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GIBBS–COX RANDOM FIELDS AND BURGERS TURBULENCE[1]

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We study the large time behavior of random fields which are solutions of a nonlinear partial differential equation, called Burgers' equation, under stochastic initial conditions. These are assumed to be of the shot noise type with the Gibbs–Cox process driving the spatial distribution of the "bumps." In certain cases, this work extends an earlier effort by Surgailis and Woyczynski, where only noninteracting "bumps" driven by the traditional doubly stochastic Poisson process were considered. In contrast to the previous work by Bulinski and Molchanov, a non-Gaussian scaling limit of the statistical solutions is discovered. Burgers' equation is known to describe various physical phenomena such as nonlinear and shock waves, distribution of self-gravitating matter in the universe and so forth.

Introduction. The simplest nondispersive waves (i.e., waves in media where the speed of propagation c is independent of the frequency of the wave) are planar hyperbolic waves described by the equation

$$u_t + cu_x = 0, \tag{I.1}$$

where $u = u(x,t)$ and where c is a constant. The obvious solution

$$u(x,t) = u_0(x - ct) \tag{I.2}$$

represents the distortionless propagation of the initial field $u_0(x) = u(x,0)$. Its straightforward nonlinear analogue is a hyperbolic conservation law expressed by the equation

$$u_t + c(u)u_x = 0, \tag{I.3}$$

with the initial condition $u_0(x) = u(x,0)$ [see, e.g., Lax (1973)]. Here the speed of propagation $c(u)$ depends on the amplitude u. The characteristic equations for the above first order partial differential equation take the form

$$\frac{dU}{dt} = 0, \qquad \frac{dX}{dt} = c(U), \tag{I.4}$$

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with the initial conditions

$$X(0) = y, \qquad U(0) = u_0(y), \tag{I.5}$$

so that

$$X(y,t) = y + c(u_0(y))t, \qquad U(y,t) = u_0(y), \tag{I.6}$$

which gives the solution

$$u = u_0(x - c(u)t) \tag{I.7}$$

in an implicit form. As long as $c' \neq 0$, we encounter a nonuniqueness problem, though. If

$$\frac{d}{dy}c(u_0(y)) < 0, \tag{I.8}$$

then for two characteristics starting at y and $y + dy$ the difference

$$X(y+dy,t) - X(y,t) = \left(1 + \frac{d}{dy}c(u_0(y))t\right)dy, \tag{I.9}$$

and the two characteristics are bound to intersect for t large enough. The uniqueness can be guaranteed only in the interval

$$t \in \left(0, \min\left(-\frac{d}{dy}c(u_0(y))\right)^{-1}\right). \tag{I.10}$$

The above analytic phenomenon is physically reflected in formation of shock waves (discontinuous solutions). One way to get around this difficulty is to take into account nonlocal interactions with the medium, such as a linear viscous dissipation, which leads to the nonlinear diffusion equation of the form

$$u_t + c(u)u_x = \nu u_{xx}. \tag{I.11}$$

It is known that if the viscosity coefficient $\nu \to 0$, then the solutions of (I.11) converge to the (generalized) solutions of (I.3) [see, e.g., DiPerna (1983a, b)].

The special case of equation (I.3) with $c(u) = cu$ gives rise to the so-called Riemann equation

$$u_t + cuu_x = 0, \tag{I.12}$$

which describes the hydrodynamic flow of noninteracting particles moving along axis x with velocity u [see, e.g., Arnold (1988)]. Softening the shock fronts in the Riemann equation by addition of a linear dissipation term (parabolic regularization) leads to Burgers' equation

$$u_t + cuu_x = \nu u_{xx}, \tag{I.13}$$

which is the main object of study in this paper. The initial condition $u_0(x)$ is assumed to be random. This is natural if one keeps in mind that Burgers' equation can be viewed as a special one-dimensional case of the Navier–Stokes equation,

$$\vec{u}_t + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\vec{\nabla}p + \nu \Delta \vec{u} + \vec{F}, \tag{I.14}$$

describing turbulent fluid flow, where the pressure p and external force field $\vec{F}$ terms are neglected. In such a flow, the velocity field appears to be random even without random initial conditions and "this contrast is the source of much of what is interesting in turbulence theory" [see Chorin (1975), page 1]. The statistical approach has been the established tool in the study of turbulence for a long time.

To describe the relationship of the Navier–Stokes equation to the Burgers equation, it is hard to improve on the following compact analysis penned some 25 years ago by Kraichnan (1968):

> The differences between Burgers' and Navier–Stokes' equations are as interesting as the similarities. The uu_x term [in (I.13)] conserves both $\int u(x,t)\,dx$ and $\int [u(x,t)]^2\,dx$, as in the incompressible Navier–Stokes equation. In both cases, the advection term tends to produce regions of steepened velocity gradients, which implies a transfer of excitation from lower- to higher-wavenumber components of the velocity field. Perhaps the sharpest difference is that Burgers' equation appears to offer no counterpart to the hierarchy of instabilities which makes the small-scale structure of high Reynolds number [small ν] turbulence chaotic and unpredictable. If the initial Reynolds number is high, Burgers' equation leads to shock fronts which coalesce on collision so that, at later times when the Reynolds number is still high, an initially complicated u field is reduced to a sparse collection of shocks, with smooth and simple variation of u between fronts. The high-wave number excitation is then associated with principally with the shocks themselves. Burgers' equation reduces initial chaos instead of increasing it [$\cdots$]. These similarities and differences make Burgers' equation a valuable vehicle for exploring the limits of applicability of statistical approximations designed for Navier–Stokes turbulence. Interest is heightened because direct numerical integration of initial ensembles of velocity fields forward in time is much more feasible for Burgers' equation than for the Navier–Stokes equation.

The interest in Burgers' turbulence remains high in the fluid dynamics and physics communities [see, e.g., Gotoh and Kreichnan (1993)]. In view of the inelastic type of particles' collisions, Burgers' equation (coupled with the continuity equation of passive tracer transport) has been also studied as a model of the evolution of self-gravitating matter. Thus, information about the time dependence of the initial fluctuations is expected to yield a theoretical model for the observed large scale structure of the universe in late nonlinear stages of the gravitational instability [see Shandarin and Zeldovich (1989), Weinberg and Gunn (1990), Gurbatov, Malakhov and Saichev (1991) and Albeverio, Molchanov and Surgailis (1994)].

Over the last 10 years there was a renewed interest in the mathematical community in the Burgers turbulence ranging from the study of propagation of chaos [see, e.g., Gutkin and Kac (1983) and Sznitman (1986)], asymmetric exclusion processes [Andjel, Bramson and Liggett (1988) and Ferrari (1992)], cellular automata [Boghosian and Levermore (1987) and Brieger and Bonomi (1992)], scale renormalization [Rosenblatt (1987)], Hausdorff dimension of the shocks set [Sinai (1992b)] to maximum principles for moving average initial data [Hu and Woyczynski (1994a, b)], and a large number of interesting problems remain unsolved.

Our main question is how do the initial random fluctuations of u propagate in the Burgers flow $u(x,t)$, $x \in \mathbf{R}$, $t > 0$, and our goal is to provide a rigorous *mathematical* study of the problem for a precisely specified initial random data and based on some relatively recent advances in the theory of random fields. Here the pioneering work was that of Bulinski and Molchanov (1991), who also elucidated the importance of the initial shot noise type data. However, we believe the present paper to be one of the first where the precise description of non-Gaussian scaling limit distributions was obtained.

More precisely, following preliminaries in Section 1, we present a general result for the scaling limit behavior for statistical solutions of the Burgers equation and show a simple application of this result for strictly stationary initial data for which the mixing coefficient satisfies an additional integrability condition. This relies on the classical work of Ibragimov and Linnik (1965).

Section 2 introduces the notion of a Gibbs–Cox random field as determined by a Gibbs measure with a pair potential $\Phi(x)$ and random fugacity $\lambda(x)$, and studies limit properties of functionals on such processes. The general concept of a Gibbs–Cox random field seems to be appearing here for the first time. However, physically, it is a very natural and familiar object, and for some special potentials it has appeared in statistical physics under the names such as the spin glass model or the Ising model with random potential (see, e.g., Campanino, Olivieri and van Enter (1987), Funaki (1991)). Intuitively speaking, it models a random distribution of points in space (like a classical Poisson process), which itself can be a random medium (reflected by a random intensity in the Cox process), with the points permitted to interact according to a certain prescribed potential (as opposed to being independent in the Cox process model).

Section 3 returns to the study of the scaling limits in Burgers' turbulence, this time with initial data which are of a shot noise type with the Gibbs–Cox process driving the spatial distribution of the "bumps." It relies on results of Sections 1 and 2. In certain cases, this work extends an earlier effort by Surgailis and Woyczynski (1993), where only noninteracting "bumps" driven by the traditional doubly stochastic Poisson (Cox) process was considered. Permitting the bumps to interact is a major step towards making the model more realistic physically. For example, in Shandarin and Zeldovitch's (1989) astrophysical work on the large scale structure of the Universe, the Burgers equation modeled the clumping of cold, sticky, but otherwise noninteracting matter, a reasonable first approximation. A natural next step would be to take the gravitational interaction of the matter particles into account. This should accelerate the clumping process. A full implementation of such a program would be difficult analytically but introduction of our Gibbs–Cox models can be thought of as a step in this direction.

The basic Theorem 4.1 gives a decomposition of possible limit random fields into Gaussian and non-Gaussian parts. Finally, Section 5 gives a complete classification of finite-dimensional distributions of possible scaling limits of Burgers' turbulence in the case considered in Section 4, but under the additional assumption that the fugacity process λ is the square of a stationary Gaussian process. For other results of a similar nature, see Giraitis, Molchanov and Surgailis (1992), Surgailis and Woyczynski (1993, 1994a, b) and Woyczynski (1993).

One of the methods we employ—the Wiener–Hermite (–Cameron–Martin) expansion of a nonlinear stochastic functional—has, of course, a long history of application to the Burgers (and Navier–Stokes) turbulence, both in the mathematical and in the fluid dynamics communities [see, e.g., Cameron and Martin (1947), Wiener (1958), Meecham and Siegel (1964), Meecham, Iyer and Clever (1975), Orszag and Bissonnette (1967), Crow and Canavan (1970), Kahng and Siegel (1970), Chorin (1974, 1975) and Fournier and Frisch (1983)] and perhaps should also be seen in the context of the statistical hydrodynamics for Burgers turbulence developed by Hopf (1952) and Kuwabara (1978), among others. However, the rigorous complete picture that we obtain in this paper for Gibbs–Cox initial random data, with full information about limiting properties of resulting solution random fields, was not available before and has to rely on more recent mathematical developments.

Finally, we would like to mention that in this paper we do not consider Burgers equations with external (possibly random) forcing, even though it is an extremely important topic. For such an equation, although the Hopf–Cole transformation works, it leads to a Schrödinger type equation (with, possibly, random potential) rather than the heat equation, and our methods do not apply directly. Many partial results in this direction can be found in papers from Kraichnan (1959) through Nakazawa (1980) to Sinai (1992a). Our methods do not seem applicable either to analysis of stochastic flows governed by more general conservation laws (I.11), where different tools are needed [see, e.g., DiPerna (1983a, b)].

1. Preliminaries. As is well known, the one-dimensional Burgers equation

$$u_t + (u^2)_x = u_{xx}, \tag{1.1}$$

with the initial condition

$$u(x,0) = u_0(x), \tag{1.2a}$$

which we will assume to be a stochastic process with parameter $x \in \mathbf{R}$, admits, as long as the velocity potential

$$U_0(y) = -\int_{-\infty}^{y} u_0(z)\,dz = o(y^2), \qquad y \to \infty, \tag{1.2b}$$

a family of solutions of the form

$$u(x,t) = -\frac{\partial}{\partial x}\log\int_{\mathbf{R}} p(x,y,t)\exp(U_0(y))\,dy, \tag{1.3}$$

where

$$p(x,y,t) = \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{(x-y)^2}{4t}\right). \tag{1.4}$$

This is obtained by the usual Hopf–Cole substitution $u = -w_x/w$, which reduces the Burgers equation (1.1) to the equation

$$\left(w_x - w\frac{d}{dx}\right)(w_t - w_{xx}) = 0, \tag{1.5}$$

and by using the usual solutions of the heat equation. Observe that rescaling both time and space variables $x \to \beta x, t \to \beta^2 t$, we obtain from (1.3) that

$$\begin{aligned} u(\beta x, \beta^2 t) &= -\frac{1}{\beta}\frac{\partial}{\partial x}\log\int_{\mathbf{R}} p(x,y,t)\exp(U_0(\beta y))\,dy \\ &= -\frac{1}{\beta}\frac{\int_{\mathbf{R}} p_x(x,y,t)\exp(U_0(\beta y))\,dy}{\int_{\mathbf{R}} p(x,y,t)\exp(U_0(\beta y))\,dy}. \end{aligned} \tag{1.6}$$

The significant feature of formula (1.6) is that the kernels $p_x(x,y,t)$ and $p(x,y,t)$ do not depend on β. This fact suggests that it is reasonable to look for a nontrivial limit behavior of rescaled solutions (1.6) which, in turn, gives a good approximation for statistical solutions of the Burgers equation for large times.

The goal of this paper is to obtain such a scaling limit for certain particular classes of initial processes (1.2). In another paper we will estimate its error as a long range predictor for solutions of (1.1).

REMARK 1.1. A slightly more general form (I.15) of the Burgers equation can be treated by a similar Hopf–Cole transformation $u = (-2\nu/c)w_x/w$. Since we are not interested in this paper in the dependence of solutions on parameters ν and c, and, in particular, in the zero viscosity limit ($\nu \to 0$), we consider only the case $\nu = 1$, $c = 2$.

2. General scaling limit behavior. We begin with a result that gives a convergence of finite-dimensional distributions of the rescaled solutions of (1.1). The assumptions involve weak convergence in the Schwartz space $\mathscr{S}'$. The general assumption in this paper is the finiteness of the exponential moments of the initial velocity potential process

$$E \exp(U_0(y)) < \infty, \qquad y \in \mathbf{R}. \tag{2.1}$$

We shall also use the following notation for the centered and rescaled exponential of the initial velocity potential process:

$$V_\beta(y) = B(\beta)\left(\exp(U_0(\beta y)) - A(\beta)\right), \qquad y \in \mathbf{R}. \tag{2.2}$$

The scaling constants $B(\beta) > 0$ and the centering constants $A(\beta)$, $\beta > 0$, will remain unspecified at this point, but their role and nature will become clear later on in this paper.

THEOREM 2.1. *Assume that the processes V_β, $\beta > 0$, converge weakly in $\mathscr{S}'(\mathbf{R})$ to a generalized process V; that is, for each $\phi \in \mathscr{S}(\mathbf{R})$,*

$$\lim_{\beta\to\infty} E \exp\left(i \int_{\mathbf{R}} V_\beta(y)\phi(y)\, dy\right) = E \exp(i\langle V, \phi\rangle), \tag{2.3}$$

and that, for a certain $a > 0$,

$$\lim_{\beta\to\infty} \int_{\mathbf{R}} \exp(U_o(\beta y))\phi(y)\, dy = a\langle 1, \phi\rangle, \tag{2.4}$$

where the convergence is in probability. Then, as $\beta \to \infty$, the finite-dimensional distributions of the two parameter random fields,

$$v_\beta(x,t) := \beta B(\beta) u(\beta x, \beta^2 t), \qquad t > 0, x \in \mathbf{R}, \tag{2.5}$$

where u is a solution (1.3) of the Burgers equation, converge to the corresponding finite-dimensional distributions of the random field

$$-a^{-1}\langle V, p_x(x,\cdot,t)\rangle. \tag{2.6}$$

PROOF. To show the convergence of finite-dimensional distributions, it suffices to prove that for each $\alpha_1,\dots,\alpha_n, t_1,\dots,t_n, x_1,\dots,x_n$, the distribution of the linear combinations

$$w_\beta = \alpha_1 v_\beta(x_1,t_1) + \cdots + \alpha_n v_\beta(x_n,t_n) \tag{2.7}$$

converges weakly, as $\beta \to \infty$, to the distribution of the random variable

$$-a^{-1}\left\langle V, \sum_{k=1}^n \alpha_k p_x(x_k,\cdot,t_k)\right\rangle. \tag{2.8}$$

Observe that because $\int_{\mathbf{R}} p_x(x,y,t)\, dy = 0$,

$$w_\beta = -a^{-1} \sum_{k=1}^n \alpha_k(1 + \varepsilon(x_k,t_k)) \int_{\mathbf{R}} p_x(x_k,y,t_k) V_\beta(y)\, dy,$$

where

$$\varepsilon(x,t) = \frac{a}{\int_{\mathbf{R}} p(x,y,t)\exp(U_0(\beta y))\,dy} - 1.$$

As $\beta \to \infty$, in view of assumption (2.4), we have that $\varepsilon(x_k, t_k) \to 0$ in probability. On the other hand,

$$\alpha_k p_x(x_k,\cdot,t_k) \in \mathscr{S}(\mathbf{R}), \qquad k = 1,\dots,n,$$

so, in view of assumption (2.3), we obtain that in distribution,

$$\begin{aligned} &\lim_{\beta\to\infty} \sum_{k=1}^{n} \alpha_k(1+\varepsilon(x_k,t_k)) \int_{\mathbf{R}} p_x(x_k,y,t_k)V_\beta(y)\,dy \\ &\quad = \left\langle V, \sum_{k=1}^{n} \alpha_k p_x(x_k,\cdot,t_k)\right\rangle. \end{aligned} \tag{2.9}$$

Hence, the distributions of w_β converge to the distribution of (2.8). □

REMARK 2.1. Condition (2.3) can be rephrased as a statement that the stochastic process $\exp(U_0) = \{\exp(U_0(y)),\ y \in \mathbf{R}\}$ has a large-scale [in the sense of Dobrushin (1980), page 169] generalized limit V with the normalizations $A(\beta)$ and $B(\beta)$. This condition will be also written

$$\exp(U_0) \in \mathbf{DA}\{V; A(\beta), B(\beta)\}.$$

It is well known [see Dobrushin (1979, 1980) for a general account of the theory] that in this case, necessarily,

$$B(\beta) = \beta^{\kappa} L(\beta)$$

for some constant $\kappa \in \mathbf{R}$ and a slowly varying (as $\beta \to \infty$) locally bounded function $L(\beta)$, $\beta > 0$. Moreover, if the centering constant $A(\beta)$ does not depend on β, the limiting generalized process V is self-similar with parameter κ, that is, for any $\beta > 0$,

$$\{\langle V, \beta^{\kappa-1}\phi(\beta^{-1}\cdot)\rangle : \phi \in \mathscr{S}\} = \{\langle V, \phi\rangle : \phi \in \mathscr{S}\}, \tag{2.10}$$

in the sense of equality of finite-dimensional distributions. In the general case, when $A(\beta)$ depends in β, (2.10) holds only for any $\beta > 0$ and

$$\phi \in \mathscr{S}_1 = \left\{\phi \in \mathscr{S} : \int_{\mathbf{R}} \phi(x)\,dx = 0\right\}.$$

Self-similar generalized Gaussian random fields indexed by $\mathscr{S}$ and $\mathscr{S}_1$ have been described by Dobrushin (1980). Below we shall denote by W' Gaussian white noise, that is, the generalized process with the characteristic functional

$$E\exp(i\langle W', \phi\rangle) = \exp\left[-\tfrac{1}{2}\int_{\mathbf{R}} \phi^2(x)\,dx\right].$$

The next general result provides sufficient conditions for certain functionals of a stationary process to satisfy the central limit theorem. Later on it will be used to provide examples of initial processes for the Burgers equation that satisfy assumptions of Theorem 2.1. Recall that if $X = \{X(y), y \in \mathbf{R}\}$ is a strictly stationary process, then the strong mixing coefficient $\alpha_X(x)$, $x > 0$, is defined as

$$\alpha_X(x) := \sup_{\substack{A \in \mathscr{F}_y^- \\ B \in \mathscr{F}_{x+y}^+}} |P(A \cap B) - P(A) \cdot P(B)|, \tag{2.11}$$

where

$$\mathscr{F}_x^- = \sigma\{X(y) \colon y < x\} \quad \text{and} \quad \mathscr{F}_x^+ = \sigma\{X(y) \colon y \geq x\}.$$

We will also denote the covariance function of such a process (whenever it exists) by

$$R_X(y) = \operatorname{Cov}(X(0), X(y)) = EX(0)X(y) - EX(0)EX(y).$$

THEOREM 2.2. *Assume that* $X = \{X(y), y \in \mathbf{R}\}$ *is a strictly stationary zero-mean process such that for some* $\delta > 0$, *the moment* $E|X(0)|^{2+\delta} < \infty$ *and*

$$\int_0^\infty (\alpha_X(x))^{\delta/(2+\delta)}\, dx < \infty. \tag{2.12}$$

Then

$$\sigma_X^2 = \int_{\mathbf{R}} R_X(x)\, dx < \infty, \tag{2.13}$$

where the integral converges absolutely and

$$X \in \mathbf{DA}\{\sigma_X W; 0, \beta^{1/2}\}.$$

In other words, for any $\phi \in \mathscr{S}(\mathbf{R})$, $z \in \mathbf{R}$,

$$\begin{aligned} \lim_{\beta \to \infty} P\left(\beta^{1/2} \int_{\mathbf{R}} X(\beta y)\phi(y)\, dy < z \right) \\ = P\left(\sigma_X \int_{\mathbf{R}} \phi(y)\, dW(y) < z \right), \end{aligned} \tag{2.14}$$

where W is the Brownian motion process.

PROOF. The absolute convergence of the integral (2.13) can be shown in a way similar to the proof of (18.5.12) in Ibragimov and Linnik (1965), using (2.12) and the estimate of the covariance function by the mixing coefficient given in that book. By inspection of the proof of Theorem 18.5.3 in Ibragimov and Linnik (1965), and also of the corresponding (multidimensional) limit

Theorem 7.3.1 in Ethier and Kurtz (1986), we obtain that, under the assumptions of the theorem,

$$\left\{ \frac{1}{\sqrt{\beta}} \int_0^{\beta x} X(y)\, dy,\ x \geq 0 \right\} \Rightarrow \{\sigma_X W(x),\ x \geq 0\}, \tag{2.15}$$

where $\Rightarrow$ denotes the convergence of finite-dimensional distributions. By (2.15), we easily get that for any $n = 1, 2, \dots,\ M = 1, 2, \dots$ and any

$$\phi_n(y) = \sum_{k=-M}^{M} \phi_k \mathbf{1}_{[k/n,(k+1)/n)}(y), \qquad |k| \leq M,$$

we have

$$\frac{1}{\sqrt{\beta}} \sum_{k=-M}^{M} \phi_k \int_{\beta k/n}^{\beta(k+1)/n} X(y)\, dy \Rightarrow \sigma_X \int_{\mathbf{R}} \phi_n(y)\, dW(y). \tag{2.16}$$

Indeed, substituting $(-M + k)/n = x_k$ so that $x_0 = 0, \dots, x_{2M+1} = (M + 1)/n$ and $\tilde{X}(y) = X(y - M/n)$, we can rewrite the left-hand side of (2.16) as

$$\frac{1}{\sqrt{\beta}} \sum_{k=0}^{2M+1} \phi_k \int_{\beta x_k}^{\beta x_{k+1}} \tilde{X}(y)\, dy.$$

As (2.15) obviously remains true for the shifted process $\tilde{X}(y)$ as well, (2.15) implies (2.16).

To finish the proof, it suffices to show that for any $\phi \in \mathscr{S}(\mathbf{R})$, the integral on the left-hand side of (2.14) can be approximated in mean square, uniformly in $\beta > 0$, by the corresponding integral with respect to a step function ϕ_n introduced above. That is obviously the case since for any $\varepsilon > 0$ there exist n, M and $\phi_k, k = -M, \dots, M$, such that

$$\sup_{y \in \mathbf{R}} |\phi(y) - \phi_n(y)| < \varepsilon,$$

which implies that

$$\sup_{y \in \mathbf{R}} |(\phi - \phi_n) * (\phi - \phi_n)| < C\varepsilon,$$

where C is a constant independent of ε. Now, in view of (2.13),

$$\begin{aligned} E\left(\beta^{1/2} \int_{\mathbf{R}} (\phi(y) - \phi_n(y)) X(\beta y)\, dy \right)^2 &= \beta^{-1} \int_{\mathbf{R}} \int_{\mathbf{R}} R_X(x - y)(\phi(x/\beta) - \phi_n(x/\beta))(\phi(y/\beta) - \phi_n(y/\beta))\, dx\, dy \\ &= \int_{\mathbf{R}} R_X(x)((\phi - \phi_n) * (\phi - \phi_n))(x/\beta)\, dx \leq C\varepsilon \int_{\mathbf{R}} |R_X(x)|\, dx < \infty. \quad \square \end{aligned}$$

An application of Theorems 2.1 and 2.2 gives the following Gaussian scaling limit result for the random field solution u from (1.3) of the Burgers equation (1.1).

COROLLARY 2.1. *Let $U_0 = \{U_0(y), y \in \mathbf{R}\}$ be a strictly stationary process satisfying the mixing conditions of Theorem 2.2, and such that for some $\delta > 0$,*

$$E\exp((2+\delta)U_0(0)) < \infty. \tag{2.17}$$

Then, as $\beta \to \infty$, the two-parameter random field in (x,t)

$$\beta^{3/2}u(\beta x, \beta^2 t) \Rightarrow -a^{-1}\sigma\int_{\mathbf{R}} p_x(x,y,t)\,dW(y), \tag{2.18}$$

where $a = E\exp(U_0(0))$,

$$R(y) = \operatorname{Cov}(\exp(U_0(0)), \exp(U_0(y)))$$

and

$$\sigma^2 = \int_{\mathbf{R}} R(y)\,dy. \tag{2.19}$$

In particular, for each $t > 0$ and $x \in \mathbf{R}$, as $\beta \to \infty$,

$$\beta^{3/2}u(\beta x, \beta^2 t) \Rightarrow N(0, 4^{-1}\pi^{-1/2}a^{-1}\sigma^2 t^{-3/2}). \tag{2.20}$$

PROOF. Applying Theorem 2.2 with $X(y) = \exp(U_0(y)) - E\exp(U_0(y))$ we can easily see that all the assumptions of Theorem 2.2 are satisfied so that for any $\phi \in \mathscr{S}(\mathbf{R})$, in distribution,

$$\lim_{\beta\to\infty}\int_{\mathbf{R}} V_\beta(y)\phi(y)\,dy = \sigma\int_{\mathbf{R}}\phi(y)\,dW(y), \tag{2.21}$$

where

$$V_\beta(y) = \beta^{1/2}(\exp(U_0(\beta y)) - a).$$

However, (2.21) is equivalent to assumption (2.3) of Theorem 2.1. On the other hand, for any $\phi \in \mathscr{S}(\mathbf{R})$, in probability,

$$\int_{\mathbf{R}}\exp(U_0(\beta y))\phi(y)\,dy = \beta^{-1/2}\int_{\mathbf{R}} V_\beta(y)\phi(y)\,dy + a\int_{\mathbf{R}}\phi(y)\,dy \to a\langle 1, \phi\rangle,$$

because the first term in the above sum converges to 0 in view of (2.21). Thus, assumption (2.4) is also satisfied and Theorem 2.1 gives statement (2.18). Statement (2.20) follows from (2.18) and the following computation of the variance:

$$E\left(\int_{\mathbf{R}} p_x(x,y,t)\,dW(y)\right)^2 = \int_{\mathbf{R}} p_x^2(x,y,t)\,dy = \left(4\sqrt{\pi}t^{3/2}\right)^{-1}. \qquad \square$$

3. Limit properties of functionals on Gibbs–Cox processes. We begin with an introduction, in some detail, of the notion of the Gibbs–Cox process. As far as we know, it appears here for the first time. However, it should be mentioned that, for some special potentials, the Gibbs–Cox process has been considered under the name spin glass model or Ising model with random potential [see, e.g., Campanino, Olivieri and van Enter (1987) and Funaki (1991)].

Let us recall the notion of a Gibbs measure on $\mathbf{R}^d$ [see, e.g., Ruelle (1969) and Spohn (1991)] and of its cluster property at low density [Spohn (1986)]. The Gibbs measure itself is determined by the pair potential $\Phi = \Phi(x)$, $x \in \mathbf{R}^d$, and the fugacity $\lambda = \lambda(x)$, $x \in \mathbf{R}^d$, which plays the role of the intensity in classical Poisson processes. We assume that the pair potential satisfies the following five conditions:

1. Φ is symmetric: $\Phi(x) = \Phi(-x)$.
2. Φ is of finite range: $\Phi(x) = 0$ for $|x| > R$ with some $R > 0$.
3. $\Phi \in C^3(\mathbf{R}^d)$.
4. Φ is superstable in the sense of Ruelle [see Ruelle (1969)].
5. $\Phi(x) \geq 0$.

In the usual references it is assumed that the fugacity λ is a constant and does not vary with $x \in \mathbf{R}^d$. In our case it is essential that space-dependent fugacity be permitted, but we impose a technical condition that λ is a function bounded by a constant determined by potential Φ:

6. $0 \leq \lambda(x) \leq z_0 < 1/(3C_0)$, where $C_0 \equiv C_0(\Phi) = e\int_{\mathbf{R}^d}(1 - e^{-\Phi(x)})\,dx < \infty$.

Let us introduce some notation. By $\mathscr{X}(\Lambda)$, $\Lambda \subset \mathbf{R}^d$, we denote the space of all $\mathbf{Z}_+$-valued Radon measures on Λ, that is, all locally finite configurations on Λ. Note that if Λ is bounded, the space $\mathscr{X}(\Lambda)$ can be identified with $\bigcup_{n=0}^{\infty}\Lambda^n$. Let $\mathscr{B}_\Lambda$, $\Lambda \subset \mathbf{R}^d$, be the σ-field of $\mathscr{X}(\mathbf{R}^d)$ generated by $\{\langle \psi, \nu \rangle;\ \psi \in C_0^\infty(\mathbf{R}^d), \operatorname{supp} \psi \subset \Lambda\}$, where $\nu \in \mathscr{X}(\mathbf{R}^d)$ and $\langle \psi, \nu \rangle = \int \psi(x)\,d\nu(x)$.

Then probability measure $\mu = \mu_\lambda$ on the space $\mathscr{X}(\mathbf{R}^d)$ is called a Gibbs measure (associated with pair potential Φ and fugacity λ) if it satisfies the so-called Dobrushin–Lanford–Ruelle (DLR) equation. More precisely, μ is a Gibbs measure if, for all bounded $\Lambda \subset \mathbf{R}^d$, its conditional probabilities with respect to $\mathscr{B}_{\Lambda^c}$ satisfy the following equation:

$$\begin{aligned}\mu(dx_1 \cdots dx_n|\mathscr{B}_{\Lambda^c})(\{y_j\}) \\ = Z_\Lambda^{-1}\frac{1}{n!}\exp\left[-\sum_{1\leq i<j\leq n}\Phi(x_i - x_j) - \sum_{i=1}^{n}\sum_{j=1}^{\infty}\Phi(x_i - y_j)\right]\prod_{i=1}^{n}\lambda(x_i)\,dx_i,\end{aligned}$$

for $(x_1,\dots,x_n) \in \Lambda^n$, $n = 0,1,2,\dots$, and μ-a.a. $\{y_j\} \in \chi(\Lambda^c)$, where

$$Z_\Lambda = \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\exp\left[-\sum_{1\leq i<j\leq n}\Phi(x_i - x_j) - \sum_{i=1}^{n}\sum_{j=1}^{\infty}\Phi(x_i - y_j)\right]\prod_{i=1}^{n}\lambda(x_i)\,dx_i$$

is a normalization constant. The term corresponding to $n = 0$ in the above sum is regarded as being equal to 1, and the integrability of Z_Λ is guaranteed by the superstability condition 4 imposed on Φ at the beginning of this section [see Ruelle (1969)].

Under conditions 1–6, the Gibbs measure $\mu = \mu_\lambda$ is uniquely determined by its potential Φ and fugacity λ. Furthermore, μ has the exponential L^2-mixing property, uniformly in λ, which is described in more detail in the next proposition, which will make use of the following notation: for $\Lambda, \Lambda_1, \Lambda_2 \subset \mathbf{R}^d$, $|\Lambda|$ = the volume of Λ,

$$\operatorname{dist}(\Lambda_1, \Lambda_2) = \inf_{\substack{x \in \Lambda_1 \\ y \in \Lambda_2}} |x - y|$$

and

$$\overline{\Lambda} = \{x \in \mathbf{R}^d; \operatorname{dist}(x, \Lambda) \le R\},$$

where $R > 0$ is the constant appearing in condition 2.

PROPOSITION 3.1. *There exist constants $C, c > 0$, depending only on z_0 and Φ, such that*

$$|E^{\mu_\lambda}(\psi_1 \psi_2)| \le \left(E^{\mu_\lambda}(\psi_1^2) E^{\mu_\lambda}(\psi_2^2)\right)^{1/2} \min\{1, C|\overline{\Lambda}_1| \exp(-c \operatorname{dist}(\Lambda_1, \Lambda_2))\},$$

for each $\mathscr{B}_{\Lambda_i}$-measurable ψ_i such that $E^{\mu_\lambda}(\psi_i) = 0$, and each bounded $\Lambda_i \subset \mathbf{R}^d$, $i = 1, 2$.

The proof of Proposition 3.1 relies on the so-called cluster expansion and can be carried out in almost the same way as the proof of Lemma 4 in Spohn (1986) which dealt with the case of constant fugacity $\lambda(x) = \lambda = \text{const}$. The fact that fugacity $\lambda(x)$ is assumed to satisfy condition 6 is sufficient to adapt the Spohn's proof to our situation. We omit the tedious details.

DEFINITION 3.1. A Gibbs–Cox random field in $\mathbf{R}^d$ is a random field generated by the Gibbs measure μ_λ whose fugacity $\lambda = \lambda(x)$ is an independent stationary ergodic random field with bounded realizations satisfying the above condition 6.

Let us make the above definition more formal. Since this paper is concerned only with the one-dimensional Burgers equation, from now on we assume that the dimension $d = 1$. Fix a pair potential satisfying conditions 1–5. Denote by $\mathbf{L} = \mathbf{L}_{z_0}$, where $z_0 < 1/3C_0$, the family of all measurable functions $\lambda = \lambda(x)$ on $\mathbf{R}$ satisfying condition 6. The stationarity and ergodicity assumptions on the process λ can now be phrased as follows:

The fugacity process $\lambda = \{\lambda(x, w);\ x \in \mathbf{R},\ w \in \mathbf{W}\}$ is defined on a probability space $(\mathbf{W}, \mathscr{F}_{\mathbf{W}}, Q)$, and satisfies the following conditions:

1′. $\lambda(x, w)$ is jointly measurable in (x, w) and $\lambda(\cdot, w) \in \mathbf{L}$ for Q-almost all w.
2′. $\lambda(x, T_y w) = \lambda(x + y, w)$ for all $x, y \in \mathbf{R}$ and a certain ergodic flow $\{T_y: \mathbf{W} \to \mathbf{W}\}_{y \in \mathbf{R}}$.

Condition (2′) can be written out more explicitly as follows:

$2'_1$. T_y is Q-invariant for all $y \in \mathbf{R}$.
$2'_2$. T_0 = identity and $T_x T_y = T_{x+y}$, $x, y \in \mathbf{R}$.
$2'_3$. $(y, w) \in \mathbf{R} \times \mathbf{W} \mapsto T_y w \in \mathbf{W}$ is measurable.
$2'_4$. $\{T_y\}_y$ is ergodic; that is, an arbitrary $\mathscr{F}_{\mathbf{W}}$-measurable bounded function which is T_y invariant for every $y \in \mathbf{R}$ is Q-a.s. constant.

Then the Gibbs–Cox process has the distribution

$$\mu = \int_W \mu_\lambda(\cdot, w)\, dQ(w),$$

and when conditioned on λ, its distribution is a Gibbs measure μ_λ. In probabilistic terms the fugacity is taken to be independent of the Gibbs measure.

EXAMPLE 3.1. The basic example of the Gibbs–Cox process is the classic Cox (sometimes called doubly stochastic Poisson) process with the random intensity measure

$$\Lambda(A) = \int_A \lambda(x)\, dx, \qquad A \subset \mathbf{R},$$

generated by a nonnegative process $\lambda = \{\lambda(x), x \in \mathbf{R}\}$ so that, in other words, if

$$N(A) := \sum_{k=1}^{\infty} \mathbf{1}_A(x_k)$$

is the number of counts in a Borel set A, then for any mutually disjoint Borel sets $A_1, \dots, A_n \subset \mathbf{R}$ and any nonnegative integers $j_1, \dots, j_n$,

$$P(N(A_1) = j_1, \dots, N(A_n) = j_n) = E\left(\prod_{k=1}^{n} \frac{(\Lambda(A_k))^{j_k}}{j_k!} \exp(-\Lambda(A_k)) \right).$$

This example allows no interactions and corresponds to the case of zero pair potential ($\Phi \equiv 0$) for the Gibbs–Cox process.

Let $\mu = \mu_\lambda$ be a Gibbs measure associated with the pair potential Φ and fugacity $\lambda = \lambda(x) \in \mathbf{L}$. We denote by P_λ, $\lambda \in \mathbf{L}$, the distribution of the point process

$$\nu = \sum_{k=1}^{\infty} \delta_{(\xi_k, \theta_k, x_k)} \in \chi := \chi(\mathbf{R}^3)$$

on $\mathbf{R}^3$, where $\{x_k\}_{k=1}^{\infty}$ is μ_λ-distributed and $\{(\xi_k, \theta_k)\}_{k=1}^{\infty}$ is $\mathbf{R}^2$-valued, i.i.d. and independent of $\{x_k\}_{k=1}^{\infty}$.

Our goal in this section is to establish a limit behavior of a special class of functionals F on the Gibbs–Cox processes introduced above.

THEOREM 3.1. *Assume that the functional $F = F(y, \nu)$, $y \in \mathbf{R}$, $\nu \in \chi$, satisfies the following four conditions*:

(a) *F is jointly measurable in (y, ν).*

(b) *There exists an $r > 0$ such that $F(y, \cdot)$ is $\mathscr{B}_{\Delta(y,r)}$-measurable in $\nu \in \chi$ for every $y \in \mathbf{R}$, where $\Delta(y, r) = \mathbf{R} \times \mathbf{R} \times (y - r, y + r)$.*

(c) *There exists a $\gamma > 2$ such that $F(y, \cdot) \in L^\gamma(dP_\lambda)$, for every $y \in \mathbf{R}$, $\lambda \in \mathbf{L}$, and $\sup_{y \in \mathbf{R}, \lambda \in \mathbf{L}} E^{P_\lambda}[|F(y)|^\gamma] < \infty$.*

(d) *$F(y + y', \nu) = F(y, \tilde{T}_{y'}\nu)$, $y, y' \in \mathbf{R}$, where $\tilde{T}_y \nu = \sum_{k=1}^{\infty} \delta_{(\xi_k, \theta_k, x_k - y)}$.*

Furthermore, let

$$F_\beta(\psi) = \beta^{-1/2}\int_{\mathbf{R}}\psi\left(\frac{y}{\beta}\right)\left(F(y) - E^{P_\lambda}[F(y)]\right)dy, \qquad \psi \in \mathscr{S}(\mathbf{R}),\ \beta \geq 1.$$

If a Hilbert space $\mathscr{E}$ is imbedded in $L^2(\mathbf{R})$ and its inclusion map is of Hilbert–Schmidt type, then, for Q-a.a. λ's, the distribution of F_β under P_λ converges weakly to that of $\sigma W'$ in $\mathscr{E}'$ as $\beta \to \infty$, where $W' = W'(x)$ is the Gaussian white noise and

$$\sigma^2 = E^Q\left[\int_{\mathbf{R}} E^{P_\lambda}[(F(0) - E^{P_\lambda}[F(0)])(F(y) - E^{P_\lambda}[F(y)])]\,dy\right].$$

The proof of Theorem 3.1 will be based on the following proposition.

PROPOSITION 3.2. *For Q-a.a. λ and all $\psi \in \mathscr{S}(\mathbf{R})$, the distribution of $F_\beta(\psi)$ under P_λ converges weakly to $N(0, \sigma^2\|\psi\|^2_{L^2(\mathbf{R})})$ as $\beta \to \infty$.*

PROOF. Fix $0 < \varepsilon < (\gamma - 2)/(2(\gamma - 1))$ and divide $F_\beta(\psi)$ into the sum

$$F_\beta(\psi) = X_\beta(\psi) + R_{\beta,1}(\psi) + R_{\beta,2}(\psi) - E^{P_\lambda}\left[R_{\beta,2}(\psi)\right]. \tag{3.1}$$

Here

$$X_\beta(\psi) = \sum_{k=-\infty}^{\infty} X_\beta^k(\psi), \tag{3.2}$$

$$X_\beta^k \equiv X_\beta^k(\psi) = \beta^{-1/2}\int_{s_k}^{t_{k+1}}\psi(y/\beta)G_\beta(y)\,dy, \tag{3.3}$$

where $t_k = t_{k,\beta} = k\beta^{1-\varepsilon/2}$, $s_k = s_{k,\beta} = t_k + \beta^{\varepsilon/4}$,

$$G_\beta(y) := 1_{\{|F(y)| \leq \beta^{1/2-\varepsilon}\}}F(y) - E^{P_\lambda}\left[1_{\{|F(y)| \leq \beta^{1/2-\varepsilon}\}}F(y)\right], \tag{3.4}$$

$$R_{\beta,1}(\psi) = \sum_{k=-\infty}^{\infty}\beta^{-1/2}\int_{t_k}^{s_k}\psi(y/\beta)G_\beta(y)\,dy, \tag{3.5}$$

$$R_{\beta,2}(\psi) = \beta^{-1/2}\int_{\mathbf{R}}\psi(y/\beta)1_{\{|F(y)| > \beta^{1/2-\varepsilon}\}}F(y)\,dy. \tag{3.6}$$

Notice that $X_\beta^k = 0$ except for finitely many k's; in fact, $X_\beta^k = 0$ for $|k| \geq a\beta^{\varepsilon/2} + 1$ if $\operatorname{supp}\psi \subset [-a, a]$. Since

$$|R_{\beta,1}(\psi)| \leq (2a\beta^{\varepsilon/2} + 1)\beta^{-1/2}\beta^{\varepsilon/4}\|\psi\|_\infty \times 2\beta^{1/2-\varepsilon}$$

and

$$E^{P_\lambda}\left[\left|R_{\beta,2}(\psi)\right|\right] \leq \beta^{1/2-(1/2-\varepsilon)(\gamma-1)}\|\psi\|_\infty \times \sup_{y\in\mathbf{R}} E^{P_\lambda}\left[|F(y)|^\gamma\right],$$

and since the right-hand sides of these two estimates tend to 0 as $\beta \to \infty$, the proof of Proposition 3.2 can be completed if one can show the weak convergence of $X_\beta(\psi)$ to $N(0, \sigma^2\|\psi\|^2_{L^2(\mathbf{R})})$ as $\beta \to \infty$ [see Ibragimov and Linnik

(1965), Lemma 18.4.1]. Equivalently, we shall prove the convergence of its characteristic functional, relying on the following proposition and lemma.

PROPOSITION 3.3.

(i) $$\lim_{\beta\to\infty} \max_k E^{P_\lambda}\Big[\big\{X_\beta^k\big\}^2\Big] = 0,$$

(ii) $$\lim_{\beta\to\infty} \sum_{k=-\infty}^{\infty} E^{P_\lambda}\Big[\big\{X_\beta^k\big\}^2\Big] = \sigma^2\|\psi\|^2_{L^2(\mathbf{R})}, \qquad Q\text{-}a.a.\ \lambda,$$

(iii) $$\lim_{\beta\to\infty} \sum_{k=-\infty}^{\infty} E^{P_\lambda}\Big[\big|X_\beta^k\big|^3\Big] = 0.$$

LEMMA 3.1. *In the above notation,*

$$\left|E^{P_\lambda}\big[\exp\big(iX_\beta(\psi)\big)\big] - \prod_{k=-\infty}^{\infty} E^{P_\lambda}\big[\exp\big(iX_\beta^k\big)\big]\right|$$
$$\le \text{const}\,\beta\exp\big[-c\beta^{\varepsilon/4}\big], \qquad \beta \ge 1.$$

PROOF. For $n \in \mathbf{Z}$, using Proposition 3.1, we have

$$\left|E^{P_\lambda}\left[\prod_{k=-\infty}^{n}\exp\big(iX_\beta^k\big)\right] - E^{P_\lambda}\big[\exp\big(iX_\beta^n\big)\big]E^{P_\lambda}\left[\prod_{k=-\infty}^{n-1}\exp\big(iX_\beta^k\big)\right]\right|$$
$$\le \text{const}\,\beta^{1-\varepsilon/2}\exp\big\{-c\beta^{\varepsilon/4}\big\}.$$

Now the conclusion easily follows. □

PROOF OF PROPOSITION 3.2 (Continued). From Proposition 3.3, noting Lemma 3.1, we obtain

$$\lim_{\beta\to\infty} E^{P_\lambda}\big[\exp\big(iX_\beta(\psi)\big)\big] = \exp\Big[-\tfrac{1}{2}\sigma^2\|\psi\|^2_{L^2(\mathbf{R})}\Big],$$

since

$$\left|\exp\big(iX_\beta^k\big) - \left\{1 + iX_\beta^k - \tfrac{1}{2}\big(X_\beta^k\big)^2\right\}\right| \le \tfrac{1}{6}\big|X_\beta^k\big|^3$$

and since $E^{P_\lambda}[X_\beta^k] = 0$. This completes the proof of Proposition 3.2. □

The proof of Proposition 3.3 itself will be based on a series of lemmas.

LEMMA 3.2. *There exists a constant C independent of $\lambda \in \mathbf{L}_{z_0}$, $\beta \ge 1$ and $y_1, y_2, y_3 \in \mathbf{R}$, and such that*

(i) $$\big|E^{P_\lambda}\big[\{F(y_1) - E^{P_\lambda}[F(y_1)]\}\{F(y_2) - E^{P_\lambda}[F(y_2)]\}\big]\big|$$
$$\le C\exp(-c|y_1 - y_2|),$$

(ii) $$\big|E^{P_\lambda}\big[G_\beta(y_1)G_\beta(y_2)\big]\big| \le C\exp(-c|y_1 - y_2|)$$

and

(iii) $$\big|E^{P_\lambda}\big[G_\beta(y_1)G_\beta(y_2)G_\beta(y_3)\big]\big| \le C\beta^{1/2-\varepsilon}\exp\big[-c\max\{|y_1-y_2|,|y_2-y_3|\}\big],$$

if $y_1 < y_2 < y_3$.

PROOF. This lemma is shown easily from Proposition 3.1 by using condition (c) on F and noting that $|G_\beta(y)| \le 2\beta^{1/2-\varepsilon}$ in view of the above condition 3. □

LEMMA 3.3. *Let*

$$\sigma^2(\lambda) := \int_{\mathbf{R}} E^{P_\lambda}\big[\{F(0) - E^{P_\lambda}[F(0)]\}\{F(y) - E^{P_\lambda}[F(y)]\}\big]\,dy, \qquad \lambda \in \mathbf{L}_{z_0}.$$

Then

$$0 \le \sup_{\lambda\in\mathbf{L}_{z_0}} \sigma^2(\lambda) < \infty \tag{3.7}$$

and

$$\lim_{\beta\to\infty}\int_{s_k-x}^{t_{k+1}-x} E^{P_\lambda}\big[G_\beta(0)G_\beta(y)\big]\,dy = \sigma^2(\lambda), \qquad s_k < x < t_{k+1},$$

uniformly in λ, x, k.

PROOF. Lemma 3.2(i) yields (3.7) directly. To compute the limit note that by Chebyshev's inequality and condition (c) on F, we have that

$$\big|E^{P_\lambda}\big[F(y_1)F(y_2); |F(y_1)| \le \beta^{1/2-\varepsilon}, |F(y_2)| \le \beta^{1/2-\varepsilon}\big] - E^{P_\lambda}\big[F(y_1)F(y_2)\big]\big| \le \text{const}\,\beta^{-(1/2-\varepsilon)(\gamma-2)},$$

$$\big|E^{P_\lambda}\big[F(y); |F(y)| \le \beta^{1/2-\varepsilon}\big] - E^{P_\lambda}\big[F(y)\big]\big| \le \text{const}\,\beta^{-(1/2-\varepsilon)(\gamma-1)}.$$

These estimates imply

$$\big|E^{P_\lambda}\big[G_\beta(y_1)G_\beta(y_2)\big] - E^{P_\lambda}\big[\{F(y_1) - E^{P_\lambda}[F(y_1)]\}\{F(y_2) - E^{P_\lambda}[F(y_2)]\}\big]\big| \le \text{const}\,\beta^{-(1/2-\varepsilon)(\gamma-2)}.$$

Hence, noting Lemma 3.2(i) and (ii), we get the conclusion. □

PROOF OF PROPOSITION 3.3. To compute $E^{P_\lambda}[\{X_\beta^k\}^2]$, set

$$Y_\beta^k = \beta^{-1/2}\psi(t_k/\beta)\int_{s_k}^{t_{k+1}} G_\beta(y)\,dy.$$

Then, using Lemma 3.2(ii), we have that

$$E^{P_\lambda}\Big[\big\{X_\beta^k - Y_\beta^k\big\}^2\Big]$$
$$\le \beta^{-1}\|\psi'\|_\infty^2\left(\frac{t_{k+1}-t_k}{\beta}\right)^2\int_{s_k}^{t_{k+1}}\int_{s_k}^{t_{k+1}}\big|E^{P_\lambda}\big[G_\beta(y_1)G_\beta(y_2)\big]\big|\,dy_1\,dy_2$$
$$\le \text{const}\ \beta^{-3\varepsilon/2}.$$

Therefore, it suffices to prove assertions (i) and (ii) of Proposition 3.3 for Y_β^k in place of X_β^k.

Let us compute $E^{P_\lambda}[\{Y_\beta^k\}^2]$. Using condition (d) on F, we get that

$$(3.8)\qquad \begin{aligned} E^{P_\lambda}\Big[\big\{Y_\beta^k\big\}^2\Big] &= \beta^{-1}\psi(t_k/\beta)^2\int_{s_k}^{t_{k+1}}\int_{s_k}^{t_{k+1}}E^{P_\lambda}\big[G_\beta(y_1)G_\beta(y_2)\big]\,dy_1\,dy_2\\ &= \beta^{-1}\psi(t_k/\beta)^2\int_{s_k}^{t_{k+1}}\Big(\sigma^2\big(\lambda(T_{y_1}w)\big)+r_\beta^k(y_1)\Big)\,dy_1,\end{aligned}$$

where

$$r_\beta^k(y_1) = \int_{s_k-y_1}^{t_{k+1}-y_1}E^{P_{\lambda(T_{y_1}w)}}\big[G_\beta(0)G_\beta(y)\big]\,dy - \sigma^2\big(\lambda(T_{y_1}w)\big).$$

Notice that Lemma 3.3 shows that

$$\lim_{\beta\to\infty}\ \sup_{k,\,y_1}\big|r_\beta^k(y_1)\big| = 0.$$

Now, in particular, assertion (i) of Proposition 3.3 with X_β^k replaced by Y_β^k is shown easily from (3.8) by noting (3.7). To prove (ii) for Y_β^k it suffices to prove that

$$(3.9)\qquad \begin{aligned}\lim_{\beta\to\infty}\sum_{k=-\infty}^{\infty}\beta^{-1}\psi(t_k/\beta)^2\int_{s_k}^{t_{k+1}}\sigma^2\big(\lambda(T_y w)\big)\,dy\\ = \sigma^2\|\psi\|_{L^2(\mathbf{R})}^2,\qquad Q\text{-}a.a.\ \lambda,\end{aligned}$$

or that

$$(3.10)\qquad \begin{aligned}\lim_{\beta\to\infty}\beta^{-1}\int_{\mathbf{R}}\psi(y)^2\sigma^2\big(\lambda(T_y w)\big)\,dy\\ = \sigma^2\|\psi\|_{L^2(\mathbf{R})}^2,\qquad Q\text{-}a.a.\ \lambda.\end{aligned}$$

In fact, (3.9) follows from (3.10) by noting (3.7), and (3.10) itself is a consequence of the individual ergodic theorem.

Finally, assertion (iii) of Proposition 3.3 follows from the estimate

$$E^{P_\lambda}\Big[\big|X_\beta^k\big|^3\Big]$$
$$\le 6\beta^{-3/2}\|\psi\|_\infty^3\int_{s_k\le y_1<y_2<y_3\le t_{k+1}}\big|E^{P_\lambda}\big[G_\beta(y_1)G_\beta(y_2)G_\beta(y_3)\big]\big|\,dy_1\,dy_2\,dy_3$$
$$\le \text{const}\ \beta^{-3/2}\times(t_{k+1}-s_k)\times\beta^{1/2-\varepsilon}\le \text{const}\ \beta^{-3\varepsilon/2},$$

which is implied by Lemma 3.2(iii). The proof of Proposition 3.3 is thus complete. □

PROOF OF THEOREM 3.1. Let $\tilde{P}_\lambda^\beta$ be the distribution of F_β on the space $\mathscr{S}'$ under P_λ. Then the family $\{\tilde{P}_\lambda^\beta\}_{\beta \geq 1}$ is tight on $\mathscr{S}'$ for all $\lambda \in \mathbf{L}$. Indeed, Lemma 3.2(i) shows

$$E^{P_\lambda}\Big[F_\beta(\psi)^2\Big] \leq \text{const}\|\psi\|^2_{L^2(\mathbf{R})}, \qquad \beta \geq 1.$$

The conclusion, therefore, follows from Proposition 3.2. □

REMARK 3.1. (i) A method similar to that described in Section 18 of Ibragimov and Linnik (1965) might also work here. However, since $F(y)$ is not stationary under P_λ, we have provided a complete proof above.

(ii) Guo (1984) and Spohn [(1991), page 87], considered special cases of F (e.g., a density field or a linear functional of ν) for which a concrete representation of $\log E^{P_\lambda}[\exp(iF_\beta(\psi)]$ is possible via the cluster expansion.

4. Scaling limits for solutions with shot noise initial process driven by a Gibbs–Cox process. In this section, we will apply Theorem 2.1 in the situation when the initial velocity potential process U_0 from (1.2b) is a shot noise driven by a Gibbs–Cox process (see Section 3). As we will see later on, the special case of the latter is a well known Cox process which is sometimes also called a doubly stochastic Poisson process [see Grandell (1976) for the theory of doubly stochastic Poisson processes].

The shot noise processes have long been a standard model for physical phenomena described by random processes (or fields) with sample paths (or surfaces) which have a smoothed out point process appearance [see, e.g., the classic paper by Rice (1945)]. The shape of their trajectories is immediately clear; they are smooth bumps of "fixed" shape which are randomly scaled and appear at random points in space. From the viewpoint of realistic physical modeling, the shot noise driven by a Gibbs–Cox process has an additional advantage of being able to describe nontrivial interactions between different bumps. This can be a useful feature in several applications and, in particular, in the stochastic Burgers flow model of evolution of matter density in the universe discussed in the Introduction [see Shandarin and Zeldovich (1989), Weinberg and Gunn (1990), and Gurbatov, Malakhov and Saichev (1991)].

Let $\mu = \mu_\lambda$ be the Gibbs measure associated with the pair potential Φ and fugacity $\lambda = \lambda(x) \in \mathbf{L}$. Then

$$U_0(x) := \sum_{k=1}^{\infty} \xi_k \varphi\left(\frac{x - x_k}{\theta_k}\right), \qquad x \in \mathbf{R}, \tag{4.1}$$

where φ is an integrable and smooth function on $\mathbf{R}$, θ is a positive random variable,

$$(\xi, \theta), (\xi_1, \theta_1), (\xi_2, \theta_2), \dots \tag{4.2}$$

are independent identically distributed random vectors and (x_i) is a Gibbs–Cox process with distribution μ_λ, independent of $\{(\xi_i, \theta_i)\}$.

We establish first a general result about the scaling limit distribution of the solution $u(x,t)$ of the Burgers equation (1.1) for the initial velocity potential U_0 given by the shot noise process (4.1). We shall assume in this section that the fugacity process λ is independent of everything else, that λ is a stationary process, and that ξ, θ and φ satisfy some boundedness assumptions. As we shall see, even under these restrictions, the scaling limit distribution of $u(x,t)$ may be non-Gaussian. This effect is in contrast to the case of the Poisson process driven shot noise when λ is nonrandom, where only Gaussian limiting distributions arise [see Bulinskii (1990) and Bulinskii and Molchanov (1991)]. The special case of the shot noise process driven by the Cox process from Example 3.1 was considered in Surgailis and Woyczynski (1993). It did not allow interaction of "bumps" of the shot noise permitted under the Gibbs–Cox model considered here. For other scaling limit results for the statistical solutions of the Burgers equation, see, for example, Rosenblatt (1987), Hu and Woyczynski (1994a, b) and Woyczynski (1993).

THEOREM 4.1. *Assume that the shot noise process* U_0 *defined in* (4.1) *satisfies the following three conditions*:

(i) $\lambda = \{\lambda(x), x \in \mathbf{R}\} \in \mathbf{L}_{z_0}$, $z_0 < 1/3C_0(\Phi)$, *is a stationary ergodic process.*

(ii) θ, $|\xi|$ *are bounded, and* φ *is smooth with a compact support.*

(iii) *There exists a generalized process* Z, *depending on the distribution of the process* λ *and normalizing constants* $A(\beta), B(\beta) > 0$, $\beta > 0$, *such that*

$$E^{P_\lambda}(e^{U_0}) \in \mathbf{DA}\{Z; A(\beta), B(\beta)\} \tag{4.3}$$

and such that

$$\lim_{\beta\to\infty} \frac{B(\beta)}{\sqrt{\beta}} = d < \infty. \tag{4.4}$$

Then, as $\beta \to \infty$,

$$\beta B(\beta) u(\beta x, \beta^2 t) \Rightarrow -a^{-1}\langle V, p_x(x,\cdot,t)\rangle, \tag{4.5}$$

where u *is a solution* (1.3) *of the Burgers equation with the initial velocity potential* U_0,

$$a = E\exp(U_0(0))$$

and

$$V = \sigma d \cdot W' + Z, \tag{4.6}$$

where

$$\sigma^2 = E^Q\left[\int_{\mathbf{R}} E^{P_\lambda}\left[\left(e^{U_0(0)} - E^{P_\lambda}[e^{U_0(0)}]\right)\left(e^{U_0(y)} - E^{P_\lambda}[e^{U_0(y)}]\right)\right] dy\right],$$

$$E\exp(i\langle Z, \phi\rangle) = \lim_{\beta\to\infty} E\exp\left(i\int_{\mathbf{R}} B(\beta)\left(E^{P_\lambda}[e^{U_0(\beta y)}] - A(\beta)\right)\phi(y)\,dy\right),$$

for each $\mathscr{S}'(\mathbf{R})$, *and* W' *is a white noise process independent of* Z.

PROOF. The proof is based on an application of the basic convergence results contained in Theorems 2.1 and 3.1. With V_β given by (2.2), we can write that

$$\begin{aligned}\langle V_\beta, \psi\rangle &= B(\beta)\beta^{-1}\int_{\mathbf{R}}\psi(\beta^{-1}y)\big(\exp(U_0(y)) - E^{P_\lambda}\exp(U_0(y))\big)\,dy\\ &\quad + B(\beta)\beta^{-1}\int_{\mathbf{R}}\psi(\beta^{-1}y)\big(E^{P_\lambda}\exp(U_0(y)) - A(\beta)\big)\,dy\\ &= I'_\beta(\psi) + I''_\beta(\psi),\end{aligned}$$

where the second integral $I''_\beta(\psi)$ is λ-measurable. Then

$$E\exp\big(i\langle V_{\beta,}\psi\rangle\big) = E\exp\big(iI''_\beta(\psi)\big)E\big(\exp\big(iI'_\beta(\psi)\big)|\Lambda\big),$$

where $E(\cdot|\Lambda)$ stands for the conditional expectation with respect to the σ-field generated by λ, and to verify (2.3) it suffices to prove that almost surely

$$\lim_{\beta\to\infty} E\big(\exp\big(iI'_\beta(\psi)\big)|\Lambda\big) = E\exp(i\sigma d\langle W', \psi\rangle)$$
$$= \exp\left(-\frac{\sigma^2 d^2}{2}\int_{\mathbf{R}}\psi^2(x)\,dx\right). \tag{4.7}$$

Indeed

$$\begin{aligned}&E\exp\big(iI''_\beta(\psi)\big)E\big(\exp\big(iI'_\beta(\psi)\big)\Lambda\big)\\ &\quad = E\exp\big(iI''_\beta(\psi)\big)\,E\exp(i\sigma d\langle W', \psi\rangle)\\ &\qquad + E\exp\big(iI''_\beta(\psi)\big)\Big(E\big(\exp\big(iI'_\beta(\psi)\big)|\Lambda\big) - E\exp(i\sigma d\langle W', \psi\rangle)\Big),\end{aligned}$$

where the first term converges to

$$E\exp(i\langle Z, \psi\rangle)\cdot E\exp(i\sigma d\langle W', \psi\rangle) = E\exp(i\langle V, \psi\rangle)$$

by assumption (iii), while the second tends to 0 in view of (4.7) and the dominated convergence theorem.

To prove (4.7), it suffices to apply Theorem 3.1 with

$$F(y, \nu) = e^{U_0},$$

which satisfies the conditions (a)–(d) on F stated in Theorem 3.1 in view of assumption (ii). □

REMARK 4.1. Notice that in the case of the shot noise driven by the usual Cox process independent of $\{(\xi_i, \theta_i)\}$, many of the above computations simplify. In particular,

$$E^{P_\lambda}\exp(U_0) = \exp(\lambda * \Psi),$$

where

$$\Psi(x) = E^{(\xi,\theta)}(\exp(\xi\phi(x/\theta)) - 1), \qquad x \in \mathbf{R}. \tag{4.8}$$

We will use this fact in what follows.

THEOREM 4.2. *Assume that conditions* (i) *and* (ii) *of Theorem* 4.1 *are satisfied, and that the shot noise is driven by the Cox process. Additionally suppose that process* λ *satisfies the strong mixing condition of Theorem* 2.2 *for some* $\delta > 0$. *Then the statement of Theorem* 4.1 *holds true with* $B(\beta) = \beta^{1/2}$ *and*

$$V = \sigma W' + \sigma_Z Z',$$

where Z' *is a white noise independent of the white noise* W' *and*

$$\sigma_Z^2 = \int_{\mathbf{R}} \operatorname{Cov}(\exp((\lambda * \Psi)(0)), \exp((\lambda * \Psi)(x)))\, dx.$$

PROOF. Similarly as in the proof of Theorem 2.2 we have that under the assumptions of the theorem,

$$e^{\lambda * \Psi} \in \mathbf{DA}\{\sigma_Z Z'; a, \beta^{1/2}\},$$

that is, the condition (iii) of Theorem 4.1 is satisfied with $d = 1$ and $Z = \sigma_Z Z'$. □

REMARK 4.2. For the Cox process we have that $C_0(\Phi) = 0$, since the pair potential $\Phi = 0$. In fact, we can prove Theorem 4.1 for the Cox process under milder conditions. More precisely, assumptions (i) and (ii) of Theorem 4.1 can be weakened to the following ones:

(i′) The process $\lambda = \{\lambda(x), x \in \mathbf{R}\}$ is a strictly stationary, nonnegative, ergodic process such that for some $c > 0$,

$$E[e^{c\lambda(0)}] < \infty.$$

(ii′) θ, $|\xi|$ and $|\phi(\cdot)|$ are bounded by constants c_θ, c_ξ and c_ϕ, respectively, where c_ξ and c_ϕ may depend on c, and ϕ has a compact support.

In particular, the boundedness assumption on $\lambda(x)$ is unnecessary in this case, as demonstrated in Theorem 3.1 of Surgailis and Woyczynski (1993).

5. Non-Gaussian scaling limits. Here we apply the results of the previous section to the situation where the initial velocity potential process is a shot noise driven by the usual Cox process with the intensity process λ equal to the square of a stationary Gaussian process. When the Gaussian process is complex (and λ is the square of its modulus), the corresponding Cox process is known as a *boson process* and was studied by Macchi (1975) [see also Daley and Vere-Jones (1988)]. In other words,

$$\lambda(x) = \zeta^2(x), \tag{5.1}$$

where $\zeta = \{\zeta(x), x \in \mathbf{R}\}$ is a stationary Gaussian process with mean $m = E\zeta(0)$ and covariance $R(x) = E(\zeta(0) - m)(\zeta(x) - m)$. A slightly more general situation $\lambda(x) = \zeta^2(x) + \lambda_0$, λ_0 a nonnegative constant, requires below only trivial changes.

According to Theorem 4.1 and Remark 4.2, the problem of the limiting distribution of $u(t, x)$ in the above situation reduces to the problem of finding the scaling limit of the exponential process $e^{\lambda * \Psi}$, with Ψ given by (4.8), that is, of finding conditions on Ψ, m and $R(x)$ such that

$$e^{\lambda * \Psi} \in \mathbf{DA}\{Z; A(\beta), B(\beta)\}, \tag{5.2}$$

for some possibly non-Gaussian generalized process Z.

The basic reference on non-central limit theorems for nonlinear functionals of Gaussian processes (and the process $e^{\lambda * \Psi}$ is such a nonlinear functional) is Dobrushin and Major (1979), which is used also in Theorem 5.1 below. However, the fact that the functional $e^{\lambda * \Psi}$ is *nonlocal* (because of the convolution) makes the application of the Dobrushin–Major theory more difficult; see Dobrushin and Major [(1979), Section 6] or Major [(1981), pages 102–103], with the main effort given to finding explicit coefficients of the Hermite expansion of $e^{\lambda * \Psi}$ in a series of Itô–Wiener integrals (Lemma 5.1).

Put

$$a = a(m) = E \exp\big((\zeta^2 * \Psi)(0)\big), \tag{5.3}$$

$$a' = a'(m) = 2E\big(\exp((\zeta^2 * \Psi)(0)) \cdot (\zeta * \Psi)(0)\big), \tag{5.4}$$

$$a'' = a''(m) = 2E\Big(\exp((\zeta^2 * \Psi)(0))\big(2((\zeta * \Psi)(0))^2 + \overline{\Psi}\big)\Big), \tag{5.5}$$

where $\overline{\Psi} = \int_{\mathbf{R}} \Psi(x)\,dx$.

THEOREM 5.1. *Let the covariance $R(x)$ be of the form*

$$R(x) = L(x) x^{-\alpha}, \tag{5.6}$$

where $0 < \alpha < 1$ and $L(\cdot)$ is a slowly varying function. Let $\Psi(x)$, $x \in \mathbf{R}$, be an integrable function such that $\|\Psi\|_1 \equiv \int |\Psi(x)|\,dx < \frac{1}{8}R(0)^{-1}$ and $\overline{\Psi} \neq 0$. Then the expectations (5.3)–(5.5) *exist and we have*:

(i) *If $m \neq 0$ and $a'(0) \neq 0$, then*

$$e^{\lambda * \Psi} \in \mathbf{DA}\big\{a(m) W'_\alpha; a(m), L^{1/2}(\beta)\beta^{\alpha/2}\big\},$$

where W'_α is the α-fractional Gaussian noise, that is, the generalized Gaussian process with

$$E \exp(i\langle W'_\alpha, \Psi\rangle) = \exp\left[-\tfrac{1}{2}\int_{\mathbf{R}^2} \psi(x)\psi(y)|x-y|^{-\alpha}\,dx\,dy\right]$$
$$= \exp\left[-\tfrac{1}{2}\int_{\mathbf{R}} |\hat{\psi}(p)|^2 |p|^{\alpha-1}\,dp\right],$$

with $\hat{\psi}$ being the Fourier transform of $\psi \in \mathscr{S}$.

(ii) *If $m = 0$, $a''(0) \neq 0$ and $0 < \alpha < \frac{1}{2}$, then*

$$e^{\lambda * \Psi} \in \mathbf{DA}\big\{\tfrac{1}{2}a''(0) W^{(2)}_\alpha; a(0), L(\beta)\beta^\alpha\big\},$$

where $W^{(2)}_\alpha = :(W'_\alpha)^2:$ is the second Wick polynomial of W_α, that is, the generalized process given by the double Itô–Wiener integral [*see, e.g., Kwapień and Woyczynski* (1992) *and Major* (1981)]

$$\langle W^{(2)}_\alpha, \psi\rangle = \int_{\mathbf{R}^2} \hat{\psi}(p_1 + p_2)|p_1|^{(\alpha-1)/2}|p_2|^{(\alpha-1)/2}\hat{W}(dp_1)\hat{W}(dp_2), \tag{5.7}$$

where $\hat{W}$ is complex Gaussian white noise (*the Fourier transform of W'*).

PROOF. Let $G(dp)$ be the spectral measure of Gaussian process ζ; that is,

$$R(x) = \int_{\mathbf{R}} e^{ixp} G(dp), \qquad x \in \mathbf{R}.$$

The process ζ itself has the spectral representation

$$\zeta(x) = m + \int_{\mathbf{R}} e^{ixp}\, M(dx), \tag{5.8}$$

where $M(dp) = \overline{M(-dp)}$ is the random spectral measure, $E|M(dp)|^2 = G(dp)$. Let $L^2(\Omega)$ be the Hilbert space of all real square integrable random variables measurable with respect to the σ-field generated by the Gaussian process. Then it is well known that any $\xi \in L^2(\Omega)$ admits the representation

$$\begin{aligned} \xi &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbf{R}^n} f^{(n)}(p_1, \dots, p_n) M(dp_1) \cdots M(dp_n) \\ &\equiv \sum_{n=0}^{\infty} \frac{1}{n!} \int f^{(n)}\, d^n M \end{aligned} \tag{5.9}$$

as a series of multiple Itô–Wiener integrals convergent in $L^2(\Omega)$, where $f^{(n)}(\cdot) \in L^2(G^n) \equiv L^2(\mathbf{R}^n; G^n)$, $n \geq 1$; $f^{(0)} \equiv \int f^{(0)}\, d^0 M \in \mathbf{R}$ and

$$\sum_{n=0}^{\infty} \frac{1}{n!} \| f^{(n)} \|^2_{L^2(G^n)} < \infty.$$

To prove the theorem, we shall need rather detailed information about the kernels of the Hermite expansion (5.9) of $\xi = \exp(\zeta^2 * \Psi(0))$, which is explicitly given in the following lemma. In the sequel we shall assume for notational convenience that the function Ψ is even, that is, $\Psi(x) = \Psi(-x)$, or its Fourier transform $\hat{\Psi}$ is real.

LEMMA 5.1. *Let* $\Psi(\cdot)$ *satisfy the conditions of Theorem* 5.1. *Then* $\exp(\zeta^2 * \Psi(0)) \in L^2(\Omega)$ *and*

$$\exp(\zeta^2 * \Psi(0)) = \sum_{s=0}^{\infty} \frac{1}{s!} \int f^{(s)}(\cdot; m)\, d^s M, \tag{5.10}$$

where $f^{(0)} = a$,

$$\begin{aligned} f^{(s)}(\cdot; m) &= a \sum_{2n+j+k=s} \frac{s!}{n!j!k!} \\ &\quad \times \left[\Gamma^{\otimes n} \otimes (2m\hat{\Psi})^{\otimes j} \otimes (4m\Gamma \circ \hat{\Psi})^{\otimes k} \right], \qquad s \geq 1, \end{aligned} \tag{5.11}$$

where the sum is taken over all integers $n, j, k \geq 0$ *such that* $2n + j + k = s$,

$$\Gamma(p_1, p_2) := \sum_{k=1}^{\infty} 2^{k-1} \hat{\Psi}^{(k)}(p_1, p_2), \qquad p_1, p_2 \in \mathbf{R}, \tag{5.12}$$

$$\begin{aligned} &\hat{\Psi}^{(k)}(p_1, p_2) \\ &\quad := \int_{\mathbf{R}^{k-1}} \hat{\Psi}(p_1 + x_1) \hat{\Psi}(-x_1 + x_2) \hat{\Psi}(-x_{k-1} + p_2) \\ &\qquad \times G(dx_1) \cdots G(dx_{k-1}), \end{aligned} \tag{5.13}$$

$k = 2, 3, \dots,$ $\hat{\Psi}^{(1)}(p_1, p_2) := \hat{\Psi}(p_1 + p_2)$, $\hat{\Psi}(p) = \int_{\mathbf{R}} e^{ipx}\Psi(x)\,dx$, $p \in \mathbf{R}$, *is the Fourier transform of* Ψ *and*

$$(5.14)\qquad (\Gamma \circ \hat{\Psi})(p) := \int_{\mathbf{R}} \Gamma(p, x)\hat{\Psi}(-x)G(dx).$$

In particular, for $m = 0$,

$$(5.15)\qquad \exp\left(\zeta^2 * \Psi(0)\right) = a \sum_{n=0}^{\infty} \frac{1}{n!} \int \Gamma^{\otimes n}\, d^{2n}M.$$

For any $m \in \mathbf{R}$, $s = 1, 2, \dots,$ $f^{(s)}(\cdot; m) \in C_b(\mathbf{R}^s)$ *and*

$$(5.16)\qquad \| f^{(s)} \|_\infty \le C\left(s! 8^s \|\Psi\|_1^s\right)^{1/2},$$

where $C = C(m, \Psi) < \infty$ *is a constant independent of* s.

Moreover, the function $a = a(m)$ *of* (5.3) *is twice continuously differentiable in* $m \in \mathbf{R}$,

$$(5.17)\qquad a'(m) = f^{(1)}(0)$$

and

$$(5.18)\qquad a''(m) = f^{(2)}(0, 0).$$

PROOF. Let us first prove the bound (5.16). Note that

$$\|\hat{\Psi}^{(k)}\|_\infty \le \|\hat{\Psi}\|_\infty^k R(0)^{k-1} \le \|\Psi\|_1^k R(0)^{k-1};$$

hence,

$$(5.19)\qquad \|\Gamma\|_\infty \le \sum_{k=1}^{\infty} s^{k-1} \|\Psi\|_1^k R(0)^{k-1} \le \tfrac{4}{3}\|\Psi\|_1.$$

In the case $m = 0$, $f^{(s)} = 0$ for s odd, and for s even,

$$\| f^{(s)} \|_\infty \le \frac{s!}{(s/2)!} a \|\Gamma\|_\infty^{s/2} \le \frac{s!}{(s/2)!} a 2^{s/2} \|\Psi\|_1^{s/2},$$

which proves (5.16) by the inequality $s! \le C2^s((s/2)!)^2$. The case $m \ne 0$ is more involved. According to the definition (5.11) and the inequality $\|\Gamma \circ \hat{\Psi}\|_\infty \le \|\Gamma\|_\infty \|\Psi\|_1 R(0) \le 2\|\Psi\|_1^2 R(0) \le \frac{1}{4}\|\Psi\|_1$ [see (5.19)],

$$\begin{aligned}
\| f^{(s)} \|_\infty &\le \sum_{2n+j+k=s} \frac{s!}{n!j!k!} \|\Gamma\|_\infty^n \|\hat{\Psi}\|_\infty^j \|\Gamma \circ \hat{\Psi}\|_\infty^k 2^{j+2k} m^{j+k} \\
&\le 2^s \sum_{2n+j+k=s} \frac{s!}{n!j!k!} \|\Psi\|_1^{n+k+j} m^{j+k} 3^{-n} 2^{-k} \\
&\le 2^s \sum_{n=0}^{[s/2]} \frac{s!}{3^n n!(s-2n)!} \|\Psi\|_1^{s-n} (3m/2)^{s-2n} \\
&\equiv 2^{3s/2} (s!)^{1/2} \|\Psi\|_1^{s/2} C(s)
\end{aligned}$$

and (5.16) follows if we show that $C(s)$ is bounded by a constant. Assume that $s = 2p$ is even (the case of s being odd requires only minor changes). Then

$$C(2p) = \frac{(2p!)^{1/2}}{2^p p!} \sum_{k=0}^{p} \frac{p!}{(2k)!(p-k)!} \|\Psi\|_1^k \left(\frac{3m}{2}\right)^{2k} 3^{-(p-k)}$$
$$\le C \sum_{k=0}^{p} \binom{p}{k} 3^{-(p-k)} \frac{\|\Psi\|_1^k (3m/2)^{2k} k^{1/2}}{4^k k!}.$$

For any $m, \|\Psi\|_1$, there exists an integer $k_0 \ge 1$ such that for all $k \ge k_0$ the last fraction is less than 2^{-k}. Therefore,

$$C(2p) \le C\left(p^{k_0} 3^{-(p-k_0)} + 1\right) \le C,$$

which proves (5.16).

Denote $A(m) = \exp(\zeta^2 * \Psi(0))$. Then

$$(5.20)\quad \begin{aligned} A(m) &= \exp\left(\zeta_0^2 * \Psi(0)\right)\exp\left(2m\zeta_0 * \Psi(0) + m^2\overline{\Psi}^2\right) \\ &= A(0)\exp\left(2m\zeta_0 * \Psi(0) + m^2\overline{\Psi}^2\right), \end{aligned}$$

where $\zeta_0(x) = \zeta(x) - m$, $x \in \mathbf{R}$, is the centered Gaussian process. Consider

$$(5.21)\quad A(0) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\zeta_0^2 * \Psi(0)\right)^n$$

By Hölder's inequality,

$$\begin{aligned} \left\|\left(\zeta_0^2 * \Psi(0)\right)^n\right\|_{L^2(\Omega)} &\le \left[\int_{\mathbf{R}} E\zeta_0^{4n}(x)|\Psi(-x)|\,dx\right]^{1/2} \|\Psi\|_1^{n-1/2} \\ &\le \|\Psi\|_1^n \left[E\zeta_0^{4n}(0)\right]^{1/2} \\ &= \|\Psi\|_1^n [4n!!]^{1/2} R(0)^n. \end{aligned}$$

Hence,

$$(5.22)\quad \|A(0)\|_{L^2(\Omega)} \le \sum_{n=0}^{\infty} \frac{[4n!!]^{1/2}}{n!} \|\Psi\|^n R(0)^n < \sum_{n=0}^{\infty} 4^n \|\Psi\|_1^n R(0)^n < \infty$$

in view of the conditions of the lemma, which shows also that the series on the right-hand side of (5.21) absolutely converges in $L^2(\Omega)$.

Using the diagram formula for powers of Itô–Wiener integrals [see, e.g., Major (1981), Theorem 5.3] one has

$$(5.23)\quad \zeta_0^2 * \Psi(0) = \int \hat{\Psi}(p_1 + p_2) M(dp_1) M(dp_2) + \overline{\Psi} R(0)$$

and

$$\left(\int \hat{\Psi}(p_1 + p_2)\, d^2M\right)^k = \sum_{l=0}^{k} \frac{k!}{(k-l)!} E\left(\int \hat{\Psi}(p_1 + p_2)\, d^2M\right)^{k-l}$$
$$\times \sum_{r_1 + \cdots + r_s = l} \frac{1}{s!} \int \left[\bigotimes_{i=1}^{s} 2^{r_1 - 1} \hat{\Psi}^{(r_i)}\right] d^{2s}M,$$

where the sum is taken over all $s = 1, 2, \dots k$ and $r_1, \dots, r_s = 1, 2, \dots$ such that $r_1 + \cdots + r_s = l$ (for $l = 0$, it equals 1 by definition) and $\hat{\Psi}^{(k)}$ is defined in (5.13). Consequently,

$$
\begin{aligned}
(\zeta_0^2 * \Psi(0))^n &= \sum_{k=0}^{n} \binom{n}{k} \left(\int \hat{\Psi}\, d^2M \right)^k (\overline{\Psi} R(0))^{n-k} \\
&= \sum_{l=0}^{n} \frac{n!}{(n-l)!} \sum_{r_1 + \cdots + r_s = l} \frac{1}{s!} \int \left[\bigotimes_{i=1}^{s} 2^{r_i - 1} \hat{\Psi}^{(r_i)} \right] \\
&\quad \times d^{2s}M\, E(\zeta_0^2 * \Psi(0))^{n-l}
\end{aligned}
\tag{5.24}
$$

Substituting (5.24) into (5.21) and changing the order of summation, one obtains

$$
\begin{aligned}
A(0) &= \sum_{l=0}^{\infty} \sum_{r_1 + \cdots + r_s = l} \frac{1}{s!} \int \left[\bigotimes_{l=1}^{s} 2^{r_i - 1} \hat{\Psi}^{(r_i)} \right] d^{2s}M \sum_{n \geq l} \frac{1}{(n-l)!} E(\zeta_0^2 * \Psi(0))^{n-l} \\
&= a(0) \sum_{l=0}^{\infty} \sum_{r_1 + \cdots + r_s = l} \frac{1}{s!} \int \left[\bigotimes_{l=1}^{s} 2^{r_i - 1} \hat{\Psi}^{(r_i)} \right] d^{2s}M \\
&= a(0) \sum_{s=0}^{\infty} \frac{1}{s!} \int \Gamma^{\otimes s}\, d^{2s}M
\end{aligned}
$$

or (5.15); the validity of the change of the order of summation is justified by (5.22) and the bounds (5.19) and (5.16).

In the general case $m \in \mathbf{R}$, the expansion (5.10) can be obtained by (5.15), (5.20) and

$$
\exp\left[2m \int \hat{\Psi}\, dM + m^2 \overline{\Psi}^2 \right] = e^{c(m)} \sum_{j=0}^{\infty} \frac{1}{j!} \int (2m\hat{\psi})^{\otimes j}\, d^jM,
\tag{5.25}
$$

where $c(m) = 2m^2 \int |\hat{\Psi}|^2\, dG + m^2 \overline{\Psi}^2$. Indeed, using the diagram formula for the product of the summands of (5.15) and (5.25), one has

$$
\begin{aligned}
A(m) &= a(0) e^{c(m)} \sum_{n,j=0}^{\infty} \sum_{r=0}^{n \wedge (j/2)} \frac{1}{r!(2r)!} E\left[\int \Gamma^{\otimes r}\, d^{2r}M \int \hat{\Psi}^{\otimes 2r}\, d^{2r}M \right] \\
&\quad \times \sum_{k=0}^{(n-r) \wedge (j-2r)} \frac{1}{k!(n-r-k)!(j-2r-k)!} \\
&\quad \times \int \left[\Gamma^{\otimes (n-r-k)} \otimes (4m\Gamma \circ \hat{\Psi})^{\otimes k} \otimes (2m\hat{\Psi})^{\otimes (j-2r-k)} \right] d^{2n+j-2k-4r}M \\
&= e^{c(m)} E\left[A(0) \exp\left(2m \int \hat{\Psi}\, dM - 2m^2 \int |\hat{\Psi}|^2\, dG \right) \right] \\
&\quad \times \sum_{s=0}^{\infty} \sum_{2n+j+k=s} \frac{1}{n!j!k!} \int \left[\Gamma^{\otimes n} \otimes (4m\Gamma \circ \hat{\Psi})^{\otimes k} \otimes (2m\hat{\Psi})^{\otimes j} \right] d^sM
\end{aligned}
$$

or the expansion (5.10); see (5.20). The change of the order of summation above can be justified using (5.16) and (5.19) similarly as in (5.15). From the above-mentioned estimates and the continuity of $\hat{\Psi}$ and $\hat{\Psi}^{(k)}, k = 1, 2, \dots$, follows the continuity of Γ, and consequently, the continuity and boundedness of $f^{(s)}$, $s = 1, 2, \dots$.

It remains to prove the relations (5.17) and (5.18) [they suggest further relations of this type, namely, $a^{(s)} = f^{(s)}(0, \dots, 0)$ for $s = 0, 1, \dots$, but we shall not pursue this point here]. According to (5.11),

$$f^{(1)} = 2m\, a(m)(\hat{\Psi} + 2\Gamma \circ \hat{\Psi}), \tag{5.26}$$

$$f^{(2)} = 2a(m)\Big(\Gamma + 2m^2\big(\hat{\Psi} \otimes \hat{\Psi} + 4\hat{\Psi} \otimes (\Gamma \circ \hat{\Psi}) + 4(\Gamma \circ \hat{\Psi}) \otimes (\Gamma \circ \hat{\Psi})\big)\Big). \tag{5.27}$$

In particular,

$$\begin{aligned} f^{(1)}(0) &= 2m\,a\big(\overline{\Psi} + 2\Gamma \circ \hat{\Psi}(0)\big) \\ &= 2m\,a\left(\overline{\Psi} + 2\sum_{k=0}^{\infty} 2^{k-1} \int \hat{\Psi}^{(k)}(0,p)\hat{\Psi}(-p)G(dp)\right) \\ &= 2m\,a\left(\overline{\Psi} + 2\sum_{k=0}^{\infty} \hat{\Psi}^{(k+1)}(0,0)\right) \\ &= 2m\,a\big(\overline{\Psi} + 2\tfrac{1}{2}(\Gamma(0,0) - \overline{\Psi})\big) \\ &= 2m\,a(m)\Gamma(0,0) \end{aligned} \tag{5.28}$$

and, similarly,

$$f^{(2)}(0,0) = 2a(m)(\Gamma(0,0) + 2m^2\Gamma^2(0,0)). \tag{5.29}$$

On the other hand, from (5.4) and (5.10) and the orthogonality of Itô–Wiener integrals,

$$a'(m) = 2a(m)\overline{\Psi} + 2\int f^{(1)}\hat{\Psi}\, dG, \tag{5.30}$$

where

$$\begin{aligned} \int f^{(1)}\hat{\Psi}\, dG &= 2m\,a(m)\left(\int |\hat{\Psi}|^2\, dG + 2\sum_{k=1}^{\infty} 2^{k-1}\hat{\Psi}^{(k+2)}(0,0)\right) \\ &= m\,a(m)\big(\Gamma(0,0) - \overline{\Psi}\big). \end{aligned}$$

Substituting the above equality into (5.30), one obtains (5.17) through (5.28).

We shall prove (5.18) for the case $m = 0$ only, as this will suffice for the proof of Theorem 5.1 and it simplifies calculations. From (5.5), (5.15) and (5.23) we have that

$$\begin{aligned} a''(0) &= 2E\left[A(0)\left(2\int \hat{\Psi}^{\otimes 2}\, d^2M + 2\int |\hat{\Psi}|^2\, dG + \overline{\Psi}\right)\right] \\ &= 8\int f^{(2)}\hat{\Psi}^{\otimes 2}\, d^2G + 2a(0)\left(2\int |\hat{\Psi}|^2\, dG + \overline{\Psi}\right) \\ &= 2a(0)\Gamma(0,0) = f^{(2)}(0,0). \end{aligned}$$

Lemma 5.1 is proved. □

PROOF OF THEOREM 5.1 (Continued). According to Lemma 5.1,

$$B(\beta)\int_{\mathbf{R}}(\exp(\zeta^2 * \Psi(\beta y)) - a(m))\psi(y)\,dy \equiv I_0(\beta) + I_1(\beta),$$

where

$$I_0(\beta) := B(\beta)\frac{1}{s_0!}\int \hat{\psi}(\beta y(p_1 + \cdots + p_{s_0}))f^{(s_0)}(p_1,\dots,p_{s_0})\,d^{s_0}M,$$

$$I_1(\beta) := B(\beta)\sum_{s=s_0+1}^{\infty}\frac{1}{s!}\int \hat{\psi}(\beta y(p_1 + \cdots + p_s))f^{(s)}(p_1,\dots,p_s)\,d^sM$$

and $s_0 \equiv s_0(m) = 1$ $(m \neq 0)$, $s_0(0) = 2$. Using the continuity and boundedness of $f^{(1)}$ and $f^{(2)}$ together with (5.17) and (5.18) and the assumptions of the theorem, similarly as in Dobrushin and Major [(1979), Theorem 3], we conclude that the distribution of $I_0(\beta)$ converges weakly to the distribution of the stochastic integrals $a'(m)\langle W_\alpha', \psi\rangle$ and $\frac{1}{2}a''(0)\langle W_\alpha^{(2)}, \psi\rangle$, respectively. It remains to show that $I_1(\beta)$ is negligible in the limit $\beta \to \infty$, that is,

$$\lim_{\beta\to\infty} EI_1^2(\beta) = 0. \tag{5.31}$$

Using bound (5.16) we have

$$EI_1^2(\beta) = B(\beta)^2\sum_{s=s_0+1}^{\infty}\frac{1}{s!}\int_{\mathbf{R}^s}\left|\hat{\psi}(\beta(p_1 + \cdots + p_s))\right|^2\left|f^{(s)}(p_1,\dots,p_s)\right|^2 d^sG$$

$$\leq B(\beta)^2\sum_{s_0+1}^{\infty}\frac{\|f^{(s)}\|_\infty^2}{s!}\int_{\mathbf{R}^s}\left|\hat{\psi}(\beta(p_1 + \cdots + p_s))\right|^2 d^sG$$

$$= B(\beta)^2 E\left(\int_{\mathbf{R}} H(\zeta_0(\beta y))\,dy\right)^2,$$

where $H(x)$, $x \in \mathbf{R}$, is a real function given by the Hermite series

$$H(x) = \sum_{s=s_0+1}^{\infty}\frac{R(0)^{s/2}\|f^{(s)}\|_\infty}{s!}H_s\left(xR(0)^{-1/2}\right), \tag{5.32}$$

with $H_s(x) = (-1)^s e^{x^2/2}d^s(e^{-x^2/2})/dx^s$, $s = 0, 1, \dots,$ being the standard Hermite polynomials. Series (5.32) converges in $L^2(\mathbf{R}, e^{-x^2/(2R(0))}\,dx)$. Indeed, from (5.16) it follows that

$$EH^2(\zeta_0) = \sum_{s=s_0+1}^{\infty}\frac{\|f^{(s)}\|_\infty^2 R(0)^s}{s!} \leq C\sum_{s=s_0+1}^{\infty} 8^s\|\Psi\|_1^s R(0)^s < \infty.$$

Hence the main result of Dobrushin and Major [(1979), Theorem 1′ for *local* functionals] applies with small changes, yielding (5.31) and the theorem itself. □

One can easily note that if $\overline{\Psi} \geq 0$, then $a'(m) \neq 0$ $(m \neq 0)$ and $a''(0) \neq 0$. Indeed from (5.5) we see that $a''(m) > 0$ for each m as long as $\overline{\Psi} \geq 0$. Moreover, $a'(0) = 0$, so that $a(m)$ is strictly convex and has minimum at $m = 0$.

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