The Annals of Applied Probability
1995, Vol. 5, No. 2, 446–460

THE HAZARD RATE TANGENT APPROXIMATION FOR BOUNDARY HITTING TIMES

BY G. O. ROBERTS AND C. F. SHORTLAND

University of Cambridge

We consider the problem of estimating first exit distributions for one-dimensional diffusion processes. We provide analytic bounds and an approximation which is shown to be accurate in a range of numerical examples involving Brownian motion.

1. Introduction. We consider the problem of estimating the distribution of

$$\tau_f = \inf_{t>0}\{t: X_t \geq f(t)\},$$

where X is a one-dimensional diffusion process and f is a given curved boundary, by using a collection of one-dimensional techniques, which provide bounds for the boundary hitting time hazard rates. When X is Brownian motion, this often leads to tight bounds, which are easily calculated.

The most general result we prove (Theorem 1) is a comparison result for boundary hitting time hazard rates (at time t) of hitting three sufficiently smooth boundaries f, g and h such that $h(s) \leq f(s) \leq g(s)$ for all $s \leq t$, and $h(t) = f(t) = g(t)$. Applications of this result are restricted to the case where boundary hitting distributions are known for suitable *enveloping* curves (see Definition 2). This paper concentrates on the case where X is Brownian motion and the enveloping curves are linear, where explicit bounds for hazard rates are easily deduced (Corollary 2). In the case where f is concave or convex, one of the enveloping curves is the tangent to f at t. The latter part of this paper investigates analytically and numerically the properties of this hazard rate tangent approximation in general.

For Brownian motion, hitting time distributions are rarely explicitly available. However, the exit time distribution to a linear boundary has a simple closed form known as the Bachelier–Lévy formula. Letting $X_0 = 0$ and $f(t) = a + bt$, $p^{a,b}(t)$, the density of τ_f, is given by

$$p^{a,b}(t) = \frac{a}{t^{3/2}}\phi\left(\frac{a+bt}{\sqrt{t}}\right), \tag{1.1}$$

where ϕ is the standard normal density function. The Bachelier–Lévy formula gives rise to an appealing simple approximation for the exit time distribution to curved boundaries. The tangent approximation [Strassen

Received December 1993; revised August 1994.
AMS 1991 *subject classifications*. Primary 60J6S, 60350.
Key words and phrases. Boundary hitting time, one-dimensional diffusion process, Brownian motion.

(1967)] approximates the density $p_f(t)$ of hitting a curved, differentiable boundary f by

$$p_f(t) \approx p^{a(t),\, b(t)}(t), \tag{1.2}$$

where $a(t) = f(t) - tf'(t)$ and $b(t) = f'(t)$. Thus p_f is approximated by the density of the linear boundary which forms the tangent to f at t. The tangent approximation is rarely accurate for boundaries which are significantly nonlinear. However, asymptotic accuracy of the approximations for sequences of concave boundaries receding to infinity can be shown [Lerche (1986)]. More importantly, the tangent approximation can be extended to more sophisticated approximations; see Durbin (1992). This approximation takes the form of an infinite expansion in which the tangent approximation forms the leading term. Durbin's work can produce extremely accurate estimates using a small number of terms in the expansion, but the calculation involved can be extremely computationally intensive.

In this contribution, we give an approximation which combines the explicitness and computational simplicity of the tangent approximation with a greater degree of accuracy. The idea is to approximate the hazard rate of f by the hazard rate of exiting a tangential linear boundary. Thus, instead of obtaining an explicit expression for the estimate of the hitting time density, we obtain an explicit form for the estimated hazard rate. This is the hazard rate tangent approximation in general, and we demonstrate that for concave or convex boundaries this approximation considerably improves on the tangent approximation (Theorem 3). Numerical investigation demonstrates the accuracy of the method for more general boundaries where analytic bounds for the hazard rate tangent approximation are not possible.

Although the numerical results are extremely good, the hazard rate tangent approximation will fail to be as accurate as Durbin's (1992) method for sufficiently many terms in the Durbin expansion. However, the hazard rate method is simpler to use and computationally straightforward. Moreover the upper and lower bounds of Theorem 1 provide an assessment of the accuracy of the hazard rate method. An application where the upper and lower bounds from Theorem 1 are essential (and turn out to be extremely close) is given in Roberts and Shortland (1994). This application concerns the evaluation of certain types of exotic financial options known as *barrier options*.

The techniques we shall use for this are exclusively one dimensional and follow from the work of Roberts (1991a, 1993). One-dimensional diffusion processes are strongly stochastically monotone [Roberts (1991a)], allowing us to prove comparison results on unconditioned and conditioned distributions of the process. (Conditioned distributions here typically refer to not exiting f before a specified time.) These comparisons in turn can be used to give estimates on hitting time hazard rates.

2. Definitions and preliminary results. We assume throughout this paper that X is a diffusion process satisfying the stochastic differential equation

$$dX_t = \sigma(t, X_t)\, dB_t + \eta(t, X_t)\, dt,$$

and we shall assume throughout that $X_0 = 0$. For technical reasons, we need to assume that η and σ satisfy the following assumptions:

(A1) $\sigma(\cdot, \cdot)$ is bounded away from zero.
(A2) η and σ are bounded continuous functions and σ is continuous in t, uniformly with respect to (t, x).
(A3) η and σ are Hölder continuous in x, uniformly with respect to (t, x) in compact subsets, and σ is Hölder continuous in x uniformly with respect to (t, x).
(A4) $\partial(\sigma^2)/\partial x$, $(\partial^2(\sigma^2))/\partial x^2$ and $\partial\eta/\partial x$ exist and satisfy the constraints of (A2) and (A3).

One important property of the diffusion process X is that of strong stochastic monotonicity, that is,

$$\frac{p_t(x_2, y)}{p_t(x_1, y)} \text{ is nondecreasing in } y \text{ for all } t,\ x_1 \le x_2,$$

where p_t is the transition density for X. We will find that this property is fundamental for the ordering results in the sequel. For further details, see Roberts (1991a).

Stochastic orderings are not necessarily preserved under conditioning: If Y_1 and Y_2 are two random variables such that Y_1 stochastically dominates Y_2, then it does not necessarily follow that $[Y_1 \mid Y_1 \in C]$ stochastically dominates $[Y_2 \mid Y_2 \in C]$. We therefore make a natural extension to a stronger stochastic ordering, which is preserved under conditioning, in some cases.

DEFINITION 1 (Strong stochastic ordering). We say the probability measure ν_2 is strongly stochastically greater than ν_1 (written $\nu_2 \ge_{\text{sst}} \nu_1$) if the Radon–Nikodym derivatives satisfy

$$\frac{\nu_2(dy)}{\nu_1(dy)} \text{ is nondecreasing in } y.$$

We will require our process X to be conditioned not to have hit our boundary f. We shall make comparisons using other boundary curves which *envelop* f.

DEFINITION 2 (Enveloping). We say that boundaries g and h envelop f from above and below, respectively, prior to t, if

$$g(s) \ge f(s) \ge h(s) \quad \text{for all } s \le t,$$

with

$$g(t) = f(t) = h(t).$$

With several boundaries now being used, we shall use subscripts and superscripts to denote the particular boundary being considered. We shall write

$$r_\lambda(t) = \lim_{\varepsilon \downarrow 0} \frac{\mathsf{P}[\tau_\lambda \le t + \varepsilon \mid \tau_\lambda > t]}{\varepsilon}$$

for the hazard rate of

$$\tau_\lambda = \inf_{t>0} \{t: X_t \ge \lambda(t)\}$$

for arbitrary C^2 function λ.

The first preliminary result finds an expression for the hazard rate in terms of the density of the conditioned process. If we denote

$$\mu_t^\lambda(x)\,dx = \mathsf{P}[X_t \in dx \mid \tau_\lambda > t],$$

then we can state the following lemma.

LEMMA 1 [Roberts (1993)]. *For an arbitrary C^2 boundary $\lambda(t)$,*

$$r_\lambda(t) = \frac{1}{2}\sigma^2(t, \lambda(t)) \lim_{x \uparrow \lambda(t)} \frac{\mu_t^\lambda(x)}{\lambda(t) - x}.$$

Intuitively, this result states that the hazard rate is proportional to the derivative of the density of the conditioned process, evaluated close to the boundary.

The second lemma uses ordered boundaries to produce strongly stochastically ordered conditioned processes. It is a special case of Theorem 2.8 of Roberts (1991a).

LEMMA 2. *Suppose that f and g are right continuous boundaries with left limits, such that $f(t) \le g(t)$ for all t. Then*

$$[X_t \mid \tau_g > t] \ge_{\mathrm{sst}} [X_t \mid \tau_f > t].$$

The intuition behind this result is that the conditioning is more severe from boundary f, and this "pushes down" the process X more than the other conditioning does. As a result, the above ordering presents itself.

3. The hazard rate bounds theorem. The two lemmas may be combined to provide a bound on the hazard rate $r_f(t)$ by using the hazard rates associated with a collection of enveloping curves $\{g_t(\cdot),\ t \ge 0\}$ and $\{h_t(\cdot),\ t \ge 0\}$, where g_t and h_t envelope f from above and below, respectively, prior to t.

THEOREM 1. *Let f be a C^2 boundary, and $\{g_t(\cdot),\ t \ge 0\}$ and $\{h_t(\cdot),\ t \ge 0\}$ be collections of C^2 functions, where g_t and h_t envelope f from above and below, respectively, prior to t. Then, for each t,*

$$r_{h_t}(t) \le r_f(t) \le r_{g_t}(t). \tag{3.1}$$

PROOF. Fix $t > 0$. First note that since f is C^2, we can find C^2 curves g_t and h_t with the desired ordering. Therefore, applying Lemma 2, we have

$$\left[X_t \mid \tau_{g_t} > t\right] \geq_{\text{sst}} \left[X_t \mid \tau_f > t\right] \geq_{\text{sst}} \left[X_t \mid \tau_{h_t} > t\right].$$

We shall consider only the first of these strong stochastic inequalities, and prove the second hazard rate inequality. The first can be proved in the same manner, using the second strong stochastic inequality.

Note Lemma 1 yields

$$r_{g_t}(t) - r_f(t) = \frac{1}{2}\sigma^2(t, g_t(t)) \lim_{x \uparrow g_t(t)} \frac{\mu_t^{g_t}(x)}{g_t(t) - x}$$
$$- \frac{1}{2}\sigma^2(t, f(t)) \lim_{x \uparrow f(t)} \frac{\mu_t^f(x)}{f(t) - x}$$

and since $f(t) = g_t(t)$, this reduces to

$$r_{g_t}(t) - r_f(t) = \frac{1}{2}\sigma^2(t, f(t)) \lim_{x \uparrow f(t)} \left[\frac{\mu_t^{g_t}(x)}{f(t) - x} - \frac{\mu_t^f(x)}{f(t) - x}\right].$$

Suppose that

$$\lim_{x \uparrow f(t)} \frac{\mu_t^{g_t}(x)}{f(t) - x} < \lim_{x \uparrow f(t)} \frac{\mu_t^f(x)}{f(t) - x},$$

so that

$$\lim_{x \uparrow f(t)} \frac{\mu_t^{g_t}(x)}{\mu_t^f(x)} < 1. \tag{3.2}$$

Then, since our strong stochastic inequality is equivalent to (see Definition 1)

$$\frac{\mu_t^{g_t}(x)}{\mu_t^f(x)} \leq \frac{\mu_t^{g_t}(y)}{\mu_t^f(y)} \quad \text{for all } x \leq y$$

[i.e., the ratio of (3.2) is an increasing function of x], we have

$$\frac{\mu_t^{g_t}(x)}{\mu_t^f(x)} < 1 \quad \text{for all } x \leq f(t).$$

That is,

$$\mu_t^{g_t}(x) < \mu_t^f(x),$$

which is impossible since both $\mu_t^{g_t}(\cdot)$ and $\mu_t^f(\cdot)$ are densities on $(-\infty, f(t)]$, and must integrate to 1, and using $f(t) = g_t(t)$.

Thus we conclude $r_{g_t}(t) \geq r_f(t)$. □

4. Remarks and corollaries. It is often more convenient, from an intuitive perspective, to use distribution functions rather than hazard functions. We can also produce bounds on the distribution function of our hitting time τ_f by exploiting the algebraic relationship between hazard rates and distribution functions.

We begin with a further definition:

DEFINITION 3. For a collection of functions $\{\lambda_t(\cdot),\ t \ge 0\}$, define

$$Q_{\lambda.}(t) = 1 - \exp\left\{-\int_0^t r_{\lambda_s}(s)\, ds\right\}.$$

COROLLARY 1. *Let f be a C^2 boundary, let P_f denote the distribution function for the first exit time across boundary $f(t)$ and let $\{g_t(\cdot),\ t \ge 0\}$ and $\{h_t(\cdot),\ t \ge 0\}$ be collections of C^2 functions, where g_t and h_t envelop f from above and below, respectively, prior to t. Then,*

$$Q_{h.}(t) \le P_f(t) \le Q_{g.}(t)$$

for all t.

This is immediate from Definition 3 and Theorem 1.

For the Brownian motion case, we can explicitly calculate the form of the hazard rate across any straight line by using the Bachelier–Lévy formula. Therefore, we select our curves g_t and h_t to be straight lines. We use the following definitions:

$$\begin{aligned}
m_t^2 &= \sup_{s<t} \frac{f(t) - f(s)}{t - s},\\
m_t^1 &= \inf_{s<t} \frac{f(t) - f(s)}{t - s},\\
c_g &= f(t) - m_t^1 t,\\
c_h &= f(t) - m_t^2 t,\\
g_t(s) &= m_t^1 s + c_g,\\
h_t(s) &= m_t^2 s + c_h.
\end{aligned}$$

Then we have $g_t(s) \ge f(s) \ge h_t(s)$ for all $s \le t$, with equality at time t. Therefore straightforward application of the Bachelier–Lévy formula (1.1),

$$P^{a,b}(t) = 1 - \Phi\left(\frac{a + bt}{\sqrt{t}}\right) + e^{-2ab}\Phi\left(\frac{bt - a}{\sqrt{t}}\right), \tag{4.1}$$

where Φ denotes the standard normal distribution function, and Theorem 1, gives the following corollary.

COROLLARY 2. *If f is a C^2 boundary and r_f denotes the hazard rate of the first exit time of Brownian motion across it, then*

$$\frac{c_h \phi((f(t))/\sqrt{t})}{t^{3/2}\left[\Phi((f(t))/\sqrt{t}) - \exp(-2c_h m_t^2)\Phi((f(t) - 2c_h)/\sqrt{t})\right]}$$
$$\leq r_f(t) \leq \frac{c_g \phi((f(t))/\sqrt{t})}{t^{3/2}\left[\Phi((f(t))/\sqrt{t}) - \exp(-2c_g m_t^1)\Phi((f(t) - 2c_g)/\sqrt{t})\right]},$$

where c_g, c_h and m_t^i are defined above.

5. Hazard rate tangent approximation. For the case of Brownian motion exiting C^2 boundaries, we introduce a new approximation for the first exit distribution. This is based on estimating the hazard rate of the first exit time by the hazard rate of the tangent at the same time point, which can be found exactly by the Bachelier–Lévy formula. We shall denote this approximation by HRT. In the case where

$$\tau_f = \inf_{t>0}\{t\colon B_t \geq f(t)\}$$

is the first exit time from a concave, C^2 boundary, the tangent to the curve at each time point is the same as our upper enveloping straight line. Consequently, the upper analytic bound and the HRT method produce identical approximations. Conversely, if f is a convex, C^2 boundary, the tangent to the curve at each time point is the same as the lower enveloping straight line. Therefore, the lower analytic bound and the HRT method produce the same approximations. In either of these cases, we can prove that the HRT technique produces more accurate approximations to the distribution function than the tangent approximation [Strassen (1967)] does.

In the remainder of this section, we shall use the following notation. Let $u_t(\cdot)$ be the tangent to f at time t and let $r_T(\cdot)$ denote the hazard rate tangent approximation (HRT) $r_T(t) = \tau_{u_t}(t)$. Therefore, $r_f(\cdot)$ is estimated by

$$r_f(t) \approx r_T(t).$$

Let p_H and P_H be the density and distribution function derived from $r_T(\cdot)$. Therefore, from Definition 3,

$$P_H(t) = Q_{u\cdot}(t),$$
$$p_H(t) = r_T(t)(1 - Q_{u\cdot}(t)).$$

We will compare the HRT approximation with the tangent approximation. Using (1.2) and adapting (4.1), we will write the tangent approximation (TA) by

$$P_T(t) = P^{a(t),\,b(t)}(t),$$
$$p_T(t) = p^{a(t),\,b(t)}(t),$$

where $a(t) = f(t) - tf'(t)$ and $b(t) = f'(t)$.

We first show that the densities produced by the HRT method and the tangent approximation for Brownian motion exit distributions are ordered if the boundary is either concave or convex.

THEOREM 2. (i) *If f is a concave, C^2 boundary, then*

$$p_H(t) \le p_T(t) \quad \text{for all } t.$$

(ii) *If f is a convex, C^2 boundary, then*

$$p_H(t) \ge p_T(t) \quad \text{for all } t.$$

PROOF. We shall prove (i) only, as (ii) follows is a similar manner, with the inequalities reversed.

Since f is concave, we have $u_t(s) \ge f(s)$ for all $s \le t$. Therefore,

$$\mathsf{P}\big[\tau_{u_t} > t\big] \ge \mathsf{P}\big[\tau_f > t\big] = 1 - P_f(t), \tag{5.1}$$

and from Corollary 1,

$$P_f(t) \le P_H(t). \tag{5.2}$$

Then, by definition,

$$\begin{aligned} p_T(t) &= r_T(t)\mathsf{P}\big[\tau_{u_t} > t\big] \\ &\ge r_T(t)\big(1 - P_f(t)\big) \quad [\text{by } (5.1)] \\ &\ge r_T(t)\big(1 - P_H(t)\big) \quad [\text{by } (5.2)] \\ &= p_H(t). \end{aligned}$$

□

The most important ordering results are for the distribution functions produced by the two approximation methods, which imply that the HRT method is superior to the tangent approximation.

THEOREM 3. (i) *If f is a concave, C^2 boundary, then*

$$P_f(t) \le P_H(t) \le P_T(t) \quad \text{for all } t.$$

(ii) *If f is a convex, C^2 boundary, then*

$$P_f(t) \ge P_H(t) \ge P_T(t) \quad \text{for all } t.$$

PROOF. This follows trivially from Corollary 1 and integrating the result of Theorem 2. □

Another advantage of the HRT method over the tangent approximation is shown by another corollary of Theorem 1.

COROLLARY 3. *If f is C^2 and concave and $\tau_f < \infty$ a.s., then*

$$P_H(t) \to 1 \quad \text{as } t \to \infty.$$

PROOF. Note that $P_f(\infty) = 1$, so that $\int_0^t r_f(s)\,ds \to \infty$ as $t \to \infty$. From Theorem 1, since f is concave,

$$r_T(t) \geq r_f(t) \quad \text{for all } t$$

and, consequently,

$$\int_0^t r_T(s)\,ds \to \infty \quad \text{as } t \to \infty.$$

The result follows from this. □

Note that this is not true for the tangent approximation, which always overestimates the density. Therefore, in this case $P_T(t) \geq 1$ for sufficiently large t is the only conclusion which can be drawn.

We can produce a partial converse to Corollary 3.

THEOREM 4. *If f is C^2 and concave, $P_H(t) \to 1$ as $t \to \infty$ and*

$$f'(t)[f(t) - tf'(t)] \to \infty \quad \text{as } t \to \infty, \tag{5.3}$$

then

$$\tau_f < \infty \quad a.s.$$

PROOF. Assume that $P_H(t) \to 1$ as $t \to \infty$, so that $\int_0^t r_T(s)\,ds \to \infty$ as $t \to \infty$. Expanding r_T, this is equivalent to

$$\int_0^t \frac{(f(s) - sf'(s))\phi((f(s))/\sqrt{s})\,ds}{A(s)}$$

where $A(s) = s^{3/2}[\Phi((f(s))/\sqrt{s}) - \exp(-2f'(s)[f(s) - sf'(s)])\Phi((2sf'(s) - f(s))/\sqrt{s})]$.

Assuming also that (5.3) holds,

$$A(s) \geq \frac{1}{2}\Phi\left(\frac{f(s)}{\sqrt{s}}\right) \geq \frac{1}{4},$$

for s sufficiently large, or else f is eventually decreasing so that the assertion is trivial. We thus deduce

$$\int_0^t \frac{f(s) - sf'(s)}{s^{3/2}}\phi\left(\frac{f(s)}{\sqrt{s}}\right) ds \to \infty \quad \text{as } t \to \infty.$$

Noting that

$$\frac{d}{ds}\left[\Phi\left(\frac{f(s)}{\sqrt{s}}\right)\right] = \frac{sf'(s) - (1/2)f(s)}{s^{3/2}}\phi\left(\frac{f(s)}{\sqrt{s}}\right),$$

we conclude

$$\int_0^t \left[\frac{(1/2)f(s)}{s^{3/2}}\phi\left(\frac{f(s)}{\sqrt{s}}\right) - \frac{d}{ds}\left[\Phi\left(\frac{f(s)}{\sqrt{s}}\right)\right]\right] ds \to \infty \quad \text{as } t \to \infty$$

and hence

$$\int_0^t \frac{f(s)}{s^{3/2}} \phi\left(\frac{f(s)}{\sqrt{s}}\right) ds \to \infty \quad \text{as } t \to \infty.$$

Using the Kolmogorov–Erdös–Feller–Petrowski theorem [see, e.g., Roberts (1991b), Corollary 5.3], we obtain the claimed result. □

REMARK. For instance, in critical cases, such as $f(t) = 1 + \sqrt{t \ln(t+1)}$, (5.3) holds and $\mathsf{P}[\tau_f < \infty]$ is known to be less than unity. In this case, HRT does not falsely approximate the distribution of τ_f by a nondefective distribution.

One way of seeing heuristically the advantages of HRT over the tangent approximation is as follows. The estimate of the density at time t,

$$p_f(t) \approx r_T(t) \exp\left\{-\int_0^t r_T(s)\, ds\right\}, \tag{5.4}$$

takes into account estimates of the density at previous time points. If the density estimates at previous time points are all overestimates, then

$$\exp\left\{-\int_0^t r_T(s)\, ds\right\} \le \exp\left\{-\int_0^t r_f(s)\, ds\right\}$$

and this has the effect of reducing the estimate of $p_f(t)$ in (5.4). Conversely, if the previous densities have all been underestimated, $\exp\{-\int_0^t r_T(s)\, ds\}$ will be larger than it should be, thus increasing the current density estimate. This negative feedback effect makes fluctuations between over- and underestimation less drastic and, intuitively, this may lead to a better approximation.

6. Numerical examples. We illustrate the analytic bounds and HRT approximation with some numerical examples, which also include the tangent approximation for comparison. We shall denote the lower bound by L, the upper bound by U (derived from Corollary 2) and the tangent approximation [given by (1.2)] by TA. For the first example, the exact distribution is found using the method of images [see Daniels (1982) or Lerche (1986)]. However, for the second and third examples, the exact exit distribution is unknown, and we have simulated these by using the empirical distribution from 200,000 sample paths.

For Figure 1, the exit boundary is

$$f(t) = \frac{\theta}{2} - \frac{t}{\theta} \log\left(\frac{1}{2}\frac{\alpha}{a} + \left[\frac{1}{4}\left(\frac{\alpha}{a}\right)^2 + \frac{1-\alpha}{a} \exp\left\{\frac{\theta^2}{t}\right\}\right]^{1/2}\right), \tag{6.1}$$

with $\theta = 20$ and $\alpha = a = 0.3$. Because this is a concave boundary, the tangent to the curve is the upper enveloping line and, consequently, the HRT method and upper analytic bound produce the same distribution. Note that the ordering of the tangent approximation and the analytic bounds are as

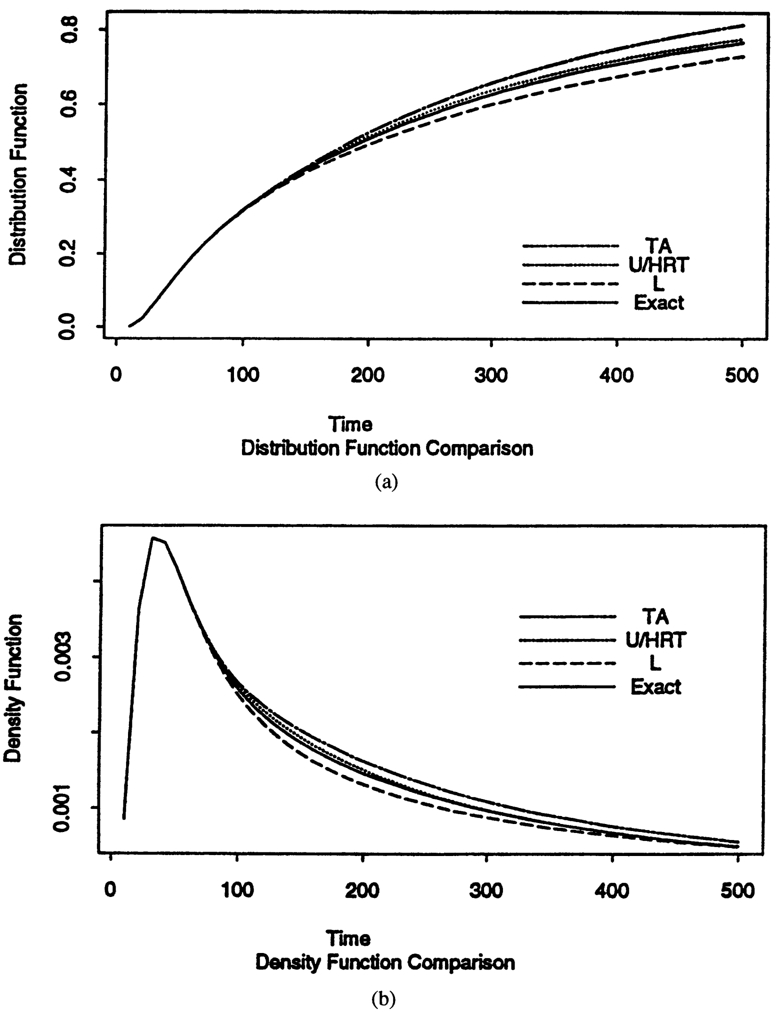

FIG. 1. *Estimates for the first hitting time of Brownian motion of the boundary given by* (6.1). (a) *Distribution function comparison*; (b) *density function comparison.*

expected (Theorem 3). Furthermore, the enveloping straight lines have fairly similar gradients and thus remain tight to the boundary. A consequence of this is that the upper and lower analytic bounds are both reasonably accurate. In fact, the lower bound is about as accurate as the tangent approximation over the range plotted.

The second example (Figure 2) is for the boundary

$$f(t) = 2 + 0.1t + 0.25\sin(t). \tag{6.2}$$

In this case, we have no theoretical justification for the HRT method to be superior to the tangent approximation, because the curve is neither concave

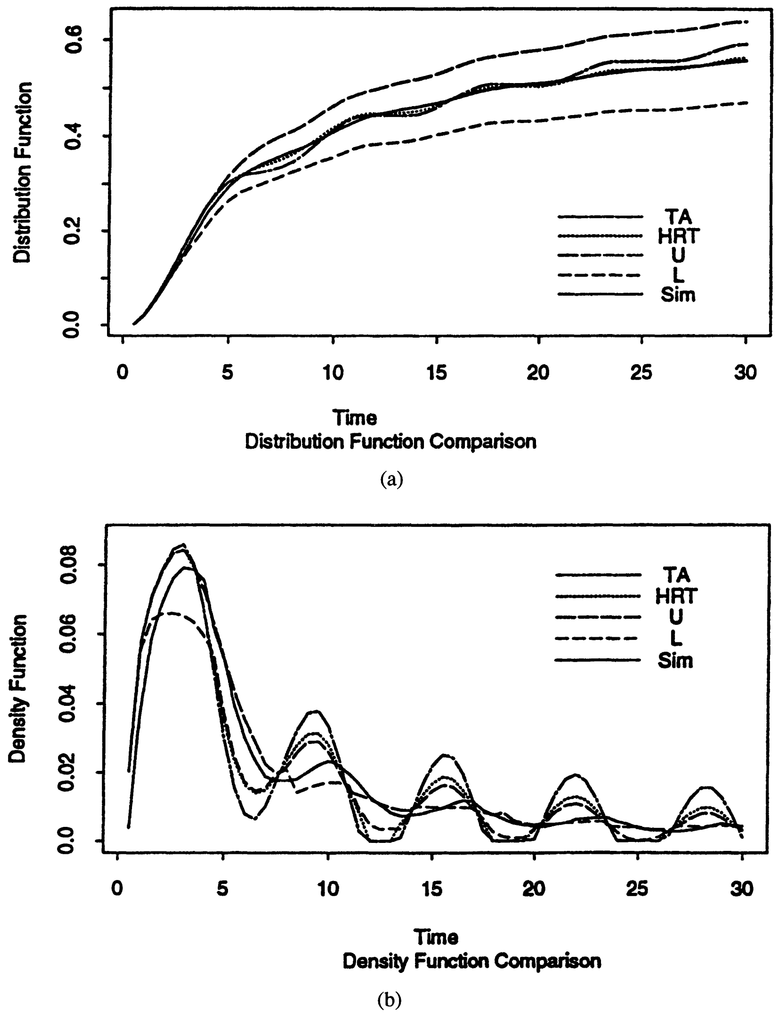

FIG. 2. *Estimates for the first hitting time of Brownian motion of the boundary given by* (6.2). (a) *Distribution function comparison*; (b) *density function comparison.*

nor convex. Because the enveloping straight lines are rarely close to the actual boundary, the quality of the upper and lower analytic bounds is fairly poor. However, both approximations remain within these analytic bounds, due to the cancellation of errors. When the curve is locally concave, both approximations overestimate the true value, and when it is locally convex, they underestimate. Thus, the successive periods of under- and overestimation cancel to some extent and produce good approximations. In fact, the HRT method is more accurate than the tangent approximation, because the use of $\int_0^t r_T(s)\,ds$ in the calculation of the distribution at time t acts as a feedback mechanism, making the fluctuations less drastic. Note also, that for some time values, the lower enveloping line is the tangent, and this intercepts the x-axis at a negative value, leading to a zero density for these times.

The final example (Figure 3) is for the boundary

$$f(t) = 6e^{-t/4} + 2e^{-4/t}, \tag{6.3}$$

which is neither concave nor convex. The most noticeable feature of this example is that the lower bound and HRT method produce virtually indistinguishable distributions. This is because the boundary is initially convex and thus the methods produce identical values over this time interval. Thereafter,

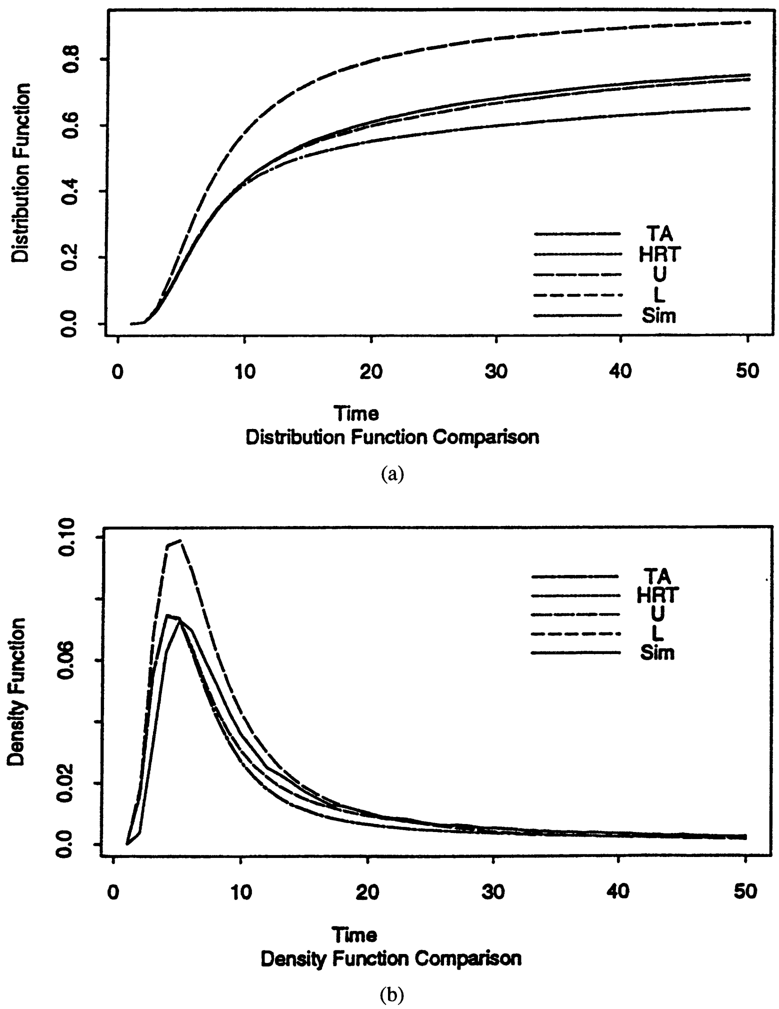

FIG. 3. *Estimates for the first hitting time of Brownian motion of the boundary given by* (6.3). (a) *Distribution function comparison*; (b) *density function comparison.*

the difference between the tangent (used in the HRT method) and lower enveloping curve is comparatively small and so similar approximations are obtained. The upper bound is fairly poor, essentially because the upper enveloping line is a poor approximation to the curve. The main cause of the inaccuracy of the tangent approximation is the underestimation when the curve is initially convex.

For further numerical examples supporting the belief that the HRT method is a good approximation technique, see Shortland (1993).

7. Discussion. We have introduced a simple approximation technique for first exit times of one-dimensional diffusions which appears to be accurate in a wide range of situations. For approximating the first exit distribution function, the HRT method has been shown to be more accurate than the tangent approximation for convex and concave boundaries, and appears to be more accurate for all choices of boundary curves. Like the tangent approximation, this approximation works best for boundaries which are almost linear, when the tangent is a good approximation to the curve.

We show the analytic bounds also work best for boundary functions which are close to linear. The accuracy is improved when the boundary is either concave or convex, since one of the analytic bounds is then the accurate HRT approximation. For other boundaries, the analytic bounds tend to be poor compared with the approximation techniques, because they produce approximations with no cancellation of errors.

The advantages of the techniques presented here lie in their ease of implementation, their accuracy in comparison to the tangent approximation and the existence of upper and lower bounds to supplement the approximation. However, in problems where the tangent approximation performs badly, HRT can also give poor results (especially in situations where tangents go below starting values). Therefore, care has to be taken in assessing the suitability of the method to particular problems. The results in Theorem 1 can be used in this assessment.

Acknowledgments. We would like to thank an Associate Editor and referee for discovering an error in the statement of Theorem 1 in an earlier draft.

REFERENCES

DANIELS, H. E. (1982). Sequential tests constructed from images. *Ann. Statist.* **10** 394–400.

DURBIN, J. (1992). The first passage density of the Brownian motion process to a curved boundary (with an appendix by D. Williams). *J. Appl. Probab.* **29** 291–304.

LERCHE, H. R. (1986). *Boundary Crossing of Brownian Motion. Lecture Notes in Statist.* **40**. Springer, Berlin.

ROBERTS, G. O. (1991a). A comparison theorem for conditioned Markov processes. *J. Appl. Probab.* **28** 74–83.

ROBERTS, G. O. (1991b). Asymptotic approximations for Brownian motion boundary hitting times. *Ann. Probab.* **19** 1689–1731.

ROBERTS, G. O. (1993). Some inequalities for one-dimensional conditioned diffusions and boundary hitting times. Unpublished manuscript.

ROBERTS, G. O. and SHORTLAND, C. (1994). Pricing barrier options. Research report 94-15, Cambridge Univ. Statistical Laboratory.

SHORTLAND, C. (1993). Some results on boundary hitting times for one-dimensional diffusion processes. Ph.D. Thesis, Univ. Nottingham.

STRASSEN, V. (1967). Almost sure behavior of sums of independent random variables and martingales. *Proc. Fifth Berkeley Symp. Math. Statist. Probab.* **2** 315–343, Univ. California Press, Berkeley.

STATISTICAL LABORATORY
16 MILL LANE
UNIVERSITY OF CAMBRIDGE
CAMBRIDGE CB2 1SB
ENGLAND