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LIMITS OF FIRST PASSAGE TIMES TO RARE SETS IN REGENERATIVE PROCESSES

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We consider limits of first passage times to indexed families of nested sets in regenerative processes. The sets are *exponentially rare,* in the sense that the probability that the process reaches an indexed set in a cycle vanishes exponentially fast in the indexing parameter. Under appropriate formulations of this hypothesis, we prove strong laws, iterated logarithm laws and limits in distribution, both for the index of the rarest set reached in a cycle and for the time to reach a set. An interesting feature of the iterated logarithm laws is an asymmetry in the normalizations for the upper and lower limits. Our results apply to (possibly delayed) wide-sense regenerative processes, as well as those with independent cycles. We illustrate our results with queueing examples.

1. Introduction and main results. We study first passage times to rare sets for discrete-time regenerative processes. A process $X = \{X_n,\ n \geq 0\}$ is (wide-sense) regenerative if there exists a renewal process $\{\tau_n,\ n \geq 0\}$ such that, for each $n \geq 0$, $\{\tau_{n+k} - \tau_n,\ k \geq 1; X_{\tau_n + j},\ j \geq 0\}$ is independent of $\{\tau_0, \dots, \tau_n\}$ and has law not depending on n. The *cycles*

$$\{X_n,\ \tau_j \leq n < \tau_{j+1}\}, \qquad j = 0, 1, \dots, \tag{1}$$

of such a process have a common distribution; they may not be independent, but are at most *one-dependent,* meaning that nonconsecutive cycles are independent. We do not exclude the delayed case in which the law of $\{X_n,\ 0 \leq n < \tau_0\}$ may differ from that of the cycles. This class of processes is sufficiently general to include, for example, Harris recurrent Markov chains; see Section V.1 of [4] for background.

Throughout, we restrict attention to the positive recurrent case in which the mean cycle length is finite. Unless otherwise indicated, *regenerative* is meant in the wide sense.

We consider first passage times to sets that are *exponentially rare.* More precisely, we consider a decreasing family $\{A_x,\ x \in \mathbb{R}_+\}$ of subsets of the state space of X, with the property that the probability that X reaches A_x in a cycle vanishes exponentially fast in x. We use three versions of this hypothesis. The weakest requires the existence of a strictly positive constant γ such that

$$\lim_{x \to \infty} (-x^{-1}) \log P\{X_n \in A_x,\ \text{for some } \tau_0 \leq n < \tau_1\} = \gamma. \tag{2}$$

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A stronger requirement is

$$b_1 e^{-\gamma x} \leq P\{X_n \in A_x, \text{ for some } \tau_0 \leq n < \tau_1\} \leq b_2 e^{-\gamma x}, \tag{3}$$

for all sufficiently large x, for some strictly positive $b_1 \leq b_2$ and γ. The strongest condition we use is the existence of b, $\gamma > 0$ for which

$$P\{X_n \in A_x, \text{ for some } \tau_0 \leq n < \tau_1\} \sim b e^{-\gamma x}, \tag{4}$$

where $\sim$ indicates that the ratio of the two quantities converges to unity as $x \to \infty$. We show that under (2), the first passage times

$$T_x \triangleq \inf\{n \geq 0: \ X_n \in A_x\}$$

obey a strong law of large numbers (SLLN) and under (3) they obey a law of the iterated logarithm (LIL). Under (4) and the additional assumption that the cycles are independent, they have a limit distribution when properly normalized.

Some remarks are in order regarding these hypotheses and their connection to specific examples. The strongest property (4) is shown by Iglehart [13] to hold for the waiting times in a GI/G/1 queue, with $A_x = \{y \in \mathbb{R}: \ y > x\}$, under a nonlattice assumption. This extends to heterogeneous multiserver queues with batch arrivals through recent results of Sadowsky and Szpankowski [23]. Bounds of the type in (3) are established for queues by Kingman [15] and Ross [22], for all $x > 0$. Asmussen and Perry [6] study (4) in some generality and establish it, in particular, for certain Markov-modulated queues. Abate, Choudhry and Whitt [1] give an example (their Example 5) of an M/G/1 queue for which (2) holds, but (3) fails. Anantharam [3] and Glasserman and Kou [10] establish instances of (2) for Jackson networks, the former based on large queues at individual nodes, the latter based on the total network population. Chang, Heidelberger, Juneja and Shahabuddin [8] prove a version of (2) for buffer overflows in ATM switches with rather general arrival processes. Some results close to (2) are established for the ALOHA protocol in Cottrell, Fort and Malgouyres [9]. Shahabuddin [25] proves a result like (4) for a general class of reliability models in which rarity arises from changes in the law of the process, rather than in the set to be reached.

Close counterparts of conditions (2)–(4) are obtained by replacing the probability in each case with $\pi(A_x)$, where π is the stationary distribution of X. (A positive recurrent regenerative process has precisely one such distribution.) Let P_0 denote the probability on sample paths of X corresponding to the nondelayed case $\tau_0 \equiv 0$ and let $\nu_x(\cdot) = P_0\{X_{T_x} \in \cdot\}$. The cycle representation of the stationary distribution of regenerative processes implies that

$$\pi(A_x) = (\mathsf{E}_0[\tau_1])^{-1} P_0\{X_n \in A_x \text{ for some } 0 \leq n < \tau_1\} \mathsf{E}_{\nu_x}\left[\sum_{n=0}^{\tau_1 - 1} \mathbf{1}_{\{X_n \in A_x\}}\right].$$

Thus, if the expected sojourn of X in A_x during a cycle given that it reaches A_x in that cycle is bounded in x, then (3) is equivalent to the counterpart condition for π. If the expected sojourn is bounded by a polynomial in x, then (2) is equivalent to its counterpart for π. Indeed, properties (2)–(4) are sometimes established as a step in proving a corresponding property for the stationary distribution. (For other aspects of the link between regeneration and exponentiality of first-passage times, see [2] and [14]; for a recent survey with particular emphasis on the Russian literature, see [17].)

The analysis of T_x can be reduced to the analysis of level-crossing times in real-valued processes by defining

$$\phi(y) = \sup\{x \geq 0\colon\ y \in A_x\}.$$

If X is regenerative, so is $\phi(X) = \{\phi(X_n),\ n \geq 0\}$. Letting T_x^{ϕ} be the first passage time for $\phi(X)$ to $A_x^{\phi} = \{y \in \mathbb{R}\colon\ y > x\}$, we see that $T_{x-\varepsilon}^{\phi} \leq T_x \leq T_x^{\phi}$ for all $\varepsilon > 0$, for all x. In this way, limit theorems for T_x^{ϕ} extend to corresponding results for T_x. Henceforth, we take X to be a real-valued regenerative process and take $A_x = A_x^{\phi}$.

Define

$$M_n = \max\{X_k\colon\ \tau_{n-1} \leq k < \tau_n\}, \qquad n = 1, 2, \ldots. \tag{5}$$

Then M_n records the index of the rarest of the sets $\{A_x\}$ visited during the nth cycle. Also define

$$M_0 = \sup\{X_k\colon\ 0 \leq k < \tau_0\},$$

taking $M_0 = -\infty$ in the nondelayed case $\tau_0 = 0$. For ease of reference we reformulate (2)–(4), specializing to the real-valued case:

(A1) $\lim_{x\to\infty}(-x^{-1})\log P\{M_1 > x\} = \gamma$.

(A2) For all sufficiently large x, $b_1 e^{-\gamma x} \leq P\{M_1 > x\} \leq b_2 e^{-\gamma x}$.

(A3) $P\{M_1 > x\} \sim b e^{-\gamma x}$, as $x \to \infty$.

Some indication of the limiting behavior of T_x is obtained from the representation ([11], page 38)

$$\mathsf{E}_0[T_x] = \frac{\mathsf{E}_0[T_x \wedge \tau_1]}{P_0\{T_x < \tau_1\}},$$

available when the cycles (1) are independent and $\tau_0 = 0$. Under (A1), we see immediately that

$$\frac{\log \mathsf{E}_0[T_x]}{\gamma x} \to 1.$$

Our main result gives a corresponding strong law and a law of the iterated logarithm, without requiring independence of cycles. To give a more complete picture of the limiting behavior, we include as well a limit in distribution which follows fairly readily from existing results. We write $\log_2$ for $\log\log$ and $\log_3$ for $\log\log_2$.

THEOREM 1.1. *As $x \to \infty$, the following hold:*

(i) *Under* (A1),

$$\frac{\log T_x}{\gamma x} \longrightarrow 1 \quad a.s. \tag{6}$$

(ii) *Under* (A2),

$$\limsup \frac{\log T_x - \gamma x}{\log_2 x} = 1 \quad a.s. \tag{7}$$

and

$$\liminf \frac{\log T_x - \gamma x}{\log x} = -1 \quad a.s. \tag{8}$$

(iii) *If the cycles* (1) *are independent and* (A3) *holds, then for all* $y \geq 0$,

$$P\{be^{-\gamma x}T_x \leq y\} \to 1 - e^{-y/c}, \tag{9}$$

where $1/c$ *is the mean cycle length.*

Perhaps the most interesting aspect of this result is the asymmetry in (7) and (8).

To prove Theorem 1.1, we first establish (in Section 2) some preliminary results on general level-crossing probabilities for regenerative processes, extending a result of Robbins and Siegmund [19] for i.i.d. sequences. We use this result to establish counterparts of (6) and (7)–(8) for the maximum of a real-valued regenerative process; the counterpart of (9) is known. These results are in Section 3. In Section 4, we prove Theorem 1.1 and in Section 5 we give examples.

2. A level-crossing theorem.

We begin with the following version of Theorem 1 of [19]:

LEMMA 2.1. *Let* $\{V_n,\ n \geq 1\}$ *be a one-dependent sequence of random variables with common distribution* H *and let* $V_n^* = \max_{1\leq j\leq n} V_j$. *If* u_n *is an ultimately increasing sequence of real numbers, then*

$$P\{V_n^* > u_n \quad i.o.\} = 0 \text{ or } 1 \tag{10}$$

according as

$$\sum_{n=1}^{\infty}(1 - H(u_n)) < \infty \text{ or } = \infty.$$

If, in addition, $n(1 - H(u_n))$ *is ultimately increasing, then*

$$P\{V_n^* \leq u_n \quad i.o.\} = 0 \text{ or } 1 \tag{11}$$

according as

$$\sum_{n=1}^{\infty}(1-H(u_n))\exp(-n(1-H(u_n)))<\infty \text{ or } =\infty.$$

PROOF. The case of i.i.d. uniform random variables is proved in [19]. Remark 2.5 of [19] extends the result to i.i.d. random variables with continuous distribution H. However, even without continuity, we may set $V_i=H^{-1}(U_i)$, where $\{U_n,\ n\geq 1\}$ are independent uniform random variables and $H^{-1}(u)=\inf\{y\colon H(y)\geq u\}$. Under this construction, the events $\{V_i\leq u_n,\ i=1,\dots,n\}$ and $\{U_i\leq H(u_n),\ i=1,\dots,n\}$ coincide for all n, as do the corresponding events with inequalities reversed. This suffices to extend the i.i.d. result to general H. For the one-dependent case, let $e_n=2n$ and $o_n=2n-1$, $n=1,2,\dots$. Let $V_n^e=\max\{V_2,V_4,\dots,V_{e_n}\}$ and $V_n^o=\max\{V_1,V_3,\dots,V_{o_n}\}$, and notice that

$$\begin{aligned} P\{V_n^e>u_{e_n}\ \text{ i.o.}\}\vee P\{V_n^o>u_{o_n}\ \text{ i.o.}\} & \\ \leq P\{V_n^*>u_n\ \text{ i.o.}\} & \\ \leq P\{V_n^e>u_{e_n}\ \text{ i.o.}\}+P\{Y_n^o>u_{o_n}\ \text{ i.o.}\}. & \end{aligned} \tag{12}$$

If $\sum_n(1-H(u_n))$ converges, then so do $\sum_n(1-H(u_{e_n}))$ and $\sum_n(1-H(u_{o_n}))$, and the first case of (10) follows from the second inequality in (12) and the result for i.i.d. sequences. If $\sum_n(1-H(u_n))$ diverges, then so does at least one of $\sum_n(1-H(u_{e_n}))$ and $\sum_n(1-H(u_{o_n}))$, and the second case of (10) follows from the first inequality in (12). The same step leads to (11). □

The extension above to discontinuous distributions is important in modeling integer-valued processes, such as queue lengths. Also, it is easy to see that Lemma 2.1 extends to m-dependent sequences for arbitrary, finite m. Theorem 2 of [16] weakens the assumption that $n(1-H(u_n))$ be ultimately increasing in the Robbins–Siegmund result.

We will need the following auxiliary result on level crossings through subsequences:

LEMMA 2.2. *Let $\{V_n,\ n\geq 1\}$ be a sequence of i.i.d. random variables with common distribution H and let $V_n^*=\max_{1\leq j\leq n}V_j$. Suppose that u_n is ultimately increasing. Then for any integer sequence $\{a_n\}$ ultimately strictly increasing to ∞,*

$$P\{V_{a_n}^*>u_n\ \text{i.o.}\}=0 \text{ or } 1, \tag{13}$$

according as $\sum_{n=1}^{\infty}(a_{n+1}-a_n)(1-H(u_{n+1}))<\infty$ or $=\infty$.

PROOF. First, we claim that

$$\sum_{n=1}^{\infty}(a_{n+1}-a_n)(1-H(u_{n+1}))<\infty \quad\Longrightarrow\quad a_n(1-H(u_n))\longrightarrow 0 \quad \text{as } n\to\infty.$$

In fact, $\forall\, \varepsilon > 0$, there must be an n_0, such that $\forall\, k \geq n_0 + 1$,

$$\sum_{n=n_0}^{k-1} (a_{n+1} - a_n)(1 - H(u_{n+1})) < \frac{\varepsilon}{2}.$$

Also, since $(1 - H(u_n))$ must goes to zero, we can find another n_1, such that $\forall\, k \geq n_1$,

$$a_{n_0}(1 - H(u_k)) < \varepsilon/2.$$

Hence for all $k \geq \max(n_1, n_0 + 1)$,

$$(a_k - a_{n_0})(1 - H(u_k)) \leq \sum_{n=n_0}^{k-1} (a_{n+1} - a_n)(1 - H(u_{n+1})) < \varepsilon/2,$$

which further implies

$$a_k(1 - H(u_k)) \leq \varepsilon/2 + a_{n_0}(1 - H(u_k)) < \varepsilon.$$

Now by interchanging the unions, we get

$$\bigcup_{n=k}^{\infty}\{V^*_{a_n} > u_n\} = \bigcup_{n=k}^{\infty}\bigcup_{j=1}^{a_n}\{V_j > u_n\} = \bigcup_{j=1}^{a_k}\{V_j > u_k\} \cup \bigcup_{i=0}^{\infty}\bigcup_{j=a_{k+i}+1}^{a_{k+1+i}}\{V_j > u_{k+i+1}\}.$$

So if $\sum_{n=1}^{\infty}(a_{n+1} - a_n)(1 - H(u_{n+1})) < \infty$, then for all large k,

$$\begin{aligned} P\left\{\bigcup_{n=k}^{\infty}\{V^*_{a_n} > u_n\}\right\} &\leq \sum_{j=1}^{a_k} P\{V_j > u_k\} + \sum_{i=0}^{\infty}\sum_{j=a_{k+i}+1}^{a_{k+1+i}} P\{V_j > u_{k+i+1}\} \\ &= a_k(1 - H(u_k)) + \sum_{i=k}^{\infty}(a_{i+1} - a_i)(1 - H(u_{i+1})) \\ &\longrightarrow 0 \quad \text{as } k \to \infty. \end{aligned}$$

Hence

$$P\{V^*_{a_n} > u_n \text{ i.o.}\} = 0.$$

If $\sum_{n=1}^{\infty}(a_{n+1} - a_n)(1 - H(u_{n+1})) = \infty$, then for all large k,

$$\begin{aligned} P\left\{\bigcup_{n=k}^{\infty}\{V^*_{a_n} > u_n\}\right\} &\geq P\left\{\bigcup_{i=0}^{\infty}\bigcup_{j=a_{k+i}+1}^{a_{k+i+1}}\{V_j > u_{k+i+1}\}\right\} \\ &= 1 - P\left\{\bigcap_{i=0}^{\infty}\bigcap_{j=a_{k+i}+1}^{a_{k+i+1}}\{V_j \leq u_{k+i+1}\}\right\} \end{aligned}$$

$$= 1 - \prod_{i=0}^{\infty} \prod_{j=a_{k+i}+1}^{a_{k+i+1}} P\{V_1 \le u_{k+i+1}\}$$

$$\ge 1 - \prod_{i=0}^{\infty} \prod_{j=a_{k+i}+1}^{a_{k+i+1}} \exp(-(1 - H(u_{k+i+1}))) \quad (\text{using } e^{-x} > 1 - x)$$

$$= 1 - \exp\left\{ -\sum_{i=k}^{\infty} (a_{i+1} - a_i)(1 - H(u_{i+1})) \right\} = 1.$$

Thus, $P\{V_{a_n}^* > u_n \text{ i.o.}\} = 1$, completing the proof. □

REMARK. An interesting open problem is to find general conditions on $\{a_n\}$ under which

$$P\{V_{a_n}^* \le u_n \quad \text{i.o.}\} = 1.$$

A nontrivial special case is treated in the Appendix as part of our analysis of the lim sup in the law of the iterated logarithm for X^*.

The following result will prove useful in accommodating delayed regenerative processes:

LEMMA 2.3. *Suppose u_n is an ultimately increasing sequence of real numbers and the integer sequence a_n is ultimately increasing to ∞. For any sequence of random variables $\{Z_n,\ n \ge 0\}$, $P\{Z_0 > u_n\} \to 0$, as $n \to \infty$, implies*

$$P\left\{ \max_{0 \le j \le a_n} Z_j \le u_n \quad \text{i.o.} \right\} = P\left\{ \max_{1 \le j \le a_n} Z_j \le u_n \quad \text{i.o.} \right\} \tag{14}$$

and

$$P\left\{ \max_{0 \le j \le a_n} Z_j > u_n \quad \text{i.o.} \right\} = P\left\{ \max_{1 \le j \le a_n} Z_j > u_n \quad \text{i.o.} \right\}. \tag{15}$$

PROOF. Notice that $P\{Z_0 > u_n\} \longrightarrow 0$ as $n \to \infty$ implies

$$P\{Z_0 > u_n \quad \text{i.o.}\} = \lim_{n\to\infty} P\left\{ \bigcup_{k=n}^{\infty} \{Z_0 > u_n\} \right\} = \lim_{n\to\infty} P\{Z_0 > u_n\} = 0.$$

So, using "f.o." for "finitely often,"

$$\begin{aligned} &P\left\{ \max_{0 \le j \le a_n} Z_j \le u_n \quad \text{i.o.} \right\} \\ &\quad \ge P\left\{ \left\{ \max_{1 \le j \le a_n} Z_j \le u_n \quad \text{i.o.} \right\} \cap \{Z_0 > u_n \text{ f.o. }\} \right\} \\ &\quad = P\left\{ \max_{1 \le j \le a_n} Z_j \le u_n \quad \text{i.o.} \right\}, \end{aligned}$$

as $P\{Z_0 > u_n \text{ f.o.}\} = 1$. This proves (14); a similar argument leads to (15). □

As in Theorem 1.1, let X be a positive recurrent, real-valued regenerative process with associated regeneration epochs $\{\tau_n,\ n \geq 0\}$. Define

$$l(n) = \sup\{j \geq 0\colon\ \tau_j \leq n\}.$$

By the strong law of large numbers for renewal processes we have

$$\frac{l(n)}{n} \longrightarrow c \quad \text{a.s.}$$

as $n \to \infty$, where $1/c$ is the mean cycle length. As in (5), let M_n be the nth cycle maximum and set $G(x) = P\{M_1 \leq x\}$. Define $X_n^* = \max_{0 \leq j \leq n} X_j$. We now have the following theorem.

THEOREM 2.4. *Suppose that u_n is an ultimately increasing sequence of real numbers, that a_n is a positive integer sequence ultimately strictly increasing to ∞ and that $P(M_0 > u_n) \to 0$ as $n \to \infty$. Then*

(i) $$P\{X_{a_n}^* > u_n \quad i.o.\} = 0 \text{ or } 1$$

according as

$$\sum_{n=1}^{\infty}(a_{n+1} - a_n)(1 - G(u_n)) < \infty \text{ or } = \infty.$$

(ii) *If $n(1 - G(u_n))$ is ultimately increasing, then*

$$P\{X_n^* \leq u_n \quad i.o.\} = 0 \text{ or } 1$$

according as

$$\sum_{n=1}^{\infty}(1 - G(u_n))\exp(-n(1 - G(u_n))) < \infty \text{ or } = \infty.$$

To lighten the notational burden, throughout the rest of the paper we distinguish between x and its integer part $[x]$ only where necessary. Furthermore, for real $t > 0$, x_t is to be understood to be $x_{[t]}$.

PROOF. It suffices to consider delayed regenerative processes with i.i.d. cycles, since the one-dependent case then follows by taking even and odd subsequences, as in Lemma 2.1. Furthermore, we can assume $c < 1$ because $c = 1$ implies $P\{\tau_{i+1} - \tau_i = 1,\ i \geq 0\} = 1$, and in this case the process degenerates to a delayed i.i.d. sequence as treated in the preceding lemmas.

Fix $0 < \delta < c$ so that $c - \delta = 1/m$, $m \geq 2$ an integer. Then, for any $k \geq 1$,

$$\begin{aligned} &P\left\{\bigcup_{n=k}^{\infty}\{X_n^* \leq u_n\}\right\} \\ &\quad \leq P\left\{\bigcup_{n=k}^{\infty}\{X_n^* \leq u_n;\ l(n) > n(c-\delta)\right\} + P\left\{\bigcup_{n=k}^{\infty}\{l(n) \leq n(c-\delta)\}\right\} \end{aligned}$$

$$\le P\left\{\bigcup_{n=k}^{\infty}\{X^*_{\tau_{n(c-\delta)}} \le u_n\}\right\} + o(1) \quad \left(\text{since } \frac{l(n)}{n} \longrightarrow c \text{ a.s.}\right)$$

$$\le P\left\{\bigcup_{n=k}^{\infty}\{M^*_{n(c-\delta)} \le u_n\}\right\} + o(1)$$

$$\le P\left\{\bigcup_{n=k(c-\delta)}^{\infty}\{M^*_n \le u_{n/(c-\delta)}\}\right\} + o(1),$$

$$= P\left\{\bigcup_{n=k(c-\delta)}^{\infty}\{M^*_n \le u_{nm}\}\right\} + o(1),$$

where $M^*_m \triangleq \max_{1\le j\le m} M_j$.

On the other hand, choosing $\delta' > 0$ so that $c + \delta' = 1$, we get

$$P\left\{\bigcup_{n=k}^{\infty}\{X^*_n \le u_n\}\right\} \ge P\left\{\bigcup_{n=k}^{\infty}\{X^*_n \le u_n;\ l(n) \le n(c+\delta')\}\right\}$$

$$\ge P\left\{\bigcup_{n=k}^{\infty}\{X^*_{\tau_{n(c+\delta')}} \le u_n\}\right\}$$

$$= P\left\{\bigcup_{n=k}^{\infty}\{M^*_n \le u_n\}\right\}.$$

By Lemmas 2.1 and 2.3, if

$$\sum_{n=1}^{\infty}(1 - G(u_{nm}))\exp(-n(1 - G(u_{nm}))) < \infty,$$

then

$$P\left\{\bigcup_{n=k(c-\delta)}^{\infty}\{M^*_n \le u_{nm}\}\right\} \longrightarrow 0 \quad \text{as } k \to \infty.$$

In particular, if

$$\sum_{n=1}^{\infty}(1 - G(u_n))\exp(-n(1 - G(u_n))) < \infty,$$

then

$$P\{X^*_n \le u_n \quad \text{i.o.}\} = 0.$$

Similarly, if

$$\sum_{n=1}^{\infty}(1 - G(u_n))\exp(-n(1 - G(u_n))) = \infty,$$

then

$$P\left\{\bigcup_{n=k}^{\infty}\{M^*_n \le u_n\}\right\} \longrightarrow 1, \quad \text{as } k \to \infty.$$

Therefore,

$$P\{X_n^* \le u_n \quad \text{i.o.}\} = 1.$$

This completes the proof of (ii).

For (i), notice that

$$(16)\qquad \begin{aligned} P\left\{\bigcup_{n=k}^{\infty}\{X_{a_n}^* > u_n\}\right\} &\le P\left\{\bigcup_{n=k}^{\infty}\{M_{a_n(c+\delta')}^* > u_n\}\right\} + o(1) \\ &= P\left\{\bigcup_{n=k}^{\infty}\{M_{a_n}^* > u_n\}\right\} + o(1) \end{aligned}$$

and

$$(17)\qquad \begin{aligned} P\left\{\bigcup_{n=k}^{\infty}\{X_{a_n}^* > u_n\}\right\} &\ge P\left\{\bigcup_{n=k}^{\infty}\{M_{[a_n(c-\delta)]}^* > u_n\}\right\} \\ &= P\left\{\bigcup_{n=k}^{\infty}\{M_{[a_n/m]}^* > u_n\}\right\}. \end{aligned}$$

So our conclusion follows from Lemmas 2.2 and 2.3, if we can show that

$$\sum_{n=1}^{\infty}(a_{n+1} - a_n)(1 - G(u_n)) = \infty \quad \Longrightarrow \quad \sum_{n=1}^{\infty}\left(\left[\frac{a_{n+1}}{m}\right] - \left[\frac{a_n}{m}\right]\right)(1 - G(u_n)) = \infty.$$

Without loss of gerneralilty, we can assume that a_n is strictly increasing and $(1 - G(u_n))$ is decreasing. If $\sum_{n=1}^{\infty}(1 - G(u_n)) < \infty$, then

$$\begin{aligned} &\sum_{n=1}^{\infty}\left(\left[\frac{a_{n+1}}{m}\right] - \left[\frac{a_n}{m}\right]\right)(1 - G(u_n)) \\ &\qquad \ge \sum_{n=1}^{\infty}\left(\frac{a_{n+1}}{m} - \frac{a_n}{m}\right)(1 - G(u_n)) - \sum_{n=1}^{\infty}(1 - G(u_n)) = \infty. \end{aligned}$$

So we only need to consider the case $\sum_{n=1}^{\infty}(1 - G(u_n)) = \infty$. Notice that, for each $k \ge 0$, there must be one n in the set $\{km + 1, km + 2, \cdots, (k + 1)m\}$, such that $[a_{n+1}/m] - [a_n/m] \ge 1$. So

$$\sum_{n=1}^{\infty}\left(\left[\frac{a_{n+1}}{m}\right] - \left[\frac{a_n}{m}\right]\right)(1 - G(u_n)) \ge \sum_{n=1}^{\infty}(1 - G(u_{nm})) = \infty,$$

using the elementary fact that if $a_n \ge 0$ is ultimately monotone, then $\sum_n a_n = \infty \Longrightarrow \sum_n a_{nm} = \infty$. This completes the proof. □

COROLLARY 2.5. *Under* (A2),

$$P\{X^*_{e^{\gamma n}n^{-a}} > n \text{ i.o.}\} = 0 \text{ or } 1$$

according as $a > 1$ *or* $a < 1$.

PROOF. This follows directly from Theorem 2.4 and the fact that

$$\sum_{n=1}^{\infty}(e^{\gamma(n+1)}(n+1)^{-a} - e^{\gamma n}n^{-a})e^{-\gamma n} = \sum_{n=1}^{\infty}\frac{1}{n^a}\left(e^{\gamma}\left(1 - \frac{1}{n+1}\right)^a - 1\right)$$

converges or diverges according as $a > 1$ or $a < 1$. □

It will become evident in the proof of Theorem 1.1 that the exponential rarity assumptions (A1)–(A3) are used exclusively to verify the series tests in Theorem 2.4. To the extent that these tests can be checked under alternative tail assumptions, versions of Theorem 1.1 can be established without (A1)–(A3).

3. Limit theorems for the maximum. In this section, we use Theorem 2.4 to get an SLLN and LIL for the maximum of a regenerative process.

PROPOSITION 3.1 (SLLN). *Suppose* (A1) *holds. Then*

$$\frac{\gamma X^*_n}{\log n} \longrightarrow 1 \quad a.s.$$

as $n \to \infty$.

PROOF. Let $F(x) = P\{\gamma M_1 \le x\}$. For all $x > 1$,

$$\begin{aligned}\sum_{n=1}^{\infty}(1 - F(x\log n)) &= \sum_{n=1}^{\infty}\exp(-x\log n + o(\log n))\\ &= \sum_{n=1}^{\infty}\frac{1}{n^x}\exp(o(\log n)) < \infty.\end{aligned}$$

So, by Theorem 2.4,

$$P\{\gamma X^*_n > x\log n \quad \text{i.o.}\} = 0;$$

that is,

$$\limsup \frac{\gamma X^*_n}{\log n} \le 1, \tag{18}$$

almost surely.

On the other hand, for all $x < 1$,

$$\begin{aligned}&\sum_{n=1}^{\infty}(1 - F(x\log n))\exp(-n(1 - F(x\log n)))\\ &\quad = \sum_{n=1}^{\infty}\frac{1}{n^x}\exp(o(\log n))\exp\{-n^{(1-x)}\exp(o(\log n))\} < \infty\end{aligned}$$

and

$$n(1 - F(x \log n)) = n^{1-x} \exp(o(\log n))$$

is ultimately increasing. Thus, again by Theorem 2.4,

$$P\{\gamma X_n^* \le x \log n \quad \text{i.o.}\} = 0;$$

that is,

$$\liminf \frac{\gamma X_n^*}{\log n} \ge 1, \tag{19}$$

almost surely. Combining (18) and (19) concludes the proof. □

PROPOSITION 3.2 (LIL). *If* (A2) *holds, then*

$$\limsup \frac{\gamma X_n^* - \log n}{\log_2 n} = 1 \quad \textit{and} \quad \liminf \frac{\gamma X_n^* - \log n}{\log_3 n} = -1,$$

almost surely.

REMARK. The triple logarithm in the lower limit (and the asymmetry in the normalization for the two cases) is rather peculiar. It is explained, in part, by the fact that the maximum tends to have a thin left tail and a relatively heavy right tail.

PROOF. First we treat the lim sup. As in the proof of Proposition 3.1, let F be the distribution of γM_1. Let the symbol $\approx$ between two series indicate that the series converge under the same conditions. Then

$$\sum_{n=1}^{\infty}(1 - F(\log n + x \log_2 n)) \approx \sum_{n=1}^{\infty} \exp(-\log n - x \log_2 n)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n(\log n)^x} < \infty \text{ or } = \infty$$

according as $x > 1$ or $x < 1$. Thus, by Theorem 2.4,

$$P\left\{\frac{\gamma X_n^* - \log n}{\log_2 n} > x \quad \text{i.o.}\right\} = 0 \text{ or } 1$$

according as $x > 1$ or $x < 1$. Hence

$$\limsup \frac{\gamma X_n^* - \log n}{\log_2 n} = 1,$$

almost surely.

At the same time,

$$\sum_{n=1}^{\infty}(1 - F(\log n - x \log_3 n)) \exp(-n(1 - F(\log n - x \log_3 n)))$$

$$\approx \sum_{n=1}^{\infty} \frac{(\log_2 n)^x}{n} \exp(-b_1(\log_2 n)^x).$$

The convergence of this series is equivalent to that of the integral

$$\begin{aligned}\int_{10}^{\infty} \frac{(\log_2 t)^x}{t} \exp(-b_1(\log_2 t)^x)\, dt \\ &= \int_{\log_2 10}^{\infty} \frac{y^x}{\exp(e^y)} \exp(-b_1 y^x) \exp(e^y) \exp(y)\, dy \qquad (y = \log_2 t) \\ &= \int_{\log_2 10}^{\infty} y^x \exp(-b_1(y^x - y))\, dy,\end{aligned}$$

which is finite or infinite according as $x > 1$ or $x < 1$. Finally, by noting that $n(1 - F(\log - x \log_3 n))$ is ultimately increasing, we conclude that the lim inf is as claimed. □

The possibility of a limit in distribution to supplement Propositions 3.1 and 3.2 is complicated by the need to distinguish lattice and nonlattice cases. In the lattice case, it may happen that no limiting distribution exists; see [18], Theorem 1.7.13, for the i.i.d. case, and the discussion in Serfozo [24] in a regenerative setting. In the nonlattice case, Iglehart's result [13] for the waiting times in a queue extends immediately to regenerative processes with independent cycles. For completeness, we include the proof. Asmussen and Perry [6] state essentially the same result; see [7], Theorem 3.2, and Rootzén [21] for related work. Notice that under (A3), M_1 is necessarily nonlattice.

PROPOSITION 3.3. *If the cycles are independent and* (A3) *holds, then*

$$P\{\gamma X_n^* - \log(ncb) \le x\} \longrightarrow \exp(-e^{-x}).$$

PROOF. Fix $0 < \delta < c$. Then, using the independence of the cycles, we get

$$\begin{aligned}P\{\gamma X_n^* - \log(ncb) \le x\} \\ &\le (P\{\gamma M_1 \le \log(ncb) + x\})^{n(c-\delta)} + P\{l(n) \le n(c-\delta)\} \\ &\to \exp\{-(1 - \delta/c)\exp(-x)\}.\end{aligned}$$

Thus,

$$\limsup P\{\gamma X_n^* - \log(ncb) \le x\} \le \exp(-(1 - \delta/c)\exp(-x)).$$

Similarly, using the fact that $P\{M_0 > x + \log n\} \to 0$ as $n \to \infty$, we have

$$\liminf P\{\gamma X_n^* - \log(ncb) \le x\} \ge \exp(-(1 + \delta/c)\exp(-x)).$$

Since $\delta > 0$ may be arbitrarily small, the result follows. □

4. Proof of Theorem 1.1.

We begin with the proof of part (i).

PROOF OF THEOREM 1.1(i). Because T_x is almost surely increasing in x, it suffices to establish the limit through integer values of x, denoted by n. From Proposition 3.1 we know that

$$\frac{\gamma X_n^*}{\log n} \longrightarrow 1 \quad \text{a.s.}$$

In particular, we can take the limit through the subsequence $\{\exp(a\gamma n)\}$, for any $a > 0$, and obtain

$$\frac{X^*_{\exp(a\gamma n)}}{an} \longrightarrow 1 \quad \text{a.s.}$$

In other words,

$$P\{X^*_{\exp\{a\gamma n\}} \leq n \quad \text{i.o.}\} = 1 \text{ or } 0$$

according as $a > 1$ or $a < 1$. So

$$P\{T_n > e^{a\gamma n} \quad \text{i.o.}\} = 1 \text{ or } 0$$

as $a > 1$ or $a < 1$. Hence,

$$\liminf \frac{\log T_n}{\gamma n} = 1$$

almost surely. By essentially the same argument, $\limsup(\log T_n)/(\gamma n) = 1$ almost surely, from which the conclusion follows. □

For the proof of part (ii), we need to be able to take limits through certain subsequences. As remarked after Theorem 2.4, general conditions for this are not available. The particular result we need is contained in the next lemma. Its proof is rather long and technical, and is therefore relegated to the Appendix.

LEMMA 4.1. *Under* (A2),

$$P\{X^*_{e^{\gamma n}(\log n)^a} \leq n \quad i.o.\} = 1$$

for all $a < 1$. The conclusion holds for $a = 1$ if, in addition, $b_2 < 1$.

PROOF OF THEOREM 1.1(ii). As in the proof of part (i), it suffices to prove the limit through integer values. From Proposition 3.2 we get, for any subsequence $\{e^{\gamma n}(\log n)^a\}$,

$$\liminf \frac{X^*_{e^{\gamma n}(\log n)^a} - n - \gamma^{-1} a \log_2 n}{\gamma^{-1}\log_2(\gamma n + a\log_2 n)} \geq -1.$$

So,

$$\liminf \frac{X^*_{e^{\gamma n}(\log n)^a} - n}{\gamma^{-1}\log_2 n} \geq a - 1,$$

and therefore

$$P\{X^*_{e^{\gamma n}(\log n)^a} \leq n \quad \text{i.o.}\} = 0 \quad \text{if } a > 1.$$

Combining this with Lemma 4.1, we get

$$P\{X^*_{e^{\gamma n}(\log n)^a} \leq n \quad \text{i.o.}\} = 1 \text{ or } 0$$

according as $a < 1$ or $a > 1$. In other words,

$$P\{T_n > e^{\gamma n}(\log n)^a \quad \text{i.o.}\} = 1 \text{ or } 0$$

according as $a < 1$ or $a > 1$. This is equivalent to

$$P\{\log T_n - \gamma n > a \log_2 n \quad \text{i.o.}\} = 1 \text{ or } 0$$

according as $a < 1$ or $a > 1$. Hence, we have

$$\limsup \frac{\log T_n - \gamma n}{\log_2 n} = 1,$$

almost surely.

For the lower limit, namely,

$$\liminf \frac{\log T_n - \gamma n}{\log n} = -1,$$

just take the subsequence $\{e^{\gamma n} n^{-a}\}$ and apply Corollary 2.5. The rest of the argument is standard and very similar to that for the upper limit, so we omit the details. □

PROOF OF THEOREM 1.1(iii). Fix a $y > 0$ and set

$$v_x = \exp\{\gamma x + \log y - \log(bc)\}.$$

Then

$$\begin{aligned} P\{bce^{-\gamma x}T_x > y\} &= P\{X^*_{v_x} \leq x\} \\ &= P\{X^*_{v_x} \leq \gamma^{-1}(\log v_x - \log y + \log(bc))\} \\ &= P\{\gamma X^*_{v_x} - \log v_x - \log(bc) \leq -\log y\} \\ &\to e^{-y} \quad \text{as } x \to \infty, \end{aligned}$$

as a consequence of Proposition 3.3. □

5. Examples and discussion.

5.1. *The first long wait in a queue.* We begin with the case of waiting times in a single-server queue, applying our main results in the stable case and noting supplementary results otherwise.

The input to the model is a sequence $\{(U_n, V_n),\ n \geq 1\}$ in which U_n is the time between the arrivals of the $(n-1)$st and nth customers, and V_n is the service time of the nth customer. Let $Y_n = V_{n-1} - U_n$ and let $S_n = Y_1 + \cdots + Y_n$, $S_0 = 0$. Then assuming the queue is initially empty, the nth waiting time W_n admits the representation

$$W_n = \sup_{0 \leq j \leq n} \{S_n - S_j\}.$$

We are interested in T_x, the index of the first customer whose waiting time exceeds x. Suppose $\{Y_n,\ n \geq 1\}$ are i.i.d. with $\mathsf{E}[Y_1] < 0$. Then $\{W_n,\ n \geq 0\}$ is a positive recurrent regenerative process (with independent cycles). In this case, we have the following corollary.

COROLLARY 5.1. *Suppose that* $\mathsf{E}[Y_1] < 0$ *and that* $\gamma > 0$ *solves* $\mathsf{E}[\exp(\gamma Y_1)] = 1$. *Then* (6)–(8) *hold with this* γ *for* T_x *the index of the first customer with a waiting time greater than* x. *If* Y_1 *is nonlattice, then* (9) *holds as well.*

PROOF. Iglehart's [13] Lemma 1 implies that (A3) holds when Y_1 is nonlattice; a similar argument shows that (A2) holds in the general case. So, the result follows directly from Theorem 1.1. □

REMARKS. (i) For the maximum wait among the first n customers, Proposition 3.1 strengthens the convergence in probability in Iglehart's [13] Corollary 2 to almost sure convergence.

(ii) The equation $\mathsf{E}[\exp(\gamma Y_1)] = 1$ typically has a solution in $(0, \infty)$ if the service-time distribution has an exponential tail.

The stable case $\mathsf{E}[Y_1] < 0$ is the most important one in queueing theory. The critical case $\mathsf{E}[Y_1] = 0$ arises in the study of cusum control charts, where T_x becomes the run length. In the unstable case $\mathsf{E}[Y_1] > 0$, T_x gives the replication length of an importance-sampling procedure that estimates rare-event probabilities associated with a stable queue by simulating an unstable queue (see, e.g., [4], pages 276–278). To give a complete picture, we briefly describe the behavior of T_x in these cases.

In the critical case, $\{W_n,\ n \geq 1\}$ is regenerative but null recurrent, so our results do not apply. The behavior of T_x is characterized by Robbins [20], who shows that $x^{-2}T_x$ has a proper limiting distribution, and $x^{-2}\mathsf{E}[T_x] \to 1$. In the unstable case, $\{W_n,\ n \geq 1\}$ is not even regenerative. However, writing $W_n = S_n - \min_{0 \leq j \leq n} S_j$ and noting that $(\min_{0 \leq j \leq n} S_j)/n \to 0$, a.s., we find that W is a *perturbed random walk*, in the sense of Gut [12]. A strong law, a central limit theorem and an iterated logarithm law now follow from Gut's Theorems 2.1, 2.3 and 2.5.

5.2. *Cell loss in an ATM switch* We consider, next, a buffered switch operating in discrete ("slotted") time. This model is a typical building block in the analysis of ATM networks (see [8] for background). The switch can transmit up to c cells in a time slot, with c a constant. The number of cells arriving in the nth slot is A_n. If we let Q_n denote the number of cells in the buffer at time slot n, then

$$Q_{n+1} = \max\{0, Q_n + A_{n+1} - c\}, \qquad n \geq 0,$$

with $Q_0 = 0$. If the buffer holds up to x cells and if T_x is the smallest n for which $Q_n > x$, then T_x is the time of the first cell loss due to buffer overflow.

To model arrival burstiness, the A_n are often taken to be increments of a Markov additive process. In particular, we suppose that there is a Markov chain $\{X_n,\ n \geq 0\}$ with finite state space $\{1, \dots, N\}$ such that

$$P(A_{n+1} \leq a, X_{n+1} = j | (X_k, A_k),\ k \leq n) = P_{X_n j}(a),$$

where $(P_{ij}(\infty))_{i,j=1}^N$ is the transition matrix of X and each $P_{ij}(\cdot)$ is a probability distribution function on the nonnegative integers. The process X represents the user environment. We take it to be irreducible and denote by π its stationary distribution. Let E_i denote the expectation operator associated with $X_0 = i$. If $\mathsf{E}_i[|A_1|] < \infty$ for all states $i = 1, \dots, N$ and if

$$\sum_i \pi_i \mathsf{E}_i[A_1 - c] < 0, \tag{20}$$

then $\{Q_n, n \geq 0\}$ is a positive-recurrent regenerative process. (See [5] for details in a closely related continuous-time model.)

For any real θ, define the nonnegative matrix $\Phi(\theta)$ by setting

$$\Phi_{ij}(\theta) = \mathsf{E}_i[\exp(\theta A_1);\ X_1 = j], \qquad i, j = 1, \dots, N.$$

Let $\phi(\theta)$ be the spectral radius of $\Phi(\theta)$. Then $\log \phi(\cdot)$ is convex. Suppose there is a solution in $(0, \infty)$, necessarily unique, to the equation $\log \phi(\gamma) - \gamma c = 0$. Then the stationary probability of the set $\{q\colon\ q > x\}$ satisfies (A1) and the upper bound in (A2); it satisfies the lower bound if, for example, the $\{A_n,\ n \geq 1\}$ are bounded (again see [5] for closely related results). The same is true (see [8]) for the probability that the buffer content reaches x in a busy cycle, where a busy cycle begins with (Q_0, X_0) having the stationary distribution of the Markov chain $\{(Q_n, X_n),\ n \geq 0\}$ conditioned on $Q_0 = 0$, and ends upon the first return of Q to the origin. In general, busy cycles are not regenerative cycles; but because X has a finite state space, there exists a state j^* such that the Markov chain $\{(Q_n, X_n),\ n \geq 0\}$ is regenerative with respect to visits to $(0, j^*)$. The bounds in [8] extend immediately to these cycles, so we have the following corollary.

COROLLARY 5.2. *If* (20) *holds, if* $\gamma > 0$ *satisfies* $\log \phi(\gamma) = \gamma c$ *and if the* $\{A_n,\ n \geq 1\}$ *are bounded, then* (A2) *holds and consequently* (6)–(8) *hold for* T_x*, the time to overflow.*

Because the A_n are integer-valued, we cannot expect (A3) to hold except possibly through integer subsequences.

Part of the interest in this type of model stems from the tractability of the key parameter γ. If the total number of arrivals A_n is the sum $A_n^1 + \cdots + A_n^k$ of arrivals from independent sources modulated by independent Markov chains $X^1, \ldots, X^k$, then γ is determined as the positive solution to the equation $\sum_i \log \phi_i(\gamma) - \gamma c = 0$, where ϕ_i is constructed as above from the ith chain. Thus, the individual sources may in effect be analyzed separately, greatly reducing the dimensionality of the problem.

APPENDIX

PROOF OF LEMMA 4.1. Clearly we only need to prove for $0 < a \leq 1$. We first consider the case $X_n^* = \max_{1 \leq j \leq n} Y_j$, where the Y_j are i.i.d. random variables for which

$$b_1 e^{-\gamma x} \leq P\{Y_1 > x\} \leq b_2 e^{-\gamma x},$$

for all sufficiently large x. By Kolmogorov's 0-1 law, we need only show that

$$P\{X^*_{e^{\gamma n}(\log n)^a} \leq n \quad \text{i.o.}\} > 0.$$

In particular, the conclusion follows if we can show that there is an m_0 such that for all $m \geq m_0$ there is an $m' > m$ for which

$$P\left\{\bigcup_{n=m}^{m'} \{X^*_{e^{\gamma n}(\log n)^a} \leq n\}\right\} \geq \left(1 - \frac{5}{16}\right)\frac{1}{16}. \tag{21}$$

Observe that for any events B_j,

$$P\left\{\bigcup_{n=m}^{m'} B_j\right\} = \sum_{j=m}^{m'} P\{B_j\} - \sum_{j=m}^{m'-1} P\left\{B_j \cap \bigcup_{t=j+1}^{m'} B_t\right\}.$$

Thus, (21) follows if we prove

$$\sum_{n=m}^{m'} P\{X^*_{e^{\gamma n}(\log n)^a} \leq n\} \geq \frac{1}{16}, \tag{22}$$

and for all $m_0 \leq m \leq n \leq m' - 1$,

$$P\left\{X^*_{e^{\gamma n}(\log n)^a} \leq n \text{ and } \bigcup_{t=n+1}^{m'} \{X^*_{e^{\gamma t}(\log t)^a} \leq t\}\right\} \leq \frac{5}{16} P\{X^*_{e^{\gamma n}(\log n)^a} \leq n\}. \tag{23}$$

First we establish (22). Pick $b_2' > b_2$ and make $b_2' \le 1$ if $a = 1$. For sufficiently small $x > 0$ we have $\log(1 - b_2 x) \ge -b_2' x$. So for sufficiently large n_0,

$$\begin{aligned}\sum_{n=n_0}^{\infty} P\{Y_1 \le n\}^{e^{\gamma n}(\log n)^a} &\ge \sum_{n=n_0}^{\infty} (1 - b_2 \exp(-\gamma n))^{e^{\gamma n}(\log n)^a}\\ &\ge \sum_{n=n_0}^{\infty} \exp\{-b_2' \exp(-\gamma n)\exp(\gamma n)(\log n)^a\}\\ &= \sum_{n=n_0}^{\infty} \exp\{-b_2'(\log n)^a\}\\ &= \infty.\end{aligned}$$

Since

$$P\{Y_1 \le t\}^{e^{\gamma t}(\log t)^a} \le (1 - b_1 \exp(-\gamma t))^{e^{\gamma t}(\log t)^a} \le \exp\{-b_1(\log t)^a\} \to 0,$$

as $t \to \infty$, we can find an $m_0 \ge 3$ such that for all $m \ge m_0$, there is an $m' > m$ for which

$$\frac{1}{16} \le \sum_{t=m}^{m'} P\{Y_1 \le t\}^{e^{\gamma t}(\log t)^a} \le \frac{1}{8} \tag{24}$$

and

$$\forall\, n \ge m, \qquad \exp\{-b_2(1 - \exp(-\gamma))(\log n)^a\} a_n \exp(b_2) \le \frac{1}{16}, \tag{25}$$

where $a_n \triangleq [a\gamma^{-1}\log\log n + \gamma^{-1}\log(2b_2/\log 2)] + 1$. In particular, (22) holds. Now we prove (23): $\forall\, m' - 1 \ge n \ge m$,

$$\begin{aligned}&P\left\{X^*_{e^{\gamma n}(\log n)^a} \le n \text{ and } \bigcup_{t=n+1}^{m'} \{X^*_{e^{\gamma t}(\log t)^a} \le t\}\right\}\\ &\quad\le \sum_{t=n+1}^{m'} P\{X^*_{e^{\gamma n}(\log n)^a} \le n,\ X^*_{e^{\gamma t}(\log t)^a} \le t\}\\ &\quad= \sum_{t=n+1}^{m'} \left(P\{X^*_{e^{\gamma n}(\log n)^a} \le n\} \prod_{j=[e^{\gamma n}(\log n)^a]+1}^{[e^{\gamma t}(\log t)^a]} P\{Y_j \le t\}\right)\\ &\quad= P\{X^*_{e^{\gamma n}(\log n)^a} \le n\}\left(\sum_{t=n+1}^{m'} \prod_{j=[e^{\gamma n}(\log n)^a]+1}^{[e^{\gamma t}(\log t)^a]} P\{Y_1 \le t\}\right),\end{aligned}$$

where the last two equalities follow from the i.i.d. property. Now

$$\begin{aligned}&\sum_{t=n+1}^{m'} \prod_{j=[e^{\gamma n}(\log n)^a]+1}^{[e^{\gamma t}(\log t)^a]} P\{Y_1 \le t\}\\ &\quad= \sum_{t=n+1}^{m'} P\{Y_1 \le t\}^{[e^{\gamma t}(\log t)^a]-[e^{\gamma n}(\log n)^a]}\end{aligned}$$

$$\leq \sum_{t=n+1}^{n+a_n} P\{Y_1 \leq t\}^{[e^{\gamma t}(\log t)^a]-[e^{\gamma n}(\log n)^a]}$$

$$+ \sum_{t=n+a_n}^{m'} P\{Y_1 \leq t\}^{[e^{\gamma t}(\log t)^a]-[e^{\gamma n}(\log n)^a]}$$

$$= \mathrm{I} + \mathrm{II} \quad \text{(say)},$$

where we use the notation $\sum_i^j \equiv 0$ for $i > j$. Then

$$\mathrm{I} \leq \sum_{t=n+1}^{n+a_n} (1 - b_2 e^{-\gamma t})^{[e^{\gamma t}(\log t)^a]-[e^{\gamma n}(\log n)^a]}$$

$$\leq \sum_{t=n+1}^{n+a_n} \exp\{-b_2 \exp(-\gamma t)(\exp(\gamma t)(\log t)^a - \exp(\gamma n)(\log n)^a - 1\}$$

$$\leq \sum_{t=n+1}^{n+a_n} \exp\{-b_2(\log t)^a + b_2 \exp(-\gamma)(\log n)^a\} \exp(b_2)$$

$$\leq \sum_{t=n+1}^{n+a_n} \exp\{-b_2(1 - \exp(-\gamma))(\log n)^a\} \exp(b_2)$$

$$= \exp\{-b_2(1 - \exp(-\gamma))(\log n)^a\} a_n \ \exp(b_2)$$

$$\leq \frac{1}{16} \quad \text{[by (25)]}.$$

Furthermore, $\forall\ t \geq n + a_n$,

$$P\{Y_1 \leq t\}^{-e^{\gamma n}(\log n)^a - 1} \leq \exp\{-b_2 \exp(-\gamma t)(-\exp(\gamma n)(\log n)^a - 1)\}$$
$$\leq \exp\{2b_2 \exp(-\gamma a_n)(\log n)^a\} \leq 2,$$

where the last inequality follows from the definition of a_n. Therefore, we have

$$\mathrm{II} \leq 2 \sum_{t=n+a_n}^{m'} P\{Y_1 \leq t\}^{e^{\gamma t}(\log t)^a}$$

$$\leq 2 \sum_{t=m}^{m'} P\{Y_1 \leq t\}^{e^{\gamma t}(\log t)^a}$$

$$\leq 2 \times \frac{1}{8} = \frac{1}{4} \quad \text{[by (24)]}.$$

The conclusion for i.i.d. sequences follows upon adding I and II.

Now consider nondelayed regenerative processes with i.i.d. cycles and $c < 1$. Choose $0 < \delta'$, such that $c + \delta' = 1$. Then

$$P\left\{\bigcup_{n=k}^{\infty}\{X^*_{a_n} \leq n\}\right\} \geq P\left\{\bigcup_{n=k}^{\infty}\{M^*_{a_n(c+\delta')} \leq n\}\right\} = P\left\{\bigcup_{n=k}^{\infty}\{M^*_{a_n} \leq n\}\right\}.$$

So, this special regenerative case follows from the i.i.d. case.

Finally, the one-dependent case follows by taking odd and even subsequences as in Lemma 2.1, and the delayed case follows in view of Lemma 2.3. □

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