The Annals of Applied Probability
1994, Vol. 4, No. 4, 953–967

AN ARBITRAGE THEORY OF THE TERM STRUCTURE OF INTEREST RATES[1]

BY KRISTIAN R. MILTERSEN

Odense Universitet

In the setting of the Heath–Jarrow–Morton model, this paper presents sufficient conditions to assure that the stochastic forward rates are strictly positive while maintaining the martingale property of the discounted bond price processes in the case where the stochastic forward rates are described as stochastic differential equations with explicitly state dependent stochastic volatility. Moreover, the stochastic development of the term structure of interest rates is generalized to be described by a class of continuous local martingales instead of Wiener processes. An example showing that this is a true extension of the Heath–Jarrow–Morton model is provided.

1. Introduction. The purpose of this paper is to present a consistent continuous-time model for the stochastic evolution of bond prices. This is the key issue in order to price contingent claims in general and contingent claims written on the term structure of interest rate dependent securities in particular. Part of a consistent model is the existence of a probability measure such that the simultaneous evolution of security prices discounted by a numeraire security is a martingale. This is a well-known sufficient condition to assure that there are no arbitrage opportunities to be exploited by trading the securities. However, an additional part of the consistent model, that many authors have set aside, is to assure nonnegative forward rates.

In this paper, the term bond is a pure default-free zero-coupon bond. That is, a security that pays one unit of account at its maturity date for sure. Using linearity of prices, any default-free (coupon) bond can be priced using the prices of pure zero-coupon bonds. Moreover, a bond with default risk can be thought of as a contingent claim on default-free bonds and is, therefore, not part of the primitives of the model.

It is well-known that current bond prices, denoted by $P(0,T)$, of all maturities, T, can be described equivalently by the current forward rate function, denoted by $f(\cdot)$, using the equation

$$P(0,T) = \exp\left(-\int_0^T f(s)\,ds\right). \tag{1.1}$$

In this paper f is referred to as the initial forward rate function.

Received August 1992; revised February 1994.
[1]Research supported by the Danish Research Academy.
AMS 1991 *subject classifications*. Primary 90A09; secondary 60G44, 60H05, 60H10, 60H30.
Key words and phrases. Term structure of interest rates, pure default-free bond, financial bond market, forward rate process, spot rate process, time changed Wiener process, continuous local martingale, stochastic differential equation (SDE).

The equivalence in (1.1) is exploited such that the stochastic evolution of bond prices of all maturities will be described by the evolution of a so-called forward rate process, which we denote $X_{(t,s)}$. That is, the sample path $s \mapsto X_{(t,s)}$ is to be interpreted as the (stochastic) forward rate function at date t.

We model the forward rate process as a solution to a stochastic differential equation (SDE) with a stochastic volatility that is explicitly state dependent; that is, the volatility is directly omega dependent besides its dependence on the present value of the forward rate. This description unifies and extends the two forward rate process descriptions in Heath, Jarrow and Morton (1992) and Morton (1988). Further, we present a theorem giving sufficient conditions on the volatility of the forward rate process that assures strict positiveness of the forward rate process.

The paper is organized as follows. Section 2 introduces the notation and the financial bond market. Section 3 defines the forward rate process and provides sufficient conditions on this forward rate process to assure no arbitrage as well as positive forward rates. Finally, proofs of the theorems are presented in the Appendix.

2. The financial bond market. This section introduces the stochastic model of the financial bond market including the securities traded on the market. Moreover, it presents a set of sufficient conditions on the price processes of the traded securities in order to avoid arbitrage between the traded securities and to avoid negative forward rates.

First of all, we set up the time horizons:

DEFINITION 2.1. Time horizons:

1. $\mathbb{I} := [0, \Gamma]$ is the time horizon in the economy, where $\Gamma \in \mathbb{R}_+$.
2. $\mathbb{I}_T := [0, T]$ is the lifetime of a bond expiring at $T \in \mathbb{I}$.
3. $\Pi := \{(t, T) \in [0, \Gamma] \times [0, \Gamma] | t \leq T\}$ is the time parameter set for the bonds.

The fundamental probability space has the form $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \in \mathbb{I}}, \mathbb{P})$, where $\{\mathscr{F}_t\}_{t \in \mathbb{I}}$ is a right continuous filtration, that is, a nondecreasing (right continuous) family of sub-σ-fields of $\mathscr{F}$, and $\mathbb{P}$ is a set of equivalent probability measures on $\mathscr{F}$. (Without loss of generality, we assume that $\mathbb{P}$ is the set of all probability measures equivalent to an arbitrary measure from $\mathbb{P}$.) Furthermore, we assume that $\mathscr{F}_0 = \sigma(\mathscr{N}(\mathbb{P}))$. That is, the fundamental probability space fulfills the usual conditions. To shorten the notation of the filtration, we denote $\{\mathscr{F}_t\}_{t \in \mathbb{I}}$ as $\mathbf{F}$. Moreover, the subfiltration $\{\mathscr{F}_t\}_{t \in \mathbb{I}_T}$ will be denoted $\mathbf{F}_T$, for $T \in \mathbb{I}$.

The model of the financial bond market includes bonds for all possible maturity dates, T, in the time horizon of the economy. The bond price process of the bond maturing at date $T \in \mathbb{I}$ is denoted $\{P(t, T)\}_{t \in \mathbb{I}_T}$ and is assumed to be adapted to the filtration $\mathbf{F}_T$. Finally, the financial bond market includes a numeraire security with price process $\{A_t\}_{t \in \mathbb{I}}$. Besides being a traded security in the whole time horizon, this security will also be used to discount future values. The process $\{A_t\}_{t \in \mathbb{I}}$ is adapted to the filtration $\mathbf{F}$. The tuple

$$(\Omega, \mathscr{F}, \mathbf{F}, \mathbb{P}, \{P(t,T)\}_{(t,T)\in\Pi}, \{A_t\}_{t\in\mathbb{I}})$$

will be termed a financial bond market. The financial bond market is open for trading in the whole time horizon, $\mathbb{I}$. The investors trade portfolios of bonds which they can buy and sell at all times in $\mathbb{I}$. They are allowed to take short positions (i.e., to have a negative position of some bonds), but they have to fulfill their obligations before the end of the time horizon (i.e., they are not allowed to go bankrupt). Furthermore, the investors can trade in the numeraire security and they are allowed to store a positive amount of the unit of account (i.e., to carry cash).

To ensure an arbitrage free market, we introduce the following notion:

DEFINITION 2.2. *Viable bond market:* A financial bond market

$$(\Omega, \mathscr{F}, \mathbf{F}, \mathbb{P}, \{P(t,T)\}_{(t,T)\in\Pi}, \{A_t\}_{t\in\mathbb{I}})$$

is said to be a viable bond market if:

1. $A_t > 0$, $\mathbb{P}$-a.e., $t \in \mathbb{I}$.
2. There exists $Q \in \mathbb{P}$ such that $\{(P(t,T))/A_t\}_{t\in\mathbb{I}_T}$ are $(\mathbf{F}_T, Q)$-martingales, for all $T \in \mathbb{I}$.
3. (a) $P(t,T) \in (0,1]$, $\mathbb{P}$-a.e., $(t,T) \in \Pi$; (b) $P(T,T) = 1$, $\mathbb{P}$-a.e., $T \in \mathbb{I}$; (c) $T \mapsto P(t,T)$: $(t,\Gamma) \to \mathbb{R}$ is differentiable, for every $\omega \in \Omega$, $t \in \mathbb{I}$; (d) $(\partial P(t,T))/\partial T \le 0$, $\mathbb{P}$-a.e., $(t,T) \in \Pi$.

Condition 2 in Definition 2.2 is a well-known sufficient condition to preclude arbitrage between all securities in the bond market. This is proved in Harrison and Kreps (1979) for investors allowed to trade a finite number of times and it is later extended in Harrison and Pliska (1981), which defines, similar to condition 2 in Definition 2.2, a security market model to be viable if an equivalent martingale measure, Q, exists. Condition 3(d) in Definition 2.2 is equivalent to the assumption of nonnegative forward rates. Combined, the four conditions in condition 3 assure that there is no arbitrage between investments in bonds and costless storage of units of account.

For the purpose of contingent claims pricing we only have to model the price processes of the securities in the financial bond market under an equivalent martingale measure. Therefore, for the rest of the paper let $Q \in \mathbb{P}$ be fixed and let us call this measure an equivalent martingale measure.

3. The construction of a viable bond market. In this section we introduce the forward rate process and we construct a financial bond market with this forward rate process as the basic modeling element. This means that we define the bond prices and the numeraire security from the forward rate process such that the discounted bond price processes are martingales under an equivalent martingale measure by construction.

The forward rate process is described as a solution to an SDE with explicitly state dependent volatility, thereby unifying the two distinct descriptions of Heath, Jarrow and Morton (1992) into a single forward rate process description. That is, this description unifies both the general Itô process description with state dependent, but forward rate independent, volatility and the SDE description with state independent, but forward rate dependent, volatility of the forward rate process. Moreover, this unification extends the SDE description from Wiener processes to time changed Wiener processes. This description facilitates an interpretation of the unbiased expectation hypothesis and it gives further insight into the mechanism that ties the forward rate process and the spot rate process together. Moreover, it gives a basis for finding conditions which almost surely make the whole forward rate process strictly positive everywhere while maintaining the martingale property of the discounted bond price processes—the main result of the paper. According to Theorem 3.5, a sufficient condition for this result is that the volatility of the forward rate process is Lipschitz in the forward rate process itself and zero when the forward rate is zero. The section includes an example demonstrating the usefulness of the results.

DEFINITION 3.1. *Forward rate process:* Suppose we have a $Q \in \mathbb{P}$, called an equivalent martingale measure, a family of K-dimensional stochastic processes $\{\{\sigma(t, s, x)\}_{(t,s)\in\Pi}\}_{x\in\mathbb{R}}$ and a K-dimensional $(\mathbf{F}, Q)$-TC Wiener process, $\{Z_t\}_{t\in\mathbb{I}}$, (cf. Definition A.1) satisfying:

1. $\sigma(t, s, x)$ is $\mathscr{F}_t$-measurable, $(t, s, x) \in \Pi \times \mathbb{R}$.
2. $(t, s) \mapsto \sigma(t, s, x)$: $\Pi \to \mathbb{R}_+^K$ has continuous sample paths, $x \in \mathbb{R}$. [If $(t, s) \mapsto \sigma(t, s, x)$ has only continuous sample paths $\mathbb{P}$-a.e., then, since $\mathscr{F}_0 \supseteq \sigma(\mathscr{N}(\mathbb{P}))$, it is possible to choose a $\mathbb{P}$-indistinguishable process, $\{\tilde{\sigma}(t, s, x)\}_{(t,s)\in\Pi}$, which has continuous sample paths for every $\omega \in \Omega$. Therefore, whenever a process has continuous sample paths in this paper, it is implicitly understood that we have chosen the one with continuous sample paths, for every $\omega \in \Omega$.]
3. σ is uniformly bounded and uniformly *Lipschitz* in the third variable over Ω [i.e., a constant, $M > 0$, exists such that $\|\sigma(u, v, x)(\omega)\| \le M$ and $\|\sigma(u, v, x)(\omega) - \sigma(u, v, y)(\omega)\| \le M|x - y|$, $x, y \in \mathbb{R}$, $(u, v) \in \Pi$, and $\omega \in \Omega$].
4. f is Lipschitz.

Then any solution, $\{X_{(t,s)}\}_{(t,s)\in\Pi}$, to the SDE,

$$X_{(t,s)} = f(s) + \int_0^t \sigma(u, s, X_{(u,s)}) \cdot dZ_u + \int_0^t \sigma(u, s, X_{(u,s)}) \cdot D\left(\int_u^s \sigma(u, v, X_{(u,v)})\, dv\right) d\langle Z\rangle_u, \qquad (t, s) \in \Pi, \quad \mathbb{P}\text{-a.e.}, \tag{3.1}$$

is called a forward rate process.

Here the multidot $(\cdot)$ represents the usual inner product of $\mathbb{R}^K$ and $\| \;\|$ represents the corresponding norm. For this definition to be proper a solution, $\{X_{(t,s)}\}_{(t,s)\in\Pi}$, to the SDE (3.1) must exist. In Morton (1988) the existence of a strong solution to the SDE (3.1) is proved for σ state independent and $\{Z_t\}_{t\in\mathbb{I}}$ as an $(\mathbf{F}, Q)$-Wiener process.

In Definition 3.1 we assume that $\{Z_t\}_{t\in\mathbb{I}}$ is an $(\mathbf{F}, Q)$-TC Wiener process which generalizes a standard $(\mathbf{F}, Q)$-Wiener process. That is, $\{Z_t\}_{t\in\mathbb{I}}$ is a continuous $(\mathbf{F}, Q)$-martingale which coordinate by coordinate can be transformed into independent $(\mathbf{F}, Q)$-Wiener processes by the same time change. We have made this generalization in order to relax the boundedness assumptions on σ in condition 3 in Definition 3.1. The exact definition of a TC Wiener process can be seen in the Appendix, Definition A.1. The following example illustrates this generalization:

EXAMPLE 3.2. Let $\{W_t\}_{t\in\mathbb{I}}$ be a one-dimensional $(\mathbf{F}, Q)$-Wiener process and let $\{Y_t\}_{t\in\mathbb{I}}$ be a solution to the SDE

$$Y_t = Y_0 + \int_0^t \rho(Y_s)\, dW_s, \qquad t \in \mathbb{I},\ \mathbb{P}\text{-a.e.},$$

with $Y_0 > 0$, $\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$ Lipschitz, bounded away from zero on any compact set in $(0, \infty)$ and $\rho(0) = 0$. This process is a continuous square integrable $(\mathbf{F}, Q)$-martingale [cf. Karatzas and Shreve (1988), Proposition 5.17, page 341]. [If $p(x) = x$, then, for $t \in \mathbb{I}$ fixed, Y_t is *lognormally distributed*, i.e.,

$$\log Y_t \sim N\big(\log Y_0 - \tfrac{1}{2}t^2, t\big).\big]$$

This process is widely used, for example, to model asset prices, since it has the convenient property of not becoming negative at any date. Moreover, the process $\{Z_t\}_{t\in\mathbb{I}}$, defined as

$$Z_t := Y_t - Y_0, \qquad t \in \mathbb{I},$$

is an $(\mathbf{F}, Q)$-TC Wiener process (cf. Definition A.1) with quadratic variation process, $\{\langle Z\rangle_t\}_{t\in\mathbb{I}}$, defined as

$$\langle Z\rangle_t = \int_0^t \rho(Z_s + 1)^2\, ds, \qquad t \in \mathbb{I}.$$

If $\{X_{(t,s)}\}_{(t,s)\in\Pi}$ is a solution to the SDE (3.1), for given f and σ fulfilling conditions 1–4 in Definition 3.1, then, of course, $\{X_{(t,s)}\}_{(t,s)\in\Pi}$ is a forward rate process with $\{Z_t\}_{t\in\mathbb{I}}$ as a TC Wiener process. However, $\{X_{(t,s)}\}_{(t,s)\in\Pi}$ is not a forward rate process with $\{Z_t\}_{t\in\mathbb{I}}$ as a Wiener process unless $\tilde{\sigma}(t, s, x)(\omega) = \sigma(t, s, x)(\omega)\rho(Z_t(\omega) + 1)$ fulfills the boundedness assumptions of Definition 3.1, that is, condition 3, which is not necessarily the case.

Since we have made this generalization to TC Wiener processes and since we have introduced explicit state dependent volatility, a new proof of the existence of a strong solution to the SDE (3.1) is needed. This existence proof is provided as Theorem A.2 in the Appendix.

Given the forward rate process, the bond prices are naturally defined as

$$P(t,T) := \exp\left(-\int_t^T X_{(t,s)}\, ds \right), \qquad (t,T) \in \Pi$$

[cf. (1.1)]. For a fixed $t \in \mathbb{I}$, the stochastic process $\{P(t,T)\}_{T \in [t,\Gamma]}$ is of bounded variation. In particular, $T \mapsto P(t,T)$: $(t,\Gamma) \to \mathbb{R}$ is differentiable, for every $\omega \in \Omega$, such that

$$X_{(t,T)} = -\frac{\partial \log P(t,T)}{\partial T} = -\frac{(\partial P(t,T))/\partial T}{P(t,T)}, \qquad (t,T) \in \Pi.$$

Moreover, the diagonal process, $\{X_{(t,t)}\}_{t \in \mathbb{I}}$, is denoted as the spot rate. The numeraire security is defined, using the spot rate, as

$$A_t := \exp\left(\int_0^t X_{(s,s)}\, ds \right), \qquad t \in \mathbb{I}.$$

This process, $\{A_t\}_{t \in \mathbb{I}}$, is also denoted the savings account, because it can be interpreted as an account initialized with one unit of account at date zero and continuously earning the spot rate of interest. Using the spot rate process the discounted bond prices are written as

$$P^*(t,T) := \frac{P(t,T)}{A_t} = \exp\left(-\int_t^T X_{(t,s)}\, ds - \int_0^t X_{(s,s)}\, ds \right), \qquad (t,T) \in \Pi.$$

Now, the construction

$$(\Omega, \mathscr{F}, \mathbf{F}, \mathbb{P}, \{P(t,T)\}_{(t,T) \in \Pi}, \{A_t\}_{t \in \mathbb{I}})$$

is a financial bond market. This financial bond market is defined with the forward rate process $\{X_{(t,s)}\}_{(t,s) \in \Pi}$ as the basic modeling element.

The issue is now under what conditions will this financial bond market be viable? The first step is to give conditions assuring that all the discounted bond price processes are martingales (cf. condition 2 in Definition 2.2):

THEOREM 3.3. *Suppose we have a forward rate process, $\{X_{(t,s)}\}_{(t,s) \in \Pi}$, from Definition 3.1 with σ fulfilling the additional condition*

5. $$E^Q\left[\exp\left(\frac{1}{2} \int_0^\Gamma \left\| \int_u^\Gamma \sigma(u, s, X_{(u,s)})\, ds \right\|^2 d\langle Z \rangle_u \right) \right] < \infty.$$

Then all the discounted bond price processes, $\{P^*(t,T)\}_{t \in \mathbb{I}_T}$, are $(\mathbf{F}_T, Q)$-martingales, for $T \in \mathbb{I}$. Furthermore, $\{P(t,T)\}_{t \in \mathbb{I}_T}$ are $\mathbf{F}_T$-semimartingales and

$$\{P(t,T)\}_{(t,T) \in \Pi} \text{ and } \left\{ E^Q\left[\exp\left(-\int_t^T X_{(s,s)}\, ds \right) \middle| \mathscr{F}_t \right] \right\}_{(t,T) \in \Pi} \tag{3.2}$$

are $\mathbb{P}$-indistinguishable.

Note that it is not necessary to make the probability measure specific when we are talking about semimartingales. The Novikov condition (i.e., condition 5 in Theorem 3.3) is not stated to assure the possibility of a change of measure, but is merely used as an integrability condition assuring that the bond price processes are martingales. It appears from Theorem 3.3 that under the measure $Q \in \mathbb{P}$, the one-dimensional process $\{X_{(t,t)}\}_{t \in \mathbb{I}}$ generates all the processes $\{P(t,T)\}_{t \in \mathbb{I}_T}$, for all $T \in \mathbb{I}$, as specified in (3.2). That is, the bond price at date t for a bond expiring at date T is a conditional expectation under an equivalent martingale measure, Q, of a known function of the future spot rates up to date T. However, this does not mean that the family of stochastic processes, $\{\{P^*(t,T)\}_{t \in \mathbb{I}_T} | T \in \mathbb{I}\}$, has a martingale multiplicity of 1, for example, generated by the spot rate process, $\{X_{(t,t)}\}_{t \in \mathbb{I}}$, since there is, in addition, a conditional expectation to be evaluated using a filtration which is not generated by the spot rate process.

OBSERVATION 3.4. Sufficient for condition 5 in Theorem 3.3 is that

$\tilde{5}$. $$E^Q\left[\exp\left(\tfrac{1}{2}M^2\Gamma^2\langle Z\rangle_\Gamma\right)\right] < \infty,$$

which is trivially true for $\{Z_t\}_{t \in \mathbb{I}}$ as an $(\mathbf{F}, Q)$-Wiener process.

To have a viable bond market we are only missing conditions ensuring positivity of the forward rate process. Therefore, define a family of K-dimensional stochastic processes

$$\{\varphi_{(t,s)}\}_{t \in \mathbb{I}_s} := \left\{-\int_t^s \sigma\left(t, v, X_{(t,v)}\right) dv\right\}_{t \in \mathbb{I}_s}, \qquad s \in \mathbb{I}.$$

If we define

$$Y_t^s := \exp\left(\int_0^t \varphi_{(u,s)} \cdot dZ_u - \tfrac{1}{2}\int_0^t \|\varphi_{(u,s)}\|^2 d\langle Z\rangle_u\right), \qquad (t,s) \in \Pi,$$

then $\{Y_t^s\}_{t \in \mathbb{I}_s}$ is an $(\mathbf{F}_s, Q)$-martingale for $s \in \mathbb{I}$ because of condition 5 in Theorem 3.3. Using this density process for each $s \in \mathbb{I}$, we can induce a new measure, Q^s, equivalent to Q by defining

$$\frac{dQ^s}{dQ} := Y_s^s, \qquad s \in \mathbb{I}.$$

Define

$$Z_t^s := Z_t - \int_0^t \varphi_{(u,s)} d\langle Z\rangle_u, \qquad t \in \mathbb{I}_s,\ s \in \mathbb{I}.$$

According to Girsanov's theorem [cf. Ikeda and Watanabe (1989), Theorem IV.4.1, page 191], $\{Z_t^s\}_{t \in \mathbb{I}_s}$ is an $(\mathbf{F}_s, Q^s)$-local martingale, $s \in \mathbb{I}$. Using (3.1) we get that

$$\begin{aligned} X_{(t,s)} &= f(s) + \int_0^t \sigma\left(u, s, X_{(u,s)}\right) \cdot dZ_u - \int_0^t \sigma\left(u, s, X_{(u,s)}\right) \cdot \varphi_{(u,s)} d\langle Z\rangle_u \\ &= f(s) + \int_0^t \sigma\left(u, s, X_{(u,s)}\right) \cdot dZ_u^s, \qquad \mathbb{P}\text{-a.e.}, (t,s) \in \Pi. \end{aligned}$$

Furthermore, if we assume that $\{Z_t\}_{t \in \mathbb{I}}$ is square integrable in each coordinate, then it follows that $\{X_{(t,s)}\}_{t \in \mathbb{I}_s}$ become square integrable $(\mathbf{F}_s, Q^s)$-martingales, for all $s \in \mathbb{I}$, such that

$$\{X_{(t,s)}\}_{t \in \mathbb{I}_s} \text{ and } \{E^{Q^s}[X_{(s,s)}|\mathscr{F}_t]\}_{t \in \mathbb{I}_s} \text{ are } \mathbb{P}\text{-indistinguishable}$$

and by continuity in s,

$$\{X_{(t,s)}\}_{(t,s) \in \Pi} \text{ and } \{E^{Q^s}[X_{(s,s)}|\mathscr{F}_t]\}_{(t,s) \in \Pi} \text{ are } \mathbb{P}\text{-indistinguishable.}$$

Hence,

$$\begin{aligned} P^*(t,T) &= \exp\left(-\int_t^T X_{(t,s)}\,ds - \int_0^t X_{(s,s)}\,ds\right) \\ &= \exp\left(-\int_0^T E^{Q^s}[X_{(s,s)}|\mathscr{F}_t]\,ds\right), \qquad \mathbb{P}\text{-a.e., } (t,T) \in \Pi. \end{aligned} \tag{3.3}$$

Equation (3.3) is compatible with the unbiased expectation hypothesis in Cox, Ingersoll and Ross [(1981), (19), page 776]. However, observe that the measure used to evaluate the conditional expectation in (3.3) is dependent on the future in terms of s. This means that, in general, the forward rate cannot be interpreted as a simple probabilistic expectation of future spot rates. The measure Q^s is introduced as the forward rate adjusted probability measure in Jamshidian (1987).

In addition, this formulation leads to the main result of this paper.

THEOREM 3.5. *Suppose we have a forward rate process,* $\{X_{(t,s)}\}_{(t,s) \in \Pi}$, *from Definition* 3.1 *with* f *and* σ *fulfilling condition* 5 *in Theorem* 3.3 *and*

1. $\sigma(u,v,0) = 0$, $\mathbb{P}$*-a.e.*, $(u,v) \in \Pi$.
2. $f(s) > 0$, $s \in \mathbb{I}$.

Then

$$Q\big(X_{(t,s)} > 0, \forall\, (t,s) \in \Pi\big) = 1.$$

That is, the forward rates remain strictly positive with probability 1. In other words, zero becomes a natural barrier of the forward rate process—a barrier the process never touches. Combining Theorems 3.3 and 3.5 infer the following.

COROLLARY 3.6. *Suppose we have a forward rate process,* $\{X_{(t,s)}\}_{(t,s) \in \Pi}$, *from Definition* 3.1 *with* f *and* σ *fulfilling condition* 5 *in Theorem* 3.3 *and conditions* 1 *and* 2 *in Theorem* 3.5. *Then*

$$\big(\Omega, \mathscr{F}, \mathbf{F}, \mathbb{P}, \{P(t,T)\}_{(t,T) \in \Pi}, \{A_t\}_{t \in \mathbb{I}}\big)$$

is a viable bond market.

It should be emphasized that the combined conditions of the volatility function, σ, that is, Definition 3.1, conditions 1–3, Theorem 3.3, condition 5, and Theorem 3.5, condition 1 are sufficient conditions, but these conditions are not always necessary. However, Heath, Jarrow and Morton [(1992), Section 6, pages 90–92] present two examples with nonstochastic volatility functions not fulfilling these combined conditions. Both of these examples give negative forward rates with strictly positive probabilities. Heath, Jarrow and Morton [(1992), Proposition 5, page 95] also prove that a state independent volatility function $\sigma(\cdot, \cdot, \cdot)$ of the form

$$\sigma(t, s, X_{(t,s)}) := \sigma \min\{X_{(t,s)}, \lambda\}$$

gives a nonnegative forward rate process with probability 1 if σ and λ are strictly positive constants. Note that this volatility function, $\sigma(\cdot, \cdot, \cdot)$, fulfills all the above stated conditions on σ.

In a continuation of Example 3.2, we can also demonstrate the relevance of our extension of the Heath–Jarrow–Morton model to explicitly state dependent volatility. Suppose that σ and f additionally fulfill conditions 1 and 2 in Theorem 3.5. Then the financial bond market is viable. This viability could not have been shown if we had only had Theorem 3.5 for state independent volatility because $\tilde{\sigma}$ of Example 3.2 is explicitly state dependent.

APPENDIX

This appendix contains the definition of a TC Wiener process and the proofs of the results in Section 3. To save space, the proofs are shortened. Detailed proofs of all lemmas and theorems can be found in Miltersen [(1992), Chapter 3].

A.1. Definition of a TC Wiener process.

Definition A.1. *TC Wiener process:* A K-dimensional continuous $(\mathbf{F}, Q)$-local martingale, $\{Z_t\}_{t \in \mathbb{I}} = \{(Z_t^1, \dots, Z_t^K)\}_{t \in \mathbb{I}}$, is called an $(\mathbf{F}, Q)$-TC Wiener process, for $Q \in \mathbb{P}$, if it satisfies:

1. $Z_0 = 0$, $\mathbb{P}$-a.e.
2. The coordinates of $\{Z_t\}_{t \in \mathbb{I}}$ are strongly orthogonal. That is, the processes $\{\langle Z^i, Z^j \rangle_t\}_{t \in \mathbb{I}}$ are $\mathbb{P}$-indistinguishable from the null process, $i \neq j$, $i, j \in \{1, \dots, K\}$, where $\langle \cdot, \cdot \rangle$ denotes the *quadratic variation process* defined in Ikeda and Watanabe [(1989), Definition 2.1, page 53].
3. $\{\langle Z^i, Z^i \rangle_t\}_{t \in \mathbb{I}}$ and $\{\langle Z^j, Z^j \rangle_t\}_{t \in \mathbb{I}}$ are $\mathbb{P}$-indistinguishable, $i, j \in \{1, \dots, K\}$. Hereafter, we use the notation $\{\langle Z \rangle_t\}_{t \in \mathbb{I}}$ for $\{\langle Z^i, Z^i \rangle_t\}_{t \in \mathbb{I}}$, for any $i \in \{1, \dots, K\}$, and call this the quadratic variation process.
4. $t \mapsto \langle Z \rangle_t$ is absolutely continuous on $[0, \Gamma]$, $\mathbb{P}$-a.e. That is, there exists $\{Y_t\}_{t \in \mathbb{I}}$, an $\mathbf{F}$-progressively measurable process, such that
$$\left\{\int_0^t Y_s \, ds\right\}_{t \in \mathbb{I}} \text{ and } \{\langle Z \rangle_t\}_{t \in \mathbb{I}} \text{ are } \mathbb{P}\text{-distinguishable.}$$
For obvious reasons, we refer to the process $\{Y_t\}_{t \in \mathbb{I}}$ as $\{(d\langle Z \rangle_t)/dt\}_{t \in \mathbb{I}}$.
5. $(d\langle Z \rangle_t)/dt > 0$, $(\mathbb{P} \times \lambda_{\mathbb{I}})$-a.e., where $\lambda_{\mathbb{I}}$ denotes the Lebesgue measure on $\mathbb{I}$.

A.2. Existence of a strong solution to SDE (3.1).

THEOREM A.2. *Suppose we have a* $Q \in \mathbb{P}$, *a family of K-dimensional stochastic volatility processes* $\{\{\sigma(t,s,x)\}_{(t,s)\in\Pi}\}_{x\in\mathbb{R}}$ *and a K-dimensional* $(\mathbf{F}, Q)$*-TC Wiener process* $\{Z_t\}_{t\in\mathbb{I}}$, *satisfying the conditions* 1–4 *in Definition* 3.1. *Then a stochastic process*, $\{X_{(t,s)}\}_{(t,s)\in\Pi}: (\Omega, \mathscr{F}_\Gamma) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$, *exists such that*:

(a) $X_{(t,s)}$ *is* $\mathscr{F}_t$*-measurable*, $(t,s) \in \Pi$.
(b) $(t,s) \mapsto X_{(x,s)}: \Pi \to \mathbb{R}$ *has continuous sample paths*.
(c) $\{X_{(t,s)}\}_{(t,s)\in\Pi}$ *is a strong solution to the SDE* (3.1).

The outline of the proof is to use *Picard–Lindelöf* iterations to ensure convergence of a solution to the SDE (3.1). Finally, the proof is completed with a localizing argument. In the special case where:

6. $t \mapsto \langle Z\rangle_t$ is uniformly Lipschitz over Ω [i.e., there exists an $L \in \mathbb{R}_+$ such that $|\langle Z\rangle_t(\omega) - \langle Z\rangle_s(\omega)| \le L|t-s|$, for all $\omega \in \Omega$, $(t,s) \in \Pi$],

the proof is a straightforward generalization of Morton [(1988), Theorem 4.6.1, page 61ff]. In this case, uniqueness of the solution to the SDE (3.1) is proven in Morton [(1988), Theorem 4.6.3, page 71ff] too. Now we give the general proof of Theorem A.2.

PROOF. Define the $\mathbf{F}$-stopping time

$$T_n := \inf\{t \in \mathbb{I} \mid \exists\, s \in \mathbb{I}_t : |\langle Z\rangle_t - \langle Z\rangle_s| > n|t-s|\}, \qquad n = 1, \dots .$$

Condition 4 in Definition A.1 assures that

$$\lim_{n\to\infty} T_n = \Gamma, \qquad \mathbb{P}\text{-a.e.}$$

Let

$$Z_t^n := Z_{t\wedge T_n}, \qquad t \in \mathbb{I},\ n = 1, \dots .$$

We have just argued for the existence and uniqueness of the solutions $\{X_{(t,s)}^n\}_{(t,s)\in\Pi}$ to the SDEs

$$X_{(t,s)}^n = f(s) + \int_0^t \sigma(u,s,X_{(u,s)}^n)\cdot dZ_u^n + \int_0^t \sigma(u,s,X_{(u,s)}^n) \cdot\left(\int_u^s \sigma(u,v,X_{(u,v)}^n)\,dv\right) d\langle Z^n\rangle_u, \quad (t,s)\in\Pi,\ \mathbb{P}\text{-a.e.},$$

for $n = 1, \dots$. It is now easy to see that, for a given $n \in \mathbb{N}$, $\{X_{(t\wedge T_m, s)}^n\}_{(t,s)\in\Pi}$ and $\{X_{(t,s)}^m\}_{(t,s)\in\Pi}$ are $\mathbb{P}$-indistinguishable, $m = 1, \dots, n$, since they are both solutions to the same SDE and this SDE has a unique solution. That is, there exists a set $A \in \mathscr{F}$, with $A \subseteq \{\lim_{n\to\infty} T_n = \Gamma\}$ and $Q(A) = 1$, such that

$$X_{(t\wedge T_m, s)}^n(\omega) = X_{(t,s)}^m(\omega), \qquad \omega \in A,\ (t,s)\in\Pi,\ m = 1,\dots,n,\ n\in\mathbb{N}.$$

Suppose we have fixed $\omega \in A$ and $(t, s) \in \Pi$, with $t < \Gamma$. Then

$$X^m_{(t,s)}(\omega) = X^n_{(t,s)}(\omega), \qquad m, n \geq \inf\{k \in \mathbb{N} | t \leq T_k(\omega)\}.$$

Therefore, the limit $\lim_{n\to\infty} X^n_{(t,s)}(\omega)$ exists. This limit also exists for $t = \Gamma$ because all the processes $\{X^n_{(t,s)}\}_{(t,s)\in\Pi}$ have continuous sample paths, for $n \in \mathbb{N}$. Given $(t, s) \in \Pi$, define

$$\text{(A.1)} \qquad X_{(t,s)}(\omega) = \begin{cases} \lim_{n\to\infty} X^n_{(t,s)}(\omega), & \omega \in A, \\ 0, & \omega \notin A. \end{cases}$$

Now, observe that, for given $m \in \mathbb{N}$ and $(t, s) \in \Pi$,

$$\text{(A.2)} \qquad \begin{aligned} X_{(t\wedge T_m, s)} &= \lim_{n\to\infty} X^n_{(t\wedge T_m, s)} \\ &= f(s) + \lim_{n\to\infty} \int_0^{t\wedge T_m} \sigma\big(u, s, X^n_{(u,s)}\big) \cdot dZ_u \\ &\quad + \lim_{n\to\infty} \int_0^{t\wedge T_m} \sigma\big(u, s, X^n_{(u,s)}\big) \cdot \left(\int_u^s \sigma\big(u, v, X^n_{(u,v)}\big)\, dv \right) d\langle Z\rangle_u, \\ &\qquad \mathbb{P}\text{-a.e.} \end{aligned}$$

Using Lebesgue's theorem two times gives

$$\text{(A.3)} \qquad \begin{aligned} &\lim_{n\to\infty} \int_0^{t\wedge T_m} \sigma\big(u, s, X^n_{(u,s)}\big) \cdot \left(\int_u^s \sigma\big(u, v, X^n_{(u,v)}\big)\, dv \right) d\langle Z\rangle_u \\ &\quad = \int_0^{t\wedge T_m} \sigma\big(u, s, X_{(u,s)}\big) \cdot \left(\int_u^s \sigma\big(u, v, X_{(u,v)}\big)\, dv \right) d\langle Z\rangle_u, \quad \mathbb{P}\text{-a.e.} \end{aligned}$$

Furthermore,

$$\int_0^{t\wedge T_m} \sigma\big(u, s, X^n_{(u,s)}\big) \cdot dZ_u \to \int_0^{t\wedge T_m} \sigma\big(u, s, X_{(u,s)}\big) \cdot dZ_u \quad \text{as } n \to \infty,$$

in Q-quadratic mean. This is seen by the argument

$$\begin{aligned} &E^Q \left| \int_0^{t\wedge T_m} \sigma\big(u, s, X_{(u,s)}\big) \cdot dZ_u - \int_0^{t\wedge T_m} \sigma\big(u, s, X^n_{(u,s)}\big) \cdot dZ_u \right|^2 \\ &\quad = E^Q \left[\int_0^{t\wedge T_m} \left\| \sigma\big(u, s, X_{(u,s)}\big) - \sigma\big(u, s, X^n_{(u,s)}\big) \right\|^2 d\langle Z\rangle_u \right] \to 0 \text{ as } n \to \infty, \end{aligned}$$

because σ is uniformly bounded and uniformly Lipschitz in the third variable over Ω. Therefore, a subsequence $\{n_k\}_{k=1}^\infty$ exists such that

$$\text{(A.4)} \qquad \lim_{k\to\infty} \int_0^{t\wedge T_m} \sigma\big(u, s, X^{n_k}_{(u,s)}\big) \cdot dZ_u = \int_0^{t\wedge T_m} \sigma\big(u, s, X_{(u,s)}\big) \cdot dZ_u, \quad \mathbb{P}\text{-a.e.}$$

Now, combining (A.2), (A.3) and (A.4) gives

$$
\begin{aligned}
X_{(t\wedge T_m, s)} &= \lim_{k\to\infty} X^{n_k}_{(t\wedge T_m, s)} \\
&= f(s) + \int_0^{t\wedge T_m} \sigma(u, s, X_{(u,s)})\cdot dZ_u \\
&\quad + \int_0^{t\wedge T_m} \sigma(u, s, X_{(u,s)})\cdot\left(\int_u^s \sigma(u, v, X_{(u,v)})\,dv\right) d\langle Z\rangle_u \\
&= f(s) + \int_0^t \sigma(u, s, X_{(u\wedge T_m, s)})\cdot dZ^m_u \\
&\quad + \int_0^t \sigma(u, s, X_{(u\wedge T_m, s)})\cdot\left(\int_u^s \sigma(u, v, X_{(u\wedge T_m, v)})\,dv\right) d\langle Z^m\rangle_u,
\end{aligned} \tag{A.5}
$$

$\mathbb{P}$-a.e.,

proving that the two processes $\{X^m_{(t,s)}\}_{(t,s)\in\Pi}$ and $\{X_{(t\wedge T_m, s)}\}_{(t,s)\in\Pi}$ are $\mathbb{P}$-indistinguishable, since they are both solutions to the same SDE, and this SDE has a unique solution.

Now, finally, we give the proof that $\{X_{(t,s)}\}_{(t,s)\in\Pi}$ is a strong solution to SDE (3.1):

$$
\begin{aligned}
X_{(t,s)} &= \lim_{n\to\infty} X^n_{(t,s)} \\
&= f(s) + \lim_{n\to\infty}\int_0^t \sigma(u, s, X_{(u\wedge T_n, s)})\cdot dZ_{u\wedge T_n} \\
&\quad + \lim_{n\to\infty}\int_0^t \sigma(u, s, X_{(u\wedge T_n, s)})\cdot\left(\int_u^s \sigma(u, v, X_{(u\wedge T_n, v)})\,dv\right) d\langle Z\rangle_{u\wedge T_n} \\
&= f(s) + \lim_{n\to\infty}\int_0^{t\wedge T_n} \sigma(u, s, X_{(u,s)})\cdot dZ_u \\
&\quad + \lim_{n\to\infty}\int_0^{t\wedge T_n} \sigma(u, s, X_{(u,s)})\cdot\left(\int_u^s \sigma(u, v, X_{(u,v)})\,dv\right) d\langle Z\rangle_u \\
&= f(s) + \int_0^t \sigma(u, s, X_{(u,s)})\cdot dZ_u \\
&\quad + \int_0^t \sigma(u, s, X_{(u,s)})\cdot\left(\int_u^s \sigma(u, v, X_{(u,v)})\,dv\right) d\langle Z\rangle_u, \qquad \mathbb{P}\text{-a.e.},
\end{aligned}
$$

using the same arguments as was used to prove (A.5).

Furthermore, $X_{(t,s)}$ is $\mathscr{F}_t$-measurable, $(t, s)\in\Pi$ and $(t, s)\mapsto X_{(t,s)}\colon \Pi\to\mathbb{R}$ has continuous sample paths due to the way it is constructed in (A.1). This completes the proof of conditions (a)–(c) in Theorem A.2. □

A.3. Proof of Theorem 3.3. The proof of Theorem 3.3 is (essentially) an application of Proposition 3 of Heath, Jarrow and Morton [(1992), page 86]. A sketch is provided.

PROOF OF THEOREM 3.3 (Sketch). To transform to the notation of Heath, Jarrow and Morton (1992), define

$$\tilde{\sigma}(t,s)(\omega) := \sigma\big(t,s,X_{(t,s)}(\omega)\big)(\omega)\sqrt{\frac{d\langle Z\rangle_t(\omega)}{dt}}, \tag{A.6}$$

$$\tilde{\alpha}(t,s)(\omega) := \sigma\big(t,s,X_{(t,s)}(\omega)\big)(\omega) \cdot\left(\int_t^s \sigma\big(t,v,X_{(t,v)}(\omega)\big)(\omega)\,dv\right)\frac{d\langle Z\rangle_t(\omega)}{dt} \tag{A.7}$$

and

$$d\tilde{W}_t := \frac{1}{\sqrt{d\langle Z\rangle_t/dt}}\,dZ_t.$$

Now, $\{\tilde{W}_t\}_{t\in\mathbb{I}}$ is a K-dimensional $(\mathbf{F},Q)$-Wiener process and SDE (3.1) can be simplified to

$$X_{(t,s)} = f(s) + \int_0^t \tilde{\sigma}(u,s)\cdot d\tilde{W}_u + \int_0^t \tilde{\alpha}(u,s)\,du, \qquad (t,s)\in\Pi,\ \mathbb{P}\text{-a.e.}$$

According to Heath, Jarrow and Morton [(1992), Proposition 3, page 86], a necessary and sufficient condition for the discounted bond price processes $\{P^*(t,T)\}_{t\in\mathbb{I}_T}$ to be simultaneous $(\mathbf{F}_T,Q)$-martingales, $T\in\mathbb{I}$, is that

$$\tilde{\alpha}(t,s) = \tilde{\sigma}(t,s)\cdot\left(\int_t^s \tilde{\sigma}(t,v)\,dv\right), \qquad \mathbb{P}\text{-a.e.}, (t,s)\in\Pi, \tag{A.8}$$

which can be checked by comparing (A.6) and (A.7). Note that ϕ_i in Heath, Jarrow and Morton [(1992), (17)] is zero, $i=1,\dots,n$, because, in this paper, we are exclusively operating under the equivalent martingale measure denoted $\tilde{Q}$ in Heath, Jarrow, and Morton [(1992), (16)]. This necessary and sufficient condition is derived under conditions C.1–C.5 in Heath, Jarrow and Morton (1992). Our assumptions in Definition 3.1 and Theorem 3.3 are sufficient to give conditions C.1–C.4. In particular, note that Heath, Jarrow and Morton [(1992), condition C.4, (12.c)] is fulfilled by the *Novikov condition* (cf. condition 5 in Theorem 3.3). Since we have no assumptions corresponding to Heath, Jarrow and Morton [(1992), condition C.5], (A.8) is not always necessary.

The second part of Theorem 3.3 follows from the equation

$$\begin{aligned} P(t,T) &= E^Q[P^*(T,T)|\mathscr{F}_t]\exp\left(\int_0^t X_{(s,s)}\,ds\right)\\ &= E^Q\left[P(T,T)\exp\left(-\int_0^T X_{(s,s)}\,ds\right)\Big|\mathscr{F}_t\right]\exp\left(\int_0^t X_{(s,s)}\,ds\right)\\ &= E^Q\left[\exp\left(-\int_t^T X_{(s,s)}\,ds\right)\Big|\mathscr{F}_t\right], \qquad \mathbb{P}\text{-a.e.}, (t,T)\in\Pi, \end{aligned}$$

using the facts that $\{P^*(t,T)\}_{t\in\mathbb{I}_T}$ is an $(\mathbf{F}_T,Q)$-martingale and that $P(T,T) = 1$, for $T\in\mathbb{I}$. The $\mathbb{P}$-indistinguishability follows by continuity of the sample paths. Furthermore, $\{P(t,T)\}_{t \in \mathbb{I}_T}$ is an $\mathbf{F}_T$-semimartingale since, according to Itô's lemma, one obtains that the product

$$P(t,T) = \exp\left(-\int_0^t X_{(s,s)}\,ds\right) P^*(t,T)$$

is an $\mathbf{F}_T$-semimartingale, because $\{\exp(-\int_0^t X_{(s,s)}\,ds)\}_{t \in \mathbb{I}}$ is of bounded variation and $\{P^*(t,T)\}_{t \in \mathbb{I}_T}$ is an $(\mathbf{F}_T, Q)$-martingale. □

A.4. Proof of Theorem 3.5. The result is derived from the following lemma.

LEMMA A.3. *Suppose that* $\{Z_t\}_{t \in \mathbb{I}}$ *is a K-dimensional* $(\mathbf{F}, Q)$*-TC Wiener process. Choose* $T \in \mathbb{I}$. *If* $\{X_t\}_{t \in \mathbb{I}_T}$ *is a solution to the SDE*

$$X_t = x + \int_0^t \sigma(s, X_s) \cdot dZ_s, \qquad t \in \mathbb{I}_T,$$

with

1. $x > 0$,
2. $\sigma(s,x)$ *is* $\mathscr{F}_s$*-measurable*, $(s,x) \in \mathbb{I}_T \times \mathbb{R}$,
3. $s \mapsto \sigma(s,x)\colon \mathbb{I}_T \to \mathbb{R}_+^K$ *has continuous sample paths*, $x \in \mathbb{R}$,
4. σ *is uniformly bounded and uniformly* Lipschitz *in the second variable over* Ω [*i.e., a constant* $M > 0$ *exists such that* $\|\sigma(s,x)(\omega)\| \le M$ *and* $\|\sigma(s,x)(\omega) - \sigma(s,y)(\omega)\| \le M|x-y|$, $x, y \in \mathbb{R}$, $s \in \mathbb{I}_T$ *and* $\omega \in \Omega$],
5. $\sigma(s,0) = 0$, $\mathbb{P}$*-a.e.*, $s \in \mathbb{I}_T$,

then

$$Q(X_t > 0, \forall\, t \in \mathbb{I}_T) = 1.$$

This lemma is proved by showing that $\log(X_{T_0})$ is Q-square integrable, where the $\mathbf{F}_T$-stopping time T_0 is defined as

$$T_0 := \inf\{t \in \mathbb{I}_T \mid X_t = 0\}$$

in the Wiener process case. Finally, the result is generalized to TC Wiener processes by a localizing argument.

PROOF OF THEOREM 3.5. By Lemma A.3 it can be seen that

$$Q^s\big(X_{(t,s)} > 0, \forall\, t \in \mathbb{I}_s\big) = 1, \qquad s \in \mathbb{I}.$$

Hence, it immediately follows that

$$Q\big(X_{(t,s)} > 0, \forall\, (t,s) \in \Pi\big) = 1,$$

since the measures Q and $\{Q^s\}_{s \in \mathbb{I}}$ are all equivalent and $\{X_{(t,s)}\}_{(t,s) \in \Pi}$ has continuous sample paths. □

Acknowledgments. The paper has been greatly improved by discussions with Darrell Duffie and Peter Ove Christensen. I am also grateful for the comments and suggestions received on presentations at Aarhus Business School, the EFA 1991 Conference in Rotterdam, Aarhus University, the E.I.A.S.M. Doctoral Tutorial in Finance in Lisbon and INSEAD from Simon Babbs, Simon Benninga, Svend Jacobsen, Peter Løchte Jørgensen, Bjørn Nybo Jørgensen, Jesper Helmuth Larsen, Lars Tyge Nielsen, Jesús Saá-Requejo and Carsten Sørensen, from the EFA discussant Günter Lassak, Bjarne Astrup Jensen, and from anonymous referees. I am especially indebted to the referee who pointed out several extensions of this paper, including the extension to a state dependent volatility of the SDE description. An earlier version of this paper was co-authored by Lars Ingemann Nielsen. However, most important of all was the help from Svend Erik Graversen, Department of Mathematics, Aarhus University, without whose help the paper would never have been written.

REFERENCES

Cox, J. C., Ingersoll Jr., J. E. and Ross, S. A. (1981). A re-examination of traditional hypotheses about the term structure of interest rates. *J. Finance* **36** 769–799.

Harrison, M. J. and Kreps, D. M. (1979). Martingales and arbitrage in multiperiod securities markets. *J. Economic Theory* **20** 381–408.

Harrison, M. J. and Pliska, S. R. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process. Appl.* **11** 215–260. [Addendum in Harrison and Pliska (1983)].

Harrison, M. J. and Pliska, S. R. (1983). A stochastic calculus model of continuous trading: Complete markets. *Stochastic Process. Appl.* **15** 313–316.

Heath, D., Jarrow, R. and Morton, A. J. (1992). Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica* **60** 77–105.

Ikeda, N. and Watanabe, S. (1989). *Stochastic Differential Equations and Diffusion Processes*, 2nd ed. North-Holland, Amsterdam.

Jamshidian, F. (1987). Pricing of contingent claims in the one-factor term structure model. Working Paper, Financial Strategies Group, Merrill Lynch Capital Markets, World Financial Center, North Tower, New York, NY 10281-1315.

Karatzas, I. and Shreve, S. E. (1988). *Brownian Motion and Stochastic Calculus*. Springer, New York.

Miltersen, K. R. (1992). A model of the term structure of interest rates. Ph.D. dissertation, Dept. Management, Odense Univ.

Morton, A. J. (1988). Arbitrage and martingales. Technical Report 821, School of Operations Research and Industrial Engineering, College of Engineering, Cornell Univ.

Department of Management
Odense University
Campusvej 55
DK-5230 Odense M
Denmark