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DIFFUSION APPROXIMATION OF NUCLEAR SPACE-VALUED STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY POISSON RANDOM MEASURES[1]

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Under suitable conditions, it is proved that limits of sequences of nuclear space-valued solutions of stochastic differential equations driven by Poisson random measures are characterized by diffusion equations. The results are applied to models of environmental pollution and to reversal potential models of neuronal behavior.

1. Introduction. Motivated by applications to neurophysiological problems, in joint work with Hardy and Ramasubramanian we studied stochastic differential equations (SDE's) on duals of countably Hilbertian nuclear spaces driven by Poisson random measures in Kallianpur, Xiong, Hardy and Ramasubramanian (1994). The stimuli received by neurons are of the form of electrical impulses and were modeled by Poisson random measures. When the pulses arrive frequently enough and the magnitudes are small enough, it is reasonable to expect that the compensated Poisson random measures are approximated by Gaussian white noises in space–time and, hence, the discontinuous processes of voltage potentials of spatially extended neurons governed by Poisson random measures are approximated by diffusion processes.

Let Φ be a countably Hilbertian nuclear space and let Φ' be its dual space. Let $(U, \mathscr{E})$ be a separable measurable space and let μ^n be a sequence of σ-finite measures on U. Let N^n be a sequence of Poisson random measures on $\mathbb{R}_+ \times U$ with intensity measures μ^n. Let A^n: $\mathbb{R}_+ \times \Phi' \to \Phi'$ and G^n: $\mathbb{R}_+ \times \Phi' \times U \to \Phi'$ be two sequences of measurable mappings on the corresponding spaces.

In this paper, we consider a sequence of SDE's

$$(1.1)\qquad X_t^n = X_0^n + \int_0^t A^n(s, X_s^n)\, ds + \int_0^t \int_U G^n(s, X_{s-}^n, u) \tilde{N}^n(du\, ds),$$

where $\{X_0^n\}$ is a sequence of Φ'-valued random variables and $\tilde{N}^n$ is the "compensated random measure" of N^n in the terminology of Jacod and Shiryaev (1987). We prove that, under suitable conditions, the sequence of

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unique solutions of the SDE's (1.1) converges in distribution to the unique solution of the diffusion equation

$$X_t = X_0 + \int_0^t A(s, X_s)\, ds + \int_0^t B(s, X_s)\, dW_s, \tag{1.2}$$

where A: $\mathbb{R}_+ \times \Phi' \to \Phi'$ and B: $\mathbb{R}_+ \times \Phi' - \mathscr{L}(\Phi', \Phi')$ are two measurable mappings and W is a Φ'-valued Wiener process.

Diffusion approximations for linear models were studied by Kallianpur and Wolpert (1984). The same authors also considered this problem for reversal potential models of point neurons, which turns out to be real-valued nonlinear SDE's [cf. Kallianpur and Wolpert (1987)]. Diffusion approximations for various models of the voltage potentials of point neurons were also studied by Tuckwell (1989). Kallianpur, Mitoma and Wolpert (1990) have obtained conditions under which the SDE (1.2) has a unique solution and have pointed out the importance of the study of the diffusion approximation for infinite-dimensional nonlinear equations.

This paper is organized as follows: In Section 2, we give some basic facts about countably Hilbertian nuclear spaces and SDE's on the duals of nuclear spaces for the convenience of the reader. Section 3 is the pivotal part of the paper. We prove that the solutions of the sequence of SDE's (1.1) converges in distribution to the unique solution of the SDE (1.2). Also we show that every diffusion process which satisfies the conditions of Kallianpur, Mitoma and Wolpert (1990) can be approximated by a sequence of Poisson random measure driven processes.

Finally, we apply our results to the diffusion approximation of environmental pollution models studied in Kallianpur and Xiong (1994). The same problem for the reversal potential model of the voltage potential of a spatially extended neuron is briefly discussed. Proposition 4.1 yields a diffusion approximation whose solution, in some cases, is distribution-valued.

2. SDE's on duals of nuclear spaces. For the convenience of the reader, we state some basic results of Kallianpur, Xiong, Hardy and Ramasubramanian (1994) and Kallianpur, Mitoma and Wolpert (1990) about SDE's on the duals of countably Hilbertian nuclear spaces either driven by Poisson random measures or by nuclear space-valued Wiener processes.

DEFINITION 2.1. We call Φ a countably Hilbertian nuclear space if Φ is a separable Fréchet space whose topology is given by an increasing sequence of Hilbertian norms $\|\cdot\|_n$, $n \geq 0$, such that if H_n is the completion of Φ with respect to the norm $\|\cdot\|_n$, then for each n there exists $m > n$ such that the canonical injection $H_m \to H_n$ is Hilbert–Schmidt.

In the above definition, the canonical injection $H_m \to H_n$ is Hilbert–Schmidt means that there exists a complete orthonormal system (CONS) $\{e_k^{m,n}\}_{k=1}^\infty$ of H_m such that $\sum_k \|e_k^{m,n}\|_n^2$ is finite.

Let H_{-n} and Φ' denote the duals of H_n and Φ, respectively. As in Kallianpur, Xiong, Hardy and Ramasubramanian (1994), we always assume that there exists a sequence $\{h_i\}$ of elements of Φ such that $\{h_i\}$ is a CONS of H_0 and is a complete orthogonal system (COS) of H_n for any $n \in \mathbb{Z}$. This assumption holds for many examples of Φ', notably the space $\mathscr{S}'(\mathbb{R}^d)$ of Schwarz distributions on $\mathbb{R}^d$.

Let $h_i^n = h_i \|h_i\|_n^{-1}$ for $n \in \mathbb{Z}$ and $i \in \mathbb{Z}^+$. Then $\{h_i^n\}$ is a CONS of H_n. For each $n > 0$, we define a linear isometry θ_n from H_{-n} onto H_n such that $\theta_n(h_i^{-n}) = h_i^n$ for any $i \in \mathbb{Z}^+$.

DEFINITION 2.2. We call W a Φ'-valued Wiener process with covariance Q if, $\forall \phi \in \Phi$, $W_.[\phi]$ is a real-valued Wiener process such that $E|W_t[\phi]|^2 = tQ(\phi, \phi)$, where Q is a continuous bilinear form on $\Phi \times \Phi$.

Let $(\Omega, \mathscr{F}, \mathscr{P}, \mathbb{F})$ be a stochastic basis. We first consider the following SDE on Φ':

$$X_t = X_0 + \int_0^t A(s, X_s)\, ds + \int_0^t \int_U G(s, X_{s-}, u) \tilde{N}(du\, ds), \tag{2.1}$$

where $N(du\, ds)$ is a Poisson random measure on $U \times \mathbb{R}_+$ with σ-finite intensity measure μ on U, X_0 is a Φ'-valued random variable and A: $R_+ \times \Phi' \to \Phi'$ and G: $R_+ \times \Phi' \times U \to \Phi'$ are two measurable mappings.

To obtain the existence and uniqueness of the solution of (2.1), we need the conditions introduced in Kallianpur, Xiong, Hardy and Ramasubramanian (1994). For simplicity of notation, we rewrite them as the following assumptions:

ASSUMPTIONS S. For any $T > 0$, there exists an index $p_0 = p_0(T)$ such that $\forall\, p \geq p_0$, $\exists\, q \geq p$ and a constant $K = K(p, q, T)$ such that:

(S1) (*Continuity*.) For every $t \in [0, T]$, $A(t, \cdot)$: $H_{-p} \to H_{-q}$ is continuous; $\forall\ t \in [0, T]$ and $v \in H_{-p}$, $G(t, v, \cdot) \in L^2(U, \mu; H_{-p})$ and, for t fixed, the map $v \to G(t, v, \cdot)$ is continuous from H_{-p} to $L^2(U, \mu; H_{-p})$.

(S2) (*Coercivity*.) For every $t \in [0, T]$ and $\phi \in \Phi \subset \Phi'$,

$$2A(t, \phi)\big[\theta_p(\phi)\big] \leq K\big(1 + \|\phi\|_{-p}^2\big). \tag{2.2}$$

(S3) (*Growth*.) For every $t \in [0, T]$ and $v \in H_{-p}$,

$$\|A(t, v)\|_{-q}^2 \leq K\big(1 + \|v\|_{-p}^2\big) \quad \text{and}$$
$$\int_U \|G(t, v, u)\|_{-p}^2\, \mu(du) \leq K\big(1 + \|v\|_{-p}^2\big). \tag{2.3}$$

(S4) (*Monotonicity*.) For every $t \in [0, T]$, $v_1, v_2 \in H_{-p}$,

$$\begin{aligned} &2\langle A(t, v_1) - A(t, v_2), v_1 - v_2\rangle_{-q} \\ &\quad + \int_U \|G(t, v_1, u) - G(t, v_2, u)\|_{-q}^2\, \mu(du) \\ &\quad \leq K\|v_1 - v_2\|_{-q}^2. \end{aligned} \tag{2.4}$$

REMARK 2.1. The left-hand side of (2.2) is well defined as $\theta_p\Phi \subset \Phi$. In fact, we only need to show that for any $p, r \geq 0$ and $\phi \in \Phi$, we have $\theta_p \phi \in \Phi_r$. As

$$\theta_p(\phi) = \theta_p\left(\sum_j \langle \phi, h_j^r\rangle_r h_j^r\right) = \theta_p\left(\sum_j \langle \phi, h_j^r\rangle_r \|h_j\|_p^{-1}\|h_j\|_r^{-1} h_j^{-p}\right)$$
$$= \sum_j \langle \phi, h_j^r\rangle_r \|h_j\|_p^{-1}\|h_j\|_r^{-1} h_j^p = \sum_j \langle \phi, h_j^r\rangle_r \|h_j\|_p^{-2} h_j^r$$

and

$$\sum_j \left(\langle \phi, h_j^r\rangle_r \|h_j\|_p^{-2}\right)^2 \leq \sum_j \langle \phi, h_j^r\rangle_r^2 = \|\phi\|_r^2 < \infty,$$

we see that $\theta_p(\phi) \in \Phi_r$.

Let $\mathbb{D}([0,T], H_{-p})$ be the space, with the usual Skorohod topology, of all mappings of $[0,T]$ to H_{-p} that are right continuous and have left limits. Let $\mathbb{D}([0,T], \Phi')$ be the space of all mappings of $[0,T]$ to Φ' that are right continuous and have left limits. The topology of $\mathbb{D}([0,T], \Phi')$ is given in Mitoma's sense [see Mitoma (1983) for more details]. The following result was proved in Kallianpur, Xiong, Hardy and Ramasubramanian (1994).

THEOREM 2.1. *Under Assumptions* S, (2.1) *has a unique Φ'-valued solution X if we have r_0 such that $E\|X_0\|_{-r_0}^2 < \infty$. Furthermore, let $p(T) = \max(r_0, p_0(T))$ and $p_1(T) \geq p(T)$ such that the canonical injection from $H_{p_1(T)} \to H_{p(T)}$ is Hilbert–Schmidt. Then $X|_{[0,T]} \in \mathbb{D}([0,T], H_{-p_1(T)})$ and*

$$E \sup_{0 \leq t \leq T} \|X_t\|_{-p_1(T)}^2 \leq \tilde{K}, \tag{2.5}$$

where $\tilde{K}$ is a finite constant that depends only on K, T and $E\|X_0\|_{-p_1(T)}^2$.

It is of interest to consider the martingale problem posed by (1.2). Let

$$\mathscr{D}_0^\infty(\Phi') = \{F\colon \Phi' \to \mathbb{R}\colon \exists\, h \in C_0^\infty(\mathbb{R}) \text{ and } \phi \in \Phi \text{ s.t. } F(v) = h(v(\phi))\}. \tag{2.6}$$

For $F \in \mathscr{D}_0^\infty(\Phi')$, we define a map $\mathscr{D}_s F\colon \Phi' \to \mathbb{R}$ by

$$\mathscr{D}_s F(v) \equiv A(s,v)[\phi]h'(v[\phi]) + \tfrac{1}{2}h''(v[\phi])Q(B(s,v)^*\phi, B(s,v)^*\phi), \tag{2.7}$$

where $B(s,v)^*\colon \Phi \to \Phi$ is the adjoint operator of $B(s,v)$. For $Z \in \mathbb{D}([0,T], \Phi')$, let

$$M^F(Z)_t \equiv F(Z_t) - F(Z_0) - \int_0^t \mathscr{D}_s F(Z_s)\, ds. \tag{2.8}$$

Let $\mathbb{C}([0,T], H_{-p})$ be the space, with the usual uniform topology, of all continuous mappings of $[0,T]$ to H_{-p}. Let $\mathbb{C}([0,T], \Phi')$ be the space of all continuous mappings of $[0, T]$ to Φ'. The topology of $\mathbb{C}([0, T], \Phi')$ is given in Mitoma's sense [see Mitoma (1983) for more details].

Let $\tilde{\mathscr{B}}_T = \mathscr{B}_T(\mathbb{C}([0, T], \Phi'))$ be the σ-field of all Borel sets in $\mathbb{C}([0, T], \Phi')$. For each $t \in [0, T]$, let $\tilde{\mathscr{B}}_t = \pi_t^{-1}\tilde{\mathscr{B}}_T$, where π_t: $\mathbb{C}([0, T], \Phi') \to \mathbb{C}([0, T], \Phi')$ is given by $(\pi_t x)_s = x_{t \wedge s}$, $\forall\ s \in [0, T]$.

DEFINITION 2.3. A probability measure P on $(\mathbb{C}([0, T], \Phi'), \tilde{\mathscr{B}}_T)$ is called a solution of the $\mathscr{D}$-martingale problem if, $\forall\ F \in \mathscr{D}_0^\infty(\Phi')$, $\{M^F(Z)_t\}$ is a P-martingale with respect to the filtration $\{\tilde{\mathscr{B}}_t\}$.

Now we summarize the results of Kallianpur, Mitoma and Wolpert [(1990), Sections 6 and 7] as a lemma to suit our purpose. First we make the following assumptions for the SDE (1.2) which is similar to the conditions given in Kallianpur, Mitoma and Wolpert (1990), but with weaker initial and continuity conditions.

ASSUMPTIONS D. For any $T > 0$ there exists an index $p_0 = p_0(T)$ such that, $\forall\ p \geq p_0$, $\exists\ q \geq p$ and a constant $K = K(p, q, T)$ such that:

(D1) (*Continuity.*) For every $t \in [0, T]$, $v \in H_{-p}$ and $v_1, v_2 \in H_{-p}$, $A(t, v) \in H_{-q}$ and $B(t, v_1)(v_2) \in H_{-p}$. Furthermore, for t fixed, $A(t, v)$ and $|Q_{B(t, v_1) - B(t, v_2)}|_{-p, -p}$ are continuous in v, v_1 and v_2, where

$$|Q_{B(t, v_1) - B(t, v_2)}|_{-p, -p} = \sum_j Q\big((B(s, v_1) - B(s, v_2))^* h_j^p, (B(s, v_1) - B(s, v_2))^* h_j^p\big). \tag{2.9}$$

(D2) (*Coercivity.*) For every $t \in [0, T]$ and $\phi \in \Phi$,

$$2A(t, \phi)\big[\theta_p(\phi)\big] \leq K\big(1 + \|\phi\|_{-p}^2\big). \tag{2.10}$$

(D3) (*Growth.*) For every $t \in [0, T]$ and $v \in H_{-p}$,

$$\begin{aligned} \|A(t, v)\|_{-q}^2 &\leq K\big(1 + \|v\|_{-p}^2\big) \quad \text{and} \\ |Q_{B(t, v)}|_{-p, -p} &\leq K\big(1 + \|v\|_{-p}^2\big). \end{aligned} \tag{2.11}$$

(D4) (*Monotonicity.*) For every $t \in [0, T]$ and $v_1, v_2 \in H_{-p}$,

$$\begin{aligned} 2\langle A(t, v_1) - A(t, v_2), v_1 - v_2\rangle_{-q} &+ |Q_{B(t, v_1) - B(t, v_2)}|_{-q, -q} \\ &\leq K\|v_1 - v_2\|_{-q}^2. \end{aligned} \tag{2.12}$$

(D5) (*Initial.*) There exists an index r_0 such that $E\|X_0\|_{-r_0}^2 < \infty$.

LEMMA 2.1. (i) *Suppose that Assumptions* (D1)–(D3) *and* (D5) *hold. If* P^* *is a solution of the* $\mathscr{D}$*-martingale problem and there exist an index* p *and a constant* $K_1(p)$ *such that*

$$E^{P^*}\left\{\int_0^T Q(B(s, Z_s)^*\phi, B(s, Z_s)^*\phi)\, ds\right\} \leq K_1(p)\|\phi\|_p^2, \tag{2.13}$$

then P^* *is a weak solution of* (1.2).

(ii) *Furthermore, if A and B also satisfy the monotonicity condition* (D4), *then the $\mathscr{D}$-martingale problem has a unique solution which is the distribution of the unique solution of* (1.2).

REMARK 2.2. In applications of the above lemma, condition (2.13) is usually verified by showing that

$$E^{P^*} \sup_{0 \le t \le T} \|Z_t\|_{-p}^2 < \infty \cdot$$

and making use of (2.11).

3. Diffusion approximation. Let $\{P^n\}$ be a sequence of probability measures on $\mathbb{D}([0,T],\Phi')$ induced by the solutions of the SDE's (1.1). In this section, we first prove that, under suitable conditions, the sequence $\{P^n\}$ is tight. Then we show that any limit point is supported on continuous path space and is a solution of the $\mathscr{D}$-martingale problem corresponding to the SDE (1.2). It follows from Lemma 2.1 that under the monotonicity condition, the $\mathscr{D}$-martingale problem has a unique solution which is the distribution of the unique solution of the diffusion equation (1.2). Finally we also show that any Φ'-valued diffusion processes given by a SDE of the form (1.2) which satisfies the Assumptions D of the previous section can be approximated by a sequence of Φ'-valued processes driven by Poisson random measures.

The following lemma has appeared in Kallianpur, Xiong, Hardy and Ramasubramanian (1994), and we give an outline of its proof for the sake of completeness.

LEMMA 3.1. *Suppose that, for each* n, (A^n, G^n, μ^n) *satisfies Assumptions* S *of Section* 2 *and that the indexes* p_0, p, q *and the constant* K *are independent of* n. *Also assume that there exists an index* r_0 *such that*

$$\sup_n E\|X_0^n\|_{-r_0}^2 < \infty. \tag{3.1}$$

Then the sequence $\{P^n\}$ *is tight in* $\mathbb{D}([0,T], H_{-p_2(T)})$, *where* $p_2(T) \ge p_1(T)$ *is such that the canonical injection from* $H_{-p_1(T)}$ *into* $H_{-p_2(T)}$ *is Hilbert–Schmidt.*

PROOF. By the assumptions and Theorem 2.1, (2.1) has a unique solution X^n taking values in $H_{-p_1(T)}$ and

$$\sup_n E_0 \sup_{0 \le t \le T} \|X_t^n\|_{-p_1(T)}^2 \le \tilde{K}\Big(K, T, \sup_n E\|X_0^n\|_{-p_1(T)}^2\Big) < \infty. \tag{3.2}$$

For any $\phi \in \Phi$ fixed, let

$$\begin{aligned} C_t^n &= \int_0^t A^n(s, X_s^n)[\phi]\, ds \quad \text{and} \\ M_t^n &= \int_0^t \int_U G^n(s, X_{s-}^n, u)[\phi] \tilde{N}^n(du\, ds). \end{aligned} \tag{3.3}$$

Then the real-valued semimartingale $X^n[\phi]$ can be written as $X_t^n[\phi] = X_0^n[\phi] + C_t^n + M_t^n$. It follows from (S3) and (3.2) that $\{C^n\}$ is $\mathbb{C}$-tight [that is, tight in $\mathbb{D}([0,T])$ and the limit points have continuous paths]. Similarly we can prove the $\mathbb{C}$-tightness for $\{\langle M^n\rangle\}$. It follows from (3.1) that $\{X_0^n[\phi]\}$ is a tight sequence of random variables. Hence, by Jacod and Shiryaev [(1987), Corollary 3.33, page 317, and Theorem 4.13, page 322] the sequence of semimartingales $X_t^n[\phi]$ is tight in $\mathbb{D}([0,T])$. So, it follows from Mitoma's argument [see Mitoma (1983) for reference] that $\{P^n\}$ is tight in $\mathbb{D}([0,T],\Phi')$.

Making use of the inequality (3.2) again, we can show that $\{P^n\}$ is uniformly $p_1(T)$-continuous in Mitoma's sense [see Mitoma (1983) for the definition] and hence, $\{P^n\}$ is tight in $\mathbb{P}([0,T], H_{-p_2(T)})$. □

To characterize the limit points of the sequence $\{P^n\}$, we introduce the following assumptions:

ASSUMPTIONS A. There exist a continuous quadratic form Q on Φ and two measurable maps $A\colon \mathbb{R}_+\times \Phi' \to \Phi'$ and $B\colon \mathbb{R}_+\times\Phi' \to \mathscr{L}(\Phi',\Phi')$ such that:

(A1) The sequence $\{X_0^n\}$ converges to an $H_{-p_2(T)}$-valued random variable X_0 in distribution.

(A2) For every $t\in[0,T]$, $p \ge p_0$ and compact subset C_0 of Φ_{-p}, we have

$$\lim_{n\to\infty}\sup_{v\in C_0}|A^n(t,v) - A(t,v)|_{-q} = 0.$$

(A3) For every $t\in[0,T]$, $\phi\in\Phi$, $a>0$, $p\ge p_0$ and compact subset C_0 of Φ_{-p}, we have

$$\lim_{n\to\infty}\sup_{v\in C_0}\mu^n\{u\colon |G^n(t,v,u)[\phi]| > a\} = 0, \tag{3.4}$$

$$\lim_{n\to\infty}\sup_{v_1,v_2\in C_0}\left|\int_U G^n(t,v_1,u)[\phi]G^n(t,v_2,u)[\phi]\,\mu^n(du) - Q(B(t,v_1)^*\phi, B(t,v_2)^*\phi)\right| = 0 \tag{3.5}$$

and

$$\lim_{M\to\infty}\sup_{\substack{v\in C_0\\ n\in\mathbb{N}}}\int_U |G^n(t,v,u)[\phi]|^2 1_{\{|G^n(t,v,u)[\phi]|\ge M\}}\mu^n(du) = 0. \tag{3.6}$$

Condition (A3), together with the conditions of Lemma 3.1, ensure that any limit process of the sequence $\{X^n\}$ has continuous paths.

LEMMA 3.2. *Let P^* be a cluster point of the sequence $\{P^n\}$ on $\mathbb{D}([0,T],H_{-p_2(T)})$. If the sequence (A^n, G^n, μ^n, X_0^n) satisfies the conditions of Lemma* 3.1 *and assumption* (A3), *then*

$$P^*\Big(\mathbb{C}\big([0,T],H_{-p_2(T)}\big)\Big) = 1. \tag{3.7}$$

PROOF. Let g be a nonnegative continuous function on $\mathbb{R}$ vanishing in a neighborhood of 0 and of ∞ [g_m, $m \in \mathbb{N}$, of (3.21) are examples of such functions]. For any $\phi \in \Phi$, let $\{F^n\}$ be a sequence of maps from $\mathbb{D}([0,T], H_{-p_2(T)})$ to $\mathbb{R}$ given by

$$F^n(Z) \equiv \sum_{0<s\leq T} g(\Delta Z_s[\phi]) - \int_0^T \int_U g(G^n(s, Z_s, u)[\phi])\mu^n(du)\, ds. \tag{3.8}$$

Without loss of generality, we assume that P^n converges to P^* weakly. Making use of a theorem of Skorohod [see Ikeda and Watanabe (1981), Theorem 2.7, page 9], there exists a probability space $(\Omega', \mathscr{F}', P')$ and $\mathbb{D}([0,T], H_{-p_2(T)})$-valued random variables ξ^n and ξ with distributions P^n and P^*, respectively, such that ξ^n tends to ξ P'-a.s.

We now divide the proof into four steps.

Step 1. First we show that

$$F^n(\xi^n) \to \sum_{0<s\leq T} g(\Delta \xi_s[\phi]) \quad \text{in probability.} \tag{3.9}$$

By the tightness of $\{P^n\}$, for any $\varepsilon > 0$, there exists a compact set C of $\mathbb{D}([0,T], H_{-p_2(T)})$ such that $P^n(C) > 1 - \varepsilon$, $\forall\, n \geq 1$. Let C_0 be a compact subset of $H_{-p_2(T)}$ and M a constant such that

$$C \subset \left\{Z \in \mathbb{D}\big([0,T], H_{-p_2(T)}\big) : Z_t \in C_0, \forall\, t \in [0,T]\right\} \tag{3.10}$$

and

$$C_0 \subset \left\{v \in H_{-p_2(T)} : \|v\|_{-p_2(T)} \leq M\right\}. \tag{3.11}$$

Let $b > 0$ be such that $g(x) = 0$ for any $|x| \leq b$. Then, $\forall\, a > 0$,

$$\begin{aligned} P'\bigg(\omega: &\int_0^T \int_U g(G^n(s, \xi_s^n, u)[\phi])\mu^n(du)\, ds > a\bigg) \\ &\leq P'(\omega: \xi^n \notin C) \\ &\quad + \frac{1}{a} E^{P'} \int_0^T \int_U g(G^n(s, \xi_s^n, u)[\phi])\mu^n(du)\, ds\, 1_C(\xi^n) \\ &\leq \varepsilon + \frac{1}{a} E^{P'} \int_0^T \mu^n\{u: |G^n(s, \xi_s^n, u)[\phi]| > b\} 1_C(\xi^n) \|g\|_\infty\, ds \\ &\leq \varepsilon + \frac{\|g\|_\infty}{a} \int_0^T \sup_{v \in C_0} \mu^n\{u: |G^n(t, v, u)[\phi]| > b\}\, ds, \end{aligned} \tag{3.12}$$

where $\|g\|_\infty$ denotes the supremum norm of the bounded function g. As

$$\begin{aligned} \sup_{v \in C_0} &\mu^n\{u: |G^n(t, v, u)[\phi]| > b\} \\ &\leq \sup_{v \in C_0} \frac{\|\phi\|_{p_2(T)}^2}{b^2} \int_U \|G^n(t, v, u)\|_{-p_2(T)}^2 \mu^n(du) \\ &\leq \frac{\|\phi\|_{p_2(T)}^2}{b^2} K(1 + M^2), \end{aligned} \tag{3.13}$$

it follows from (A3) and the bounded convergence theorem that

$$\limsup_{n\to\infty} P'\left(\omega\colon \int_0^T\int_U g(G^n(s,\xi_s^n,u)[\phi])\mu^n(du)\,ds > a\right) \le \varepsilon; \tag{3.14}$$

that is,

$$\int_0^T\int_U g(G^n(s,\xi_s^n,u)[\phi])\mu^n(du)\,ds \to_{P'} 0. \tag{3.15}$$

Note that

$$\sum_{0<s\le T} g(\Delta\xi_s^n[\phi]) \to \sum_{0<s\le T} g(\Delta\xi_s[\phi]) \quad P'\text{-a.s.} \tag{3.16}$$

This proves (3.9).

Step 2. The sequence $\{F^n(\xi^n)\}_{n\in\mathbb{N}}$ is uniformly integrable.

For each n, let p^n and D^n be the point process and jump set, respectively, corresponding to the Poisson random measure N^n. As X^n is a solution of the SDE (1.1), then

$$\begin{aligned} F^n(X^n) &= \sum_{0<s\le T} g(G^n(s,X_s^n,p^n(s))[\phi]1_{D^n}(s)) \\ &\quad - \int_0^T\int_U g(G^n(s,X_s^n,u)[\phi])\mu^n(du)\,ds \\ &= \int_0^T\int_U g(G^n(s,X_{s-}^n,u)[\phi])\tilde{N}^n(du\,ds). \end{aligned} \tag{3.17}$$

As G^n satisfies (S3) uniformly for n, it follows from (2.5) that

$$\begin{aligned} &\sup_n E^{P'}|F^n(\xi^n)|^2 \\ &\quad = \sup_n E|F^n(X^n)|^2 \\ &\quad = \sup_n E\int_0^T\int_U g(G^n(s,X_s^n,u)[\phi])^2\mu^n(du)\,ds \\ &\quad \le \sup_n E\int_0^T\int_U K_g(G^n(s,X_s^n,u)[\phi])^2\mu^n(du)\,ds \\ &\quad \le K_g \sup_n E\int_0^T\int_U \|G^n(s,X_s^n,u)\|_{-p_1(T)}^2\|\phi\|_{p_1(T)}^2\mu^n(du)\,ds \\ &\quad \le K_g\|\phi\|_{p_1(T)}^2 \sup_n E\int_0^T K\left(1+\|X_s^n\|_{-p_1(T)}^2\right)ds \\ &\quad \le K_g\|\phi\|_{p_1(T)}^2 K(1+\tilde{K})T, \end{aligned} \tag{3.18}$$

where $K_g \equiv \sup\{(g(x)/x)^2\colon x\in\mathbb{R}\}$ is finite. This proves the assertion of step 2.

Step 3. We have

$$E^{P^*} \sum_{0<s\le T} g(\Delta Z_s[\phi]) = 0. \tag{3.19}$$

It follows from (3.17) that $E(F^n(X^n)) = 0$ for any $n \in \mathbb{N}$. Hence

$$\begin{aligned} E^{P^*} \sum_{0<s\le T} g(\Delta Z_s[\phi]) &= E^{P'} \sum_{0<s\le T} g(\Delta \xi_s[\phi]) \\ &= \lim_{n\to\infty} E^{P'}(F^n(\xi^n)) \\ &= \lim_{n\to\infty} E(F^n(X^n)) = 0. \end{aligned} \tag{3.20}$$

Step 4. Equation (3.7) holds.

Let $\{g_m\}$ be a sequence of continuous functions on $\mathbb{R}$ vanishing in a neighborhood of 0 and ∞ such that $\{g_m(x)\}$ increases to x^2 as m tends to ∞. For example, we take

$$g_m(x) = \begin{cases} 0, & \text{if } |x| \le 1/(m+1) \\ & \quad \text{or } |x| \ge m+1, \\ x^2, & \text{if } 1/m \le |x| \le m, \\ ((m+1)|x|-1)/m, & \text{if } 1/(m+1) \le |x| \le 1/m, \\ m^2(m+1-|x|), & \text{if } m \le |x| \le m+1. \end{cases} \tag{3.21}$$

Making use of the monotone convergence theorem, we have

$$E^{P^*} \sum_{0<s\le T} |\Delta Z_s[\phi]|^2 = 0, \qquad \forall \phi \in \Phi. \tag{3.22}$$

Taking $\phi = h_j^{p_2(T)}$, $j = 1, 2, \dots,$ and adding, we have

$$E^{P^*} \sum_{0<s\le T} \|\Delta Z_s\|^2_{-p_2(T)} = 0. \tag{3.23}$$

This proves (3.7) and hence finishes the proof of the lemma. □

LEMMA 3.3. *Under the conditions of Lemma* 3.1, *if Assumptions* A *hold*, *then* P^* *is a solution of the* $\mathscr{D}$*-martingale problem.*

PROOF. For $F \in \mathscr{D}_0^\infty(\Phi')$, let $\mathscr{L}_s^n F$ be a map for Φ' to $\mathbb{R}$ given by

$$\begin{aligned} \mathscr{L}_s^n F(v) &= A^n(s,v)[\phi]h'(v[\phi]) \\ &\quad + \int_U \{h(v[\phi] + G^n(s,v,u)[\phi]) - h(v[\phi]) \\ &\qquad - G^n(s,v,u)[\phi]h'(v[\phi])\}\mu^n(du). \end{aligned} \tag{3.24}$$

For $Z \in \mathbb{D}([0,T], \Phi')$, let

$$M_n^F(Z)_t = F(Z(t)) - F(Z(0)) - \int_0^t \mathscr{L}_s^n F(Z(s))\,ds. \tag{3.25}$$

Let ξ^n, ξ, C, C_0, M and P' be as in the proof of Lemma 3.2. Note that

$$\begin{aligned} |M_n^F(\xi^n)_t - M^F(\xi)_t| &\le I_1^n(t) + I_1^n(0) \\ &\quad + \int_0^t I_2^n(s)\,ds + \left|\int_0^t I_3^n(s)\,ds\right|, \end{aligned} \tag{3.26}$$

where

$$I_1^n(s) = |h(\xi_s^n[\phi]) - h(\xi_s[\phi])|, \tag{3.27}$$

$$\begin{aligned} I_2^n(s) = |&A^n(s, \xi_s^n)[\phi]h'(\xi_s^n[\phi]) \\ &-A(s, \xi_s)[\phi]h'(\xi_s[\phi])| \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} I_3^n(s) = \int_U \{&h(\xi_s^n[\phi] + G^n(s, \xi_s^n, u)[\phi]) - h(\xi_s^n[\phi]) \\ &-G^n(s, \xi_s^n, u)[\phi]h'(\xi_s^n[\phi])\}\mu^n(du) \\ &-\tfrac{1}{2}h''(\xi_s[\phi])Q(B(s, \xi_s)^*\phi, B(s, \xi_s)^*\phi). \end{aligned} \tag{3.29}$$

Now we prove that $\forall\ t \in [0, T]$,

$$E^{P'}|M_n^F(\xi^n)_t - M^F(\xi)_t| \to 0 \quad \text{as } n \to \infty. \tag{3.30}$$

It follows from the uniform continuity of h'' that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|h''(x) - h''(y)| < \varepsilon$ whenever $|x - y| < \delta$. Letting

$$\begin{aligned} D_n \equiv \Big\{u: \Big|\int_0^1\int_0^1 \alpha\big(&h''(\xi_s^n[\phi] + \alpha\beta G^n(s, \xi_s^n, u)[\phi]) \\ &-h''(\xi_s^n[\phi])\big)\, d\alpha\, d\beta\Big| > \varepsilon\Big\}, \end{aligned} \tag{3.31}$$

we have

$$\mu^n(D_n)1_C(\xi^n) \le \sup_{v \in C_0} \mu^n\{u: |G^n(s, v, u)[\phi]| > \delta\} \to 0. \tag{3.32}$$

Note that

$$\begin{aligned} &|I_3^n(s)|1_C(\xi^n)1_C(\xi) \\ &\le \Big|\int_U \Big\{\int_0^1\int_0^1 \alpha\big(h''(\xi_s^n[\phi] + \alpha\beta G^n(s, \xi_s^n, u)[\phi]) - h''(\xi_s^n[\phi])\big)\, d\alpha\, d\beta\Big\} \\ &\qquad\qquad \times G^n(s, \xi_s^n, u)[\phi]^2 \mu^n(du)\Big| 1_C(\xi^n) \\ &\quad + \tfrac{1}{2}|h''(\xi_s^n[\phi]) - h''(\xi_s[\phi])| \sup_{v \in C_0} \int_U G^n(s, v, u)[\phi]^2 \mu^n(du) \\ &\quad + \tfrac{1}{2}|h''(\xi_s[\phi])| \sup_{v \in C_0} \Big|\int_U G^n(s, v, u)[\phi]^2 \mu^n(du) \\ &\qquad\qquad - Q(B(s, v)^*\phi, B(s, v)^*\phi)\Big| \\ &\quad + \tfrac{1}{2}|h''(\xi_s[\phi])|\,|Q(B(s, \xi_s^n)^*\phi, B(s, \xi_s^n)^*\phi) \\ &\qquad\qquad - Q(B(s, \xi_s)^*\phi, B(s, \xi_s)^*\phi)|. \end{aligned} \tag{3.33}$$

It follows from the continuity of h'' that the second term tends to 0 P'-a.s. Condition (A3) implies that the third term converges to 0 for all $\omega \in \Omega'$. It follows from (3.5) and the continuity of G^n that $Q(B(s,v)^*\phi, B(s,v)^*\phi)$ is continuous in v and hence the fourth term tends to 0 P'-a.s. Note that the first term is dominated by

$$\begin{aligned}
&\sup_{v\in C_0}\int_U \varepsilon G^n(s,v,u)[\phi]^2\mu^n(du)\\
(3.34)\qquad &+\|h''\|_\infty \sup_{v\in C_0}\int_{D_n} G^n(s,v,u)[\phi]^2\mu^n(du)\\
&\le \varepsilon K(1+M^2)\|\phi\|^2_{p_2(T)} + \|h''\|_\infty \sup_{v\in C_0}\int_{D_n} G^n(s,v,u)[\phi]^2\mu^n(du).
\end{aligned}$$

From (3.6) and (3.32) we have

$$(3.35)\qquad \sup_{v\in C_0}\int_{D_n} G^n(s,v,u)[\phi]^2\mu^n(du) \to 0.$$

Hence by (3.33)–(3.35),

$$(3.36)\qquad \limsup_{n\to\infty} |I_3^n(s)|1_C(\xi^n)1_C(\xi) \le \varepsilon K(1+M^2)\|\phi\|^2_{p_2(T)}.$$

As

$$\begin{aligned}
&|I_3^n(s)|1_C(\xi^n)1_C(\xi)\\
&\le \tfrac{1}{2}\|h''\|_\infty\bigg\{\int_U G^n(s,\xi_s^n,u)[\phi]^2\mu^n(du)1_C(\xi^n)\\
&\qquad + Q(B(s,\xi_s)^*\phi, B(s,\xi_s)^*\phi)1_C(\xi)\bigg\}\\
&\le \tfrac{1}{2}\|h''\|_\infty\bigg\{K\big(1+\|\xi_s^n\|^2_{-p_2(T)}\big)\|\phi\|^2_{p_2(T)}1_C(\xi^n)\\
&\qquad + \lim_{m\to\infty}\int_U |G^m(t,\xi_s,u)[\phi]|^2\mu^m(du)1_C(\xi)\bigg\}\\
&\le K(1+M^2)\|h''\|_\infty\|\phi\|^2_{p_2(T)},
\end{aligned}$$

it follows from Fatou's Lemma and (3.36) that

$$\begin{aligned}
&\limsup_{n\to\infty} P'\left(\left|\int_0^t I_3^n(s)\,ds\right| > a\right)\\
(3.37)\qquad &\le 2\varepsilon + \limsup_{n\to\infty}\frac{1}{a}E^{P'}\int_0^t |I_3^n(s)|1_C(\xi^n)1_C(\xi)\,ds\\
&\le 2\varepsilon + \frac{\varepsilon KT}{a}(1+M^2)\|\phi\|^2_{p_2(T)}.
\end{aligned}$$

Hence, $|\int_0^t I_3^n(s)\,ds|$ converges to 0 in probability. Similarly we can prove that $|\int_0^t I_2^n(s)\,ds|$ converges to 0 in probability. Furthermore, it is easy to see that

$I_1^n(t)$ and $I_1^n(0)$ tend to 0 P'-a.s. So, by (3.26), $M_n^F(\xi^n)_t$ tends to $M^F(\xi)_t$ in probability.

As X^n is a solution of (1.1), it follows from Itô's formula that

$$M_n^F(X^n)_t = \int_0^T \int_U (h(X_{s-}^n[\phi] + G^n(s, X_{s-}^n, u)[\phi]) - h(X_{s-}^n[\phi]))\tilde{N}^n(du\, ds) \tag{3.38}$$

and hence

$$\begin{aligned} E^{P'}|M_n^F(\xi^n)_t|^2 &= E|M_n^F(X^n)_t|^2 \\ &= E\int_0^T \int_U (h(X_{s-}^n[\phi] + G^n(s, X_{s-}^n, u)[\phi]) - h(X_{s-}^n[\phi]))^2 \mu^n(du)\, ds \\ &\le K(\tilde{K} + 1)T\|h'\|_\infty^2 \|\phi\|_{p_1(T)}^2. \end{aligned} \tag{3.39}$$

Thus for every $t \in [0, T]$, $\{M_n^F(\xi^n)_t\}$ is uniformly integrable and hence (3.30) holds.

It follows from (3.38) that, for every $n \in \mathbb{N}$, $\{M_n^F(X^n)_t\}$ is a martingale and hence $\{M_n^F(\xi^n)_t\}$ is a P'-martingale. Passing to the limit, we see that $M^F(\xi)_t$ is a P'-martingale and hence, $M^F(Z)_t$ is a P^*-martingale. Therefore, P^* is a solution of the $\mathscr{D}$-martingale problem. □

THEOREM 3.1. *Suppose that* (A^n, G^n, μ^n) *satisfies Assumptions* A *and also satisfies Assumptions* S *uniformly in* n. *In addition, let the initial condition* (3.1) *hold. Then* P^n *converges to* P^* *weakly and* P^* *is the distribution of the unique solution of the SDE* (1.2).

PROOF. It follows from Lemma 3.3 that P^* is a solution of the $\mathscr{D}$-martingale problem. Also, by Assumptions A and S and passing to the limit, we see that (A, B, Q) and the initial value X_0 satisfy Conditions D of Section 2. Furthermore,

$$\begin{aligned} &E^{P^*}\left\{\int_0^T Q(B(s, Z_s)^*\phi, B(s, Z_s)^*\phi)\, ds\right\} \\ &\quad = E^{P^*}\left\{\int_0^T \lim_{n\to\infty} \int_U |G^n(t, Z_s, u)[\phi]|^2 \mu^n(du)\, ds\right\} \\ &\quad \le E^{P^*}\left\{\int_0^T K(1 + \|Z_s\|_{-p_1(T)}^2)\|\phi\|_{p_1(T)}^2\, ds\right\} \\ &\quad \le TK(1 + \tilde{K})\|\phi\|_{p_1(T)}^2. \end{aligned} \tag{3.40}$$

Hence, by Lemma 2.1, P^* is the distribution of the unique solution of the SDE (1.2). By uniqueness, $\{P^n\}$ converges weakly to P^*. □

The next theorem will show that Φ'-valued diffusion processes which are of the form of (1.2) can be approximated by processes driven by Poisson random measures.

THEOREM 3.2. *Under Assumptions* D, *the* SDE (1.2) *has a unique solution which can be approximated by a sequence of processes driven by Poisson random measures.*

PROOF. As $Q(\phi, \phi)$ is continuous in $\phi \in \Phi$, we see that, for any $\varepsilon > 0$, there exists an index r_0 and $\delta > 0$, such that $Q(\phi, \phi) \le \varepsilon$ whenever $\phi \in \Phi$ and $\|\phi\|_{r_0} \le \delta$. For any $\phi \in \Phi$, let $\tilde{\phi} = \delta\phi/\|\phi\|_{r_0}$. Then $\|\tilde{\phi}\|_{r_0} = \delta$ and hence

$$Q(\phi, \phi) \le \frac{\varepsilon}{\delta^2}\|\phi\|^2_{r_0}. \tag{3.41}$$

It is then easy to see that Q can be extended to a continuous nonnegative-definite symmetric bilinear form on $H_{r_0} \times H_{r_0}$. That is, there exists an operator $Q^{1/2}_{r_0}$ on H_{r_0} such that

$$Q(\phi, \psi) = \left\langle Q^{1/2}_{r_0}\phi, Q^{1/2}_{r_0}\psi\right\rangle_{r_0}. \tag{3.42}$$

Let $U = \{1, 2, \dots\}$, $\mu^n(\{k\}) = n^2$, $X^n_0 = X_0$, $A^n(s, v) = A(s, v)$ and

$$G^n(s, v, k)[\phi] = \frac{1}{n}\left\langle Q^{1/2}_{r_0}B(s, v)^*\phi, h^{r_0}_k\right\rangle_{r_0}. \tag{3.43}$$

Now, we only need to verify the conditions of Theorem 3.1. From (3.43), we have

$$\begin{aligned}
\int_U \|G^n(t, v, u)\|^2_{-p}\mu^n(du) &= \sum_j \int_U \left(G^n(t, v, u)[h^p_j]\right)^2 \mu^n(du)\\
&= \sum_j \sum_k \left\langle Q^{1/2}_{r_0}B(t, v)^*h^p_j, h^{r_0}_k\right\rangle^2_{r_0}\\
&= \sum_j \left\|Q^{1/2}_{r_0}B(t, v)^*h^p_j\right\|^2_{r_0}\\
&= \sum_j Q\left(B(t, v)^*h^p_j, B(t, v)^*h^p_j\right)\\
&= |Q_{B(t, v)}|_{-p, -p} \le K\left(1 + \|v\|^2_{-p}\right).
\end{aligned} \tag{3.44}$$

Similarly

$$\int_U \|G^n(t, v, u) - G^n(t, v', u)\|^2_{-p}\mu^n(du) \le K\|v - v'\|^2_{-p}. \tag{3.45}$$

Hence (A^n, G^n, μ^n) satisfies Assumptions S uniformly in n.

Next, let C_0 be any compact subset of H_{-p}. Note that

$$\begin{aligned} n^2\|G^n(t,v,k)\|_{-p}^2 & \\ \le \sum_{r=1}^{\infty} n^2\|G^n(t,v,r)\|_{-p}^2 & \\ = \int_U \|G^n(t,v,u)\|_{-p}^2 \mu^n(du) \le K\big(1+\|v\|_{-p}^2\big). \end{aligned} \tag{3.46}$$

Hence, for $n \ge (\|\phi\|_p/a)\sqrt{K\big(1+\sup_{v\in C_0}\|v\|_{-p}^2\big)}$, we have

$$\begin{aligned} &\sup_{v\in C_0} \mu^n\{u: |G^n(t,v,u)[\phi]| > a\} \\ &\le \sup_{v\in C_0} \mu^n\Big\{u: K\big(1+\|v\|_{-p}^2\big)\|\phi\|_p^2 > (na)^2\Big\} = 0. \end{aligned} \tag{3.47}$$

This proves (3.4); (3.6) can be shown in a similar manner. For (3.5), we note that

$$\begin{aligned} &\int_U G^n(t,v_1,u)[\phi]G^n(t,v_1,u)[\phi]\,\mu^n(du) \\ &= \sum_{k=1}^{\infty} \Big\langle Q_{r_0}^{1/2}B(t,v_1)^*\phi, h_k^{r_0}\Big\rangle_{r_0} \Big\langle Q_{r_0}^{1/2}B(t,v_2)^*\phi, h_k^{r_0}\Big\rangle r_0 \\ &= Q(B(t,v_1)^*\phi, B(t,v_2)^*\phi). \end{aligned} \tag{3.48}$$

Hence (A^n, G^n, μ^n) also satisfies Assumptions A. □

REMARK 3.1. The conditions in Theorem 3.2 for a unique solution of (1.2) are weaker than those imposed by Kallianpur, Mitoma and Wolpert (1990). More specifically, they require that, in addition to Assumptions D, $A(t,v)$ and $B(t,v)$ be jointly continuous in $(t,v) \in \mathbb{R}_+ \times \Phi'$ and

$$E\Big\{\big(1+\|X_0\|_{-r_0}^2\big)\big[\log\big(1+\|X_0\|_{-r_0}^2\big)\big]^2\Big\} < \infty.$$

4. Applications. In this section, we apply our results to various models of environmental pollution problems and also to a stochastic reversal potential model of spatially extended neurons.

Stochastic environmental pollution models have been studied by various authors [Curtain (1975), Kwakernaak (1974) and Kallianpur and Xiong (1994)]. All models investigated so far are of Poisson random deposit of chemical.

For the convenience of the reader, we briefly describe two kinds of pollution models [see Kallianpur and Xiong (1994) for more details]. Let L be a second order differential operator given by

$$-Lf(x) = D\sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}(x) - \sum_{i=1}^{d} V_i \frac{\partial f}{\partial x_i}(x), \qquad x \in \mathscr{X} = [0,l]^d, \tag{4.1}$$

where D is the diffusion coefficient and $V = (V_1, \dots, V_d)$ is the drift vector.

In the absence of random deposits, the chemical concentration $u(t, x)$ at time t and location x should satisfy the partial differential equation

$$\frac{\partial}{\partial t}u(t,x) = -Lu(t,x) - \alpha u(t,x), \tag{4.2}$$

where α is the leakage rate. We also impose Neumann boundary conditions:

$$\frac{\partial}{\partial x_i}u(t,0) = \frac{\partial}{\partial x_i}u(t,l) = 0, \qquad i = 1,2,\dots,d. \tag{4.3}$$

The chemicals are deposited at sites in $\mathscr{X}$ at random times $\tau_1(\omega) < \tau_2(\omega) < \cdots$ and locations $\kappa_1(\omega), \kappa_2(\omega), \dots$ with positive random magnitudes $A_1(\omega), A_2(\omega), \dots$. Taking these random deposits into account, we may convert (4.2) formally into the SPDE

$$\begin{aligned}\frac{\partial}{\partial t}&u(t,x,\omega)\\ &= -Lu(t,x,\omega) - \alpha u(t,x,\omega) + \sum_j A_j(\omega)\delta_{\kappa_j(\omega)}(x)\delta_{\tau_j(\omega)}(t)\end{aligned} \tag{4.4}$$

with Neumann boundary conditions (4.3), where δ_x is the Dirac measure at x.

For $A \subset \mathscr{X}$ and $B \subset \mathbb{R}_+$, let

$$N([0,t] \times A \times B) = \sum_{j:\, \tau_j \le t} 1_B\big(A_j(\omega)\big)1_A\big(\kappa_j(\omega)\big). \tag{4.5}$$

We make the further assumption that $\tau_1, \tau_2, \dots$ are the jump times of a Poisson process and that (κ_j, A_j), $j = 1,2,\dots$, are i.i.d. random variables so that N is a Poisson random measure on $\mathbb{R}_+ \times \mathscr{X} \times \mathbb{R}_+$ with intensity measure μ on $\mathscr{X} \times \mathbb{R}_+$.

Suppose there is a mechanism to clean up the environment when the chemical concentration at x exceeds a fixed level $\xi(x)$. In this case, the real effect of the chemical deposit depends on the magnitude $A_i(\omega)$ and the tolerance level $\xi(x)$. For the sake of mathematical simplicity, we assume that at time $\tau_i(\omega)$ the pollutant is uniformly deposited over the whole $\mathscr{X}$ instead of at a point $\kappa_i(\omega)$ as in model (4.4). We also assume that the real effect of the chemical deposit at $x \in \mathscr{X}$ is proportional to the difference between the chemical concentration $u(\tau_i-, x)$ and the tolerance level $\xi(x)$. Then (4.4) can be modified and written formally as the SPDE

$$\begin{aligned}\frac{\partial}{\partial t}&u(t,x,\omega)\\ &= -Lu(t,x,\omega) - \alpha u(t,x,\omega)\\ &\quad + \sum_j A_j(\omega)\big(\xi(x) - u(\tau_j(\omega)-,x,\omega)\big)\delta_{\tau_j(\omega)}(t)\end{aligned} \tag{4.6}$$

with Neumann boundary conditions (4.3).

The basic pollution model (4.4) and the pollution model (4.6) with a tolerance level are then understood, respectively, as the integral equations

$$u_t[\phi] = u_0[\phi] + \int_0^t (u_s[-L\phi] - \alpha u_s[\phi])\,ds + \int_0^t \int_{\mathscr{X}} \int_0^\infty a\phi(x)\rho(x)N(ds\,dx\,da) \tag{4.7}$$

and

$$u_t[\phi] = u_0[\phi] + \int_0^t (u_s[-L\phi] - \alpha u_s[\phi])\,ds + \int_0^t \int_0^\infty a(\xi[\phi] - u_{s-}[\phi])N'(ds\,da), \tag{4.8}$$

where u_t is regarded as a Φ'-valued process and Φ is a countably Hilbertian nuclear space constructed below in terms of the operator L. Also N is the Poisson random measure on $\mathbb{R}_+ \times \mathscr{X} \times \mathbb{R}_+$ with intensity measure μ on $\mathscr{X} \times \mathbb{R}_+$ as in (4.5) and N' is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity measure μ' on $\mathbb{R}_+$ which can be constructed similarly as in (4.5).

Let ρ be a function on $\mathscr{X}$ given by

$$\rho(x) = \exp\left(-\sum_{i=1}^d \frac{V_i}{D} x_i\right). \tag{4.9}$$

Let

$$H = \left\{ h \in L^2(\mathscr{X}, \rho(x)\,dx) \colon \left.\frac{\partial h}{\partial x_i}\right|_{x_i=0} = \left.\frac{\partial h}{\partial x_i}\right|_{x_i=l} = 0, i = 1, \dots, d \right\}. \tag{4.10}$$

Then L is a nonnegative-definite and self-adjoint operator on the separable Hilbert space H with discrete spectrum. Let λ_j, ϕ_j, $j = 0, 1, 2, \dots,$ be the eigenvalues and eigenvectors, respectively, of L. For $r_1 > d/4$, it is known that

$$\sum_j (1 + \lambda_j)^{-2r_1} < \infty. \tag{4.11}$$

For $r \in \mathbb{R}$ and $h \in H$, let

$$\|h\|_r^2 \equiv \sum_j \langle h, \phi_j \rangle^2 (1 + \lambda_j)^{2r} \tag{4.12}$$

and

$$\Phi \equiv \{h \in H \colon \|h\|_r < \infty, \forall\, r \in R\}, \tag{4.13}$$

where $\langle\cdot,\cdot\rangle$ is the inner product on H. For each r, let H_r be the completion of Φ with respect to the norm $\|\cdot\|_r$. Let Φ' be the union of all H_r, $r \in \mathbb{R}$. Note that $H_0 = H$ and $\langle\cdot,\cdot\rangle_0 = \langle\cdot,\cdot\rangle$. Then Φ is a countably Hilbertian nuclear space and Φ' is its dual space. It might be of interest to consider diffusion models for the pollution processes. A natural way to study this problem is to regard diffusion models as the limiting case of Poisson models.

To this end, consider a sequence of SDE's on Φ',

$$u_t^n[\phi] = u_0^n[\phi] - \int_0^t u_s^n[(\alpha + L)\phi]\,ds + \int_0^t \int_{\mathscr{X}} \int_0^\infty a\phi(x)\rho(x) N^n(da\,ds), \tag{4.14}$$

where N^n is a sequence of Poisson random measures on $\mathbb{R}_+ \times \mathscr{X} \times \mathbb{R}_+$ with intensity measure μ^n on $\mathscr{X} \times \mathbb{R}_+$. We study the diffusion approximation for the centered processes $\tilde{u}^n = u^n - Eu^n$. It is easy to see that $\tilde{u}^n$ satisfies the following SDE on Φ':

$$\tilde{u}_t^n[\phi] = \tilde{u}_0^n[\phi] - \int_0^t \tilde{u}_s^n[(\alpha + L)\phi]\,ds + \int_0^t \int_{\mathscr{X}} \int_0^\infty a\phi(x)\rho(x) \tilde{N}^n(dx\,da\,ds), \tag{4.15}$$

where $\tilde{N}^n$ is the compensated random measure of N^n, that is, $\tilde{N}^n(dx\,da\,ds) = N^n(dx\,da\,ds) - \mu^n(dx\,da)\,ds$.

To obtain a unique solution of (4.15) with a uniform bound, we assume that there exist two indexes r_0 and r_2 and a finite constant K, all independent of n and such that

$$E\|\tilde{u}_0^n\|_{-r_0}^2 \le K \quad \text{and} \quad \int_{\mathscr{X}} \int_0^\infty |a\phi(x)\rho(x)|^2 \mu^n(dx\,da) \le K\|\phi\|_{r_2}^2, \quad \forall\, \phi \in \Phi. \tag{4.16}$$

To get a diffusion approximation result, we make the following assumptions:

ASSUMPTIONS L. For each $\varepsilon > 0$ and $\phi \in \Phi$:

(L1) $\tilde{u}_0^n \to \tilde{u}_0$ as $n \to \infty$ in distribution as Φ'-valued random variables.
(L2) $\mu^n(\mathscr{X} \times [\varepsilon, \infty)) \to 0$ as $n \to \infty$.
(L3) The limit of $\int_{\mathscr{X}} \int_0^\infty |a\phi(x)\rho(x)|^2 \mu^n(dx\,da)$ exists as n tends to ∞.
(L4) As $m \to 0$,

$$\sup_n \int_{\mathscr{X}} \int_0^\infty |a\phi(x)\rho(x)|^2 1_{\{|a\phi(x)\rho(x)| \ge M\}} \mu^n(dx\,da) \to 0. \tag{4.17}$$

We denote the limit of (L3) by $Q(\phi, \phi)$. It is easy to see that Q determines a continuous nonnegative definite bilinear form on $\Phi \times \Phi$ under condition (4.16).

PROPOSITION 4.1. *Let $A^n(t, v) = A(t, v) = -(\alpha + L')v$ and $G^n(t, v, x, a)[\phi] = a\phi(x)\rho(x)$. Under Condition* (4.16) *and the Assumptions* L, *the conditions of Theorem* 3.1 *hold and hence, $\tilde{u}^n$ converges weakly to a Φ'-valued diffusion process $\tilde{u}$ governed by the SDE*

$$\tilde{u}_t = \tilde{u}_0 - \int_0^t (\alpha + L')\tilde{u}_s\,ds + W_t, \tag{4.18}$$

where W is a Φ'-valued Wiener process with covariance Q. Further, if Q is given by

$$(4.19)\qquad Q(\phi,\psi)=\int_{\mathscr{X}}\phi(x)\psi(x)\,dx,\qquad \forall\,\phi,\psi\in\Phi,$$

then (4.18) can be formally written as the SPDE

$$(4.20)\qquad \frac{\partial}{\partial t}\tilde{u}(t,x)=-(\alpha+L)\tilde{u}(t,x)+\dot{W}_{t,x},\qquad \tilde{u}(0,x)=\tilde{u}_0(x),$$

where $\dot{W}_{t,x}$ is Gaussian white noise in space–time.

REMARK 4.1. (i) If $r_0=r_2=0$ and μ^n are finite measures on $\mathscr{X}\times\mathbb{R}_+$, it follows from Kallianpur and Xiong (1994) that $\tilde{u}^n$ are H_0-valued processes. Therefore, $\tilde{u}^n$ can be regarded as random fields. That is, for each n, there exists a real-valued measurable function $\overline{u}^n(t,x,\omega)$ on $\mathbb{R}_+\times\mathscr{X}\times\Omega$ such that $\forall\, t\ge 0$ and $\phi\in\Phi$, we have

$$\tilde{u}_t^n(\omega)[\phi]=\int_{\mathscr{X}}\overline{u}^n(t,x,\omega)\phi(x)\rho(x)\,dx\quad\text{a.s.}$$

The process $\overline{u}^n(t,x,\omega)$, depending on t and x is called the random field corresponding to the H_0-valued process $\tilde{u}^n$.

(ii) If $d>1$ and Q is given by (4.19), then the limit process $\tilde{u}$ is not H_0-valued [see Walsh (1984)].

Now we consider a sequence of pollution models with a tolerance level of the form of (4.8):

$$(4.21)\quad u_t^n=u_0^n-\int_0^t(L'u_s^n+\alpha^n u_s)\,ds+\int_0^t\int_0^\infty a(\xi-u_{s-}^n)N^n(da\,ds),$$

where $\xi\in H$ is the (nonrandom) tolerance level and N^n is a sequence of Poisson random measures on $\mathbb{R}_+\times\mathbb{R}_+$ with intensity measure μ^n on $\mathbb{R}_+$.

ASSUMPTIONS E. The assumptions we need in this case are:

(E1) $\alpha^n+\int_0^\infty a\mu^n(da)\to\alpha$ and $\int_0^\infty a^2\mu^n(da)\to\beta^2$.
(E2) For any $\varepsilon>0$, $\mu^n\{a\colon a>\varepsilon\}\to 0$.
(E3) There exists a sequence c^n such that $c^n\alpha^n\to\gamma$.
(E4) $\sup_n\int_M^\infty a^2\mu^n(da)\to 0$ as $M\to\infty$.

For any ϕ and ψ in Φ, let $Q(\phi,\psi)=\langle\phi,\psi\rangle$. Let $A\colon\Phi'\to\Phi'$ and $B\colon\Phi'\to\Phi'$ be given by

$$(4.22)\qquad A(v)=-L'v-\alpha v+\gamma\xi\quad\text{and}\quad B(v)^*\phi=\beta v[\phi]\phi_0,$$

where ϕ_0 is the unique eigenvector of $-L$ corresponding to the eigenvalue $\lambda_0=0$.

Let $V_t^n=c^n u_t^n$. We have the following diffusion approximation result for $\{V^n\}$.

PROPOSITION 4.2. *Suppose that we have r_0 such that $\sup_n E\|V_0^n\|_{-r_0}^2 < \infty$ and $\{V_0^n\}$ converges to a Φ'-valued random variable V_0 in distribution. Then V^n converges in distribution to the solution of the diffusion equation on Φ':*

$$V_t = V_0 + \int_0^t A(V_s)\,ds + \int_0^t B(V_s)\,dW_s, \tag{4.23}$$

where W is a Φ'-valued Wiener process with covariance Q.

PROOF. Note that

$$V_t^n = V_0^n + \int_0^t A^n(V_s^n)\,ds + \int_0^t\int_0^\infty G^n(V_{s-}^n, a)\tilde{N}^n(da\,ds), \tag{4.24}$$

where

$$\begin{aligned} A^n(v) &= -L'v - \alpha^n v + (c^n\xi - v)\int_0^\infty a\mu^n(da) \\ \text{and}\quad G^n(v,a) &= (c^n\xi - v)a. \end{aligned} \tag{4.25}$$

We show that $c^n \to 0$. In fact, we note that

$$\begin{aligned} \int_0^\infty a^2\mu^n(da) &= \int_0^\varepsilon a^2\mu^n(da) + \int_\varepsilon^M a^2\mu^n(da) + \int_M^\infty a^2\mu^n(da) \\ &\le \varepsilon\int_0^\infty a\mu^n(da) + M^2\mu^n\{a: a > \varepsilon\} + \sup_n \int_M^\infty a^2\mu^n(da). \end{aligned} \tag{4.26}$$

Then

$$\beta^2 \le \varepsilon \liminf_{n\to\infty} \int_0^\infty a\mu^n(da) + \sup_m \int_M^\infty a^2\mu^m(da). \tag{4.27}$$

Letting $M \to \infty$, we have

$$\beta^2 \le \varepsilon \liminf_{n\to\infty} \int_0^\infty a\mu^n(da). \tag{4.28}$$

Letting $\varepsilon \to 0$, we have

$$\liminf_{n\to\infty} \int_0^\infty a\mu^n(da) = \infty. \tag{4.29}$$

As $\alpha^n + \int_0^\infty a\mu^n(da) \to \alpha$ and $c^n\alpha^n \to \gamma$, we have $c^n \to 0$. It is then easy to see that Assumptions A of Section 3 hold for (A^n, G^n, μ^n) and the proposition follows from Theorem 3.1. □

Now we show that the limiting process is in fact in H_0 and can be regarded as the solution of a stochastic partial differential equation.

THEOREM 4.1. *Suppose that $\xi \in H_0$ and V_0 is an H_0-valued random variable such that $E\|V_0\|_0^2 < \infty$. Then $V \in C([0,T], H_0)$. Let $V(t,\cdot) = V_t$. Then*

$$V(t,x) = V(0,x) - \int_0^t (L'V(s,x) + \alpha V(s,x) - \gamma\xi(x))\, ds + \int_0^t \beta V(s,x)\, dB_s, \tag{4.30}$$

where B is a one-dimensional Brownian motion.

PROOF. It follows from (4.22) and (4.23) that, for $\phi \in \Phi$ such that $L\phi = \lambda\phi$,

$$\begin{aligned} V_t[\phi] &= V_0[\phi] + \int_0^t A(V_s)[\phi]\, ds + \int_0^t \langle B(V_s)^*\phi, dW_s\rangle_0 \\ &= V_0[\phi] - \int_0^t ((\alpha+\lambda)V_s[\phi] - \gamma\xi[\phi])\, ds \\ &\quad + \int_0^t \beta V_s[\phi]\, dW_s[\phi_0]. \end{aligned} \tag{4.31}$$

Making use of Itô's formula, we have

$$\begin{aligned} V_t[\phi]^2 &= V_0[\phi]^2 - \int_0^t 2V_s[\phi]((\alpha+\lambda)V_s[\phi] - \gamma\xi[\phi])\, ds \\ &\quad + \int_0^t 2\beta V_s[\phi]^2\, dW_s[\phi_0] + \int_0^t \beta^2 (V_s[\phi])^2\, ds. \end{aligned} \tag{4.32}$$

From the Burkholder–Davis–Gundy inequality [see Dellacherie and Meyer (1982), (90.1), page 285] we have

$$\begin{aligned} f(r) &\equiv E \sup_{0\le t\le r} V_t[\phi]^2 \\ &\le EV_0[\phi]^2 + \int_0^r \left((2|\alpha| + 1 + \beta^2)EV_s[\phi]^2 + \gamma^2\xi[\phi]^2\right) ds \\ &\quad + 4\beta E\sqrt{\textstyle\int_0^r V_s[\phi]^4\, ds} \\ &\le EV_0[\phi]^2 + (2|\alpha| + 1 + \beta^2)\int_0^r f(s)\, ds + \gamma^2\xi[\phi]^2 r \\ &\quad + 4\beta E\left(\sup_{0\le t\le r} |V_T[\phi]| \sqrt{\textstyle\int_0^r V_s[\phi]^2\, ds}\right) \\ &\le EV_0[\phi]^2 + (2|\alpha| + 1 + \beta^2)\int_0^r f(s)\, ds + \gamma^2\xi[\phi]^2 r \\ &\quad + \tfrac{1}{2}f(r) + 32\beta^2\int_0^r E(V_s[\phi])^2\, ds. \end{aligned} \tag{4.33}$$

That is,

$$f(r) \le 2EV_0[\phi]^2 + 2(2|\alpha| + 1 + 33\beta^2) \times \int_0^r f(s)\,ds + 2\gamma^2\xi[\phi]^2 r. \tag{4.34}$$

Gronwall's inequality then yields

$$E \sup_{0\le t\le T} V_t[\phi]^2 \le \left(2EV_0[\phi]^2 + 2\gamma^2\xi[\phi]^2 T\right) \times \exp\left(2(2|\alpha| + 1 + 33\beta^2)T\right). \tag{4.35}$$

Letting $\phi = \phi_j$ and adding, we have

$$E \sum_{j=0}^{\infty} \sup_{0\le t\le T} V_t[\phi_j]^2 \le \left(2E\|V_0\|_0^2 + 2\gamma^2\|\xi\|_0^2 T\right) \times \exp\left(2(2|\alpha| + 1 + 33\beta^2)T\right). \tag{4.36}$$

The continuity of $V_t[\phi_j]$ is obvious. It follows from (4.36) that $V \in C([0,T], H_0)$. (4.30) easily follows upon setting $B_t = W_t[\phi_0]$. □

In recent years it has been well recognized in the neurophysiological literature that a neuron cell is spatially extended. That is, a realistic description of neuronal activity would have to take into account synaptic inputs that occur randomly in time as well as at different locations on the neuron's surface. It is of interest, therefore, to consider diffusion approximation for reversal potential models of voltage potentials of spatially extended neurons. One motivation for such models is to regard the Poisson events as the openings (and consequent closings) of various ion-specific passages through the membrane. During the open period, ions of the appropriate type pass into or out of the cell through such a passage at a rate depending on the difference between an equilibrium potential and the voltage potential.

A general result has been obtained in Kallianpur and Wolpert (1987) when the neuron can be regarded as a single point and the importance of the investigation for spatially extended neurons is indicated. We derive such a result as an application of our diffusion approximation theorem in this setup. A similar result was obtained by Baldwin (1990) in his dissertation.

For the convenience of the reader, we describe the reversal potential model briefly. We refer to Kallianpur and Wolpert (1987) and to Hodgkin and Huxley (1952a–d), who originated the term "equilibrium" potential, for a more detailed description.

Let $\mathscr{X}$ and L be given by (4.1), where $\mathscr{X}$ represents the membrane of the neuron. More general $\mathscr{X}$ and L can be treated similarly. Let $\xi(x,t)$ be the nerve membrane potential at time t and at a point x. In the absence of stimuli, ξ will satisfy the cable equation

$$\frac{\partial}{\partial t}\xi(x,t) = -L\xi(x,t) - \alpha\xi(x,t). \tag{4.37}$$

If the stimulus arriving at time t at a point x is $I(x,t)$, then ξ will satisfy

$$\frac{\partial}{\partial t}\xi(x,t) = -L\xi(x,t) - \alpha\xi(x,t) + I(x,t). \tag{4.38}$$

The stimuli received by the neuron are pulses of electrical current of such short duration that we may consider them to be impulses. They can be either positive (excitation) or negative (inhibition). Suppose that there are excitatory (resp. inhibitory) ions with equilibrium potentials $\eta_e \in \Phi'$ (resp. $\eta_i \in \Phi'$) arriving according to Poisson stream N_e (resp. N_i) with random magnitudes $A_e^k \geq 0$, $k = 1,2,\dots,$ with common distribution F_e on $[0,\infty)$ (resp. $A_i^k \leq 0$, $k = 1,2,\dots,$ with common distribution F_i on $(-\infty,0]$). Let N_e and N_i be independent Poisson processes with parameters of f_e and f_i, respectively. The random variables A_e^k, A_i^k, N_e and N_i are all taken to be mutually independent. Let $\{\tau_k\}$ and $\{\tau'_k\}$ be the jump instants of the processes N_e and N_i, respectively.

Then ξ can be regarded as a Φ'-valued process and characterized by the following reversal potential model

$$\xi_t = \xi_0 - \int_0^t (\alpha + L')\xi_s\, ds + \sum_{k=1}^{N_e(t)} (\eta_e - \xi_{\tau_k -})A_e^k + \sum_{k=1}^{N_i(t)} (\xi_{\tau'_k -} - \eta_i)A_i^k. \tag{4.39}$$

Let $U \equiv \Phi' \times \mathbb{R}$ and

$$N(\Lambda \times B \times [0,t]) \equiv \sum_{k=1}^{N_e(t)} 1_B(A_e^k)1_\Lambda(\eta_e) + \sum_{k=1}^{N_i(t)} 1_B(A_i^k)1_\Lambda(\eta_i) \tag{4.40}$$

for any $t \geq 0$, $B \in \mathscr{B}(\mathbb{R})$ and $\Lambda \in \mathscr{B}(\Phi')$. Then N is a Poisson random measure on $\Phi' \times \mathbb{R} \times \mathbb{R}_+$ with intensity measure

$$\mu(\Lambda \times B) = f_e 1_\Lambda(\eta_e)F_e(B) + f_i 1_\Lambda(\eta_i)F_i(B) \tag{4.41}$$

for any $\Lambda \in \mathscr{B}(\Phi')$ and $B \in \mathscr{B}(\mathbb{R})$. Equation (4.39) is then rewritten as

$$\xi_t = \xi_0 - \int_0^t (\alpha + L')\xi_s\, ds + \int_0^t \int_{\Phi'} \int_{\mathbb{R}} f(\xi_{s-}, \eta, a) N(d\eta\, da\, ds), \tag{4.42}$$

where

$$f(v,\eta,a) = \begin{cases} (\eta - v)a, & \text{if } a \geq 0, \\ (v - \eta)a, & \text{if } a < 0, \end{cases} \tag{4.43}$$

for $v \in \Phi'$, $\eta \in \Phi'$, $a \in \mathbb{R}$.

Now we consider a sequence of SDE's on Φ' of the form (4.42):

$$\xi_t^n = \xi_0^n - \int_0^t (\alpha^n + L')\xi_s^n\, ds + \int_0^t \int_{\Phi'} \int_{\mathbb{R}} f(\xi_{s-}^n, \eta, a) N^n(d\eta\, da\, ds), \tag{4.44}$$

where α^n is a sequence of real numbers and $N^n(d\eta\, da\, ds)$ is a sequence of Poisson random measures on $\Phi' \times \mathbb{R} \times [0,\infty)$ given by (4.40) with f_e, f_i, F_e and F_i replaced by f_e^n, f_i^n, F_e^n and F_i^n, respectively. The intensity measure μ^n are given by (4.41) with f_e, f_i, F_e and F_i replaced by f_e^n, f_i^n, F_e^n and F_i^n, respectively.

To characterize the limiting behavior of ξ^n as $n \to \infty$, we make the following assumptions:

ASSUMPTIONS R.

(R1) $\alpha^n + f_e^n a_e^n - f_i^n a_i^n \to \alpha$ and $f_e^n b_e^n + f_i^n b_i^n \to \beta^2$ in $\mathbb{R}$, where $a_e^n = \int_0^\infty a F_e^n(da)$, $b_e^n = \int_0^\infty a^2 F_e^n(da)$ and a_i^n and b_i^n are defined similarly.
(R2) For any $\varepsilon > 0$, $f_e^n F_e^n\{a\colon a > \varepsilon\} + f_i^n F_i^n\{a\colon a < -\varepsilon\} \to 0$.
(R3) There exists a sequence c^n such that $c^n f_e^n a_e^n \to \gamma_e$ and $c^n f_i^n a_i^n \to \gamma_i$.
(R4) $\sup_n (f_e^n \int_M^\infty a^2 F_e^n(da) + f_i^n \int_{-\infty}^{-M} a^2 F_i^n(da)) \to 0$ as $M \to \infty$.

In analogy with Theorem 4.1, we have the following diffusion approximation result for $V_t^n \equiv c^n \xi_t^n$.

THEOREM 4.2. *Suppose that we have r_0 such that $\sup_n E\|V_0^n\|_{-r_0}^2 < \infty$ and $\{V_0^n\}$ converges to a Φ'-valued random variable V_0 in distribution. Then V^n converges in distribution to a Φ'-valued process V. Further, suppose that η_e and $\eta_i \in H_0$ and V_0 is an H_0-valued random variable such that $E\|V_0\|_0^2 < \infty$. Then $V \in C([0,T], H_0)$ and $V(t,\cdot) = V_t$ satisfies*

$$\begin{aligned} V(t,x) &= V(0,x) \\ &\quad + \int_0^t (-(\alpha + L')V(s,x) + \gamma_e \eta_e(x) - \gamma_i \eta_i(x))\, ds \\ &\quad + \int_0^t \beta V(s,x)\, dB_s, \end{aligned} \tag{4.45}$$

where B is a one-dimensional standard Brownian motion.

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