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ANALYSIS OF SOME NETWORKS WITH INTERACTION

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A la mémoire de Claude Kipnis

In this paper we study a stochastic network model introduced recently in the analysis of neural networks. In this model the interaction between the nodes of the network is local: with each node is associated some real number (the *inhibition* in the language of neural networks) which is decreasing linearly with time. When this number reaches 0, it sends out some random input to its neighbors (a spike) and restarts with some random value. The state of our network is described as a Markov process. We are interested in the stability of this network, that is, under which conditions the associated Markov process is ergodic. As we will see, when the network is not stable, some of the nodes die, that is, almost surely after a given time, their inhibition never returns to 0 and grows arbitrarily. When these stability conditions are not satisfied, we analyze the set of nodes which are likely to die. We consider networks with a finite number of nodes and two kinds of topologies, the fully connected network and related graphs, and the linear network where the nodes are located on a line. A quantity ρ is associated with this network and the stability properties of the network depend only on it. For the fully connected network, we prove that if $\rho < 1$, then the network is stable, and in this case we give the explicit expression for the invariant measure of the Markov process associated with this model. When $\rho > 1$ this network is shown to be not stable. For the stability of the linear network of size N, there is a critical value for ρ which is $1/2$ if N is odd and $1/(2\cos \pi/(N+1))$ if N is even. We prove that if ρ is strictly less than this critical value, then the network is stable, and if ρ is strictly greater, it is not. In this last case, the set of possible asymptotic states is analyzed.

1. Introduction and description of the model. We consider a locally interacting process on a finite network. This mathematical model was introduced in [2] to analyze the behavior of some neural cells in the cortex. A nonnegative real number is associated with every node: its inhibition. The inhibition represents the duration of time after which the node will modify the state of other nodes if no interaction takes place meanwhile. As long as the inhibition of a node is strictly positive, it decreases linearly with time and the node does not modify the behavior of its neighbors. When its inhibition is 0, the node fires, that is, the inhibition of its neighbors is increased by some random quantity (the node is also said to send out a spike). Throughout this

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paper, we will use the picturesque terminology of neural networks in the analysis of our model.

Our network is a finite graph $\mathscr{G}$ with N nodes. The stochastic process $(X_t(i))_{1 \le i \le N} \in \mathbb{R}_+^N$, $t \in \mathbb{R}_+$, describing the state of our network evolves as follows. From an initial configuration $(X_0(i),\ 1 \le i \le N)$, every coordinate decreases linearly in time with slope 1 until the first time T_1 when the state of some node i_1 reaches 0. At this moment, the component with index i_1 restarts with a new value, an exponentially distributed random variable G_{i_1} (with parameter λ_{i_1}). At the same time, the node i_1 sends out a spike to all its neighbors, that is, increases the value of their components by the same amount, a random variable with distribution F_{i_1}. Starting from this new configuration $(X_{T_1}(i),\ 1 \le i \le N)$, we iterate this procedure until the time T_2 of the second spike and so on. A right-continuous Markov process $(\mathbf{X}_t)_{t \ge 0}$ is thus defined in this way. Since its evolution between two successive spikes is deterministic (linear decay with rate 1), there is a natural embedded Markov chain at the instants of spikes: $(\mathbf{X}_n)_{n \in \mathbb{N}} = (X_{T_n}(i),\ 1 \le i \le N)_{n \in \mathbb{N}}$. We shall say that the network is stable if the Markov process $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ is ergodic.

The first results on this model were obtained in [2]. It was assumed that the F_i's were Dirac measures at some $\theta > 0$. It was proved that the Markov chain $(\mathbf{X}_n)_{n \in \mathbb{N}}$ is Lebesgue-irreducible, aperiodic and that if $\theta < \min_i(E(G_i))/(V_i)$, it is positive recurrent [V_i is the number of neighbors of the node i and $E(\cdot)$ denotes the expected value]. As we will see in the following, this condition is, in general, not necessary for the stability of the network.

We begin with the case of a completely connected graph (i.e., every pair of nodes is linked by an edge of the graph). For this topology, we prove that the network is stable if $\rho = \max_i \rho_i = \max_i E(\theta_i)/E(G_i) < 1$, where θ_i is some random variable with distribution F_i. Under this stability condition, we give the explicit expression for the Laplace transform of the invariant measure of the Markov process associated with this model. When $\rho > 1$, the network is not stable. In this case its asymptotic behavior is fairly simple to describe: only one node does not die; that is, after some time all the other nodes do not fire any more.

For the case of a linear network of size N, the behavior is quite interesting. We slightly modify the interaction between nodes: at the moment of a spike, instead of sending the same random variable to all its neighbors, a node sends out independent identically distributed random variables. We assume that the F_i's (resp. G_i) are exponentially distributed with parameter μ (resp. λ). We prove the following results:

1. If N is odd, the network is stable if $\rho = \lambda/\mu < 1/2$ and not stable if $\rho > 1/2$.
2. If N is even, then the network is stable if $\rho < 1/(2\cos \pi/(N+1))$ and not stable if $\rho > 1/(2\cos \pi/(N+1))$.

The boundary cases. Throughout this paper, our stability results are of the following type: "There is some constant C such that if $\rho < C$, then the network is stable, and if $\rho > C$, it is not." Our analysis does not cover the boundary cases $\rho = C$, which are not known to us. It is nevertheless reasonable to conjecture that, in these cases, a null recurrent phenomenon occurs.

The proof of ergodicity involves the second vector field associated with Markov processes that was introduced by Malyshev and Menshikov [5] in the analysis of the ergodicity of multidimensional random walks on $\mathbb{N}^n$. To our knowledge, this is one of the first applications of the second vector field to study the ergodicity of Markov processes in state spaces of dimension greater than or equal to 5.

When the network is not stable, it has different possible asymptotic distributions, depending on its initial state. We call these distributions the stable states of the network. We give a list of possible stable states, which depend mainly on the position of ρ among the $1/(2\cos\pi/(2p+1))$ for $p \in \mathbb{N}$.

2. The completely connected network. We assume in this section that the graph $\mathcal{G}$ is fully connected, that is, every pair of vertices is connected. For $1 \le i \le N$, we define $\rho_i = \lambda_i E(\theta_i)$, θ_i being some generic random variable with distribution F_i.

THEOREM 1. *If* $\max_{1 \le i \le N} \rho_i < 1$, *the Markov chain* $(\mathbf{X}_n)_{n \in \mathbb{N}}$ *admits the invariant probability measure given by*

$$\pi_N = \sum_{i=1}^{N} p_i \mu_i,$$

where, for $1 \le i \le N$,

$$p_i = \frac{\lambda_i}{1-\rho_i} \left(\sum_{j=1}^{N} \frac{\lambda_j}{1-\rho_j} \right)^{-1}$$

and μ_i *is the distribution of the random vector* $\mathbf{G} + (W_i + \theta_i)\mathbf{e}_i$. *The vector* $\mathbf{e}_i$ *has all components* 1 *except the ith one which is* 0, *and the independent random variables* $\mathbf{G}, W_i, \theta_i$ *satisfy*:

(a) $\mathbf{G} = (G_1, \dots, G_N)$ *is a vector of independent exponentially distributed random variables with parameters* $\lambda_1, \dots, \lambda_N$.

(b) W_i *has the same distribution as the stationary waiting time of an* $M/G/1$ *queue with arrival rate* λ_i *and service distribution* F_i, *that is*,

$$W_i =_{(d)} \max(W_i + \theta_i - G_i, 0), \tag{1}$$

where $=_{(d)}$ *stands for equality of distributions.*

REMARK. The Laplace transform of π_N can be made explicit since for $i \le N$, it is well known (see, e.g., [1]) that the Laplace transform of W_i is given by

$$\tilde{W}_i(\xi) = \frac{1-\rho_i}{1-\lambda_i\left(1-\tilde{F}_i(\xi)\right)/\xi}, \qquad \xi \in \mathbb{R}_+, \tag{2}$$

denoting by $\tilde{D}$ the Laplace transform of a distribution D.

PROOF. If P denotes the transition probabilities of the Markov chain, all we have to prove is that $\pi_N P = \pi_N$. We will show the equality of the Laplace transforms, $\widetilde{\pi_N P} = \widetilde{\pi_N}$. We fix $1 \le i \le N$ and we compute $\widetilde{\mu_i P}$, that is, the Laplace transform of the distribution of $\mathbf{X}_1$ if the initial distribution of $\mathbf{X}_0$ is μ_i. Thus the initial inhibition state is $X_0(i) = G_i$ for node i and $X_0(j) = G_j + W_i + \theta_i$ for $j \ne i$, where W_i, θ_i, G_k, $1 \le k \le N$ are independent. The node with the smallest inhibition state will thus send a spike first. There are two possibilities:

(i) If $G_i < W_i + \theta_i$, node i sends out a spike first and $X_1(i) = G_i'$, $X_1(j) = G_j + (W_i + \theta_i - G_i) + \theta_i'$ for $j \ne i$, where G_i' and θ_i' are other independent random variables with respective distributions G_i and F_i.

(ii) If $G_i \ge W_i + \theta_i$, then $X_0(j) \ge W_i + \theta_i$ for every $1 \le j \le N$. Due to the deterministic evolution of $(X_t)_{t \in \mathbb{R}_+}$ between spikes, $P(\mathbf{x}, \cdot) = P(\mathbf{y}, \cdot)$ if $y_j = x_j - a$ $(1 \le j \le N)$ with $a \le \min_{1 \le j \le N} x_j$. Therefore $\mathbf{X}_1$ has the same distribution as if the initial state was $\mathbf{X}_0'$with $X_0'(j) = G_j$ for $j \ne i$ and $X_0'(i) = G_i - (W_i + \theta_i)$. Because of the usual properties of exponential distributions, the distribution of $\mathbf{X}_0'$ conditioned on $\{G_i \ge W_i + \theta_i\}$ is simply ν, the distribution of $\mathbf{G}$.

Thus, for $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N) \in \mathbb{R}_+^N$,

$$\widetilde{\mu_i P}(\boldsymbol{\xi}) = E\left(1_{\{G_i < W_i + \theta_i\}} \exp\left(-\xi_i G_i' - \sum_{j \ne i} \xi_j\big(G_j + (W_i + \theta_i - G_i) + \theta_i'\big)\right)\right) + P(G_i \ge W_i + \theta_i)\widetilde{\nu P}(\xi). \tag{3}$$

We are left with the computation of $\widetilde{\nu P}(\boldsymbol{\xi})$. In this case, we assume that $\mathbf{X}_0 = \mathbf{G}$; hence, if $G_j = \min_{1 \le k \le N}\{G_k\}$, $X_1(j) = G_j'$ and $X_1(k) = G_k - G_j + \theta_j'$ for $k \ne j$. We obtain

$$\widetilde{\nu P}(\boldsymbol{\xi}) = \sum_{j=1}^{N} \frac{\lambda_j}{\sum_{k=1}^N \lambda_k} \tilde{\mu}_j'(\boldsymbol{\xi}),$$

where μ_j' is the distribution of the random vector $\mathbf{G}' + \theta_j' \mathbf{e}_j$. Because of (1) and (2), we have $P(W_i + \theta_i \le G_i) = P(W_i = 0) = 1 - \rho_i$. Thus, by (1) we can rewrite (3) as

$$\widetilde{\mu_i P} = \tilde{\mu}_i - (1 - \rho_i)\tilde{\mu}_i' + (1 - \rho_i) \sum_{j=1}^{N} \frac{\lambda_j}{\sum_{k=1}^N \lambda_k} \tilde{\mu}_j'.$$

At this point it is clear that a sufficient condition for the stationarity of $\pi_N = \sum_{i=1}^N p_i \ \mu_i$ is that $p_1, \dots, p_N$ satisfy

$$-\sum_{i=1}^{N} p_i(1 - \rho_i)\tilde{\mu}_i' + \sum_{i=1}^{N} p_i(1 - \rho_i)\left(\sum_{j=1}^{N} \frac{\lambda_j}{\sum_{k=1}^N \lambda_k} \tilde{\mu}_j'\right) = 0$$

or that for every i, $1 \le i \le N$,

$$p_i(1 - \rho_i) = \frac{\lambda_i}{\sum_{k=1}^N \lambda_k} \sum_{j=1}^{N} p_j(1 - \rho_j).$$

Hence, $p_i = C(\lambda_i/(1-\rho_i))$. The normalization relation $\sum_{i=1}^N p_i = 1$ (π_N is a probability) gives the value of C. Our theorem is proved. □

We can now state our stability result for the fully connected network.

PROPOSITION 2. *When* $\max_{1\le j\le N}\rho_j < 1$, *the Markov chain* $(\mathbf{X}_n)_{n\in\mathbb{N}}$ *is Harris ergodic.*

PROOF. We obtain this proposition by proving the coupling property for the Markov process $(\mathbf{X}_t)_{t\ge 0}$. Recall [1] that a Markov process $(Y_s)_{s\in S}$ (where S is either $\mathbb{N}$ or $\mathbb{R}_+$) has the coupling property if, for any probability distributions μ_0^1 and μ_0^2, one can construct a probability space and two stochastic processes $(Y_s^1)_{s\in S}$ and $(Y_s^2)_{s\in S}$ on it such that:

(a) The process $(Y_s^i)_{s\in S}$ is Markov with the same transition probabilities as $(Y_s)_{s\in S}$ for $i=1,2$.
(b) The distribution of Y_0^i is μ_0^i for $i=1,2$.
(c) There exists some random time after which the two processes are identical.

Coupling for the Markov process $(\mathbf{X}_t)_{t\ge 0}$ clearly implies coupling for the embedded Markov chain $(\mathbf{X}_n)_{n\in\mathbb{N}}$. The coupling property and the existence of an invariant measure are sufficient to prove Harris ergodicity (see [1], Proposition 3.13, page 157). We shall prove that starting from any initial condition, there exists some random time when the distribution of the process is a product of independent exponential distributions. Hence, for any starting distribution, the Markov process couples with the process which starts with independent exponentially distributed components. Consequently, the coupling property will be true. Notice, nevertheless, that for the stationary distribution π_N, the components are not independent.

If $(x_1,\dots,x_N)$ is the initial state of the network, let us assume that x_q is the smallest of the x_i's so that node q will be the first to send out a spike to all the other nodes. Until time x_q, all the coordinates decrease at rate 1. Let $(\theta_q^i)_i$ be the sequence of spikes sent out by node q and let $(G_q^i)_i$ be the sequence of new values of node q after sending spikes (independent exponentially distributed random variables with parameter λ_q). If we define

$$\nu^q = \inf\left\{k\ge 1\colon \sum_{j=1}^{k}\left(\theta_q^j - G_q^j\right) < 0\right\},$$

ν^q is finite a.s. because $\rho_q \le 1$. As long as $t \le S_1 = x_q + \sum_{j=1}^{\nu^q}\theta_q^j$,

$$X_i(t) = x_i + \sum_{j=1}^{k}\theta_q^j - t \quad \text{for } i \ne q,$$

where k is such that $t \in [x_q + \sum_1^{k-1}G_q^j, x_q + \sum_1^k G_q^j[$. Up to time S_1, node q is the only one to fire. At that time, because of the memoryless property of

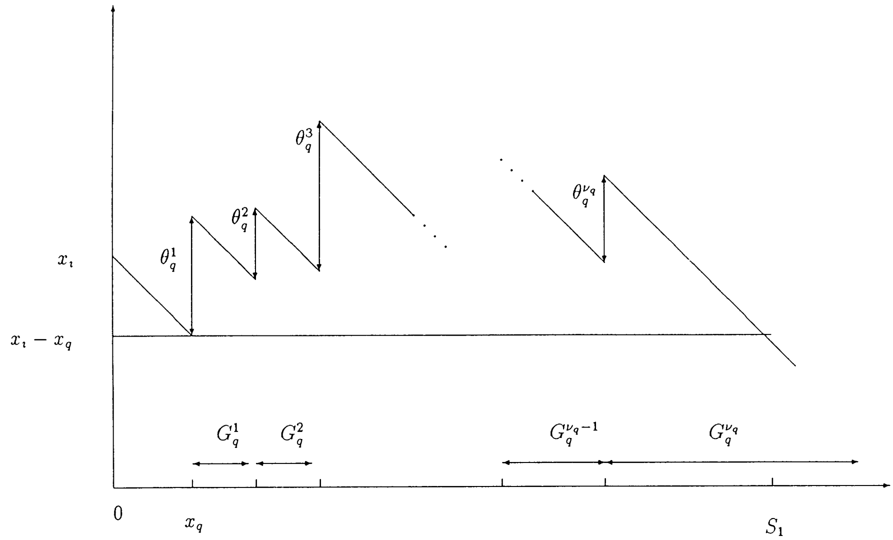

FIG. 1. *State of node $i \neq q$.*

exponentially distributed random variables, $X_q(S_1)$ is exponentially distributed and for $j \neq q$, $X_j(S_1) = x_j - x_q$ (see Figure 1).

We repeat this procedure and construct a nondecreasing sequence $(S_k)_{k \in \mathbb{N}}$. For $k \geq 1$, the state of the network at time S_{k+1} is defined by

$$X_i(S_{k+1}) = X_i(S_k) - \min\{X_j(S_k), 1 \leq j \leq N\}, \qquad i \neq i_0, \tag{4}$$

where i_0 is the index which achieves the minimum of the coordinates of $X(S_k)$ and $X_{i_0}(S_{k+1})$ is an independent exponential variable (with parameter λ_{i_0}) independent of the $X_j(S_{k+1})$, $j \neq i_0$. We shall say that i_0 is at the origin of the kth period. Using again the memoryless property of exponential distributions and (4), it is then easy to see that if node i has fired at least once before S_k, then its state at the beginning of the kth period is an exponential random variable, independent of the other components. If a node j never fired up to time $S_{(k+1)N}$, then its state at that time is stochastically smaller than x_j minus a sum of k independent exponential variables with parameter $\lambda = \max_i \lambda_i$ (there exists a node which is at the origin of at least $k+1$ periods before time $S_{(k+1)N}$). Hence, after a while, all the nodes will have fired at least once; thus there exists some random time S_p when the coordinates of the Markov process are independent exponentially distributed random variables. The coupling property is proved. □

THEOREM 3. *If we order the ρ's so that*

$$\rho_1 \leq \rho_2 \leq \cdots \leq \rho_N$$

and if for some $p \leq N - 1$, $\rho_p < 1 < \rho_{p+1}$, then almost surely, there exists some $i > p$ such that the inhibitions of all nodes different from i converge to

$+\infty$ *with probability* 1. *Hence, with probability* 1, $N-1$ *nodes are "dead" after a while.*

PROOF. We keep the same notation as in the proof of Proposition 2. When node q begins to fire, the variable ν^q is the number of consecutive spikes sent out by node q before being possibly interrupted by the other nodes.

On the event $A_q = \{\nu^q = +\infty\} = \{\min_k \sum_{j=1}^{k}(\theta_q^j - G_q^j) > 0\}$, node q will forever continue to send out spikes to the other nodes. In this case, the inhibition of node $i \neq q$ just before the kth spike of node q will be

$$x_i - x_q + \sum_{j=1}^{k-1} \left(\theta_q^j - G_q^j\right)$$

and thus will converge to $+\infty$ with probability 1.

In the case $\rho_q > 1$, then $P(A_q) > 0$ (a random walk with positive drift stays above 0 forever with positive probability); thus, with some positive probability ν^q can be infinite. Hence if a node with index greater than p begins to fire, with positive probability it will kill all the other nodes.

To prove that, let us assume that at least two nodes fire infinitely often. Using the same argument as in Proposition 2, it is easy to see that at least one of these nodes, q say, has an index greater than p; hence, $\rho_q > 1$. According to our assumption, node q does not fire continuously but it fires infinitely often. It implies that it restarts firing after every period of the other node. At these moments, with probability $P(A_q)$, the length of the period is infinite. Because of the independent of the length of periods initiated by node q, it is easy to see that with probability 1 the node q will never stop firing after a given time. This contradicts our assumption. We conclude that with probability 1, one node with index greater than p fires forever without being interrupted by other nodes. Our theorem is proved. □

REMARKS. (a) The sufficient stability condition of [2] is, in this case, $\max_{1 \le j \le N} \rho_j < 1/(N-1)$, which is quite strong for this topology.

(b) The assumption of exponentially distributed variables is not essential for stability properties (see [3]).

(c) The above results for stability can be generalized to a larger class of graphs, the N-partite complete graphs (see [3]). A graph is N-partite and complete if its vertex set can be decomposed into a partition $\bigcup_1^N H_k$ such that:

(i) Two nodes of H_k are not connected for $k \in \{1, \dots, N\}$.
(ii) Two nodes from different H's are connected.

3. The linear network. In this section we study the one-dimensional topology for the network where every node has two neighbors except at both ends. We assume that the θ_i's (resp. G_i's) are identically distributed (resp. exponentially distributed with parameter λ). We know that if $\rho = \lambda E(\theta_1) <$

1/2, then it is stable ([2]). The following proposition shows that the condition $\rho < 1/2$ is indeed necessary for the stability when N is odd.

PROPOSITION 4. *If $\rho > 1/2$ and if the number of nodes is odd, then the linear network is unstable.*

PROOF. Let $N = 2k + 1$, and for $i = 1, \dots, k+1$, let $Q_i(t) = \sum_{t_i^j \le t} \theta_i^j$, where $P_i = (t_i^j)_{j \ge 0}$ is a Poisson process with parameter λ. We assume that all the P_i's and θ_i^j's are independent. For $i = 1, \dots, k+1$, $(\theta_i^j)_{j \ge 0}$ is the sequence of spikes sent by node $2i - 1$ and $(t_{i+1}^j - t_i^j)_j$ are the exponential variables associated with this node. Using the fact that $2\rho > 1$ and the law of large numbers, we get that $R_i = \inf\{Q_i(t) + Q_{i+1}(t) - t,\ t \ge 0\}$ is finite almost surely for $i = 1, \dots, k$; hence, there exists an $x_0 > 0$ such that $P(R_i > -x_0, 1 \le i \le k+1) > 0$.

Now take the initial state $x_{2i+1}(0) = 0$ and $x_{2i}(0) = x_0$. For $i = 1, \dots, k$, as long as $x_{2i}(s) > 0$ for all $s \le t$ and $i \le k$, then $x_{2i}(t) = x_0 + Q_i(t) + Q_{i+1}(t) - t$. This last equality will be true for all t if $x_{2i}(0) > -R_i$ for all i. In this case, $x_{2i}(t)$ converges to $+\infty$. Hence, on a set of positive probability, the components of $(\mathbf{Q}(t))_{t \ge 0}$ with an even index converge to infinity. In other words, for all i odd, nodes i and $i + 2$ "kill" node $i + 1$ with positive probability. Our proposition is proved. □

As we will see in Theorem 5, the condition $\rho < 1/2$ is not necessary when the number of nodes is even. From now on, we consider the linear network with an even number of nodes and assume that $1 > \rho > 1/2$, so that for some $\theta \in]0, \pi/3[$, $\rho = 1/(2\cos\theta)$.

For technical convenience, we will change the dynamics of our process: The single spike sent out to neighbors is replaced by i.i.d. spikes. Thus a node sends out independent exponentially distributed (μ) spikes to its neighbors. There is no change in the way a node "resets" its own value when its inhibition reaches 0. A firing node restarts with an exponentially distributed random variable with parameter λ (in particular $\rho = \lambda/\mu$). We can then describe the state of the network as a Markov process $(\mathbf{X}(t))_{t \ge 0}$ on the countable state space $\mathbb{N}^N$. The dynamics of the process $(\mathbf{X}(t))_{t \ge 0}$ are described by the following transitions:

$$\mathbf{x} \to \mathbf{x} - \boldsymbol{\delta}_i \qquad \text{with rate } \mu \text{ if } x_i \neq 0,$$
$$\mathbf{x} \to \mathbf{x} + \boldsymbol{\delta}_{i+1} + \boldsymbol{\delta}_{i-1} \qquad \text{with rate } \lambda \text{ if } x_i = 0,$$

for $i = 1, \dots, N$, assuming $x_0 = x_{N+1} = +\infty$ for convenience, if N is the number of nodes of the network. The vector $\boldsymbol{\delta}_i$ is the ith unit vector: $\boldsymbol{\delta}_i(j) = 1_{\{j = i\}}$.

The inhibition at a given node is the residual sum of the spikes received from other nodes and the residual initial inhibition, that is, the spike it sent to itself the last time it fired. The total inhibition being decreased at speed 1, we can assume that the part due to the other nodes is decreased first when it is not 0. Using again the memoryless property of exponential variables, the value of the inhibition of the ith node can then be represented as the sum of two independent variables: an exponential r.v. with parameter λ, the residual value of the initial inhibition, and a sum of x_i i.i.d. exponential r.v.'s (μ), the residual sum of the spikes received since the last firing.

Hence we kept the base features of our model: The inhibition of every coordinate is still decreased at rate 1 and when there is a spike, r.v.'s with independent exponential distributions (with parameters μ) are sent out to the neighbors. The main result of this section is the following theorem:

THEOREM 5. *For a linear network with $2N$ nodes, the Markov process $(\mathbf{X}(t))_{t\geq 0}$ is ergodic if $\rho < \rho_c(N) = 1/(2\cos\pi/(2N+1))$ and transient if $\rho > \rho_c(N)$.*

REMARK. The sequence of critical constants $(\rho_c(N))_N$ is nonincreasing and converging to $1/2$, which is the critical constant for the "odd" case.

We begin by the following simple lemma (proved in [2]).

LEMMA 6. *If the network with N nodes is ergodic and the quantity y_i^t denotes the sum of the spikes sent out by node i to node $i-1$ up to time t for $1 \leq i \leq N$, then $y_i(N) = \lim_{t\to+\infty}(y_i^t/t)$ satisfies*

$$\rho y_{i-1}(N) + y_i(N) + \rho y_{i+1}(N) = \rho, \tag{5}$$

$$i = 1,\dots,N, \text{ with } y_0(N) = y_{N+1}(N) = 0.$$

The useful properties of $y_1(N)$ are given in Proposition 10 in the Appendix.

PROOF. The existence of the limit is simply a consequence of an ergodic theorem for ergodic Markov processes. Clearly, y_i^t has the same distribution as $\bar{y}_i^t$, the sum of the inhibitions sent out by node i to node $i+1$; hence, we have also $y_i(N) = \lim_{t\to+\infty}(\bar{y}_i^t/t)$. If a node spends time s in state 0, it will send in the limit, for s large, the inhibition ρs to its neighbors. Now if we note that, up to an asymptotically negligible term, the time spent by i in state 0 up to time t is $t - y_{i+1}^t - \bar{y}_{i-1}^t$, then our lemma is proved. □

PROOF OF THEOREM 5. We proceed by induction on N to prove the following property: If $\rho_c(N) < \rho < \rho_c(N-1)$, all the networks with an even number of nodes larger than $2N$ are transient and for $\rho < \rho_c(N)$, the network with $2N$ nodes is ergodic. For $N = 1$, the property is trivially true since:

(i) If $\rho < 1$, the network with two nodes is stable according to the previous section. This network is indeed completely connected and the change we made on the dynamics of the process does not affect the model since there is only one neighbor for every node.

(ii) When $\rho > 1$, the network with $2k$ nodes, $k \geq 1$, is unstable by the same argument as in Proposition 4. If the inhibition of nodes $1, 3, \dots, 2k-1$ is sufficiently large, then each of them will be killed by its right neighbor because of $\rho > 1$ (in this case, one node is sufficient to kill another one).

Now let us assume that our property is satisfied for networks with $2N$ or fewer nodes.

(a) $\rho_c(N+1) < \rho < \rho_c(N)$. Let $k \geq N+1$. We want to prove that a network with $2k$ nodes is unstable. According to the induction hypothesis, the network with $2N$ nodes is ergodic. Using Lemma 6, the vector $(y_i(2N))_{1 \leq i \leq 2N}$ is a solution of (8) and according to the hypothesis on ρ, ($\theta \in]\pi/(2N+3), \pi/(2N+1)[$) and Proposition 10, we have $y_1(2N) + \rho > 1$. If in the picture of the network of $2k$ nodes (Figure 2), the inhibition of black nodes is sufficiently large, then with some positive probability none of them will fire any more. For the black nodes 2 and $2N+3$, it is a consequence of the relation $y_1(2N) + \rho > 1$, and for the others, it is simply because $\rho > 1/2$. The proof is similar to that of Proposition 4 and therefore will be omitted. Hence in the network with $2k$ nodes, $k \geq N+1$ is transient when $\rho_c(N+1) < \rho < \rho_c(N)$.

(b) $\rho < \rho_c(N+1)$. We want to prove that $(\mathbf{X}(t))_{t \geq 0}$ is an ergodic Markov process in $\mathbb{N}^{2N+2}$. Our main tools, in the proof of ergodicity are the ergodicity criteria of [5]. The results of [5] are stated in the context of Markov chains. Although our study fits that context [by looking at the embedded Markov chain $(\mathbf{X}_n)_{n \in \mathbb{N}}$ at the instants of jumps, for example], we found it more convenient to work with processes and adjust the notation. We translate some of the notation of [5] to our case.

DEFINITION. A *face* Δ is a subset of $\{1, \dots, 2N+2\}$. For each face Δ we consider the same stochastic process on $\mathbb{N}^{2N+2}$ but with the restriction that the nodes whose indices are in Δ do not send spikes to their neighbors. The nodes in Δ only receive inputs from their neighbors outside of Δ. The induced Markov process on $\mathbb{N}^{2N+2}$ associated with Δ is denoted by $(X_\Delta(t))_{t \geq 0}$.

If we remove the components of $X_\Delta(t)$ which are in Δ, then we will have a Markov process. We shall say that the face is ergodic (resp. transient) if this Markov process is.

The second vector field. For some face Δ, we can write $\{1, \dots, 2N+2\} - \Delta$ as $\bigcup_{i \in \Delta} L_i = \bigcup_{i \in \Delta} R_i$, where the L_i's and R_i's are (possibly empty) intervals such that if $L_i \neq \emptyset$ (resp. $R_i \neq \emptyset$), then $i - 1 \in L_i$ (resp. $i + 1 \in R_i$). For convenience, we assume that the fictitious nodes at both ends of the network, 0 and $2N+3$, are always in Δ. The Markov process associated with Δ is

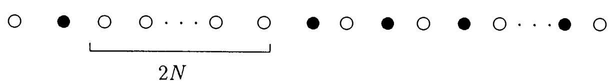

FIG. 2.

basically the concatenation of the independent processes $(\mathbf{X}_{L_i})_{i \in \Delta}$, where $\mathbf{X}_J$ is the Markov process associated with a network of size $|J|$.

We define $y_1(J) = y_1(|J|)$ if the interval J has the cardinality $|J|$ even, $y_1(J) = 0$ if $J = \varnothing$ and ρ if $|J|$ is odd. When $|J|$ is even, $y_1(J)$ is simply the output at equilibrium of the first node of a network of size $|J|$ (cf. Lemma 6). This leads us to the following definition for the *drift vector* v^Δ:

(a) For $i \in \Delta$, $v_i^\Delta = y_1(L_i) + y_1(R_i) - 1$.
(b) If $|L_i|$ is even or 1, then $v_j^\Delta = 0$ for all $j \in L_i$.
(c) If $L_i = \{a_1, a_1 + 1, \dots, a_1 + 2k\}$, then $v_{a_1+2j}^\Delta = 0$, $j = 0, \dots, k$, and $v_{a_1+2j+1} = 2\rho - 1$, $j = 0, \dots, k-1$.

If Δ is some ergodic face, then for every $i \in \Delta$, $\mathbf{X}_{L_i}$ is ergodic; hence, according to Proposition 4 and the induction hypothesis, $|L_i| = 0, 1$ or $2k$, with $k \le N$. In this case, the only coordinates of v^Δ which are not zero are those with indices in Δ. According to our assumption on ρ ($\theta \in [0, \pi/(2N+3)[$) and Proposition 10 in the Appendix, it is easily verified that for $i \in \Delta$ such that $|L_i|$ or $|R_i| \neq 1$, then $v_i^\Delta < 0$ and that $v_i^\Delta = 2\rho - 1 > 0$ otherwise. In this case, the vector v^Δ has the following intuitive meaning: If we freeze the nodes of Δ and let the other nodes reach a stationary state, then for $i \in \Delta$, v_i^Δ is simply the stationary drift of the ith component, that is,

$$\lim_{t \to +\infty} \frac{1}{t}\big((\text{sum of the inputs received by } i \text{ up to time } t) - t\big).$$

For the embedded Markov chain of $(\mathbf{X}_\Delta(n))_{n \in \mathbb{N}}$ at the instants of jumps of $(X_\Delta(t))_{t \ge 0}$, the ith component of the stationary drift can be expressed as

$$\lim_{n \to +\infty} \frac{1}{n}\big(\text{number of upward jumps of } i - \text{number of downward jumps of } i, \text{ up to the } n\text{th jump of } (X_\Delta(t))_{t \ge 0}\big)$$

and it is easy to see that the stationary drift for the discrete case is simply $(\mu/\lambda^\Delta) v^\Delta$, where λ^Δ is the intensity of jumps of $\mathbf{X}_\Delta(t)$.

If Δ is not ergodic, then at least one of the intervals among the L_i's or R_i's, J say, has an odd cardinality [using again the induction hypothesis and the monotonicity of the $\rho_c(N)$, $N \in \mathbb{N}$]. In this case, $v_i^\Delta = 2\rho - 1 > 0$ when i is the 2nd, 4th, ... node of J.

Let

$$B^\Delta = \big\{x \in \mathbb{R}_+^{2N+2}\colon x_i = 0 \text{ if } i \notin \Delta \text{ and } x_i > 0 \text{ otherwise}\big\}.$$

DEFINITION. The second vector field $(v(x))_{x \in \mathbb{R}^{2N+2}}$ of the Markov process $(\mathbf{X}(t))_{t \ge 0}$ (see [5] and [4]) is defined by $v(x) = v^\Delta$ for $x \in B^\Delta$ and $v(\mathbf{0}) = \mathbf{0}$. When $\Delta = \{1, \dots, 2N+2\}$, v^Δ is simply the usual drift vector $v_i^\Delta = -1$ for all $i \le 2N+2$ and $(v(x))_{x \in B^\Delta}$ is the first vector field.

The associated dynamical system. We can now define the deterministic dynamical system $(\Gamma^x(t))_{t \geq 0}$ associated with the second vector field, for $x \in \mathbb{R}^{2N+2}$:

$$\Gamma^x(0) = x, \tag{6}$$
$$\frac{d\Gamma^x}{dt}(t) = v(\Gamma^x(t))$$

and $\tau(x)$ the hitting time of $\mathbf{0}$ by Γ^x,

$$\tau(x) = \inf\{t : \Gamma^x(t) = \mathbf{0}\} \quad (\text{with } \inf\varnothing = +\infty). \tag{7}$$

From the definition of the second vector field, it is easy to recognize that $\Gamma^x(t)$ is nonnegative for all $t \geq 0$ and that its rate of growth is bounded by 1 in absolute value. Denote by Δ_t the face such that $\Gamma^x(t) \in B^{\Delta_t}$ with $\Delta_0 = \Delta$. The travel of Γ^x on the faces can then be decomposed as follows:

Step 1. Let us call an island a subset of $\Delta\{i, i+1, \dots, i+p\}$ with $p > 0$. The number of nodes in islands of Δ_t is decreasing to 0 (simply because $\rho < 1$). Consequently, if there are no more islands at time t, this will be true for all $s \geq t$. Moreover, after a while the nodes at both ends do not belong to Δ_t any more.

Step 2. We assume that Step 1 is achieved. An additional interval L_i with an even cardinality cannot be created by one of the only possible operations: (a) splitting, when a transient face is reached or (b) merging, when a node disappears between two L_i's whose cardinalities have distinct parities $(1, 2k$ or vice versa). Hence the number of i such that $|L_i|$ is even is nonincreasing.

Step 3. There exists at least one L_i whose cardinality is $2k$ with $k > 0$. Let S be the set of L_i's with an even cardinality. It is easy to see that the cardinality of S is odd. Between two consecutive elements L and L' of S (i.e., there is no element of S between them), the L_i's have cardinality 1. According to our hypothesis on ρ, if L and L' do not vanish, these L_i's are swallowed by L or L' and when the last node of Δ in the middle finally collapses, the whole space created then has an odd cardinality and thus is split into L_i's of size 1. Hence at the end, there will remain only one element of S which will swallow all the remaining L_i's with cardinality 1. Point $\mathbf{0}$ is finally reached; hence, $\tau(x)$ is finite for all $x \in \mathbb{R}_+^{2N+2}$.

We defined the second vector field and its associated dynamical system when the dimension of the state space is even ($2N+2$, here). When the dimension is odd ($2N+1$, say), we can define in the same way the second vector field and the dynamical system Γ' without any change. The corresponding hitting time τ' will be, in this case,

$$\tau'(x) = \inf\{t : \Gamma'^x(t) \in B^{\{2,4,6,\dots,2N\}}\}.$$

To describe the travel of Γ' on the faces, Steps 1 and 2 do not change but Step 3 has to be modified in the following way:

Step 3′. With the notation of Step 3, we remark that the cardinality of S is necessarily even (possibly 0). As before, two consecutive elements of S finally collapse and split into L_i's of size 1 if they are not destroyed before. Hence all the elements of S will disappear, all the remaining L_i's will have cardinality 1 and the nodes at both ends are not in Δ (we assume that Step 1 is achieved). We conclude that $\tau'(x)$ is finite for all $x \in \mathbb{R}_+^{2N+1}$.

LEMMA 7. *The dynamical systems* $(\Gamma^x(t))_{t\ge 0}$, $(\Gamma'^x(t))_{t\ge 0}$ *do not visit the same face twice and there exists some constant* $K > 2$ *such that* $|\tau(x)| < K\|x\|$ *for* $x \in \mathbb{R}_+^{2N+2}$ *and* $|\tau'(x)| < K\|x\|$ *for* $x \in \mathbb{R}_+^{2N+1}$.

PROOF. For the first part of the lemma, we can proceed by induction. The case $N = 0$ is trivial. We assume that all the dynamical systems $(\Gamma_{2k}^x(t))_{t\ge 0}$ and $(\Gamma'^x_{2k+1}(t))_{t\ge 0}$, $k \le N$, do not visit the same face twice (the subscript $2k$ or $2k + 1$ refers to the number of nodes of dynamical system). We prove that this is also the case for $(\Gamma_{2N+2}^x(t))_{t\ge 0}$ and $(\Gamma'^x_{2N+3}(t))_{t\ge 0}$.

Assume that some face Δ is visited twice by $(\Gamma_{2N+2}^x(t))_{t\ge 0}$. This face cannot have an island. Otherwise this island must remain between the two visits to Δ, since no island can be created. However, the complement of this island consists of one or two intervals of size $\le 2N$. According to the induction hypothesis, the dynamical system cannot visit the same face twice on this (or these) interval(s). Applying the same reasoning, Δ cannot contain either node 1 or $2N + 2$ or some $|L_i|$ with an even cardinality (since following Steps 2 and 3, the elements of S can only grow or disappear as time goes on). Thus the result is proved for the case $2N + 2$. The method is strictly the same for $(\Gamma'^x_{2N+3}(t))_{t\ge 0}$. The first part of our lemma is proved.

To complete the proof of the lemma, we just have to remark that the duration of any of the face transitions described in one of the Steps 1, 2, 3 and 3′ above is proportional to the values of the components of the starting point and that there is only a bounded number of them. □

We can now state the continuity property of τ.

PROPOSITION 8. *The function* τ *defined by* (7) *is finite and Lipschitz.*

PROOF. If $x, y \in \mathbb{R}_+^{2N+2}$ are on the same face, as long as the two dynamical systems $\Gamma^x(t)$ and $\Gamma^y(t)$ remain on this face, the difference $\|\Gamma^x(t) - \Gamma^y(t)\|$ does not change.

Hence we can assume that x and y are not on the same face and denote this by $\alpha = \|x - y\|$. There exists some $p \le 4N + 4$ such that the interval $]\alpha K^p, \alpha K^{p+1}]$ does not contain any of the $4N + 4$ coordinates of x and y. If T is the subset of all $i \le 2N + 2$ such that $x_i > \alpha K^{p+1}$ or equivalently (since $K > 2$), $y_i > \alpha K^{p+1}$, we can write $T^c = \{1, \dots, 2N + 2\} - T$ as a union of intervals $\bigcup_{J\in\mathcal{J}} J$. For all $t < K^{p+1}\alpha$, the coordinates of $\Gamma^x(t)$ and $\Gamma^y(t)$ with index in T never reach zero since the drift of the coordinates is always larger than -1. Using this property, it is easy to see that, on the time interval $[0, \alpha K^{p+1}[$, the T^c-components of the dynamical system $\Gamma^x(t)$ [resp. $\Gamma^y(t)$] behave as a concatenation of the dynamical subsystems $\Gamma_J^x(t)$ [resp.

$\Gamma_J^y(t)]$, $J \in \mathscr{J}$. According to Lemma 7, for any $J \in \mathscr{J}$, we know that $\Gamma_J^x(t)$ and $\Gamma_J^y(t)$ will have reached the same face at time αK^{p+1}, either the empty face or the face with the even components of J, depending on the parity of $|J|$. Hence at some time $t_0 \le \|x - y\| K^{4N+5}$, $\Gamma^x(t)$ and $\Gamma^y(t)$ will have reached the same face and $\|\Gamma^x(t_0) - \Gamma^y(t_0)\| < 3\|x - y\| K^{4N+5}$ (remember that the rate of growth of Γ is bounded by 1 in absolute value). Now we use the fact that there is only a bounded number of changes of faces to get the desired Lipschitz property. □

The function τ thus satisfies Condition B of [5], page 17:

(a) $\tau \ge 0$.
(b) $\tau(x) - \tau(y) \le K_1 \|x - y\|$, $x, y \in \mathbb{N}^{2N+2}$.
(c) For any ergodic face Δ and all $x \in B^\Delta \cap \{x_i > C,\ i \in \Delta\}$:

$$\tau(x + v(x)) - \tau(x) \le -\delta.$$

If we choose C such that $C > \|v^\Delta\|$ for all ergodic faces Δ, then (c) is satisfied with $\delta = 1$ because $\Gamma^x(1) = x + v(x)$ for $x \in B^\Delta \cap \{x_i > C,\ i \in \Delta\}$.

As we said before, we worked with the process rather than the embedded Markov chain for which the results of [5] may apply. However, as we have already remarked, the second vector field for the Markov chain is proportional to the vector field for the Markov process (with the proportionality constant depending only on the face considered). Hence the variable τ can be modified so that Condition B is satisfied for the embedded Markov chain. It implies that $(\mathbf{X}_n)_n$ is ergodic; hence, our Markov process is also ergodic. Our proof by induction is finished as well as the proof of Theorem 5. □

REMARKS. (a) The second vector field of the Markov process is essential in our proof of ergodicity. It is not necessary for the stability proof of [2]. In this case, the sum of the components of the vector is a natural Lyapounov function.

(b) Our second vector field is not multivalued on the transient faces. In general, the second vector field is generally multivalued in order to get the appropriate continuity properties of the associated dynamical system. On the transient faces, its values are all the possible drift vectors pointing to an ergodic face (see [4]).

(c) The behavior of the network depends on the parity of its size. A similar phenomenon has been analyzed for a queueing network with local interaction in [7], but in this case the critical ρ's are not explicitly known.

Stable states of the linear network. Our purpose here is the asymptotic behavior of the linear network with N nodes. We enlarge our state space to allow the value $+\infty$ for the components of $(\mathbf{X}(t))_{t \ge 0}$ so that we deal with measures defined on $(\mathbb{N} \cup \{+\infty\})^N$. We do not consider the special cases where ρ is one of the $1/(2\cos \pi/(2N + 1))$, $N \in \mathbb{N}$. As in the proof of Theorem 5, for every subset Δ of $\{1,\dots,N\}$, we define the interval L_i for $i \in \Delta$ such that

$$\{1,\dots,N\} - \Delta = \bigcup_{i \in \Delta} L_i.$$

We set $L_0 = \{1,\dots,N\}$ when $\Delta = \varnothing$. For every Δ such that $|L_i| = 0, 1, 2k$, $k \geq 1$ for all i, we define the distribution π^Δ on $(\mathbb{N} \cup \{+\infty\})^N$ by

$$\pi^\Delta = \prod_{i \in \Delta} \delta_{+\infty} \prod_{L_i \neq \varnothing} \pi_{|L_i|},$$

where π_k is the invariant measure of the linear network with k nodes and $\delta_{+}\infty$ is the Dirac measure at infinity. Notice that this definition is meaningful only on some range of ρ. The distribution π^Δ is a concatenation of linear subnetworks in equilibrium and Δ is simply the set of dead nodes for π^Δ.

PROPOSITION 9. *Any distribution of* $\mathscr{E} = \{\pi^\Delta\colon \Delta \in \mathscr{S}\}$ *is a possible limiting distribution for* $(\mathbf{X}(t))_{t \geq 0}$. *The set* $\mathscr{S}$ *of admissible* Δ*'s is defined by*:

(a) $\rho < 1/2$. $\mathscr{S} = \varnothing$.

(b) $\rho > 1$. $\mathscr{S}$ *is the set of all* Δ*'s such that if* $i \in \Delta$, *then* $i - 1$ *or* $i + 1 \notin \Delta$ *and for all* $i < N$, *then either* $i \in \Delta$ *or* $i + 1 \in \Delta$.

(c) $1/2 < \rho < 1$. *We have* $\rho_c(K+1) < \rho < \rho_c(K)$ *for some* $K \geq 1$.

(i) N *is even and* $\leq 2K$: $\mathscr{S} = \{\varnothing\}$.

(ii) N *is odd and* $\leq 2K + 1$: $\mathscr{S} = \{\Delta_0\}$ *with* $\Delta_0 = \{1, 3, 5, \dots, N\}$.

(iii) $N \geq 2K + 2$: $\mathscr{S}$ *is the set of all* Δ*'s such that* (1) $1, N \notin \Delta$, (2) *for all* $i \in \Delta$, $i - 1 \in L_i$, $i + 1 \in L_{i+1}$ *and* $|L_i| = 1$ *or* $2K$, *and* (3) *if* $|L_i| = 2K$, *then* $|L_{i-1}| = |L_{i+1}| = 1$.

PROOF. As in the proof of Proposition 4, the method consists in using ergodic theorems for ergodic Markov processes. □

REMARKS. (a) We conjecture that every limiting distribution of $(\mathbf{X}(t))_{t \geq 0}$ is a convex combination of the π^Δ with $\Delta \in \mathscr{S}$, that is, $\mathscr{E}$ is the set of extremal measures of the set of limiting distributions of $(\mathbf{X}(t))_{t \geq 0}$.

(b) When $\rho < 1$, the dead nodes (if any) are isolated.

(c) In the definition of $\mathscr{S}$ in (iii), notice that if there is a linear subnetwork of size $2k$, then the neighboring subnetworks must be isolated nodes. The reason is that two subnetworks of respective sizes $2k$ and k' can be neighbors only if their output can "kill" the node between them. If $\rho_c(K+1) < \rho < \rho_c(K)$, this is possible only if $y_1(2k) + y_1(k') > 1$. The case $k < K$ is impossible because $y_1(k') < \rho$ and $y_1(2k) + \rho < 1$ ($\rho < \rho_c(k)$); hence, $k = K$. If $k' > 1$, then $k' = 2K$ for the same reason, but according to Proposition 10 of the Appendix, $2y_1(2K) < 1$; hence, $k' = 1$.

(d) Following (c), Figure 3 is an example of a stable configuration of our network (when $\rho < 1$), but that cannot be reached from any initial state with

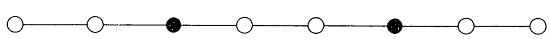

FIG. 3.

finite components (the components represented by black nodes have the value $+\infty$).

APPENDIX

Let $(y_i(N))_{1 \le i \le N}$ be a solution of the following linear system:

$$(8) \qquad \begin{aligned} 2y_1 \cos\theta + y_2 &= 1, \\ y_1 + 2y_2 \cos\theta + y_3 &= 1, \\ y_2 + 2y_3 \cos\theta + y_4 &= 1, \\ &\cdots \\ y_{i-1} + 2y_i \cos\theta + y_{i+1} &= 1, \\ &\cdots \\ y_{N-1} + 2y_N \cos\theta &= 1. \end{aligned}$$

PROPOSITION 10. *Suppose N is an even integer.*

(a) *If* $f(\theta) = y_1(N) + 1/(2\cos\theta) - 1$, *then* $f(0) = -1/(2(N+1))$, *f is negative on the interval* $[0, \pi/(N+3)]$ *and positive on* $]\pi/(N+3), \pi/(N+1)[$.
(b) *For* $\theta \in [0, \pi/(N+1)[$, $y_1(N) < 1/2$.

PROOF. Standard calculations with the above recurrence give easily that for $0 < \theta < \pi/(N+1)$, we have

$$y_1(N) + \frac{1}{2(1+\cos\theta)}\left(1 + \frac{\sin N\theta - \sin\theta}{\sin(N+1)\theta}\right).$$

Thus, f can be written as

$$f(\theta) = \frac{((1 - 2\cos^2\theta)\sin(N+1)\theta + \cos\theta(\sin N\theta - \sin\theta))}{2(1+\cos\theta)\cos\theta\sin(N+1)\theta}.$$

If $p(\theta)$ is the numerator of f, then

$$p(\theta) = \sin(N+1)\theta - \sin(N+3)\theta - \sin 2\theta.$$

We remark that $p(\theta_0) = 0$ for $\theta_0 = k\pi/N + 3$, k odd and $k \le N + 3$, or for $\theta_0 = k\pi/N + 1$, k odd, $k \le N + 1$ and

$$\frac{p(\theta)}{\sin\theta} = U_N(x) - U_{N+2}(x) - U_1(x),$$

where $x = \cos\theta$ and U_N is the Nth Chebyshev polynomial of the second kind. The right-hand side of the above expression is a polynomial of degree less than or equal to $N + 2$; hence, we have all the zeroes of p in $[0, \pi]$. It is now easy to complete the proof of (a).

To prove (b), notice that $y_1(N)$ can be written as

$$y_1(N) = \frac{1}{2(1+\cos\theta)}\left(1 + \frac{\sin(N+1)\theta\cos\theta - \cos(N+1)\theta\sin\theta - \sin\theta}{\sin(N+1)\theta}\right)$$
$$= \frac{1}{2} - \frac{\sin\theta}{2(1+\cos\theta)}\frac{\cos(N+1)\theta/2}{\sin(N+1)\theta/2}.$$

Hence our proof of (b) is complete. □

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