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ON WEAK CONVERGENCE OF CONDITIONAL SURVIVAL MEASURE OF ONE-DIMENSIONAL BROWNIAN MOTION WITH A DRIFT

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We consider a one-dimensional Brownian motion with a constant drift, moving among Poissonian obstacles. In the case where the drift is below some critical value we characterize the limiting distribution of the process under the conditional probability measure that the particle has survived up to time t. Unlike the situation where the drift equals zero, we show in particular that in the presence of a constant drift, the process in natural scale feels the "boundary."

1. Introduction. The "kinetic description" of a diffusing particle in a medium that contains randomly distributed static traps is known in the physical literature as the trapping problem. It serves, for instance, as a model to describe the so-called diffusion controlled chemical reactions. In particular, a question of interest is the large t behaviour of the number of particles not yet trapped until time t, as well as the mean squared displacement of an untrapped particle, (see, for instance, [5], [8] and, for a review, [2]). The situation, when the particle "feels" some external force, that is, when the particle has a drift, was also studied in the physical literature. For instance, a one-dimensional system was analyzed by Movaghar, Pohlman and Würtz in [9], where they found the existence of a threshold for the external force, above which the decay rate of the density of untrapped particles undergoes a transition. A discrete analogue was studied by Kang and Redner [7]. Simulation in higher dimension were performed by Grassberger and Procaccia [6]. On the mathematical side, Eisele and Lang (see [3]), and Sznitman (see [14]) investigated the survival probability in arbitrary dimension. For particles in the absence of drift, the study of the limiting distribution of the process under the conditional probability measure that the particle has survived up to time t has been carried out in dimension $d = 1, 2$ (see [13], respectively, [15]).

The goal of this article is to investigate the one-dimensional situation when the particle feels a constant drift h. We let $\mathbb{P}$ denote the law of a Poisson point process with constant intensity ν and $(X_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$ with constant drift h, starting in 0. Let $C_t(X_\bullet)$ be the image of the path $X_\bullet$ up to time t and W_0^h the law of $(X_t)_{t\geq 0}$. We say that the Brownian motion gets killed when it hits a Poisson point. Now denote by T the hitting time of these

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random points. Then

$$S^h(t) = W_0^h \otimes \mathbb{P}[T > t] \tag{1}$$

is the probability that the Brownian motion survives up to time t. Because $\{T > t\} = \{N(C_t) = 0\}$ we see that

$$S^h(t) = E_0^h[\mathbb{P}[N(C_t) = 0]] = E_0^h[\exp\{-\nu|C_t(X_\bullet)|\}]. \tag{2}$$

This shows that the probability measure

$$\frac{1}{S^h(t)} \exp\{-\nu|C_t(X_\bullet)|\}\, dW_0^h, \tag{3}$$

where S_t^h is the normalization, is the conditional measure of Brownian motion with drift h starting in 0 given it survives up to time t, the "survival measure." An interesting fact is that the model exhibits a transition between the small and large drift regime. Indeed, from the Girsanov theorem we may rewrite (1) as

$$S^h(t) = e^{-h^2 t/2}\mathbb{E} \otimes E_0[e^{hX_t}, T > t], \tag{4}$$

where E_0 denotes expectation with respect to Wiener measure starting from 0, and if we introduce $\tilde{S}^h(t) := S^h(t)e^{h^2 t/2}$, then one has

$$\lim_{t\to\infty} t^{-1} \log \tilde{S}^h(t) = \tfrac{1}{2}(|h| - \nu)^2, \qquad |h| > \nu \tag{5}$$

(see [3], Theorem 2), respectively,

$$\lim_{t\to\infty} t^{-1/3} \log \tilde{S}^h(t) = -\tfrac{3}{2}(\pi(\nu - |h|))^{2/3}, \qquad 0 \le |h| < \nu \tag{6}$$

(see [14], Theorem 4.1)

The last two formulae reflect a transition between "localized" behaviour of the surviving particle when the drift is below some critical value, and "delocalized" behaviour when the drift is bigger than this critical value. In the following text we will only deal with the localized behaviour. Our main aim is to show:

THEOREM A. *For $|h| \in (0, \nu)$ the following hold:*

(i) *The limiting distribution of $(1/t^{1/3})X_{\bullet t^{2/3}}$ under (3) as t goes to infinity is given as the taboo measure starting from 0 with taboo interval $(0, c_0)$, where $c_0 = (\pi^2/(\nu - |h|))^{1/3}$ (and the taboo measure is defined below).*

(ii) *Let $a \in (0, \infty)$. The limiting distribution of $X_\bullet$ under (3) as t goes to infinity is given as a mixture of Bessel-3 processes under which $(X_t)_{t\ge 0}$ starts in 0 and never hits $-a$. The density of the mixture is given by $h^2 a e^{-|h|a}$.*

This result should be contrasted to the case where $h = 0$: the limiting distribution of $X_\bullet$ under the conditional survival measure up to time t is just standard Wiener measure. This means that the process $X_\bullet$ has not enough time to feel the "boundary" of the trap-free region whence it starts. In other words, when $h = 0$ the scaled process $(1/t^{1/3})X_{\cdot t^{2/3}}$ feels the boundary of the trap-free region, but not the unscaled process $X_\cdot$. Theorem A shows that in the presence of a constant drift $h \in (0, \nu)$ the particle starts near the left end of the interval and feels the boundary even in natural true scale.

2. The result. We define $\Omega := C([0,\infty),\mathbb{R})$ and $(X_t)_{t\geq 0}$ the coordinate process on Ω, equipped with the usual metric, which induces the uniform convergence on bounded intervals and which makes Ω a complete separable metric space. W_a is the Wiener measure on Ω, starting from $a \in \mathbb{R}$, and we denote by E_a the expectation with respect to W_a. We let $\mathscr{F}_t := \sigma(X_s;\ 0 \leq s \leq t)$ and $\mathscr{F}_\infty := \sigma(\bigcup_{t\geq 0}\mathscr{F}_t)$. For $h > 0$ we denote by W_0^h the measure on $(\Omega, \mathscr{F}_\infty)$ under which the process $(X_t)_{t\geq 0}$ is a Brownian motion with drift h, starting in 0, and by E_0^h the expectation with respect to this measure. For $t \in [0,\infty)$ we let $C_t(X_\cdot) := \{X_s;\ 0 \leq s \leq t\}$ be the image of the path $X_\cdot$ up to time t, and $|C_t(X_\cdot)|$ its length. Let $f\colon \Omega \longrightarrow \mathbb{R}$ be bounded and measurable. We want to study the limiting distribution of $(1/t^{1/3})X_{\cdot t^{2/3}}$ (respectively, $X_\cdot$) under the probability measure

$$dQ_t^h = \frac{1}{S^h(t)} \exp\{-\nu|C_t(X_\cdot)|\}\, dW_0^h, \tag{7}$$

where $S^h(t)$ is the normalization. If we put for $s \geq 0$,

$$A_s^f := E_0[f(X_\cdot)\exp(hsX_s)\exp\{-\nu s|C_s(X_\cdot)|\}] \tag{8}$$

and define the functions $f_1(X_\cdot) := f(X_\cdot)$ and $f_2(X_\cdot) := f(sX_\cdot/s^2)$, then using the scaling property of Brownian motion and the Girsanov transformation, we see that the study of the limiting distribution of $(1/t^{1/3})X_{\cdot t^{2/3}}$ (respectively, $X_\cdot$) under Q_t^h is the same as the study of convergence of the following expectations:

$$E_{P_s}[f_1] := \frac{A_s^{f_1}}{A_s^1} \quad \left(\text{respectively, } E_{Q_s}[f_2] := \frac{A_s^{f_2}}{A_s^1}\right), \tag{9}$$

where $s := t^{1/3}$. Before we state Theorem 1, we want to introduce some notation and recall some definitions.

For real numbers a, c with $0 < a < c$, we denote by $P_a^{(0,c)}$ the probability measure on $(\Omega, \mathscr{F}_\infty)$, under which the coordinate process $(X_t)_{t\geq 0}$ is a taboo process starting in a with taboo interval $(0, c)$, that is, for $B \in \mathscr{F}_t$,

$$P_a^{(0,c)}[B] = \frac{e^{\mu t}}{\phi(a)} E_a[1_B 1_{\{\mathscr{T}_{(0,c)} > t\}}\phi(X_t)], \tag{10}$$

where $a \in (0, c)$, $\mathscr{T}_{(0,c)} := \inf\{t \geq 0; X_t \notin (0,c)\}$ is the exit time from $(0, c)$, $\mu := \pi^2/2c^2$ and

$$\phi(x) = \begin{cases} \sqrt{\dfrac{2}{c}}\sin\left(\dfrac{\pi x}{c}\right), & x \in (0,c), \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

Let us comment on the definition of the taboo measure. We will restrict ourselves to the one-dimensional case although an analogue result holds for arbitrary dimensions.

Let I be an interval and $a \in I$. Then P_a^I is constructed as the weak limit of the conditional probability measure $W_a[\cdot|\mathscr{T}_I > t]$ as t tends to infinity. This means you condition your Brownian particle, starting from a, not to leave the interval I until time t, where t gets large. Using an eigenfunction expansion and Lemma 1 gives you the explicit formula (10). In particular we have the following proposition.

PROPOSITION 1. *Let $a \in (0,c)$. Then $\{P_a^{(0,c)};\ a \in (0,c)\}$ is tight. Furthermore there exists a probability measure $P_0^{(0,c)}$ on $(\Omega, \mathscr{F}_\infty)$, such that $P_a^{(0,c)}$ converges weakly to $P_0^{(0,c)}$ as $a \downarrow 0$.*

Although this is quite a classical result, we were not able to find it in the literature, and we provide a proof in the Appendix for the reader's convenience. Let us introduce some more notation.

For $a > 0$ we define $\tilde{E}_a$, the expectation with respect to the measure on $(\Omega, \mathscr{F}_\infty)$ under which the coordinate process $(X_t)_{t\geq 0}$ is a Bessel-3 process starting in a. In particular, we have for $B \in \mathscr{F}_t$,

$$\tilde{E}_a[1_B] = \frac{1}{a} E_a[1_B X_{t\wedge \mathscr{T}_0}], \tag{12}$$

where $\mathscr{T}_0 := \inf\{t > 0;\ X_t = 0\}$ and $t \wedge s = \min\{t, s\}$ (see [12], page 419). Observe that A_s^f in (8) depends on h only through the absolute value $|h|$ due to symmetry of Brownian motion. Thus it is clear that we can restrict ourselves to the case where $h \in (0, \nu)$. In what follows, we will work with t instead of $s = t^{1/3}$, to simplify the notations. We are now ready to state Theorem 1.

THEOREM 1. *For $\nu > h > 0$, the following hold:*

(i) *P_t converges weakly to $P_0^{(0,c_0)}$ as $t \to \infty$, where $c_0 := (\pi^2/(\nu - h))^{1/3}$, and $P_0^{(0,c_0)}$ is the probability measure from Proposition* 1.

(ii) *Q_t converges weakly to a probability measure Q on $(\Omega, \mathscr{F}_\infty)$, as $t \to \infty$, where*

$$Q[B] = h^2 \int_0^\infty a e^{-ha} \tilde{E}_a[1_B(X_\bullet - a)]\, da, \tag{13}$$

with $B \in \mathscr{F}_\infty$ and the notations from (12).

In particular this result shows that when we are in the "localized regime," that is, when the drift is sufficiently small, even the process $X_\bullet$ feels the boundary of the trap-free region whence the particle starts. This is in contrast to the case without drift, where the motion of the surviving process in natuaral scale is at long times essentially that of a Brownian particle in the absence of obstacles.

Before starting the proof of Theorem 1 we need some preparation. The first lemma is a reduction step, for it says that it is enough to investigate the quantities in (9) for paths up to a fixed time $u \geq 0$. More precisely, for $u \geq 0$ let Π_u: $\Omega \ni \omega \mapsto \omega|_{[0,u]}$ be the restriction on $[0,u]$. We then have for probability measures L_t on $(\Omega, \mathscr{F}_\infty)$ (see, for instance, Proposition 3.2.4 in [4]) the following lemma.

LEMMA 1. *$\{L_t;\ 0 \leq t < \infty\}$ is tight $\Leftrightarrow$ $\{L_t \circ \Pi_u^{-1}; 0 \leq t < \infty\}$ is tight, $\forall u \in \mathbb{N}$.*

If we denote by $\Omega_u := C([0,u],\mathbb{R})$, then Lemma 1 tells us that it is enough to show for $f \in C_b(\Omega_u)$ and $u \geq 0$ that $E_{Q_t}[f \circ \Pi_u]$ and $E_{P_t}[f \circ \Pi_u]$ converge to $E_Q[f \circ \Pi_u]$ and $E_{P_0^{(0,c_0)}}[f \circ \Pi_u]$, respectively, $(f \circ \Pi_u \in C_b(\Omega))$.

We also want to reexpress (8). Let f: $\Omega \to \mathbb{R}$ be bounded, measurable and $\nu > h > 0$. Denote by $M_t := \max_{0\leq s\leq t} X_s$ and by $m_t := \min_{0\leq s\leq t} X_s$. Now $|C_t(X_\bullet)| = M_t - m_t$, which gives us

$$\exp\{-\nu t|C_t(X_\bullet)|\} = \nu^2 t^2 \int_{-\infty}^{m_t} \int_{M_t}^{\infty} \exp(-\nu t(b-a))\, db\, da.$$

However, then, because $C_t(X_\bullet) \subset (a,b) \Leftrightarrow \mathscr{T}_{(a,b)} > t$, and $X_0 = 0$ a.s., we get a.s.,

$$\int_{-\infty}^{m_t} \int_{M_t}^{\infty} \exp(-\nu t(b-a))\, db\, da = \int_{-\infty}^{0} \int_{0}^{\infty} \exp(-\nu t(b-a)) 1_{\{\mathscr{T}_{(a,b)}>t\}}\, db\, da.$$

Putting $f^a := f(X_\bullet - a)$ we finally get

$$\begin{aligned}
A_t^f &= \nu^2 t^2 E_0\left[f(X_\bullet)\exp(htX_t)\int_0^\infty\int_0^\infty \exp(-\nu t(a+b)) 1_{\{\mathscr{T}_{(-a,b)}>t\}}\, da\, db\right] \\
&= \nu^2 t^2 \int_0^\infty dc \exp(-\nu tc) \int_0^c da E_0[f(X_\bullet)\exp(htX_t)1_{\{\mathscr{T}_{(-a,c-a)}>t\}}] \\
&= \nu^2 t^2 \int_0^\infty dc \exp\left(-t\left\{c(\nu-h) + \frac{\pi^2}{2c^2}\right\}\right) \\
&\quad \times \int_0^c da \exp\left(t\frac{\pi^2}{2c^2}\right) E_0[f(X_\bullet)\exp(ht(X_t-c))1_{\{\mathscr{T}_{(-a,c-a)}>t\}}] \qquad (14) \\
&= \nu^2 t^2 \int_0^\infty dc \exp(-tg(c)) \\
&\quad \times \int_0^c da \exp(t\mu_1(c))\exp(-hta)E_a[f^a \exp(ht(X_t-c))1_{\{\mathscr{T}_{(0,c)}>t\}}],
\end{aligned}$$

where $g(c) := c(\nu - h) + \mu_1(c)$, with $c \in (0,\infty)$, and $\mu_1(c) := \pi^2/2c^2$ is the lowest eigenvalue of $-\frac{1}{2}(d^2/dx^2)$, with Dirichlet boundary conditions on $(0,c)$.

Before we give the strategy of the proof we want to make the following comments.

REMARKS.

1. In fact, the proof of Theorem 1 will provide an asymptotic equivalent of A_t^f as t tends to infinity, for a suitable class of functions f, take for instance $f > 0$, $\mathscr{F}_u$ measurable, $u \geq 0$. For such functions with $f_1 = f(X_\bullet)$, respectively, $f_2 = f(tX_{\bullet/t^2})$, one finds

$$A_t^{f_1} \sim \frac{2\pi^2}{t^2 c_0^3 h^3} \sqrt{\frac{2\pi}{t\ddot{g}(c_0)}} \exp(-tg(c_0)) E_0^{(0,c_0)}[f(X_\bullet)] \tag{15}$$

and

$$A_t^{f_2} \sim \frac{2\pi^2}{t^2 c_0^3 h^3} \sqrt{\frac{2\pi}{t\ddot{g}(c_0)}} \exp(-tg(c_0)) \times \int_0^\infty a \exp(-ha) \tilde{E}_a[f(X_\bullet - a)]\, da. \tag{16}$$

2. We define the semigroup

$$(R^t v)(x) := E_x[v(X_t) 1_{\{\mathscr{T}_{(0,c)} > t\}}] \tag{17}$$

for bounded measurable v, $t \geq 0$, $c > 0$. If we choose a complete orthonormal system $(\phi_j)_{j\geq 1}$ of $L^2(0,c)$, where the ϕ_j are eigenfunctions of $-\frac{1}{2}(d^2/dx^2)$ with Dirichlet boundary conditions on $\partial(0,c)$, the corresponding eigenvalues $(\mu_j)_{j\geq 1}$ form a strictly increasing sequence of positive real numbers. If we denote by $c_v = \langle v, \phi_1 \rangle_{L^2(0,c)}$, and $\|\cdot\|$ the $L^2(0,c)$ norm, then the self-adjointness of R^t (see, for instance, Chapter 2, Theorem 4.3 in [11]) and Parseval's identity give us

$$\|R^t(v - c_v\phi_1)\| \leq e^{-\mu_2 t}\|v\|. \tag{18}$$

Indeed, if we observe that with Itô's formula $(R^t\phi_j)(x) = e^{-\mu_j t}\phi_j(x)$, we get by symmetry

$$\begin{aligned}
\|R^t(v - c_v\phi_1)\| &= \left(\sum_{j\geq 1} |\langle R^t(v - c_v\phi_1), \phi_j\rangle|^2\right)^{1/2} \\
&= \left(\sum_{j\geq 2} |\exp(-\mu_j t)\langle v, \phi_j\rangle|^2\right)^{1/2} \\
&\leq \exp(-\mu_2 t)\|v\|.
\end{aligned}$$

Here then briefly is the strategy for the proof of Theorem 1. The first step will now be to prove that for the leading asymptotic, the main contribution in (14) comes from values of c near the global minimum

$$c_0 := \left(\frac{\pi^2}{\nu - h}\right)^{1/3} \tag{19}$$

of the function $g(c)$: we are going to split the integral over c in one over $[c_0 - \delta, c_0 + \delta]$, $\delta > 0$, and in one over its complement, and we are going to show that the leading term comes from the integral over $[c_0 - \delta, c_0 + \delta]$. After having seen that for the leading asymptotic in (14) we can restrict ourselves to the case where c belongs to some compact interval that contains c_0, we have to understand the asymptotics of

$$E_a[f(X_\bullet - a)\exp(ht(X_t - c))1_{\{\mathscr{T}_{(0,c)} > t\}}]. \tag{20}$$

In order to get in (20) a term where the semigroup from (17) appears and where we could use (18), we have to apply the strong Markov property. Now Lemma 1 tells us that we can assume that f is $\mathscr{F}_u$ measurable. This enables us to use the Markov property in a profitable way. It is actually at this point where the scaling of the process becomes relevant, because f_1 will be $\mathscr{F}_u$ measurable and f_2 will be $\mathscr{F}_{u/t^2}$ measurable. Applying Laplace's method we can then find the leading asymptotic behaviour of $A_t^{f_1}$ and $A_t^{f_2}$, as t tends to infinity, for a suitable class of functions.

The following lemma is a first reduction step in the study of the asymptotics of A_t^f as t tends to infinity. It says that we can restrict ourselves to the case where c belongs to some compact interval.

LEMMA 2. *Let $f\colon \Omega \longrightarrow \mathbb{R}$ be bounded measurable, $\nu > h > 0$ and*

$$c_0 := \left(\frac{\pi^2}{\nu - h}\right)^{1/3} = \arg\min_{0<c<\infty} g(c), \tag{21}$$

where $g(c)$ was defined in (14). *Let $\delta > 0$ be such that $c_0 - \delta > 0$, and define $A := \mathbb{R}^+ \setminus [c_0 - \delta, c_0 + \delta]$. We then have*

$$\begin{aligned} &\lim_{t\to\infty} \exp(tg(c_0))\nu^2 t^2 \int_A dc \exp(-tg(c)) \int_0^c da \exp(-t(ha - \mu_1(c))) \\ &\qquad \times E_a[f^a \exp(ht(X_t - c))1_{\{\mathscr{T}_{(0,c)} > t\}}] = 0. \end{aligned} \tag{22}$$

PROOF. Let $\nu > h > 0$ and choose some real number $\delta > 0$ such that $c_0 - \delta > 0$. Observe that $g(c)$ is a strictly convex function. Thus we can find $l(c) = \alpha c + \beta$, where $\alpha > 0$, $\beta \in \mathbb{R}$, $l(c) \le g(c)$, for all $c \in (0, \infty)$, and $g(c_0 + \delta) = l(c_0 + \delta)$. Denote by $\tilde{M}_\delta := \max\{e^{-tg(c_0-\delta)}, e^{-tg(c_0+\delta)}\} > 0$, and $\tilde{m}_\delta := \tilde{M}_\delta^{-1}$. For some constant $\kappa \in (0, \infty)$, independent of $t \ge 1$, we get the following estimates:

$$
\begin{aligned}
&\left|\nu^2 t^2 \int_A dc \exp(-tg(c)) \int_0^c da \exp(-t(ha-\mu_1(c)))\right.\\
&\qquad\qquad \left.\times E_a[f^a \exp(ht(X_t-c))1_{\{\mathcal{T}_{(0,c)}>t\}}]\right|\\
&\le \nu^2 t^2 \|f\|_\infty \int_A dc \exp(-t(g(c)-\mu_1(c))) \int_0^c da E_a[1_{\{\mathcal{T}_{(0,c)}>t\}}]\\
&\le \nu^2 t^2 \|f\|_\infty \int_A dc \exp(-t(g(c)-\mu_1(c)))\sqrt{c}\|R^t 1\|\\
&\le \nu^2 t^2 \|f\|_\infty \tilde{M}_\delta \int_A dc \exp(-tg(c))\tilde{m}_\delta c\\
&\le \nu^2 t^2 \|f\|_\infty \tilde{M}_\delta \left\{\int_0^{c_0-\delta} dc \exp(-t(g(c)-g(c_0-\delta)))c\right.\\
&\qquad\qquad \left. + \int_{c_0+\delta}^{\infty} dc \exp(-t\alpha(c-c_0-\delta))c\right\}\\
&\le \nu^2 t^2 \kappa \|f\|_\infty \tilde{M}_\delta,
\end{aligned}
\tag{23}
$$

where we used that $\|R^t 1\| \le \sqrt{c}e^{-\mu_1 t}$. However, $e^{tg(c_0)}\tilde{M}_\delta$ goes to zero exponentially fast as $t \longrightarrow \infty$. This completes the proof of Lemma 2. □

Thus Lemma 2 tells us that we are left with the study of the asymptotic behaviour of

$$
\begin{aligned}
&\nu^2 t^2 \int_{c_0-\delta}^{c_0+\delta} dc \exp(-tg(c)) \int_0^c da \exp(-t(ha-\mu_1(c)))\\
&\quad \times E_a[f^a \exp(ht(X_t-c))1_{\{\mathcal{T}_{(0,c)}>t\}}].
\end{aligned}
\tag{24}
$$

We now want to get an expression in which the semigroup R^t from (17) appears. We have to use the Markov property, but conditioning on two different σ-algebras corresponding to the different scaling of the process. As we mentioned already, it is enough to look at functions $f_u := f \circ \Pi_u$, where $u \ge 0$ and $f \in C_b(\Omega_u)$.

Let us fix $u \ge 0$ and denote by

$$
\psi(x) = \exp(ht(x-c)) \quad \text{and} \quad c_f = \int_0^c f(y)\phi(y)\,dy,
\tag{25}
$$

where ϕ is defined in (11). We have to treat the following two cases [cf. (9)]:

(i) $\quad f_{1,u}^a(X_\bullet) = f_u(X_\bullet - a)$ which is $\mathcal{F}_u$-measurable;

(ii) $\quad f_{2,u}^a(X_\bullet) = f_u(tX_{\bullet/t^2} - at)$ which is $\mathcal{F}_{u/t^2}$-measurable.

For the second case, calling $Y_\cdot := tX_{\bullet/t^2}$, and remembering that $(R^t\phi)(x) = e^{-\mu_1(c)t}\phi(x)$, we get for all $t \in \mathbb{R}^+$ with $t^3 > u$, using the strong Markov property,

$$
\begin{aligned}
(26)\quad & E_a[f^a_{2,u}(X_\bullet)\exp(ht(X_t-c))1_{\{\mathscr{T}_{(0,c)}>t\}}] \\
&= E_0[f_u(Y.)\exp(ht(X_t+a-c))1_{\{\mathscr{T}_{(-a,c-a)}>t\}}] \\
&= E_0[f_u(Y_\bullet)1_{\{\mathscr{T}^{Y.}_{(-at,ct-at)}>u\}} \\
&\qquad \times E_{(1/t)Y_u}[\exp(ht(X_{(t-u/t^2)}+a-c))1_{\{\mathscr{T}^{X.}_{(-a,c-a)}>t-u/t^2\}}]] \\
&= E_0[f_u(X_\bullet)1_{\{\mathscr{T}^{X_\bullet}_{(-at,ct-at)}>u\}} \\
&\qquad \times E_{(1/t)X_u}[\exp(ht(X_{(t-u/t^2)}+a-c))1_{\{\mathscr{T}^{X_\bullet}_{(-a,c-a)}>t-u/t^2\}}]] \\
&= E_{at}[f_u(X_\bullet-at)1_{\{\mathscr{T}_{(0,ct)}>u\}} \\
&\qquad \times E_{(1/t)X_u}[\exp(ht(X_{(t-u/t^2)}-c))1_{\{\mathscr{T}_{(0,c)}>t-u/t^2\}}]] \\
&= E_{at}[f_u(X_\bullet-at)1_{\{\mathscr{T}_{(0,tc)}>u\}}\phi(X_u/t)\exp(-\mu_1(c)(t-u/t^2))c_\psi] \\
&\quad + E_{at}[f_u(X_\bullet-at)1_{\{\mathscr{T}_{(0,tc)}>u\}}\{R^{t-u/t^2}(\psi-c_\psi\phi)(X_u/t)\}].
\end{aligned}
$$

For the first case we get, by the same calculations,

$$
\begin{aligned}
(27)\quad & E_a[f^a_{1,u}\exp(ht(X_t-c))1_{\{\mathscr{T}_{(0,c)}>t\}}] \\
&= E_a[f_u(X_\bullet-a)1_{\{\mathscr{T}_{(0,c)}>u\}}\phi(X_u)\exp(-\mu_1(c)(t-u))c_\psi] \\
&\quad + E_a[f_u(X_\bullet-a)1_{\{\mathscr{T}_{(0,c)}>u\}}\{R^{t-u}(\psi-c_\psi\phi)(X_u)\}].
\end{aligned}
$$

Our next step to show is that when we consider (24) and use the identities we obtained in (26) and (27), the last terms (involving the semigroup) multiplied with $e^{tg(c_0)}$ are going to zero exponentially fast, as $t \longrightarrow \infty$. To this end we will show the next lemma.

LEMMA 3. *Define the function*

$$
\begin{aligned}
(28)\quad F(\hat{t},t) := \nu^2 t^2 \int_{c_0-\delta}^{c_0+\delta} dc\exp(-tg(c)) \int_0^c da\exp(-t(ha-\mu_1)) \\
\times E_a[R^{\hat{t}}(v-c_v\phi)(X_u)],
\end{aligned}
$$

where $v \in L^2(0,c)$ *and* $\hat{t}$ *is some positive real number. Let* $u \geq 0$ *be fixed. We then have:*

(i) $$\lim_{t\to\infty}\exp(tg(c_0))F(t-u,t)=0,$$

(ii) $$\lim_{t\to\infty}\exp(tg(c_0))F\left(t-\frac{u}{t^2},t\right)=0.$$

PROOF. Let us denote by $p(t,a,y) = (2\pi t)^{-1/2}\exp\{-(a-y)^2/2t\}$. For some arbitrary $v \in L_2(0,c)$ we have the estimates

$$\begin{aligned}\left|\int_0^c da E_a[(R^t v)(X_u)]\right| &\le \int_0^c da\, E_a[|(R^t v)(X_u)|]\\ &= \int_0^c da \int_{-\infty}^{\infty} dy\, p(u,a,y)|(R^t v)(y)|\\ &= \int_0^c da \int_0^c dy\, p(u,a,y)|(R^t v)(y)|\\ &\le c^{3/2}\|R^t v\|.\end{aligned}$$

However, then with $\mu_2 := 2\pi^2/c^2$, for all positive real numbers $\hat{t}$ and a constant $\kappa := (c_0+\delta)^{3/2}$, we find

$$\begin{aligned}&\left|\nu^2 t^2 \int_{c_0-\delta}^{c_0+\delta} dc\, \exp(-tg(c)) \int_0^c da \exp(-t(ha-\mu_1))E_a[R^{\hat{t}}(v-c_v\phi)(X_u)]\right|\\ &\quad\le \nu^2 t^2 \int_{c_0-\delta}^{c_0+\delta} dc \exp(-t(g(c)-\mu_1))c^{3/2}\exp(-\mu_2\hat{t})\|v\| \qquad (29)\\ &\quad\le \kappa\nu^2 t^2\|v\| \int_{c_0-\delta}^{c_0+\delta} dc \exp(-t\{c(\nu-h)+\mu_2\})\exp(-\mu_2(\hat{t}-t)),\end{aligned}$$

where we used (18).

With $c_1 := (4\pi^2/(\nu-h))^{1/3}$ and $k(c) := c(\nu-h)+\mu_2$, we find for $c \in [c_0-\delta, c_0+\delta]$:

$$k(c) \ge k(c_1) = \frac{3}{2}c_1(\nu-h) > \frac{3}{2}c_0(\nu-h) = g(c_0) \qquad (30)$$

Observe that $\forall t \ge 0$ we have $\|\psi\| \le \sqrt{c}$. Now for the first case we have $\hat{t} = t-u$, as for the second case $\hat{t} = t-u/t^2$. But since $e^{\mu_2 u}$ and $e^{\mu_2(u/t^2)}$ are both bounded for $c \in [c_0-\delta, c_0+\delta]$ and t large enough, estimates (29) and (30) finishes the proof of Lemma 3. □

We are now ready to begin the proof of Theorem 1.

PROOF OF THEOREM 1. We want to apply Laplace's method to the expression of A_t^f that we obtained in (14), where f belongs to a suitable class of functions. So let $u \ge 0$, $f \in C_b(\Omega_u)$ and $f_u := f \circ \Pi_u$, where we use the same notation as in Lemma 1. To show part (i) of the theorem we have to study the case [cf. (9), resp. (14)] with $f_{1,u}^a(X_\bullet) = f_u(X_\bullet - a)$ as for part (ii) we have $f_{2,u}^a(X_\bullet) = f_u(tX_{\bullet/t^2} - at)$. Thus inserting (27), resp. (26), in (24) and using the estimates from Lemmas 2 and 3, we see that we are left with the expressions

$$\begin{aligned}&\nu^2 t^2 \int_{c_0-\delta}^{c_0+\delta} dc \exp(-tg(c)) \int_0^c da \exp(-hta)\exp(\mu_1(c)u)c_\psi \qquad (31)\\ &\qquad\times E_a[f_u(X_\bullet - a)1_{\{\mathcal{T}_{(0,c)}>u\}}\phi(X_u)],\end{aligned}$$

respectively, for the second case,

$$(32)\qquad \nu^2 t^2 \int_{c_0-\delta}^{c_0+\delta} dc\,\exp(-tg(c)) \int_0^c da\,\exp(-hta)\exp(\mu_1(c)u/t^2)c_\psi \times E_{at}[f_u(X_{\bullet} - at)1_{\{\mathscr{T}_{(0,ct)}>u\}}\phi(X_u/t)],$$

where c_ψ was defined in (25).

We first show part (i), so we begin with the expression (31): Observe that

$$(33)\qquad \begin{aligned} c_\psi &= \int_0^c \exp(ht(y-c))\phi(y)\,dy = \frac{1}{t}\int_0^{tc}\exp(-h(y-tc))\phi\left(\frac{y}{t}\right)dy \\ &= \frac{1}{t}\int_0^{tc}\exp(-hb)\phi\left(\frac{b}{t}\right)db, \end{aligned}$$

where we used symmetry of ϕ with respect to $c/2$. However, then (31) multiplied with t^2 becomes

$$(34)\qquad \begin{aligned} &\nu^2 t^3 \int_{c_0-\delta}^{c_0+\delta} dc\,\exp(-tg(c)) \int_0^{tc} db\,\exp(-hb)\phi(b/t) \\ &\qquad \times \int_0^c da\,\exp(-hta)\exp(\mu_1(c)u)E_a[f_u(X_{\bullet}-a)1_{\{\mathscr{T}_{(0,c)}>u\}}\phi(X_u)] \\ &\quad = \nu^2 \int_{c_0-\delta}^{c_0+\delta} dc\,\exp(-tg(c))\Phi(t,c), \end{aligned}$$

where

$$(35)\qquad \begin{aligned} \Phi(t,c) = &\int_0^{tc} db\,\exp(-hb)t\phi(b/t) \\ &\times \int_0^{tc} da\,\exp(-ha)t\phi(a/t)E_{a/t}^{(0,c)}[f_u(X_{\bullet}-a/t)] \end{aligned}$$

and $E_a^{(0,c)}$ denotes expectation with respect to the taboo measure $P_a^{(0,c)}$ from (10). Let us denote by $I := [c_0-\delta, c_0+\delta]$. Because I is compact we get

$$(36)\qquad \lim_{t\to\infty}\frac{t}{b}\phi(b/t) = \sqrt{\frac{2}{c}}\frac{\pi}{c}$$

uniformly in $c \in I$ and bounded, positive b.

We now want to show that

$$(37)\qquad \lim_{t\to\infty} E_{a/t}^{(0,c)}[f_u(X_{\bullet}-a/t)] = E_0^{(0,c)}[f_u(X_{\bullet})]$$

uniformly in $c \in I$, and bounded $a \in (0,\kappa)$, where κ is some constant. So let us fix $c^* \in I$ and let $y \in (0,c)$ and denote for the moment the function $\phi(x)$ from (11) as $\phi^{(0,c)}(x)$. Observe that $\phi^{(0,c^*)}(x) = \sqrt{c/c^*}\phi^{(0,c)}(cx/c^*)$ and $\mu(c^*) = \mu(c)(c/c^*)^2$. We then get, from the scaling property of Brownian motion,

$$(38)\qquad E_{y(c^*/c)}^{(0,c^*)}\left[f_u\left(\frac{c}{c^*}X_{\bullet/(c/c^*)^2} - y\right)\right] = E_y^{(0,c)}[f_u(X_{\bullet}-y)].$$

We claim that

$$\lim_{y\downarrow 0} E_y^{(0,c^*)}\left[f_u\left(\frac{c}{c^*}X_{\bullet/(c/c^*)^2}-\frac{c}{c^*}y\right)\right]=E_0^{(0,c^*)}\left[f_u\left(\frac{c}{c^*}X_{\bullet/(c/c^*)^2}\right)\right] \tag{39}$$

uniformly in $c\in I$. From this, (38) and the fact that for all a: $a/t\le\kappa/t$, (37) now follows. So let us show (39).

Let $\varepsilon\in(0,1)$. Thanks to Proposition 1 we can find $K\subset\subset\Omega$ such that for all $y\in(0,c^*)$,

$$E_y^{(0,c^*)}[1_K]>1-\varepsilon.$$

However, then

$$\begin{aligned}&\left|E_y^{(0,c^*)}\left[f_u\left(\frac{c}{c^*}X_{\bullet/(c/c^*)^2}-\frac{c}{c^*}y\right)\right]-E_0^{(0,c^*)}\left[f_u(\frac{c}{c^*}X_{\bullet/(c/c^*)^2})\right]\right|\\ &\quad\le\varepsilon\|f\|_\infty+\left|E_y^{(0,c^*)}\left[f_u\left(\frac{c}{c^*}X_{\bullet/(c/c^*)^2}-\frac{c}{c^*}y\right)1_K\right]-E_0^{(0,c^*)}\left[f_u\left(\frac{c}{c^*}X_{\bullet/(c/c^*)^2}\right)\right]\right|\\ &\quad\le\varepsilon\|f\|_\infty+\left|E_y^{(0,c^*)}\left[f_u\left(\frac{c}{c^*}X_{\bullet/(c/c^*)^2}-\frac{c}{c^*}y\right)1_K-f_u\left(\frac{c}{c^*}X_{\bullet/(c/c^*)^2}\right)\right]\right|\\ &\qquad+\left|E_y^{(0,c^*)}\left[f_u\left(\frac{c}{c^*}X_{\bullet/(c/c^*)^2}\right)\right]-E_0^{(0,c^*)}\left[f_u\left(\frac{c}{c^*}X_{\bullet/(c/c^*)^2}\right)\right]\right|.\end{aligned}$$

The last term tends to zero as $y\downarrow 0$ uniformly in $c\in I$, because $\{f_u((c/c^*)X_{\bullet/(c/c^*)^2})\}_{c\in I}$ is equicontinuous at each $\omega\in\Omega$ because $I\times\Omega\ni(c,\omega)\mapsto(c/c^*)X_{\bullet/(c/c^*)^2}=(c/c^*)\omega(\bullet/(c/c^*)^2)\in\Omega$ is continuous and I is compact. Thus Theorem 6.8 in [10] applies and together with Proposition 1, we see that we get the limit uniform in $c\in I$. Now f_u is uniformly continuous on the image of $I\times K$ under $(c,\omega)\mapsto(c/c^*)\omega(\bullet/(c/c^*)^2)$, which gives us, for sufficiently small y,

$$\begin{aligned}&\left|E_y^{(0,c^*)}\left[f_u\left(\frac{c}{c^*}X_{\bullet/(c/c^*)^2}-\frac{c}{c^*}y\right)1_K-f_u\left(\frac{c}{c^*}X_{\bullet/(c/c^*)^2}\right)\right]\right|\\ &\quad\le\varepsilon\|f\|_\infty+E_y^{(0,c^*)}\left[\sup_{c\in I}\left|f_u\left(\frac{c}{c^*}X_{\bullet/(c/c^*)^2}-\frac{c}{c^*}y\right)-f_u\left(\frac{c}{c^*}X_{\bullet/(c/c^*)^2}\right)\right|1_K\right]\\ &\quad\le\varepsilon(\|f\|_\infty+1).\end{aligned}$$

This shows (39). Furthermore the preceding calculations show that

$$E_0^{(0,c)}[f_u(X_\bullet)]=E_0^{(0,c^*)}\left[f_u\left(\frac{c}{c^*}X_{\bullet/(c/c^*)^2}\right)\right]. \tag{40}$$

Finally Lebesgue's theorem gives us

$$\begin{aligned}\lim_{t\to\infty}\Phi(t,c)&=\frac{2\pi^2}{c^3}\int_0^\infty db\exp(-hb)b\int_0^\infty da\exp(-ha)E_0^{(0,c)}[f_u(X_\bullet)]\\ &=:\Phi(\infty,c),\end{aligned} \tag{41}$$

where the limit is uniform in $c \in I$. Furthermore, (40) shows, that $I \ni c \mapsto \Phi(\infty, c)$ is continuous in c, thanks to the continuity of f_u. Choosing $\delta > 0$ small enough we can make the expression $|\Phi(t,c) - \Phi(\infty, c_0)|$ arbitrarily small, for all t big enough, and $c \in I$. In the case where $\Phi(\infty, c_0) \neq 0$, for instance, when $f_u > 0$, the proof of Theorem 18 (page 39ff. in [1]) works as well for the time dependent case with $\Phi(t,c)$, respectively, $\Phi(\infty, c_0)$, and this gives us that the asymptotic behaviour of (34), as $t \longrightarrow \infty$, is

$$
\begin{aligned}
t^2 A_t^{f_{1,u}} &\sim \nu^2 \int_{c_0-\delta}^{c_0+\delta} dc \exp(-tg(c))\Phi(t,c) \\
&= \nu^2 \sqrt{\frac{2\pi}{t\ddot{g}(c_0)}} \exp(-tg(c_0))\Phi(\infty, c_0)(1+o(1)),
\end{aligned} \tag{42}
$$

where $f_{1,u}(X_\bullet) = f_u(X_\bullet)$. Remember we have to investigate $A_t^{f_{1,u}}/A_t^1$. Because it is enough to work with, $f_u > 0$, (41) and (42) together with the estimates from Lemma 2 [resp. Lemma 3(i)] completes the proof of part (i) of Theorem 1.

Let us now show part (ii), so we look at the expression we obtained in (32). We have, by the same calculations as before, that (32) multiplied with t^2 becomes

$$
\nu^2 \int_{c_0-\delta}^{c_0+\delta} dc \exp(-tg(c))\Psi(t,c), \tag{43}
$$

where

$$
\begin{aligned}
\Psi(t,c) = \exp(\mu_1 u/t^2) &\int_0^{tc} db \exp(-hb) t\phi(b/t) \\
\times &\int_0^{tc} da \exp(-ha) E_a[f_u(X_\bullet - a) 1_{\{\mathscr{T}_{(0,tc)} > u\}} t\phi(X_u/t)].
\end{aligned} \tag{44}
$$

Using Lebesgue's theorem we get

$$
\begin{aligned}
\lim_{t\to\infty} \Psi(t,c) &= \frac{2\pi^2}{c^3} \int_0^\infty db\, b \exp(-hb) \int_0^\infty da\, a \exp(-ha) \frac{1}{a} \\
&\quad \times E_a[f_u(X_\bullet - a) 1_{\{\mathscr{T}_{(0,\infty)} > u\}} X_u] \\
&=: \Psi(\infty, c)
\end{aligned} \tag{45}
$$

uniformly in $c \in I$ because for all $c \in I$ we have

$$
0 \le 1_{\{\mathscr{T}_{(0,\infty)} > u\}} - 1_{\{\mathscr{T}_{(0,tc)} > u\}} \le 1_{\{\mathscr{T}_{(0,\infty)} > u\}} - 1_{\{\mathscr{T}_{(0,t(c_0-\delta))} > u\}}
$$

and the last term goes to zero W_a-a.s. with $t \longrightarrow \infty$. Furthermore, we see that $\Psi(\infty, c)$ is continuous in $c \in I$. Now we have $1_{\{\mathscr{T}_{(0,\infty)} > u\}} X_u = X_{u \wedge \mathscr{T}_0}$ W_a-a.s., where $\mathscr{T}_0$ is the hitting time of 0, and we get for $a > 0$,

$$
\frac{1}{a} E_a[f_u(X_\bullet - a) 1_{\{\mathscr{T}_{(0,\infty)} > u\}} X_u] = \tilde{E}_a[f_u(X_\bullet - a)], \tag{46}
$$

where $\tilde{E}_a$ is the expectation from (12). As before we finally obtain the asymptotic behaviour of (43) as $t \longrightarrow \infty$ for functions f_u for which $\Psi(\infty, c_0) \neq 0$:

$$\begin{aligned} t^2 A_t^{f_{2,u}} &\sim \nu^2 \int_{c_0-\delta}^{c_0+\delta} dc \exp(-tg(c))\Psi(t,c) \\ &= \nu^2 \sqrt{\frac{2\pi}{t\ddot{g}(c_0)}} \exp(-tg(c_0))\Psi(\infty, c_0)(1+o(1)), \end{aligned} \tag{47}$$

where $f_{2,u} = f_u(tX_{\cdot/t^2})$. Because it is enough to work with $f_u > 0$, (45), (46) and (47), together with the Lemma 1 and the estimates from Lemma 2 [resp. Lemma 3(ii)] completes the proof of part (ii) of Theorem 1. □

APPENDIX

Here we give a proof of Proposition 1 for the reader's convenience. We are first going to show the tightness of $\{P_a^{(0,c)}; a \in (0,c)\}$. So let $\varepsilon > 0, T > 0$ be arbitrary. Define $A_{\varepsilon,\delta}^{[0,T]} := \{\sup_{0\le s,t\le T, |s-t|<\delta} |X_s - X_t| > \varepsilon\}$, where δ is some positive real number. It suffices to check that

$$\lim_{\delta\downarrow 0} \limsup_{a\downarrow 0} P_a^{(0,c)}\Big[\sup_{\substack{0\le s,t\le T\\ |s-t|<\delta}} |X_s - X_t| > \varepsilon\Big] = 0. \tag{48}$$

To check (48) we use the strong Markov property. Choose some fixed $\rho \in (0, \varepsilon)$. For all $a \in (0, \rho)$ we have

$$\begin{aligned} P_a^{(0,c)}\big[A_{\varepsilon,\delta}^{[0,T]}\big] &\le P_a^{(0,c)}\Big[\Big\{\sup_{\substack{0\le s,t\le T\\ |s-t|<\delta}} |X_{s+\mathscr{T}_\rho} - X_{t+\mathscr{T}_\rho}| > \varepsilon - \rho\Big\}\Big] \\ &= P_\rho^{(0,c)}\big[A_{\varepsilon-\rho,\delta}^{[0,T]}\big]. \end{aligned}$$

However, by continuity of the paths, the last term goes to zero, as $\delta \downarrow 0$. This proves the asserted tightness. The next lemma will show the uniqueness of the limit point:

LEMMA 4. *If $E_a^{(0,c)}$ denotes the expectation with respect to $P_a^{(0,c)}$, where $a \in (0,c)$, we have*

$$\lim_{\rho\downarrow 0} \lim_{a\downarrow 0} E_a^{(0,c)}[\mathscr{T}_\rho] = 0. \tag{49}$$

Before we show Lemma 4, we want to give the proof of Proposition 1.

PROOF OF PROPOSITION 1. Because of the tightness, it is enough to show that $\lim_{a\downarrow 0} E_a^{(0,c)}[f]$ exists for arbitrary $f \in C_b(\Omega)$.

Denote by $\Omega \ni \omega \mapsto \vartheta_t(\omega) := \omega(.+t)$ the shift. Let $\varepsilon \in (0,1)$ and choose $K \subset\subset \Omega$, such that for all $a \in (0,c)$,

$$P_a^{(0,c)}[K] > 1 - \varepsilon. \tag{50}$$

Then pick $\delta > 0$, such that $|f - f \circ \vartheta_u|1_K \le \varepsilon$, for $0 \le u \le \delta$. Pick $\rho > 0$ and $a_0 \le \rho$ such that for $a \le a_0$,

$$E_a^{(0,c)}[\mathscr{T}_\rho] < \varepsilon\delta. \tag{51}$$

Let a and a' be smaller than a_0. Using the Markov property we find

$$\begin{aligned}&|E_a^{(0,c)}[f] - E_{a'}^{(0,c)}[f]|\\ &\quad \le |E_a^{(0,c)}[f] - E_a^{(0,c)}[f \circ \vartheta_{\mathscr{T}_\rho}]| + |E_{a'}^{(0,c)}[f \circ \vartheta_{\mathscr{T}_\rho}] - E_{a'}^{(0,c)}[f]|.\end{aligned}$$

Now

$$\begin{aligned}&E_a^{(0,c)}\big[|f - f \circ \vartheta_{\mathscr{T}_\rho}|\big]\\ &\quad = E_a^{(0,c)}\big[|f - f \circ \vartheta_{\mathscr{T}_\rho}|1_{\{\mathscr{T}_\rho > \delta\}}\big] + E_a^{(0,c)}\big[|f - f \circ \vartheta_{\mathscr{T}_\rho}|1_{\{\mathscr{T}_\rho \le \delta\}}\big].\end{aligned}$$

Because of the boundedness of the function f and thanks to (51) we have

$$\begin{aligned}E_a^{(0,c)}\big[|f - f \circ \vartheta_{\mathscr{T}_\rho}|1_{\{\mathscr{T}_\rho > \delta\}}\big] &\le \frac{2\|f\|_\infty}{\delta} E_a^{(0,c)}[\mathscr{T}_\rho]\\ &\le 2\|f\|_\infty \varepsilon.\end{aligned}$$

Furthermore, using (50),

$$\begin{aligned}E_a^{(0,c)}\big[|f - f \circ \vartheta_{\mathscr{T}_\rho}|1_{\{\mathscr{T}_\rho \le \delta\}}\big] &\le 2\varepsilon\|f\|_\infty + E_a^{(0,c)}\big[|f - f \circ \vartheta_{\mathscr{T}_\rho}|1_{\{\mathscr{T}_\rho \le \delta\}}1_K\big]\\ &\le \varepsilon(2\|f\|_\infty + 1)\end{aligned}$$

and the claim of Proposition 1 follows. □

It remains to show Lemma 4.

PROOF OF LEMMA 4. Let $0 < a^* < a < \rho < c$. We want to compute $E_a^{(0,c)}[\mathscr{T}_{(a^*,\rho)}]$, where $\mathscr{T}_{(a^*,\rho)}$ is the exit time from (a^*, ρ), and then let a^* go to 0 from above. Now

$$E_a^{(0,c)}[\mathscr{T}_{(a^*,\rho)}] = \int_{a^*}^{\rho} G_{(a^*,\rho)}(a, y)m(dy), \tag{52}$$

where $m(dy)$ is the speed measure and G is the Green's function (see [12], Theorem 3.6). These quantities can be expressed in terms of the scale function $s(x)$, which is defined as follows: Let $\alpha \in (0, c/2)$ be fixed. For $x \in (0, c)$ the drift of the taboo process is

$$b(x) = \frac{d}{dx} \log \phi(x), \tag{53}$$

where $\phi(x)$ is the function defined in (11). We then have

$$s(x) = \int_\alpha^x \exp\left\{-2\int_\alpha^y b(z)\,dz\right\} dy = \phi(\alpha)^2 \int_\alpha^x \frac{1}{\phi(y)^2}\,dy. \tag{54}$$

For $x \in (0, c)$ the speed measure is given by

$$m(dx) = 2(\dot{s}(x))^{-1} = 2\frac{\phi(x)^2}{\phi(\alpha)^2}\,dx \tag{55}$$

and the Green's function

$$G_{(a^*,\rho)}(a, y) = \frac{(s(a \wedge y) - s(a^*))(s(\rho) - s(a \vee y))}{s(\rho) - s(a^*)}, \tag{56}$$

where $a \vee b := \max\{a, b\}$. With $\lim_{a^* \downarrow 0} s(a^*) = -\infty$, Lebesgue's theorem and remembering that $E_a^{(0,c)}[\mathscr{T}_{(0,\rho)}] = E_a^{(0,c)}[\mathscr{T}_\rho]$ because of $P_a^{(0,c)}[\mathscr{T}_\rho > \mathscr{T}_0] = 0$, we get

$$\begin{aligned} E_a^{(0,c)}[\mathscr{T}_\rho] &= \frac{2}{\phi^2(\alpha)}\int_0^\rho (s(\rho) - s(a \vee y))\phi(y)^2\,dy \\ &= \frac{2}{\phi(\alpha)^2}\left[\int_0^a (s(\rho) - s(a))\phi(y)^2\,dy + \int_a^\rho (s(\rho) - s(y))\phi(y)^2\,dy\right]. \end{aligned} \tag{57}$$

Let $a \in (0, \alpha)$ be small enough. We have $\int_0^a \phi(y)^2\,dy \le a\phi(a)^2$. Because $\alpha \in (0, c/2)$, (54) gives us $|\phi(a)^2 s(a)| \le \phi(\alpha)^2(\alpha - a)$; hence,

$$\left|\int_0^a s(a)\phi(y)^2\,dy\right| \le a\phi(\alpha)^2\alpha. \tag{58}$$

Thus

$$\begin{aligned} \lim_{a \downarrow 0} E_a^{(0,c)}[\mathscr{T}_\rho] &= \frac{2}{\phi(\alpha)^2}\int_0^\rho (s(\rho) - s(y))\phi(y)^2\,dy \\ &= 2\int_0^\rho \left(\int_y^\rho \frac{1}{\phi(z)^2}\,dz\right)\phi(y)^2\,dy. \end{aligned} \tag{59}$$

Because for small enough ρ we have by the mean value theorem $z/\phi(z) \le c\sqrt{c}/\pi$, for all $z \in (0, \rho)$, we get

$$\begin{aligned} \int_y^\rho \frac{1}{\phi(z)^2}\,dz &\le \frac{c^3}{\pi^2}\int_y^\rho \frac{1}{z^2}\,dz \\ &\le \frac{c^3}{\pi^2}\frac{1}{y}. \end{aligned}$$

Thus

$$\int_0^\rho \left(\int_y^\rho \frac{1}{\phi(z)^2}\,dz\right)\phi(y)^2\,dy \le \frac{c^3}{\pi^2}\int_0^\rho \frac{1}{y}\phi(y)^2\,dy.$$

Finally,

$$\begin{aligned} \int_0^\rho \frac{1}{y}\phi(y)^2\,dy &\le \phi(\rho)\int_0^\rho \frac{\phi(y)}{y}\,dy \\ &\le \phi(\rho)\rho\sqrt{\frac{2}{c}}\frac{\pi}{c} \end{aligned}$$

for sufficiently small ρ. Now the claim of Lemma 4 follows. □

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