s)\exp((t - \tau)\mathbf{A})| \\ & \leq 2\sqrt{\sum_{k=0}^s \frac{1}{\pi_k}} \sum_{j=1}^s \exp(-\sigma_j(t - \tau)). \end{aligned}$$

Combining all of these results, we obtain

$$(6.8) \quad \begin{aligned} |\mathbf{p}^*(t) - \mathbf{p}(t)| & \leq 2\mu_+ \beta_s(\bar{\rho}) |\rho_+ - \rho_-| \\ & \times \sqrt{\sum_{k=0}^s \frac{1}{\pi_k}} \sum_{j=1}^s \frac{\exp(-\sigma_j t) - \exp(-\mu_+ t)}{\mu_+ - \sigma_j}, \end{aligned}$$

where  $\bar{\rho} = \max(\rho_+, \rho_-)$ . Moreover, we get from the other bounds,

$$\begin{aligned} |(\mathbf{p}^*(t) - \mathbf{p}(t))\mathbf{K}| & \leq \mu_+ |\rho_+ - \rho_-| e^{-\mu_+ t} \int_0^t \beta_s(m_\infty(\tau)) \\ & \quad \times (s - m_\infty(\tau)(1 - \beta_s(m_\infty(\tau)))) d\tau \\ & \leq \mu_+ |\rho_+ - \rho_-| t e^{-\mu_+ t} \beta_s(\bar{\rho}) (s - \underline{\rho}(1 - \beta_s(\underline{\rho}))), \end{aligned}$$

where  $\underline{\rho} = \min(\rho_+, \rho_-)$ .

In addition to these error estimates, we get the following stochastic dominance results by applying Proposition 5.1:

$$(6.9) \quad \rho_- \leq \rho_+ \quad \Rightarrow \quad \mathbf{p}^*(t) \leq_{st} \mathbf{p}(t) \quad \text{for all } t \geq 0$$

and

$$(6.10) \quad \rho_- \geq \rho_+ \quad \Rightarrow \quad \mathbf{p}^*(t) \geq_{st} \mathbf{p}(t) \quad \text{for all } t \geq 0.$$

**7. Proving the main theorem.**

LEMMA 7.1. *If  $\mathbf{x} = \sum_{\alpha \in C} x_\alpha \mathbf{e}_\alpha$  and we define*

$$(7.1) \quad \pi(\mathbf{k}, \mathbf{x}) \equiv \frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!} \bigg/ \sum_{j=0}^s \frac{|\mathbf{x}|^j}{j!},$$

then it follows that

$$(7.2) \quad \frac{\partial}{\partial x_\alpha} \pi(\mathbf{k}, \mathbf{x}) = \pi(\mathbf{k} - \mathbf{e}_\alpha, \mathbf{x}) \operatorname{sgn}(k_\alpha) - \pi(\mathbf{k}, \mathbf{x}) \left( 1 - \sum_{|\mathbf{j}|=s} \pi(\mathbf{j}, \mathbf{x}) \right)$$

and

$$(7.3) \quad \pi(\mathbf{k}, \mathbf{x}) x_\alpha = \pi(\mathbf{k} + \mathbf{e}_\alpha, \mathbf{x}) (k_\alpha + 1).$$

PROOF OF THEOREM 4.1. We first observe that  $p^*(\mathbf{k}, t) = \pi(\mathbf{k}, \mathbf{m}_\infty(t))$  and

$$(7.4) \quad \beta_s(m_\infty(t)) = \sum_{|\mathbf{k}|=s} \pi(\mathbf{k}, \mathbf{m}_\infty(t)).$$

We then apply the identities of Lemma 7.1 and get

$$\begin{aligned} & \frac{d}{dt} p^*(\mathbf{k}, t) \\ &= \sum_{\alpha \in C} \left[ p^*(\mathbf{k} - \mathbf{e}_\alpha, t) \operatorname{sgn}(k_\alpha) - p^*(\mathbf{k}, t) (1 - \beta_s(m_\infty(t))) \right] \cdot \frac{d}{dt} m_\infty^\alpha(t) \\ &= \sum_{\alpha \in C} \left[ p^*(\mathbf{k} - \mathbf{e}_\alpha, t) \operatorname{sgn}(k_\alpha) \frac{d}{dt} m_\infty^\alpha(t) \right] \\ & \quad - p^*(\mathbf{k}, t) (1 - \beta_s(m_\infty(t))) \frac{d}{dt} m_\infty(t) \\ &= \sum_{\alpha \in C} \left[ p^*(\mathbf{k} - \mathbf{e}_\alpha, t) \operatorname{sgn}(k_\alpha) \right. \\ & \quad \times \left( \lambda_\alpha(t) + \sum_{\beta \in C} \mu_\beta(t) p_{\beta\alpha}(t) m_\infty^\beta(t) - \mu_\alpha(t) m_\infty^\alpha(t) \right) \\ & \quad \left. - p^*(\mathbf{k}, t) \sum_{\alpha \in C} \lambda_\alpha(t) - \mu_\alpha(t) q_\alpha(t) m_\infty^\alpha(t) \right] \\ & \quad + p^*(\mathbf{k}, t) \beta_s(m_\infty(t)) \frac{d}{dt} m_\infty(t) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\alpha \in C} \left[ \lambda_{\alpha}(t) \operatorname{sgn}(k_{\alpha}) p^{*}(\mathbf{k} - \mathbf{e}_{\alpha}, t) + \mu_{\alpha}(t)(k_{\alpha} + 1) q_{\alpha}(t) p^{*}(\mathbf{k} + \mathbf{e}_{\alpha}, t) \right. \\
 &\quad \left. + \sum_{\beta \in C} \mu_{\beta}(t)(k_{\beta} + 1) p_{\beta\alpha}(t) \operatorname{sgn}(k_{\alpha}) p^{*}(\mathbf{k} - \mathbf{e}_{\alpha} + \mathbf{e}_{\beta}, t) \right. \\
 &\quad \quad \left. - \mu_{\alpha}(t) k_{\alpha} p^{*}(\mathbf{k}, t) - \lambda_{\alpha}(t) p^{*}(\mathbf{k}, t) \right] \\
 &\quad + p^{*}(\mathbf{k}, t) \beta_s(m_{\infty}(t)) \frac{d}{dt} m_{\infty}(t) \\
 &= \sum_{\alpha \in C} \left[ \lambda_{\alpha}(t) \operatorname{sgn}(k_{\alpha}) p^{*}(\mathbf{k} - \mathbf{e}_{\alpha}, t) + \mu_{\alpha}(t)(k_{\alpha} + 1) q_{\alpha}(t) p^{*}(\mathbf{k} + \mathbf{e}_{\alpha}, t) \right. \\
 &\quad \left. + \sum_{\beta \in C} \mu_{\beta}(t)(k_{\beta} + 1) p_{\beta\alpha}(t) \operatorname{sgn}(k_{\alpha}) p^{*}(\mathbf{k} - \mathbf{e}_{\alpha} + \mathbf{e}_{\beta}, t) \right. \\
 &\quad \quad \left. - (\lambda_{\alpha}(t) + \mu_{\alpha}(t) k_{\alpha}) p^{*}(\mathbf{k}, t) \right] \\
 &\quad + p^{*}(\mathbf{k}, t) \beta_s(m_{\infty}(t)) \frac{d}{dt} m_{\infty}(t).
 \end{aligned}$$

The above relation holds for all  $\mathbf{k}$ , but we can also write it as

$$\begin{aligned}
 \frac{d}{dt} p^{*}(\mathbf{k}, t) &= \sum_{\alpha \in C} \left[ \lambda_{\alpha}(t) \operatorname{sgn}(k_{\alpha}) p^{*}(\mathbf{k} - \mathbf{e}_{\alpha}, t) \right. \\
 &\quad \left. + \sum_{\beta \in C} \mu_{\beta}(t)(k_{\beta} + 1) p_{\beta\alpha}(t) \operatorname{sgn}(k_{\alpha}) p^{*}(\mathbf{k} - \mathbf{e}_{\alpha} + \mathbf{e}_{\beta}, t) \right. \\
 &\quad \quad \left. - \mu_{\alpha}(t) k_{\alpha} p^{*}(\mathbf{k}, t) \right] \\
 &\quad - p^{*}(\mathbf{k}, t) (1 - \beta_s(m_{\infty}(t))) \frac{d}{dt} m_{\infty}(t).
 \end{aligned}$$

This last equation resembles the forward equations for  $p(\mathbf{k}, t)$  when  $|\mathbf{k}| = s$ . Recasting these results in operator form gives us

$$(7.5) \quad \frac{d}{dt} \mathbf{p}^{*}(t) = \mathbf{p}^{*}(t) \mathbf{A}(t) + \frac{d}{dt} m_{\infty}(t) \sum_{|\mathbf{k}|=s} p^{*}(\mathbf{k}, t) (\mathbf{p}^{*}(t) - \mathbf{e}_{\mathbf{k}}).$$

We can then write the solution to this inhomogeneous ordinary differential equation as

$$(7.6) \quad \begin{aligned}
 \mathbf{p}^{*}(t) &= \mathbf{p}^{*}(0) \mathbf{E}_{\mathbf{A}}(t) \\
 &\quad + \int_0^t \frac{d}{d\tau} m_{\infty}(\tau) \sum_{|\mathbf{k}|=s} p^{*}(\mathbf{k}, \tau) (\mathbf{p}^{*}(\tau) - \mathbf{e}_{\mathbf{k}}) \mathbf{E}_{\mathbf{A}}(\tau, t) d\tau.
 \end{aligned}$$

Combining the above with (2.5) yields the desired (4.2).  $\square$

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