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AN ERROR ANALYSIS FOR THE NUMERICAL CALCULATION OF CERTAIN RANDOM INTEGRALS: PART 1

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For a wide sense stationary random field $\Phi = \{\phi(x)\colon x \in R^2\}$, we investigate the asymptotic errors made in the numerical integration of line intergrals of the form $\int_\Gamma f(x)\phi(x)d\sigma(x)$. It is shown, for example, that if f and Γ are smooth, and if the spectral density $\rho(\lambda)$ satisfies $\rho(\lambda) \approx k|\lambda|^{-4}$ as $\lambda \to \infty$, then there is a constant c' with $N^3 E|\int_\Gamma f(x)\phi(x)d\sigma(x) - \sum \beta_j \varphi(x_j)|^2 \geq c'N^{-3}$ for all finite sets $\{x_j\colon 1 \leq j \leq N\}$ and all choices of coefficients $\{\beta_j\}$. And, if any fixed parameterization $x(t)$ of Γ is given and the integral $\int_0^1 f(x(t))\phi(x(t))|x'(t)|dt$ is numerically integrated using the midpoint method, the exact asymptotics of the mean squared error is derived. This leads to asymptotically optimal designs, and generalizes to other power laws and to nonstationary and nonisotropic fields.

1. Introduction. This paper presents a detailed analysis of the asymptotic mean square errors that arise when stochastic line integrals of the form

$$\int_\Gamma f(y)\phi(y)\,d\sigma(y) \tag{1.1}$$

are computed by means of a generalized midpoint method. In (1.1) and throughout this paper, $\Phi = \{\phi(x)\}$ is a continuous random field, Γ is a simple smooth curve of finite length, $d\sigma(y)$ denotes arc length and $f(x)$ is a continuous function defined on Γ.

Our attention was drawn first to problems of this type that arose in the theory of least mean square prediction for Gaussian Markov fields that satisfy linear elliptic partial differential equations. See [9], [11] and [15]. When such a field $\{\phi(y)\}$ defined on R^2 is observed in the complement of a domain D with smooth boundary Γ, the prediction $\{\hat{\phi}(x) : x \in D\}$ may be identified with the solution of an associated Dirichlet problem with stochastic boundary values. As such, these predictions have representations as generalized Poisson integrals over the boundary curves Γ. Problems of optimal sample design and computation of the predictions thus reduce to problems of optimal design and computation of these integrals.

This paper contains a nearly complete analysis of the errors in several distinct settings, and is organized as follows.

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First, in Section 2, the simplest and most illustrative example, the Whittle field on R^2, which satisfies

$$(I - \Delta)\phi(x) = \dot{W}(x), \tag{1.2}$$

where $\dot{W}(x)$ is a two-dimensional white noise, is introduced together with the associated prediction problem. This is followed in Section 3 with the introduction of a generalized midpoint method for evaluating line integrals of parametrized curves $\Gamma = \{x(t) : 0 \le t \le 1\}$. Theorems 3.1, 3.2 and 3.3 contain statements of our main results for the Whittle field. In particular, Theorem 3.3 asserts that the mean squared error that arises when (1.1) is evaluated using an n point generalized midpoint method equals

$$cn^{-3} + o(n^{-3}), \tag{1.3}$$

where

$$c = (2\pi)^{-2}\zeta(3) \int_0^1 |f(x(t))|^2 |x'(t)|^4\, dt, \tag{1.4}$$

and $\zeta(\cdot)$ is the Riemann zeta function. We remark in passing that we were surprised by the occurrence of the zeta function here and in the extensions of Theorem 3.3 that we present in Sections 7 and 8.

These theorems are proved in Sections 4 and 5. The exact constant in (1.4) allows for the comparison of methods, and Section 6 discusses the improvements obtained with the generalized midpoint methods over the classical midpoint method.

Section 7 extends the results obtained for the Whittle field to a class of stationary nonisotropic and nonstationary second-order Gaussian Markov fields. Finally, Section 8 extends these results to non-Markovian stationary Gaussian fields satisfying asymptotic spectral power laws at infinity.

Subsequent papers will extend this analysis to surface integrals that arise in predicting solutions of stochastic elliptic problems in higher dimensions and to the line integrals arising from the parabolic stochastic heat equation.

Our work here is related to earlier work on centered sampling for one-dimensional integrals, especially Matheron [7], Schoenfelder [12] and also Stein [13]. Our ongoing work on multidimensional integrals is closely related to [13].

2. The Whittle field. Letting $\{\dot{W}(x)\}$ denote a white noise field on R^2, the stationary mean zero Gaussian field $\Phi = \Phi(k, \beta) = \{\phi(x)\}$ satisfying the stochastic partial differential equation

$$(k^2 - \Delta)\phi(x) = \beta \dot{W}(x), \qquad x \in R^2, \tag{2.1}$$

was introduced by Whittle [15] and will be referred to as the Whittle field. Φ is perhaps the simplest continuous autoregressive spatial field, and it arises naturally when the Poisson equation

$$(k^2 - \Delta)u_0(x) = f(x)$$

is perturbed with a white noise forcing term $\beta\dot{W}(x)$,

$$(k^2 - \Delta)u(x) = f(x) + \beta\dot{W}(x).$$

The resulting perturbation $u(x) - u_0(x)$ of u_0, that is, the field ϕ, will thus naturally occur wherever the Poisson equation occurs. Φ and related fields have been widely used to model variations of spatial fields from trend surfaces. For examples in the context of hydrological and geological data, see [5] and [6].

The covariance function

$$R(x, y) = E\phi(x)\phi(y)$$

of $\{\phi(x)\}$, when interpreted as the kernel of an integral operator $\mathbf{R}$ acting on $L^2(R^2)$, is easily seen to satisfy

$$\mathbf{R} = \beta^2(k^2 - \Delta)^{-2}. \tag{2.2}$$

Scaling arguments can be used to reduce this to the special case when $k = 1$ and $\beta^2 = 4\pi$, for which

$$R(x, y) = \pi^{-1}\int_{R^2} \exp\{i(x - y, \lambda)\}\rho(\lambda)\,d\lambda, \tag{2.3}$$

with $\rho(\lambda) = (1 + |\lambda|^2)^{-2}$.

The integral (2.3) for R can be explicitly evaluated ([1], page 376, 9.6.25) and gives

$$R(x, y) = |x - y|K_1(|x - y|), \tag{2.4}$$

where K_1 is the modified Bessel function of the second kind and order 1. From (2.4) the asymptotic expression

$$R(x, y) \approx \left(1 - |x - y|^2 \log\left(\frac{1}{|x - y|}\right)\right), \qquad |x - y| \to 0, \tag{2.5}$$

follows ([1], page 379, 9.8.5), and it is easily deduced that with probability 1 the field $\phi(x)$ is nondifferentiable a.e., but that for each $\alpha \in (0, 1)$, $\phi(x)$ satisfies a locally uniform Lipschitz condition of order α.

For purposes of illustration we include a simulated graph of $\phi(x)$ in Figure 1.

The prediction problem for Φ. See [11] and [16]. Given a domain $D \subseteq R^2$ with smooth compact boundary Γ and complementary domain D^c, the least mean square error prediction of $\phi(x)$ for x in D given the σ-field $\sigma\{\phi(y) : y \in D^c\}$ is the conditional expectation

$$\hat{\phi}_D(x) \equiv E\{\phi(x) \mid \sigma\{\phi(y) : y \in D^c\}\}.$$

The field Φ enjoys a germ field Markov property [9], which implies that $\hat{\phi}_D$ is identifiable with

$$\hat{\phi}_D(x) = E\{\phi(x) \mid g(\Gamma)\}.$$

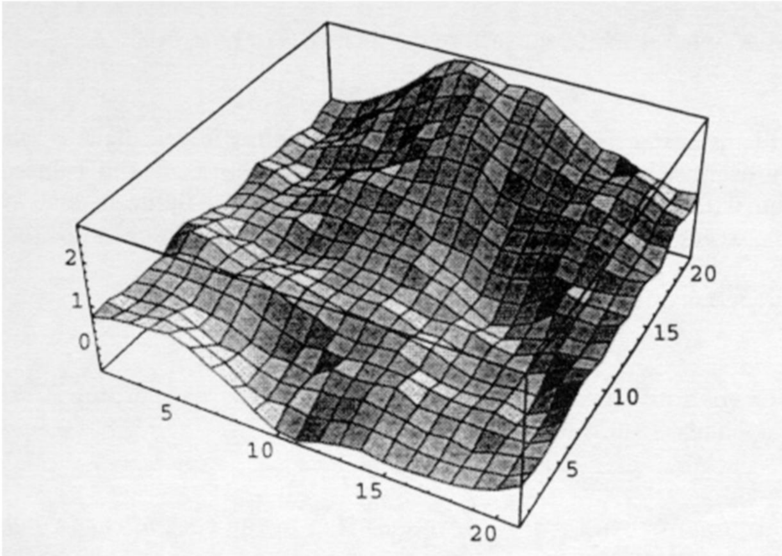

FIG. 1. *Whittle field with $k = 1$ and $\beta^2 = 4\pi$.*

The "germ field" $g(\Gamma)$ occurring here is defined as

$$g(\Gamma) = \bigcap_{\varepsilon>0} \sigma\{\phi(y) : y \in \Gamma_\varepsilon\}.$$

The function $\hat{\phi}_D(x)$ is characterized as the solution of the boundary value problem [11]

$$\begin{aligned} (I-\Delta)^2\hat{\phi}_D(x) &= 0, \qquad && x \in D^2, \\ \hat{\phi}_D(y) &= \phi(y), && y \in \Gamma, \\ \partial_{\mathbf{n}}\hat{\phi}_D(y) &= \partial_{\mathbf{n}}\phi(y), && y \in \Gamma. \end{aligned} \tag{2.6}$$

Here the normal derivative $\partial_{\mathbf{n}}\phi(y)$ is a distributional derivative defined by

$$\int_\Gamma f(y)\,\partial_{\mathbf{n}}\phi(y)\,d\sigma(y) = \lim_{h\to 0} h^{-1}\int_\Gamma f(y)[\phi(y+h\mathbf{n}) - \phi(y)]\,d\sigma(y), \tag{2.7}$$

where $\mathbf{n} = \mathbf{n}(y)$ is a smooth unit vector normal to Γ at y. It follows from (2.6) that $\hat{\phi}_D$ has a Poisson representation,

$$\hat{\phi}_D(x) = \int_\Gamma p_0(x,y)\phi(y)\,d\sigma(y) + \int_\Gamma p_1(x,y)\,\partial_{\mathbf{n}}\phi(y)\,d\sigma(y), \tag{2.8}$$

in which the Poisson kernels $p_0(x,y)$ and $p_1(x,y)$ are smooth functions of $x \in D$ and $y \in \Gamma$ that are, in principle, computable.

Thus, the effective numerical computation of $\hat{\phi}_D(x)$ reduces to the numerical computation of these two integrals, and although the integral $\int p_1(x,y)\,\partial_{\mathbf{n}}\phi(y)\,d\sigma(y)$ involves the normal derivative, it is operationally defined in (2.7) as a limit of integrals of the form $\int f(y)\phi(y)\,d\sigma(y)$. This leads to a design problem of a familiar type. Let the field ϕ be observed at a finite number of sites $\{\phi(x_j) : j = 1, \dots, n\}$, $x_j \notin D$, and let weights $\{\lambda_j\}$ be given. Then form the sum $\sum_{j=1}^{n} \lambda_j \phi(x_j)$ as an approximation to the integral $\int_\Gamma f(y)\phi(y)\,d\sigma(y)$. The basic questions here are: Where should the observation sites $\{x_j\}$ be located and what should the coefficients $\{\lambda_j\}$ be to minimize the expected square error

$$E\left\{\int_\Gamma f(y)\phi(y)\,d\sigma(y) - \sum_{k=1}^{n} \lambda_j \phi(x_j)\right\}^2 ?$$

3. On the midpoint method. It is known that when computing a wide variety of random integrals, midpoint methods are asymptotically optimal, or are nearly so; see [2]. That this is so in the present case follows by combining the next two theorems. Theorem 3.1 was established in [15] and [11].

THEOREM 3.1. *Let $\Gamma \subseteq R^2$ be a smooth curve of length l, and let $f(x)$ be a smooth nonzero function defined on Γ. For each fixed n, divide Γ into n disjoint arcs of length l/n and let $\{x_j\}$ denote the midpoints of these arcs. Then for each smooth function $f(x)$ defined on Γ,*

$$E\left[\int_\Gamma f(y)\phi(y)\,d\sigma(y) - ln^{-1}\sum_{j=1}^{n} f(x_j)\phi(x_j)\right]^2 = O(n^{-3}) \quad \text{as } n \to \infty. \tag{3.1}$$

THEOREM 3.2. *Let $\Gamma \subseteq R^2$ be a smooth curve of length $l > 0$ and let $f(x)$ be a continuous nonzero function defined on Γ. There exists a constant $c > 0$ so that: Given any collection of points $\{x_j : 0 \le j \le n\}$ and constants $\{\lambda_j : 0 \le j \le n\}$,*

$$E\left\{\int_\Gamma f(y)\phi(y)\,d\sigma(y) - \sum_{j=1}^{n} \lambda_j \phi(x_j)\right\}^2 \ge cn^{-3}. \tag{3.2}$$

Our main results in this paper are refinements of Theorem 3.1. In particular, for the midpoint method and for a class of generalized midpoint methods, we calculate an exact asymptotic constant in place of the "big O" condition of (3.1).

The generalized midpoint method. Let $\{x(t),\ 0 \le t \le 1\}$ be a C^2-parameterization of Γ, and assume that $u(t) = x'(t)$ is continuous and never vanishes. For n fixed and $1 \le j \le n$, we denote the arc $x[(j-1)/n, j/n]$ with A_j. Dividing A_j into two segments of equal length, we let $x_j = x(t_j)$ denote the midpoint of A_j. Finally, we let λ_j be the length of A_j and we set

$$I(f, \phi, \Gamma, n) = \sum_{j=1}^{n} f(x_j)\phi(x_j)\lambda_j. \tag{3.3}$$

THEOREM 3.3. *For each smooth curve Γ of finite length $l > 0$ and each smooth function f defined on Γ,*

$$\lim_{n\to\infty} n^3 E\left\{I(f,\phi,\Gamma,n) - \int_\Gamma f(y)\phi(y)\,d\sigma(y)\right\}^2 = c\int_0^1 |f(x(t))|^2 |x'(t)|^4\,dt, \tag{3.4}$$

where c is the constant $c = (2\pi)^{-2}\zeta(3)$.

4. Proof of Theorem 3.2. The Markov property of ϕ will be used to replace the approximation $\Sigma\lambda_j\phi(x_j)$ of $\int f(y)\phi(y)\,d\sigma(y)$ with an approximation for which the errors made in approximating $\int_\Gamma f(y)\phi(y)\,d\sigma(y)$, which result from disjoint subarcs of Γ, are independent, or nearly so. We begin by introducing some notation. Let $D = D(y,r) = \{x : |x-y| < r\}$ be a disk containing an arc $A = D \cap \Gamma$. Write $\hat{\phi}(x)$ for the conditional expectation

$$\hat{\phi}(x) = \hat{\phi}_D(x) = E\{\phi(x) \mid \sigma\{\phi(y) : y \notin D\}\}$$

and observe that

$$E\left\{\int_A f(x)\phi(x)\,d\sigma(x) \mid \sigma\{\phi(x) : x \notin D\}\right\} = \int_A f(x)\hat{\phi}(x)\,d\sigma(x).$$

The error $\int_A f(x)\phi(x)\,d\sigma(x) - \int_A f(x)\hat{\phi}(x)\,d\sigma(x)$ will be denoted with $e(f,\phi,A,D)$.

We will also find it convenient to denote with $\mathscr{D}$ the class of all disks $D(y,r)$ that are centered at points $y \in \Gamma$ and that are dissected by Γ, that is, $D \setminus \Gamma$ consists of two components. Finally, for $D \in \mathscr{D}$, we let A' stand for the diameter of D that is tangent to Γ at the center y.

The following technical results will be essential.

LEMMA 4.1. *There exists a constant k with $0 < k$ such that for all $D = D(y,r)$ in $\mathscr{D}$,*

$$E\{e(1,\phi,A',D)\}^2 \geq kr^4 \tag{4.1}$$

and

$$E\{e(f,\phi,A,D)\}^2 \geq kf^2(y)E\{e(1,\phi,A',D)\}^2. \tag{4.2}$$

Assuming momentarily the validity of Lemma 4.1, we now proceed with the proof of Theorem 3.2. Let $\{D_j = D(y_j,r_j) : 1 \leq j \leq n\}$ be any finite collection of disjoint disks in $\mathscr{D}$. By the Markov property of Φ, the errors $\{e_j = e(f,\phi,A_j,D_j)\}$ are mutually independent and are independent of the σ-field $G = \sigma\{\phi(x) : x \notin \cup D_j\}$. Thus, if $\{\phi(x_j)\}$ is a set of observations for which the points $\{x_j\}$ lie outside of $\cup D_j$ and if $\{\lambda_j\}$ is any set of constants, we have

$$
\begin{aligned}
&E\left\{\int_\Gamma f(x)\phi(x)\,d\sigma(x) - \Sigma\lambda_j\phi(x_j)\right\}^2 \\
&\quad\geq E\left\{\int_\Gamma f(x)\phi(x)\,d\sigma(x) - E\left\{\int_\Gamma f(x)\phi(x)\,d\sigma(x) \mid \phi(x_j), 1 \leq j \leq n\right\}\right\}^2 \\
&\quad\geq E\left\{\int_\Gamma f(x)\phi(x)\,d\sigma(x) - E\left\{\int_\Gamma f(x)\phi(x)\,d\sigma(x) \mid G\right\}\right\}^2 \\
&\quad\geq E\left\{\sum_{j=1}^n e(f,\phi,A_j,D_j)\right\}^2 = \sum_{j=1}^n Ee(f,\phi,A_j,D_j)^2.
\end{aligned}
$$

However, from Lemma 4.1 we have for each j that

$$Ee(f,\phi,A_j,D_j)^2 \geq kf^2(y_j)E\{e(1,\phi,A_j',D_j)\}^2 \geq k^2 f^2(y_j)r_j^4,$$

and thus

$$E\left\{\int_\Gamma f(x)\phi(x)\,d\sigma(x) - \sum_{j=1}^n \lambda_j\phi(x_j)\right\}^2 \geq k^2\sum_{j=1}^n f^2(y_j)r_j^4. \tag{4.3}$$

To complete the proof we will use Lemma 4.2.

LEMMA 4.2. *Let $\Gamma \subseteq R^2$ be a smooth, bounded simple curve of length $l > 0$. There exists an integer $N > 0$ such that for any $n \geq N$ and any collection of points $\{x_j : 1 \leq j \leq n\}$, there are disjoint disks $\{D(y_j, r_j)\} = \{x : |x - y_j| < r_j \mid 1 \leq j \leq 2n\}$ in $\mathscr{D}$ which are disjoint of the set $\{x_k : 1 \leq k \leq n\}$ and are such that the sum of the radii r_j satisfies*

$$\sum_{j=1}^{2n} r_j \geq \frac{l}{3}. \tag{4.4}$$

Because the minimum of the sum

$$\sum_{j=1}^{2n} r_j^4, \quad \textit{with } \Sigma r_j \geq \frac{l}{3},$$

occurs when $r_j = l/6n$ holds for each j, the desired inequality (3.2) holds for $n \geq N$ with $c = 2k^2\min\{f(x)^2 : x \in \Gamma\}6^{-4}$.

This completes the proof except for Lemmas 4.1 and 4.2.

PROOF OF LEMMA 4.1. We begin with (4.1). Let $R_D(x,y)$ and $R_{D,0}(x,y)$ be the Greens functions of $(I-\Delta)^2$ and Δ^2, respectively. It is known [11] that

$$E\{e(1,\phi,A',D)\}^2 = \int_{A'}\int_{A'} R_D(x,x')\,d\sigma(x)\,d\sigma(x)$$

and that there is a constant K such that for all $r \leq 1$,

$$\int_{A'}\int_{A'} R_D(x,x')\,d\sigma(x)\,d\sigma(x') \leq \int_{A'}\int_{A'} R_{D,0}(x,x')\,d\sigma(x)\,d\sigma(x')$$
$$\leq K\int_{A'}\int_{A'} R_D(x,x')\,d\sigma(x)\,d\sigma(x').$$

An elementary scaling argument shows that the inner integral is homogeneous of degree 4 in r, that is,

$$\int_{A'}\int_{A'} R_{D,0}(x,x')\,d\sigma(x)\,d\sigma(x') = \text{ const. } r^4,$$

which proves (4.1).

For the proof of (4.2), it again suffices to consider the case of an arbitrarily small radius r. We write

$$\int_A f(x)\phi(x)\,d\sigma(x) = f(y)\int_{A'}\phi(x)\,d\sigma(x) + \int_A [f(x)-f(y)]\phi(x)\,d\sigma(x)$$
$$+ f(y)\left\{\int_A \phi(x)\,d\sigma(x) - \int_{A'}\phi(x)\,d\sigma(x)\right\}.$$

From this and (4.1), together with the fact that conditional expectations decrease L^2 norms, it suffices to show that

$$E\left\{\int_A [f(x)-f(y)]\phi(x)\,d\sigma(x)\right\}^2 = O(r^4) \quad \text{as } r \to 0, \tag{4.5}$$

and

$$E\left\{\int_A \phi(x)\,d\sigma(x) - \int_{A'}\phi(x')\,d\sigma(x')\right\}^2 = O(r^4) \quad \text{as } r \to 0. \tag{4.6}$$

However,

$$\begin{aligned} &E\left\{\int_A [f(x)-f(y)]\phi(x)\,d\sigma(x)\right\}^2 \\ &\quad = \int_A\int_A [f(x)-f(y)][f(x')-f(y)]R_D(x,x')\,d\sigma(x)\,d\sigma(x'). \end{aligned} \tag{4.7}$$

Because for all x and x' in D, $|R_D(x,x')| \leq$ const. r^2 (see [11]), $|[f(x)-f(y)][f(x')-f(y)]|$ is uniformly $o(1)$ and $\sigma(A) = O(r)$, (4.7) is seen to be uniformly $O(r^4)$ as $r \to 0$, and (4.5) is thus established.

To establish (4.6), we note that our assumption that Γ has bounded curvature implies there is a constant c such that for all D in $\mathscr{D}$, the curves A and A' may be simultaneously parametrized so that $x'(t)$ maps $[0,2r]$ linearly onto A' with $d\sigma/dt = 1$ and $x(t)$ maps $[0,2r]$ onto A with $|x(t)-x'(t)| \leq cr^2$ and

$h(t) = d\sigma/dt$ satisfying $|h(t) - 1| \le cr^2$ hold for all t. Then

$$
\begin{aligned}
E\left\{\int_A \phi(x)\,d\sigma(x) - \int_{A'} \phi(x')\,d\sigma(x')\right\}^2
&= E\left\{\int_0^{2r} [\phi(x(t))h(x(t)) - \phi(x'(t))]\,dt\right\}^2 \\
&\le 2E\left\{\int_0^{2r} \phi(x(t))[h(x(t)) - 1]\,dt\right\}^2 \\
&\quad + 2E\left\{\int_0^{2r} [\phi(x(t)) - \phi(x'(t)]\,dt\right\}^2.
\end{aligned}
$$

Applying Schwarz's inequality to each of the last integrals gives

$$
\begin{aligned}
E\left\{\int_0^{2r} \phi(x(t))[h(x(t)) - 1]\,dt\right\}^2 &\le E\int_0^{2r} \phi(x(t))^2\,dt \int_0^{2r} [h(x(t)) - 1]^2\,dt \\
&\le 8c^2r^6
\end{aligned}
$$

and

$$
\begin{aligned}
&E\left\{\int_0^{2r} [\phi(x(t)) - \phi(x'(t))]\,dt\right\}^2 \qquad (4.8)\\
&\quad\le \int_0^{2r} 1\,dt \int_0^{2r} e[\phi(x(t)) - \phi(x'(t))]^2 dt \\
&\quad\le \text{ const. } r\int_0^{2r} |x(t) - x'(t)|^2 \log\left(\frac{1}{|x(t) - x'(t)|}\right) dt \quad \text{[by (2.5)]} \qquad (4.9)\\
&\quad\le \text{ const. } r^6 \log\left(\frac{1}{2}\right),
\end{aligned}
$$

thereby completing the proof. □

REMARK. Although it will not be pursued here, the preceding proof may be modified to extend Theorem 5.2 to a broad class of locally nondeterministic fields. Definitions and related arguments using local nondeterminism occur in [10] and [8].

5. Proof of Theorem 3.3. The proof turns on the fact that if f is constant and the curve Γ is a line segment parametrized by arc length, then an exact integral expression may be obtained for $E\{I(f,\phi,\Gamma,n) - \int f(y)\phi(y)\,d\sigma(y)\}^2$, and this expression is amenable to a precise analysis. By approximating Γ with a polygonal curve and then replacing the integral along Γ with one along the approximating polygonal curve, we may obtain (3.4).

We now describe this approximation. We begin by extending the function $f(x)$ onto a domain Γ_0 that contains the curve Γ and for which the condition $\partial_{\mathbf{n}} f(y) = 0$ holds on Γ. For each of the arcs A_j defined in the modified midpont method, we let L_j be the line segment joining the ends of A_j. The midpoint of L_j will be denoted with y_j. We choose an N so large that each segment L_j is contained in Γ_0 whenever $n \geq N$. Finally, we set $\Gamma' = \cup L_j$, and we let μ_j equal the length of L_j. The sum that corresponds to the midpoint method as applied to the polygonal curve will be written as

$$aI(f, \phi, \Gamma, n) = \sum_{j=1}^{n} f(y_j)\phi(y_j)\mu_j.$$

The accuracy of this approximation is estimated in Lemma 5.1.

LEMMA 5.1. *With the foregoing notation there are constants K_1 and K_2 depending only on the length and the curvature of Γ and on the C^2-norm of the function f, such that for all $n \geq N$,*

$$E\{aI(f, \phi, \Gamma, n) - I(f, \phi, \Gamma, n)\}^2 \leq K_1 n^{-4} \log(n), \tag{5.1}$$

$$E\left\{\int_{\Gamma} f(y)\phi(y)\, d\sigma(y) - \int_{\Gamma'} f(y)\phi(y)\, d\sigma(y)\right\}^2 \leq K_2 n^{-4} \log(n). \tag{5.2}$$

Assuming the validity of Lemma 5.1, Theorem 3.3 is seen to be equivalent to Theorem 5.2.

THEOREM 5.2.

$$\begin{aligned} \lim_{n\to\infty} n^3 E\{aI(f, \phi, \Gamma, n) - \int_{\Gamma'} f(y)\phi(y)\, d\sigma(y)\}^2 \\ = c\int_0^1 |f(x(t))|^2 |x'(t)|^4\, dt, \end{aligned} \tag{5.3}$$

where $c = (2\pi)^{-2}\zeta(3)$.

PROOF OF LEMMA 5.1. To establish (5.1), we first break the quantity $aI(f, \phi, \Gamma, n) - I(f, \phi, \Gamma, n)$ into three sums:

$$\begin{aligned} aI(f, \phi, \Gamma, n) - I(f, \phi, \Gamma, n) &= \sum_j \{f(y_j)\phi(y_j)\mu_j - f(y_j)\phi(y_j)\lambda_j\} \\ &\quad + \sum_j \{f(y_j)\phi(y_j)\lambda_j - f(x_j)\phi(y_j)\lambda_j\} \\ &\quad + \sum_j \{f(x_j)\phi(y_j)\lambda_j - f(x_j)\phi(x_j)\lambda_j\} \\ &= S_1 + S_2 + S_3, \end{aligned}$$

where, of course,

$$S_1 = \sum_j \{f(y_j)\phi(y_j)\mu_j - f(y_j)\phi(y_j)\lambda_j\},$$

$$S_2 = \sum_j \{f(y_j)\phi(y_j)\lambda_j - f(x_j)\phi(y_j)\lambda_j\}$$

and

$$S_3 = \sum_j \{f(x_j)\phi(y_j)\lambda_j - f(x_j)\phi(x_j)\lambda_j\}.$$

Then

$$\{aI(f,\phi,\Gamma,b) - I(f,\phi,\Gamma,n)\}^2 \le 3[S_1^2 + S_2^2 + S_3^2].$$

We estimate each piece on the right. First, $ES_1^2 \le \sup|f(y_j)|^2 E\phi(y_j)^2 \cdot \sum_{j=1}^n (\mu_j - \lambda_j)^2$. However, $E\phi(y_j)^2 = 1$, and because Γ has bounded curvature, there is a constant κ so that $0 \le \lambda_j - \mu_j \le \kappa(\mu_j)^3$, provided only that μ_j is sufficiently small. Because $x(t)$ is continuously differentiable with a nonvanishing derivative, $\mu_j \simeq n^{-1}$ so we have $ES_1^2 \simeq O(n^{-5})$.

Second, $ES_2^2 \le \sum_j [f(y_j) - f(x_j)]^2 \sum_j \lambda_j^2 = O(n^{-4})$, and finally,

$$\begin{aligned} ES_3^2 &\le E\sum_j \{\phi(y_j) - \phi(x_j)\}^2 \sum_j \{f(x_j)\lambda_j\}^2 \\ &\le \text{const. } n^{-1} \sum_j E\{\phi(y_j) - \phi(x_j)\}^2 \\ &\le \text{const. } n^{-1} \sum_j |y_j - x_j|^2 \log\left(\frac{1}{|y_j - x_j|}\right) \\ &\le \text{const. } n^{-1} \sum_j n^{-4}\log(n) = \text{ const. } n^{-4}\log(n). \end{aligned}$$

Here we have used Schwarz's inequality between the first and second lines.

The proof of (5.2) is similar:

$$\begin{aligned} &E\left\{\int_\Gamma f(y)\phi(y)\,d\sigma(y) - \int_{\Gamma'} f(y)\phi(y)\,d\sigma(y)\right\}^2 \\ &\quad = E\left\{\sum_j \int_{A_j} f(y)\phi(y)\,d\sigma(y) - \int_{L_j} f(y)\phi(y)\,d\sigma(y)\right\}^2. \end{aligned}$$

Setting $e(j) = \int_{A_j} f(x)\phi(x)\,d\sigma(x) - \int_{L_j} f(y)\phi(y)\,d\sigma(y)$, we have by Schwarz's inequality that

$$E\left\{\int_\Gamma f(y)\phi(y)\,d\sigma(y) - \int_{\Gamma'} f(y)\phi(y)\,d\sigma(y)\right\}^2 \le n\sum_{j=1}^n Ee(j)^2. \tag{5.4}$$

Fixing the index j, stationarity and isotropy of Φ may be used to replace L_j with a segment $[0,h]$ of the real line. Parametrizing this segment with t

and the corresponding curve A_j with $t \to (t, x(t))$ gives

$$
\begin{aligned}
e(j) &= \int_0^h [f(t,x(t))\phi(t,x(t)) - f(t,0)\phi(t,0)]\,dt \\
&= \int_0^h [f(t,x(t)) - f(t,0)]\phi(t,x(t))\,dt \\
&\quad + \int_0^h f(t,0)[\phi(t,x(t)) - \phi(t,0)]\,dt.
\end{aligned}
$$

However, Γ has bounded curvature and there are constants $\{\kappa_j\}$ for which $|x(t) - t| \le \kappa_1 h^2$, $|f(x(t)) - f(t)|^2 \le \kappa_2 h^4$ and $[f(t)]^2 E|\phi(x(t)) - \phi(t)|^2 \le \kappa_3 h^4 \log(h^{-1})$ hold for all $n \ge N$ and all $0 \le t \le h$. Then

$$
\begin{aligned}
Ee(j)^2 &\le 2E\left\{\int_0^h [f(t,x(t)) - f(t,0)]\phi(t,x(t))\,dt\right\}^2 \\
&\quad + 2E\left\{\int_0^h f(t,0)[\phi(t,x(t)) - \phi(t,0)]\,dt\right\}^2,
\end{aligned}
$$

and by Schwarz's inequality we have

$$
\begin{aligned}
Ee(j)^2 &\le 2h\int_0^h [f(t,x(t)) - f(t,0)]^2 E[\phi(t,x(t))]^2\,dt \\
&\quad + 2h\int_0^h [f(t,0)]^2 E[\phi(t,x(t)) - \phi(t,0)]^2\,dt \\
&\le 2\kappa_2 h^6 + 2\kappa_3 h^6 \log(h^{-1}) \\
&\le \kappa_4 h^6 \log(h^{-1}).
\end{aligned}
$$

Substituting into (5.4) gives the desired result. □

We now turn our attention to

$$
\begin{aligned}
&E\left\{\int_{\Gamma'} f(t)\phi(t)\,d\sigma(y) - aI(f,\phi,\Gamma,n)\right\} \\
&\quad = E\left\{\sum_j \int_{L_j} f(y_j)[\phi(y) - \phi(y_j)]\,d\sigma(y)\right\}^2 \\
&\quad = \sum_j\sum_k E\int_{L_j} f(y_j)[\phi(y) - \phi(y)_j]\,d\sigma(y) \\
&\qquad \times \int_{L_k} f(y_k)[\phi(y) - \phi(y_k)]\,d\sigma(y).
\end{aligned} \tag{5.5}
$$

We derive an explicit Fourier representation for the summands in (5.5). We observe that the correspondence $\phi(x) \to \exp i\langle x, \lambda\rangle$ extends by linearity to a unique unitary U map from that subspace of $L^2(\Omega, P)$ that is spanned by the random variables $\phi(x)$ onto the space $L^2(R^2, \rho(\lambda)\,d\lambda)$. Here, as before, $\rho(\lambda) = (1 + |\lambda|^2)^{-2}$.

To compute the image

$$U\int_{L_j}[\phi(y)-\phi(y_j)]\,d\sigma(y),$$

we introduce the following notation: The endpoints of the segment L_j are x_j and x_{j+1}, and we let $u_j=(x_{j+1}-x_j)/|x_{j+1}-x_j|$ be the unit vector giving the direction of L_j. Then $y_j=(x_{j+1}+x_j)/2$ is the midpoint of L_j and $l_j=|x_{j+1}-x_j|$ is the length of L_j. Thus,

$$\begin{aligned}U\int_{L_j}[\phi(y)-\phi(y_j)]\,d\sigma(y)&=\int_{L_j}\{\exp i\langle y,\lambda\rangle\exp i\langle y_j,\lambda\rangle\}\,d\sigma(y)\\&=\exp i\langle y_j,\lambda\rangle\left\{\frac{\sin(u_j\cdot\lambda l_j/2)}{u_j\cdot\lambda/2}-l_j\right\}\end{aligned}$$

and

$$\begin{aligned}&E\int_{L_j}f(y_j)[\phi(y)-\phi(y_j)]\,d\sigma(y)\int_{L_k}f(y_k)[\phi(y)-\phi(y_k)]\,d\sigma(y)\\&\quad=f(y_j)f(y_k)\pi^{-1}\int_{R^2}\exp i\langle y_j-y_k,\lambda\rangle\left\{\frac{\sin(u_j\cdot\lambda l_j/2)}{u_j\cdot\lambda/2}-l_j\right\}\\&\qquad\times\left\{\frac{\sin(u_k\cdot\lambda l_k/2)}{u_k\cdot\lambda/2}-l_k\right\}\rho(\lambda)\,d\lambda.\end{aligned}\tag{5.6}$$

For notational convenience we denote this integral with $\mathrm{Cov}(j,k,n)$. Making the change of variables $\xi=l_j\lambda$ transforms (5.6) into

$$\begin{aligned}\mathrm{Cov}(j,k,n)&=l_j^3l_k\pi^{-1}\int_{R^2}\exp i\left\langle\frac{y_j-y_k}{l_j},\xi\right\rangle\left\{\frac{\sin(u_j\cdot\xi/2)}{u_j\cdot\xi/2}-1\right\}\\&\qquad\times\left\{\frac{\sin(u_k\cdot\xi l_k/2l_j)_{-1}}{u_k\cdot\xi l_k/2l_j}\right\}(l_j^2+|\xi|^2)^{-2}\,d\xi.\end{aligned}\tag{5.7}$$

To proceed, we fix a point $x=x(t)$ on Γ and let j be chosen so that x lies on the arc A_j. Then fixing the value of $m=j-k$ and noting that as $n\to\infty$, $nl_j\to|x'(t)|$, $u_j\to u(t)=(x'(t))/|x'(t)|$, $u_k\to u(t)$ and $(y_j-y_k)/l_j\to mu(t)$, it follows that

$$\begin{aligned}&\lim_{n\to\infty}n^4\,\mathrm{Cov}(j,k,n)\\&\quad=|x'(t)|^4\pi^{-1}\int_{R^2}\exp im\langle u(t),\xi\rangle\left\{\frac{\sin(u(t)\cdot\xi/2)}{(u(t)\cdot\xi/2)}-1\right\}^2|\xi|^{-4}\,d\xi.\end{aligned}$$

However, this last integral is independent of the direction of $u(t)$ and may be written as

$$\pi^{-1}\int_{R^1}\int_{R^1}\cos(mp)\left[\frac{\sin(p/2)}{p/2}-1\right]^2|p^2+q^2|^{-2}\,dp\,dq$$
$$=(2\pi)^{-1}\int_{-\infty}^{\infty}|1+r^2|^{-2}\,dr\int_0^{\infty}\cos(2mp)\left[\frac{\sin(p)}{p}-1\right]^2p^{-3}\,dp$$
$$=\frac{1}{4}\int_0^{\infty}\cos(2mp)\left[\frac{\sin(p)}{p}-1\right]^2p^{-3}\,dp.$$

We summarize the result of this argument with Lemma 5.3.

LEMMA 5.3. *Fix a point $x=x(t)$ on Γ and an integer $m\geq 0$. Let x belong to the arc A_j. (Note that A_j and j both depend on n.) Then, for $k=j-m$,*

$$\lim_{n\to\infty} n^4 f(y_j)f(y_k)\operatorname{Cov}(j,k,n)=\chi(m),$$

where

$$\chi(m)=f(x(t))^2|x'(t)|^4\frac{1}{4}\int_0^{\infty}\cos(2mp)\left[\frac{\sin(p)}{p}-1\right]^2p^{-3}\,dp. \tag{5.8}$$

We now consider the sum $n^3\sum_{j=1}^n\sum_{k=1}^n f(y_j)f(y_n)\operatorname{Cov}(j,k,n)$, which we write as

$$n^{-1}\sum_{j=1}^n\sum_{k=1}^n n^4 f(y_j)f(y_k)\operatorname{Cov}(j,k,n).$$

Fixing a t in $(0,1)$ and the corresponding j for which $x(t)$ is in A_j we write

$$S(t,n)=\sum_{k=1}^n n^4 f(y_j)f(y_k)\operatorname{Cov}(j,k,n)$$
$$=\sum_{m=1-j}^{n-j} n^4 f(y_j)f(y_{j+m})\operatorname{Cov}(j,j+m,n).$$

Because $n^4 f(y_j)f(y_k)\operatorname{Cov}(j,j+m,n)\to\chi(m)$ as $n\to\infty$, it is reasonable to expect the next lemma.

LEMMA 5.4.

$$\lim_{n\to\infty}S(t,n)=\sum_{-\infty}^{\infty}\chi(m).$$

PROOF. By Lebesgue's dominated convergence theorem it will suffice to show there exists a sequence $r(m)$ satisfying $\Sigma r(m)<\infty$ and satisfying $|\operatorname{Cov}(j,m,n)|\leq r(m)$ for all j, m and n. We will in fact show there is a constant such that

$$|\operatorname{Cov}(j,m,n)| \le \text{ const. } m^{-2}\log(|m|) \tag{5.9}$$

holds.

From our hypotheses on Γ and the construction of the sequence $\{y_j\}$, it follows that there is a finite positive constant c for which,

$$\left|\frac{(y_j - y_k)}{l_j}\right| \ge c|j-k| \quad \text{and} \quad c^{-1} \le \left|\frac{l_k}{l_j}\right| \le c \quad \text{hold for all } j,k \text{ and } n.$$

The desired result will thus follow from the formula (5.7) and the next lemma.

LEMMA 5.5. *For each constant $c > 1$, there exists a constant C such that*

$$\begin{aligned} &\left|\int_{R^2} \exp i\langle y,\xi\rangle \left\{\frac{\sin(u\cdot\xi)}{(u\cdot\xi)} - 1\right\}\left\{\frac{\sin(v\cdot\xi)}{(v\cdot\xi)} - 1\right\}(l^2 + |\xi|^2)^{-2}\,d\xi\right| \\ &\qquad \le C|y|^{-2}\log(|y|^{-1}) \end{aligned} \tag{5.10}$$

holds for all scalars l with $0 < l < 1$, for all y in R^2 and for all vectors u and v with $c^{-1} \le |u| \le c$ and $c^{-1} \le |v| \le c$.

This inequality (5.10) follows from laborious but elementary computations that we present only in outline. First, we introduce the function

$$F(u,v,l,\xi) = \left\{\frac{\sin(u\cdot\xi)}{(u\cdot\xi)} - 1\right\}\left\{\frac{\sin(v\cdot\xi)}{(v\cdot\xi)} - 1\right\}(l^2 + |\xi|^2)^{-2}.$$

Denoting the $L^1(R^2)$ norm with $\|\ \|_1$, it can be shown that there are constants k_1 and k_2 so that each second derivative $g(\xi) = \partial\,\partial F$ satisfies

$$\|g(\xi)\|_1 \le k_1 \log(l^{-1}) \tag{5.11}$$

and

$$\|(F(u,v,l,\cdot) - F(u,v,l',\cdot)\|_1 \le k_2 l^2 \log(l^{-1}), \tag{5.12}$$

provided only that $0 < l' < l < 1$, $c^{-1} \le |u| \le c$ and $c^{-1} \le |v| \le c$. From (5.11) it follows that the Fourier transform $\hat{F}(l,y)$ of $F(u,v,l,\cdot)$ satisfies

$$|\hat{F}(l,y)\| \le k_1 \log(l^{-1}) y^{-2}, \tag{5.13}$$

which gives

$$|\hat{F}(l',y)| \le k_1 y^{-2}\log(|y|) \quad \text{provided } |y| \ge (l')^{-1}. \tag{5.14}$$

If $|y| \le (l')^{-1}$ we note that (5.12) implies that

$$|\hat{F}(l',y) - \hat{F}(l,y)| \le k_2 l^2 \log(l^{-1}) \quad \text{provided that } 0 < l' < l < 1. \tag{5.15}$$

Then, choosing $l = |y|^{-1}$ and applying the inequality

$$|\hat{F}(l',y)| \le |\hat{F}(l,y)| + |\hat{F}(l',y) - \hat{F}(l,y)|$$

we observe that (5.12) together with (5.14) implies the desired inequality

$$|\hat{F}(l', y)| \leq k_1 \log(|y|)|y|^{-2} + k_2|y|^{-2}\log(|y|),$$

thereby completing the proof. □

The sum $\Sigma\psi(m)$ may now be evaluated using the Poisson summation formula, [4]. Namely, $\int_0^\infty \cos(2mp)[\sin(p)/p - 1]^2 p^{-3}\,dp = \frac{1}{2}\hat{h}(2m)$, where $h(p) = [\sin(p)/p - 1]^2|p|^{-3}$. Thus,

$$\tfrac{1}{2}\sum_{-\infty}^{\infty}\hat{h}(2m) = \frac{\pi}{2}\sum_{-\infty}^{\infty}h(n\pi) = \pi\sum_{n=1}^{\infty}(n\pi)^{-3} = \pi^{-2}\zeta(3)$$

Applying this to Lemma 5.3 gives

$$\lim_{n\to\infty} S(t,n) = |x'(t)|^4 f^2(x(t))\pi^{-1}\zeta(3) \simeq 0.385 f^2(x(t))|x'(t)|^4$$

and, by Lemma 5.4, this convergence is uniformly bounded. Hence

$$\begin{aligned}\lim_{n\to\infty} n^3 \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f(y_j)f(y_k)\operatorname{Cov}(j,k,n) &= \lim_{n\to\infty} n^{-1}\sum_{j=1}^{n}\sum_{k=1}^{n} n^4 f(y_j)f(y_k)\operatorname{Cov}(j,k,n)\\ &= (2\pi)^{-2}\zeta(3)\int_0^1 f(x(t))^2|x'(t)|^4\,dx,\end{aligned}$$

which completes the proof of Theorem 3.3. □

6. On the constant $\int_0^1 f(x(t))^2\,|x'(t)|^4\,dt$. The constant $\int_0^1 f(x(t))^2 \cdot |x'(t)|^4\,dx$ that determines the asymptotic error for the modified midpoint methods is a function of the spacing chosen for the points $\{x_j\}$ or, what is the same thing, the parameterization chosen for the curve Γ. Ideally, a parameterization should be chosen that minimizes this integral. That is, we seek the parameterization $\{x(t) : 0 \leq t \leq 1\}$ for which

$$\int_0^1 f(x(t))^2|x'(t)|^4\,dt \leq \int_0^1 f(y(t))^2|y'(t)|^4\,dt \tag{6.1}$$

holds for all parameterizations $\{y(0) : 0 \leq t \leq 1\}$.

This is an elementary variational problem for which the Euler–Lagrange differential equation (see, e.g., [3]) reduces to

$$\frac{\partial}{\partial t} f(x(t))^2|x'(t)|^4 = 0. \tag{6.2}$$

Thus $f(x(t))^2|x'(t)|^4$ must be a constant, and the problem (6.1) is soluble with elementary means. In fact, $|x'(t)| = c|f(x(t))|^{-1/2}$ with c such that

$$\int_0^1 |x'(t)|\,dt = L = \text{ length of } \Gamma. \tag{6.3}$$

This can be readily implemented numerically.

REMARK. We observe that the optimal spacings for the $\{x_j\}$ depend upon the particular function $f(x)$ that is being integrated. A choice that is optimal for a particular f will be far from optimal for a different function g when the ratio $(f(x))/(g(x))$ varies substantially along Γ.

In particular, because the Poisson kernels $p_0(x,y)$ and $p_1(x,y)$, which arise in the solution of our original prediction problem, become singular when x approaches the boundary curve Γ, it will be impossible to choose the $\{x_j\}$ to provide simultaneous near-optimal estimation of the $\phi(x)$ for all x in D.

As a simple illustration, consider the curve $\Gamma = [0,1]$. When $f(x) = 1$ the optimal choice for the $\{x_j\}$ is the uniform spacing on [0, 1]. However, when $f(x) = 1 + x$, the differential equation (6.3) is

$$(1+x)^{1/2}\,dx = c\,dt.$$

The general solution is

$$\tfrac{2}{3}(1+x)^{3/2} = ct + a,$$

where a is a constant. However, $x(0) = 0$ and $x(1) = 1$ imply that $a = \frac{2}{3}$ and $c = \frac{2}{3}(2^{3/2} - 1) = 1.219$. Hence,

$$\int_0^1 f(x(t))^2 |x'(t)|^4\,dt = c^4 = (1.219)^4 = 2.208.$$

If, on the other hand, corresponding to the standard midpoint method, one simply parametrized Γ with $y(t) = t$, then

$$\int_0^1 f(y(t))^2 |y'(t)|^4\,dt = \int_0^1 (1+t)^2\,dt = \frac{7}{3} = 2.333.$$

The ratio of the standard deviations of the errors associated with the optimal generalized midpoint method and the standard midpoint method would thus asymptotically equal $(2.333/2.208)^{1/2} = 1.025$. A 2.5% decrease in efficiency is, in this case, not significant.

7. Extensions of nonisotropic Markov fields of order 2. This section extends Theorems 3.2 and 3.3 first to general nonisotropic stationary Gaussian fields that are Markovian of order 2, and then to the nonstationary case.

If the stationary mean 0 Gaussian field Ψ on R^2 is Markovian of order 2, then it is known [9] that Ψ has absolutely continuous spectrum with a spectral density $\rho(\lambda) = \rho(\lambda_1, \lambda_2)$ that is the reciprocal of a nonvanishing elliptic polynomial of order 4:

$$\rho^{-1}(\lambda) = \Delta(\lambda_1, \lambda_2) = \sum_{i=0}^{2}\sum_{j=0}^{2} a_{ij}\lambda_1^i\lambda_2^j, \tag{7.1}$$

with

$$k^{-1} < \Delta(\lambda_1, \lambda_2) < k(1 + |\lambda|^2)^2, \tag{7.2}$$

for some constant $k > 0$ and for all λ in R^2.

If Φ denotes the Whittle field, then condition (7.2) implies there is another constant K for which the inequality

$$K^{-1}E[\Sigma b_j\phi(x_j)^2] \leq E[\Sigma b_j\psi(x_j)]^2 \leq KE[\Sigma b_j\phi(x_j)]^2 \tag{7.3}$$

holds for all finite linear combinations $\Sigma b_j\phi(x_j)$ and $\Sigma b_j\psi(x_j)$.

That Theorems 3.1 and 3.2 hold for the field Ψ follows directly from condition (7.3). The extension of Theorem 3.3 to Ψ, however, requires that the directional dependence of Ψ be explicitly accounted for. This may be done as follows.

Define the function $\rho_0(\theta)$ as

$$\rho_0(\theta) = \lim_{r\to\infty} r^{-4}\rho(r\cos(\theta), r\sin(\theta)).$$

Observe this limit exists uniformly in θ and that ρ_0 is the trigonometric polynomial

$$\rho_0(\theta) = \sum_{j=0}^{4} a_{4-j,j}\cos^j(\theta)\sin^{4-j}(\theta).$$

The proof of Theorem 3.3 given in Section 5 can remain largely unchanged except for details. In particular, the technical estimates given in Lemmas 5.1, 5.3, and 5.4 all remain valid. The explicit Fourier calculations in the derivation of constant the c in Theorem 3.3 remain unchanged until (5.7), which becomes

$$\begin{aligned}\mathrm{Cov}(j,k,n) = l_k l_j^{-1}\int_{R^2}&\exp i\left\langle\frac{(y_j-y_k)}{l_j},\xi\right\rangle\left\{\frac{\sin(u_j\cdot\xi/2)}{(u_j\cdot\xi/2)}-1\right\}\\ &\times\left\{\frac{\sin(u_k\cdot\xi l_k/2l_j)}{(u_k\cdot\xi/2)}-1\right\}\rho\left(\frac{\xi}{l_j}\right)d\xi.\end{aligned} \tag{7.4}$$

The subsequent limit argument gives

$$\lim_{n\to\infty} n^4\,\mathrm{Cov}(j,j+m,n) = \chi(m,x'(t)),$$

where

$$\chi(m,x'(t)) = |x'(t)|^4\int_{R^2}\cos(m\langle u(t),\xi\rangle)\left\{\frac{\sin(u(t)\cdot\xi/2)}{(u(t)\cdot\xi/2)}-1\right\}^2|\xi|^{-4}\rho_0\left(\frac{\xi}{|\xi|}\right)d\xi.$$

Note that $\chi(m,x'(t))$ contains an explicit dependence on the direction $u(t) = x'(t)/|x'(t)|$ that was absent in the isotropic case. The integral for $\chi(m,x'(t))$ can be evaluated using a change of variables adapted to the direction of $u(t)$. In particular, if $u = u(t) = (\cos(\theta(t)),\sin(\theta(t)))$ and $v = v(t) = (-\sin(\theta(t)),\cos(\theta(t)))$, we set $\xi = pu+qv$ and define τ and t by $\tan(\tau) = q/p = t$. Then as a function of the angle α, $\rho_0(\xi/|\xi|) = \rho_0(\alpha+\theta)$ and $\chi(m,x'(t))$ becomes

$$\begin{aligned}&\chi(m,x'(t))\\ &\quad= |x'(t)|^4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos(mp)\left\{\frac{\sin(p/2)}{p/2}-1\right\}^2|p|^{-3}(1+t^2)^{-2}\rho_0(\tau+\theta)\,dp\,dt.\end{aligned}$$

Introducing

$$c(\theta) = \int_{-\infty}^{\infty} (1+t^2)^{-2} \rho_0(\tau+\theta)\, dt$$

or

$$c(\theta) = \tfrac{1}{2} \int_0^{2\pi} \cos^2(\tau) \rho_0(\tau+\theta)\, d\tau \tag{7.5}$$

and the sequence

$$g(m) = \frac{1}{2} \int_0^{\infty} \cos(2mp) \left\{ \frac{\sin(p)}{p} - 1 \right\}^2 |p|^{-3}\, dp, \tag{7.6}$$

we have

$$\chi(m, x'(t)) = |x'(t)|^4 c(\theta(t)) g(m).$$

Using the same estimates used in Section 5, this formula leads to Theorem 7.1.

THEOREM 7.1. *Suppose the spectral density ρ is continuous and satisfies condition* (7.2). *Then for each smooth curve Γ and each smooth function $f(x)$ defined on Γ, the modified midpoint method gives*

$$\begin{aligned} &\lim_{n\to\infty} n^3 \left\{ I(f,\psi,\Gamma,n) - \int_\Gamma f(y)\psi(y)\, d\sigma(y) \right\}^2 \\ &\quad = \kappa \int_0^1 c(\theta(t)) |f(x(t))|^2 |x'(t)|^4\, dt, \end{aligned} \tag{7.7}$$

where κ is the constant $\kappa = \frac{1}{2}\pi^{-2}\zeta(3)$.

REMARK. The formula

$$c(\theta) = \tfrac{1}{2} \int_0^{2\pi} \cos^2(\tau) \rho_0(\tau+\theta)\, d\tau$$

leads to several useful expressions for c. First, in terms of the Fourier series of ρ_0, if

$$\rho_0(\tau) = \frac{a_0}{2} + \Sigma a_{2n} \cos(2n\tau) + b_{2n} \sin(2n\tau),$$

then

$$c(\theta) = \frac{\pi}{4} \{a_0 + a_2 \cos(2\theta) + b_2 \sin(2\theta)\}. \tag{7.8}$$

This can also be calculated in terms of singularity of the covariance function $R(x,y)$ of Ψ; see the next lemma.

LEMMA 7.2. *For each unit vector* $u = (\cos(\theta(t)), \sin(\theta(t)))$ *and for all* x *in* R^2,

$$2c(\theta) = \lim_{t\to 0} E|\psi(x+tu) - \psi(x)|^2/t^2 \log\left(\frac{1}{|t|}\right). \tag{7.9}$$

PROOF. For each nonzero t,

$$E|\psi(x+tu) - \psi(x)|^2 = \int_{R^2} |\exp it(u,\lambda) - 1|^2 \rho(\lambda)\,d\lambda.$$

Breaking this integral into integrals over $\{\lambda : |\lambda| \le 1/|t|\}$ and $\{\lambda : |\lambda| > 1/|t|\}$, we have by (7.2) that

$$\int_{\{\lambda:|\lambda|>1/|t|\}} |\exp it(u,\lambda) - 1|^2 \rho(\lambda)\,d\lambda = O(|t^3|).$$

For the integral over $\{\lambda : |\lambda| \le 1/|t|\}$ we use a Taylor approximation to obtain

$$\begin{aligned}&\int_{\{|\lambda|\le 1/|t|\}} |\exp it(u,\lambda) - 1|^2 \rho(\lambda)\,d\lambda \\ &\quad = t^2 \int_{\{|\lambda|\le 1/|t|\}} (u,\lambda)^2 \rho(\lambda)\,d\lambda + O(t^4) \int_{\{|\lambda|\le 1/|t|\}} (u,\lambda)^4 \rho(\lambda)\,d\lambda,\end{aligned}$$

which by (7.3) equals

$$t^2 \int_{\{|\lambda|\le 1/|t|\}} (u,\lambda)^2 \rho(\lambda)\,d\lambda + O(t^2).$$

However,

$$\int_{\{|\lambda|\le 1/|t|\}} (u,\lambda)^2 \rho(\lambda)\,d\lambda = \int_0^{2\pi} \int_0^{1/|t|} r^3 \cos(\alpha-\theta)^2 \rho(r\cos(\alpha), r\sin(\alpha))\,dr\,d\alpha,$$

and, from the definition of ρ_0, this is asymptotic to

$$\log\left(\frac{1}{|t|}\right) \int_0^{2\pi} \cos(\alpha-\theta)^2 \rho_0(\cos(\alpha), \sin(\alpha))\,d\alpha,$$

which completes the proof. □

Translating (7.9) in terms of the covariance function $R(x,y) = E\psi(x)\psi(y)$ gives Corollary 7.3.

COROLLARY 7.3. *For each unit vector* $u = \cos((\theta(t)), \sin(\theta(t)))$ *and for all* x *in* R^2,

$$c(\theta) = \lim_{t\to 0} \frac{1}{2}\{R(x,x) - R(x+tu, x)\}/t^2 \log\left(\frac{1}{|t|}\right). \tag{7.10}$$

The nonstationary case. We conclude this section by stating an extension of Theorem 7.1 for nonstationary Markov fields.

THEOREM 7.4. *Let $A\{A(x) = \Sigma a_j(x)D^j\}$ be a linear fourth-order elliptic operator with smooth real coefficients defined on a domain D in R^2. Assume that A is formally symmetric on $L^2(D, dx)$ and satisfies for some $k > 0$ the inequality*

$$\int_D Af(x)f(x)\,dx \geq k\int_D |f(x)|^2\,dx,$$

whenever f is in $C_0^\infty(D)$. Then Friedricks' extension of A is an invertible, positive, self-adjoint integral operator $\mathbf{R}$ on $L^2(D, dx)$, and the kernel $R(x, y)$ of $\mathbf{R}$ is the covariance operator of a continuous mean zero Gaussian field $\Psi = \{\psi(x) : x \in D\}$. For each x in D and each unit vector $u = (\cos(\theta), \sin(\theta))$, the limit

$$c(x,\theta) = \lim_{t\to 0}\frac{1}{2}\{R(x,x) - R(x+tu,x)\}/t^2\log\left(\frac{1}{|t|}\right)$$

exists and is jointly continuous in x and u. Moreover, for each smooth parametrized curve $\Gamma = \{x(t) : 0 \leq t \leq 1\}$ in d and each smooth function $f(x)$ defined on Γ, the modified midpoint method gives

$$\lim_{n\to\infty} n^3 E\left\{I(f,\Gamma,n) - \int_\Gamma f(y)\phi(y)\,d\sigma(y)\right\}^2 = \kappa\int_0^1 c(x(t),\theta(t))|f(x(t))|^2|x'(t)|^4\,dt, \tag{7.11}$$

where κ is the constant $\kappa = \frac{1}{2}\pi^{-3}\zeta(3)$.

Sketch of the proof. The existence of the field Ψ is discussed in [12]. To prove (7.11) the essential idea is to localize by breaking Γ into short segments on which the coefficients of A are almost constant. Errors that arise from disjoint arcs may be seen to be asymptotically independent. On short arcs the field can be approximated with a stationary field, and it is easy to use (7.10) to derive upper and lower bounds for the expected squared errors in terms of the principle part of the operator A. A passage to the limit then yields (7.11).

8. Non-Markovian fields. The spectral methods used to prove the earlier results for stationary Markov fields may easily be adapted to derive analogous results for non-Markov stationary fields when the spectral density satisfies an appropriate asymptotic power law at infinity. We outline this extension here. We only consider fields defined on R^2, but this restriction is not essential, as the interested reader will easily be able to establish.

We assume that $\alpha \in (1, \frac{5}{2})$ is fixed and that Φ is a weakly stationary field on R^2 with covariance function

$$R(x, y) = \operatorname{Cov}(\phi(x), \phi(y)) = \int_{R^2} \exp\{i(x - y, \lambda)\}\rho(\lambda)\,d\lambda,$$

where $\rho(\lambda) = \rho(\lambda_1, \lambda_2)$ is a continuous spectral density such that the limit

$$\lim_{r\to\infty} r^{-2\alpha}\rho(r\cos(\theta), r\sin(\theta)) = \rho_0(\theta) \tag{8.1}$$

exists uniformly in θ. The restrictions on α will be discussed later in this section, but we note here that the Markovian case discussed earlier corresponds to $\alpha = 2$.

The results of Section 7 are modified by defining the function $c_\alpha(\theta)$,

$$\begin{aligned} c(\theta) = c_\alpha(\theta) &= \int_{-\infty}^{\infty} (1 + t^2)^{-\alpha}\rho_0(\tau + \theta)\,dt \\ &= \tfrac{1}{2}\int_0^{2\pi} \cos^{2(\alpha-1)}(\tau)\rho_0(\tau + \theta)\,d\tau, \end{aligned} \tag{8.2}$$

where again $\tan(\tau) = p/q = t$.

We can now state the principle result.

THEOREM 8.1. *Suppose the spectral density ρ is continuous and satisfies condition* (8.1) *with $1 < \alpha < \frac{5}{2}$. Then for each smooth curve Γ and each smooth function $f(x)$ defined on Γ, the modified midpoint method gives*

$$\begin{aligned} &\lim_{n\to\infty} n^{2\alpha-1}E\{I(f, \phi, \Gamma, n) - \int_\Gamma f(y)\phi(y)\,d\sigma(y)\}^2 \\ &\quad = \kappa\int_0^1 c(\theta(t))|f(x(t))|^2|x'(t)|^{2\alpha}\,dt, \end{aligned} \tag{8.3}$$

where κ is the constant $\kappa = 2^{3-2\alpha}\pi^{2-2\alpha}\zeta(2\alpha - 1)$.

PROOF. The proof follows that of Theorem 7.1 closely, but it is necessary to modify several of the approximation lemmas to account for the fact that $\alpha \neq 2$. The essential changes are given in the following text.

With the same notation as used in Section 5, Lemma 5.1 must be replaced with Lemma 8.2.

LEMMA 8.2. *There are constants K_j depending only on the length and curvature of Γ and on the C^2-norm of the function f, the density ρ and the constant α such that for all large n:*

If $1 < \alpha < 2$, then

$$E\{aI(f, \phi, \Gamma, n) - I(f, \phi, \Gamma, n)\}^2 \le K_1 n^{-4(\alpha-1)}, \tag{8.4}$$

$$E\left\{\int_\Gamma f(y)\phi(y)\,d\sigma(y) - \int_{\Gamma'} f(y)\phi(y)\,d\sigma(y)\right\}^2 \le K_2 n^{-4(\alpha-1)}. \tag{8.5}$$

If $\alpha = 2$, *then*

$$E\{aI(f,\phi,\Gamma,n) - I(f,\phi,\Gamma,n)\}^2 \leq K_1 n^{-4}\log(n), \tag{8.6}$$

$$E\left\{\int_\Gamma f(y)\phi(y)\,d\sigma(y) - \int_{\Gamma'} f(y)\phi(y)\,d\sigma(y)\right\}^2 \leq K_2 n^{-4}\log(n), \tag{8.7}$$

and if $2 < \alpha$,

$$E\{aI(f,\phi,\Gamma,n) - I(f,\phi,\Gamma,n)\}^2 \leq k_1 n^{-4}, \tag{8.8}$$

$$E\left\{\int_\Gamma f(y)\phi(y)\,d\sigma(y) - \int_{\Gamma'} f(y)\phi(y)\,d\sigma(y)\right\}^2 \leq K_2 n^{-4}. \tag{8.9}$$

We define, in addition, the sequence $g(m) = g_\alpha(m)$:

$$g(m) = 2^{2\alpha-3}\int_0^\infty \cos(2mp)\left\{\frac{\sin(p)}{p} - 1\right\}^2 |p|^{1-2\alpha}\,dp. \tag{8.10}$$

Proceeding, the analysis in Sections 5 and 7 can go forward until (7.4), which we multiply by $(l_j)^{-2\alpha}$, and note that l_j is asymptotic to $|x'(t)|/n$ to obtain

$$\lim_{n\to\infty} n^{2\alpha}\operatorname{Cov}(j, j+m, n) = \chi(m, x'(t)),$$

where

$$\begin{aligned}
&\chi(m, x'(t))\\
&\quad = |x'(t)|^{2\alpha}\int_{R^2} \exp im\langle u(t),\xi\rangle\left\{\frac{\sin(u(t)\cdot\xi/2)}{(u(t)\cdot\xi/2)} - 1\right\}|\xi|^{-2\alpha}\rho_0\left(\frac{\xi}{|\xi|}\right)d\xi\\
&\quad = |x'(t)|^{2\alpha}c(\theta(t))g(m),
\end{aligned}$$

with

$$\begin{aligned}
c(\theta) &= \int_{-\infty}^{\infty}(1+t^2)^{-2}\rho_0(\alpha+\theta)\,dt\\
&= \tfrac{1}{2}\int_0^{2\pi}\cos^2(\alpha)\rho_0(\alpha+\theta)\,d\alpha
\end{aligned}$$

and

$$g(m) = 2^{2\alpha-3}\int_0^\infty \cos(2mt)\left\{\frac{\sin(t)}{t} - 1\right\}^2 |t|^{1-2\alpha}\,dt.$$

In Sections 5 and 7, the proof was completed with a dominated convergence argument, in which the key ingredient was Lemma 5.5. This route is not directly applicable in the present case because the decay estimate (5.9) was derived from smoothness properties of ρ that are not available in the present case. To overcome this, we first proceed with a substitute technical lemma.

LEMMA 8.3. *Let $\alpha \in (1, \frac{5}{2})$, fix $c > 1$ and suppose that the spectral density $\rho(\lambda)$ is twice continuously differentiable and satisfies*

$$\rho(\lambda) = 0 \quad \textit{for } |\lambda| < 1 \tag{8.11}$$

and

$$\rho(\lambda) = \frac{\rho_0(\theta)}{(1+|\lambda|^2)^\alpha} \quad \textit{for } |\lambda| > 2, \tag{8.12}$$

where $\rho_0(\theta) \geq 0$ is a function of the angle θ. Set $\delta = \min\{2, 6-2\alpha\}$. Then there is a constant C for which the inequalities

$$\left| \int_{R^2} \exp i\langle y, \xi\rangle \left\{ \frac{\sin(u \cdot \xi)}{(u \cdot \xi)} - 1 \right\} \left\{ \frac{\sin(v \cdot \xi)}{(v \cdot \xi)} - 1 \right\} l^{-2\alpha} \rho\left(\frac{\xi}{l}\right) d\xi \right|$$

$$\leq C|y|^{-\delta} \quad \textit{if } \alpha \neq 2 \tag{8.13}$$

$$\leq C|y|^{-2} \log(|y|^{-1}) \quad \textit{if } \alpha = 2 \tag{8.14}$$

hold for all scalars l with $0 < l < 1$, for all y in R^2 and for all vectors u and v with $c^{-1} \leq |u| \leq c$ and $c^{-1} \leq |v| \leq c$.

PROOF. The proof is unchanged for $\alpha = 2$, and for $a < 2$, the proof follows simply by noting that there is a constant for which

$$\|g\|_1 \leq K$$

holds for each function $g = \partial\,\partial F$ that is a second derivative of the function

$$F(u, v, l, \xi) = \left\{ \frac{\sin(u \cdot \xi/2)}{u \cdot \xi/2)} - 1 \right\} \left\{ \frac{\sin(v \cdot \xi)}{(v \cdot \xi)} - |v| \right\} l^{-2\alpha\rho(\xi/l)}.$$

The proof for $\alpha > 2$ follows the same outline used in Lemma 5.5. Namely, for $g = \partial\,\partial F$ it is elementary to show that

$$\|g(\xi)\|_1 \leq k_1 l^{4-2\alpha} \tag{8.15}$$

and

$$\|F(u, v, l, \cdot) - F(u, v, l', \cdot)\|_1 \leq k_2 l^{6-2\alpha}, \tag{8.16}$$

provided only that $0 < l' < l < 1$, $c^{-1} \leq |u| \leq c$ and $c^{-1} \leq |v| \leq c$. Following the route mapped out in the proof of Lemma 5.5 completes the proof. □

Applying this as Lemma 5.5 was used in Section 5, for each $t \in (0, 1)$ we fix j with $j/n \leq t < (j+1)/n$,

$$\lim_{n\to\infty} n^{2\alpha} \sum_{k=1}^{n} f(y_j) f(y_k) \operatorname{Cov}(j, k, n)$$

$$= \lim_{n\to\infty} n^{2\alpha} \sum_{m=1-j}^{n-j} f(y_j) f(y_{j+m}) \operatorname{Cov}(j, j+m, n)$$

$$= f^2(x(t)) \sum_{-\infty}^{\infty} \chi(m, x'(t))$$

$$= f^2(x(t)) |x'(t)|^{2\alpha} c(\theta(t)) \sum_{-\infty}^{\infty} g(m).$$

As occurred earlier in Section 5, the sum $\Sigma g(m)$ is evaluated using the Poisson summation formula, to give (8.3).

This also completes the proof of Theorem 8.1 in the special case of densities of the form given in (8.11) and (8.12). To complete the proof of Theorem 8.1 in the general case we use Lemma 8.4.

LEMMA 8.4. *For fixed $\alpha \in (2, \frac{5}{2})$, let Φ_1, Φ_2 and Φ_3 be continuous stationary Gaussian fields on R^2 with spectral measures $R_1(d\lambda)$, $R_2(d\lambda)$ and $R_3(d\lambda)$, respectively. Suppose, in addition, that*

$$\int_{R^2} |\lambda|^4 R_3(d\lambda) < \infty. \tag{8.17}$$

If

$$R_1(d\lambda) + R_3(d\lambda) \geq R_2(d\lambda), \tag{8.18}$$

then for each smooth curve Γ and smooth function f on Γ,

$$\liminf_{n\to\infty} n^{(2\alpha-1)} \left[E_{\Phi_1} \left\{ I(f, \phi_1, \Gamma, n) - \int_\Gamma f(y)\phi_1(y)\, d\sigma(y) \right\}^2 - E_{\Phi_2} \left\{ I(f, \phi_2, \Gamma, n) - \int_\Gamma f(y)\phi_2(y)\, d\sigma(y) \right\}^2 \right] \geq 0.$$

PROOF. Assuming as we may, that Φ_1 and Φ_3 are independent, it follows from (8.17) that

$$E_{\Phi_1+\Phi_3} \left\{ I(f, \phi_1 + \phi_3, \Gamma, n) - \int_\Gamma f(y)[\phi_1(y) + \phi_3(y)]\, d\sigma(y) \right\}^2$$
$$\geq E_{\Phi_2} \left\{ I(f, \phi_2, \Gamma, n) - \int_\Gamma f(y)\phi_2(y)\, d\sigma(y) \right\}^2$$

holds for all n. However, the assumption (8.16) implies that the process $\psi(t) = f(x(t))\phi_3(x(t))$ is twice mean squared differentiable, and in this case it is elementary to show that

$$\limsup_{n\to\infty} n^4 E \left\{ I(f, \phi_3, \Gamma, n) - \int_\Gamma f(y)\phi_3(y)\, d\sigma(y) \right\}^2$$
$$= \frac{64}{9} \int_0^1 \int_0^1 E\psi''(t)\psi''(s)\, dt\, ds < \infty.$$

Thus

$$\liminf_{n\to\infty} n^{(2\alpha-1)} E_{\Phi_1}\left\{ I(f,\phi_1,\Gamma,n) - \int_\Gamma f(y)\phi_1(y)\,d\sigma(y)\right\}^2$$
$$= \liminf_{n\to\infty} n^{(2\alpha-1)} E_{\Phi_1+\Phi_3}\left\{ I(f,\phi_1+\phi_3,\Gamma,n) \right.$$
$$\left. - \int_\Gamma [f(y)\phi_1(y)+\phi_3(y)]\,d\sigma(y)\right\}^2,$$

and the result follows. □

The proof of Theorem 8.1 is completed by observing that for any spectral density ρ satisfying the conditions of Theorem 8.1, for each $\varepsilon > 0$ it is possible to find smooth functions $\rho_1(\theta)$ and $\rho_2(\theta)$ and $g_1(\lambda)$ and $g_2(\lambda)$ satisfying the following conditions:

$$0 \le \rho_1(\theta) \le \rho_0(\theta) \le \rho_2(\theta) \le \rho_1(\theta)+\varepsilon, \quad 0 \le g_1(\lambda) \text{ and } 0 \le g_2(\lambda), \tag{8.19}$$

and

$$\int_{R^2} |\lambda|^4 [g_1(\lambda)+g_2(\lambda)]\,d\lambda < \infty, \tag{8.20}$$

$$\frac{\rho_1(\theta)}{(1+|\lambda|^2)^\alpha} \le \rho(\lambda)+g_1(\lambda), \tag{8.21}$$

$$\rho(\lambda) \le \frac{\rho_2(\theta)}{(1+|\lambda|^2)^\alpha} + g_2(\lambda). \tag{8.22}$$

Condition (8.17) implies that the fields with spectral densities $g_1(\lambda)$ and $g_2(\lambda)$ are twice mean square differentiable. By Lemma 8.4 and (8.18) and (8.19), we can sandwich the asymptotic errors

$$n^{(2\alpha-1)} E\left\{ I(f,\phi,\Gamma,n) - \int_\Gamma f(y)\phi(y)\,d\sigma(y)\right\}^2$$

for the field Φ with spectral density ρ between the errors for the fields with densities $(\rho_1(\theta))/(1+|\lambda|^2)^\alpha$ and $(\rho_2(\theta))/(1+|\lambda|^2)^\alpha$. These last two asymptotic errors are computable from Lemma 8.3, and by (8.16) the resulting bounds are arbitrarily close to the right-hand side of (8.3). □

REMARK. We conclude with a remark concerning the restriction $1 < \alpha < \frac{5}{2}$. The condition $1 < \alpha$ is simply a reflection that the density $\rho(\lambda)$ must be integrable unless Φ is a generalized field. The condition that $\alpha < \frac{5}{2}$ is, however, an expression of the fact that all of the technical estimates in Lemmas 8.2–8.4 break down at $\alpha = \frac{5}{2}$. This represents an essential transition. When $\alpha = \frac{5}{2}$, a closer analysis will show that midpoint methods give an asymptotic error of the form const. $n^{-4}\log(n)$; for $\alpha > \frac{5}{2}$, the midpoint methods can be shown to be less than optimal. In fact, they produce asymptotic squared errors of the order of n^{-4} for all $\alpha > \frac{5}{2}$.

To optimally approximate integrals of the form $\int_\Gamma f(y)\phi(y)\,d\sigma(y)$ when α is greater than $\frac{5}{2}$ it is necessary to make more refined approximations of Γ than the piecewise linear approximations used here, and it is necessary to adjust the weights used at the ends of Γ, in a manner analogous to those used in Simpson's method. With these adjustments it is possible to extend the statement and proof of Theorem 8.1 to include values of α greater than $\frac{5}{2}$. A similar situation with multidimensional integrals is discussed by Stein [13].

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