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IMPULSE CONTROL OF PIECEWISE DETERMINISTIC MARKOV PROCESSES

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This paper concerns the optimal impulse control of piecewise deterministic Markov processes (PDPs). The PDP optimal (full) control problem with dynamic control plus impulse control is transformed to an equivalent dynamic control problem. The existence of an optimal full control and a generalized Bellman–Hamilton–Jacobi necessary and sufficient optimality condition for the PDP full control problem in terms of the value function for the new dynamic control problem are derived. It is shown that the value function of the original PDP optimal full control problem is Lipschitz continuous and satisfies a generalized quasivariational inequality with a boundary condition. A necessary and sufficient optimality condition is given in terms of the value function for the original full control problem.

1. Introduction. This paper deals with optimal impulse control of piecewise deterministic Markov processes (abbreviated PDPs). Such processes, first explicitly introduced by Davis (1984), are continuous time homogeneous Markov processes consisting of a mixture of deterministic motion and random jumps. The optimal control theory of PDPs has been developed by Vermes (1985), Davis (1986), Soner (1986), Dempster (1991), Dempster and Ye (1990, 1991, 1992) and Ye (1990). The optimal stopping problem for PDPs has been studied by Gugerli (1986) and Costa and Davis (1988). The optimal impulse control problem for PDPs has recently been studied by Gątarek (1990, 1991, 1992), Costa and Davis (1989) and Lenhart (1989). In their papers, the optimal *PDP impulse control problem* is formulated as follows. At a *stopping time* $\boldsymbol{\tau}$, the state is moved from x to $x + \xi \in E^0$ ($\subset \mathbb{R}^n$), the interior of the state space of the process, with *impulse* $\xi \in U \subset \mathbb{R}^n$ and a *cost* $c(x, \xi)$ is incurred when the impulse ξ is applied while the process is in state x. An *impulse control* (*strategy*) π is a sequence of stopping times and impulses,

$$\pi := \{\boldsymbol{\tau}_1, \boldsymbol{\xi}_1, \boldsymbol{\tau}_2, \boldsymbol{\xi}_2, \dots\},$$

where $\boldsymbol{\tau}_i \to \infty$ almost surely as $i \to \infty$. The controlled PDP $\mathbf{x}^\pi$ satisfies $\mathbf{x}^\pi(\boldsymbol{\tau}_i^+) = \mathbf{x}^\pi(\boldsymbol{\tau}_i^-) + \boldsymbol{\xi}_i$.

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The associated *expected cost* to be minimized is

$$J_x(\pi) := \mathbb{E}_x\left[\int_0^\infty e^{-\delta t} l_0(\mathbf{x}^\pi(t))\,dt + \sum_{i=1}^\infty e^{-\delta \tau_i} c(\mathbf{x}^\pi(\tau_i^-), \xi_i)\right].$$

To solve this optimal impulse control problem, Costa and Davis take the *value improvement* approach while the others take the (*quasi-*) *variational inequality* approach.

Since we will relate our approach to quasivariational inequalities, we now illustrate this latter approach. Under certain assumptions, Gątarek (1990, 1991, 1992) and Lenhart (1989) characterized the value function as the unique viscosity solution of the following quasivariatioal inequality:

$$(AV - \delta V + l_0) \wedge (MV - V) = 0 \quad \text{for } x \in E^0, \tag{1}$$

$$V(x) = \int_{E^0} V(y) Q_\partial(dy; x) \quad \text{for } x \in \partial E, \tag{2}$$

where E^0 and ∂E denote respectively the interior and the boundary of the state space E of $\mathbf{x}^\pi$,

$$AV(x) := \nabla V(x) f(x) + \lambda(x) \int_{E^0} (V(y) - V(x)) Q_0(dy; x)$$

and

$$MV(x) := \inf_{v \in U} \{c(x, v) + V(x + v)\}.$$

The approach taken to the optimal PDP impulse control problem in this paper is different from the approaches in the previous literature in two respects: the very general formulation of the problem and the characterization of optimality given.

By applying an impulse control action v in state x, instead of being moved to state $x + v \in E^0$, the state x will be moved to state $\mathbf{y} \in E^0$, which is a random variable with a given transition measure $Q_\delta(\cdot; x, v)$. Since a deterministic change to state $x + v$ can be considered to be a random variable with distribution $1_{\{x+v\}}(\cdot)$, the 1-atom measure concentrated on $x + v$, our problem formulation generalizes the formulation of the PDP optimal impulse control problem considered in the literature. It is similar to the concept of interventions introduced by Yushkevich (1983) and we will therefore use the words "intervention" and "impulse control" interchangeably. We will also call a stopping time an *intervention epoch* (or *moment*). For each such stochastic intervention, we will introduce a cost $l_\delta(x, v)$ of intervening with a control action v from the compact set U_δ when the process is in state x. The combination of Q_δ, l_δ and U_δ allows considerable flexibility of representation. For example, since the state space $E \in \mathbb{R}^n$ is assumed compact, we can model the common requirement of state dependent deterministic interventions $v \in U_\delta(x)$ which move the process to $y := x + v \in E^0$ at cost $c(x, y) \geq \varepsilon > 0$ by setting $Q_\delta(\cdot; x, v) := 1_{\{x+v\}}(\cdot) 1_{U_\delta(x)}(v)$, $l_\delta(x, v) := c(x, x + v)$ and

$U_\delta := \bigcup_{x \in E} U_\delta(x)$. Further, we may introduce intervention sets F_i to which the process may always be returned by defining $U_\delta(x) := \bigcup_i [F_i - x]$.

In Section 2, we formulate the PDP optimal full control problem. Unlike the usual formulation of impulse control problems with no dynamic control, the problem considered here includes not only impulse control, but also dynamic control. We have also generalized the usual impulse control problem by allowing interventions to occur even at jump epochs and at time $t = 0$.

Other than the more general formulation, the novelty of this paper is to transform the original PDP control problem with both dynamic control and impulse control to a new PDP control problem with only dynamic control. This is done in Section 3. The new problem is equivalent to the original problem in that they both have the same expected cost, the data for the new problem are obtained from the original problem and the control strategy (dynamic plus impulse control) of the original problem can be recovered from the corresponding control strategy (dynamic control only) of the new problem. This approach was first taken by Dempster and Solel (1987) and Solel (1986) to formulate stochastic scheduling as a PDP optimal control problem.

In Section 4, we set up the generalized Bellman–Hamilton–Jacobi (BHJ) equation in terms of the value function of the new dynamic control problem and provide a necessary and sufficient optimality condition for the PDP optimal full control problem.

In Section 5, we show that the BHJ equation in terms of the value function for the new dynamic control problem is the quasivariational inequality in terms of the value function for the original PDP full control problem by giving the relation between the value function for the original problem and that for the new problem.

2. The PDP optimal full control problem. First we give a precise definition of a PDP. Let $E \subset \mathbb{R}^n$ be a state space with nonempty interior E^0 and smooth boundary. We shall assume that there exists a point $x_0 \in E^0$ from which E is star-shaped and that lines $[x_0, z] \subset E$ intersect the boundary ∂E at a unique point.

More generally, E may be taken to be a *union* of sets in $\mathbb{R}^n$, or even manifolds, whose boundaries have suitable smoothness properties. In the practically important situation in which the state space is a (possibly countably infinite) union of disjoint bounded sets in $\mathbb{R}^n$, it is sufficient that each such component have the strongly star-shaped property assumed here for E. (We shall make use of this straightforward extension in Section 3 in order to keep the running cost of our transformed problem bounded.) Indeed, this property is required only to ensure that the boundary jump cost defined on that portion of ∂E in each component set can be extended in a Lipschitz continuous manner to the entire set so that the value function of the underlying deterministic control problem [see Dempster and Ye (1991)] defined on each component is Lipschitz continuous on it. In the case of manifolds, the strongly star-shaped property is required for the pre-image of the atlas of smooth local coordinate maps for each disjoint piece of the manifold.

A *piecewise deterministic process* (PDP) taking values in E is determined by its three *local characteristics*:

1. A Lipschitz continuous *vector field* $f\colon E \to \mathbb{R}^n$, which determines a *flow* (or *integral curve*) $\phi(t, x)$ in E such that for $t \geq 0$,

$$\frac{\partial}{\partial t}\phi(t,x) = f(\phi(t,x)), \qquad \phi(0,x) = x \quad \text{for all } x \in E^0.$$

With the convention $\inf\varnothing := \infty$, we define the *boundary hitting time*

(3) $$t_*(x) := \inf\{t > 0\colon \phi(t,x) \in \partial E\}.$$

2. A *jump rate* $\lambda\colon E^0 \to \mathbb{R}_+ := [0,\infty)$ such that for each $x \in E^0$ there is an $\varepsilon > 0$ such that

(4) $$\int_0^{\varepsilon} \lambda(\phi(s,x))\, ds < \infty,$$

that is, the process does not manifest *point (jump) explosions*. (In the sequel we will, for simplicity, assume λ bounded.)

3. A *transition measure* $Q\colon E^0 \cup \Gamma^* \to \mathbb{P}(E^0)$, where $\mathbb{P}(E^0)$ denotes the set of probability measures on E^0 with the relative weak* topology and

(5) $$\Gamma^* := \{z \in \partial E\colon \exists\, t > 0, \exists\, x \in E^0 \text{ s.t. } z = \phi(t,x)\}$$

denotes the *active boundary* which flows may reach. Note that on reaching Γ^* the process *necessarily* jumps back to E^0.

From these characteristics a right-continuous sample path x_t of the process $\{\mathbf{x}_t\colon t > 0\}$ starting at $x \in E^0$ may be constructed as follows. Define $x_t := \phi(t, x)$ for $0 \leq t < T_1$, where T_1 is the realization of the *first jump time* $\mathbf{T}_1$ with generalized negative exponential distribution determined by

$$P_x[\mathbf{T}_1 > t] = \begin{cases} \exp\left[-\int_0^t \lambda(\phi(s,x))\, ds\right], & t < t_*(x), \\ 0, & t \geq t_*(x). \end{cases}$$

Having realized $\mathbf{T}_1 = T_1$ [possibly at $T_1 = t_*(x)$], we have $x_{T_1^-} := \phi(T_1, x)$ and the *post-jump state* $\mathbf{x}_{T_1}$ has distribution given by

$$P_x\left[\mathbf{x}_{\mathbf{T}_1} \in A | \mathbf{T}_1 = T_1\right] = Q(A; \phi(T_1, x))$$

on the Borel sets A of E^0.

We may now restart the process at $\mathbf{x}_{T_1} = x_{T_1}$ according to the same recipe, and proceeding recursively we obtain a sequence of jump time realizations $T_1, T_2, \ldots$ between which x_t follows the integral curves of f. Considering this construction as generic yields the process $\{\mathbf{x}_t\colon t \geq 0, \mathbf{x}_0 = x\}$ and the sequence of its jump times $\mathbf{T}_1, \mathbf{T}_2, \ldots$. Our jump rate assumption (4) implies that $P_x[\mathbf{T}_{k+1} > \mathbf{T}_k] = 1$ and we now further *assume* that $P_x[\mathbf{T}_n \uparrow \infty] = 1$ for all $x \in E^0$.

As shown by Davis (1984), $\{\mathbf{x}_t\}$ is a temporally homogeneous strong Markov process with right continuous, left-limited sample paths.

The *dynamic control problem* arises when the local characteristics f, λ, Q of $\{\mathbf{x}_t\}$ depend on a *control action* v from a compact set U. We assume that $v \in U_0 \subset \mathbb{R}^m$ if $x \in E^0$ and $v \in U_\partial \subset \mathbb{R}^l$ if $x \in \partial E$. Therefore, we shall distinguish the transition measure $Q_0(dy; x, v)$, for $x \in E^0$, $v \in U_0$, describing jumps from interior points, from $Q_\partial(dy; x, v)$, for $x \in \partial E$, $v \in U_\partial$, describing jumps from boundary points.

Impulse control is required if one wishes to take actions which can cause an immediate change in the state of the process (i.e., a jump). We shall term the times that such a decision is taken *intervention epochs* and denote them by $\{\boldsymbol{\tau}_i\}$. At an intervention epoch, upon applying an impulse control $v \in U_\delta \subset \mathbb{R}^k$, the state x is moved to state $\mathbf{y}$, which is a random variable with intervention transition measure $Q_\delta(dy; x, v)$ and the process restarts at $\mathbf{y}$ as before.

We make the following *assumptions* throughout:

(A1) The *control sets* U_0, U_∂ and U_δ are compact.

(A2) The vector field f: $E \times U_0 \to \mathbb{R}^n$ is bounded, continuous and Lipschitz continuous in $x \in E$ uniformly in $v \in U_0$.

(A3) The jump rate λ: $E^0 \times U_0 \to \mathbb{R}_+$ is bounded, continuous and Lipschitz continuous in $x \in E^0$ uniformly in $v \in U_0$.

(A4) The map Q_0: $E^0 \times U_0 \to \mathbb{P}(E^0)$ is bounded, continuous relative to the weak* topology on $\mathbb{P}(E^0)$ and Lipschitz continuous in $x \in E^0$ [i.e., for all $\theta \in C(E^0)$ the map $(x, v) \mapsto \int_{E^0} \theta(y) Q_0(dy; x, v)$ is continuous and Lipschitz in x] uniformly in $v \in U_0$. The map Q_∂ is bounded, continuous and Lipschitz continuous in $x \in \partial E$ uniformly in $u \in U_\partial$. The map Q_δ: $E \times U_\delta \to \mathbb{P}(E^0)$ is bounded, continuous and Lipschitz continuous in $x \in E$ uniformly in $u \in U_\delta$.

An *interior control* is defined by $u_0(\tau_t, z_t)$ where u_0: $\mathbb{R}_+ \times E^0 \to U_0$ is a (separately) measurable function, τ_t is the *time elapsed since the last jump* (either a *process jump*, i.e., a jump determined by the local characteristics of the process or one caused by intervention) and z_t is the *post-jump state* (i.e., the state at the last jump time). Interior controls are *piecewise open loop* in that at each jump time T and post-jump state z, we choose a measurable function $t \mapsto u_{0t}(z) := u_0(t - T, z) \in U_0$.

A *boundary control* is a measurable feedback function u_∂: $\partial E \to U_\partial$.

A *dynamic control* is a pair (u_0, u_∂) involving an interior control and a boundary control.

An *impulse control* involves a sequence of stopping times $\{\boldsymbol{\tau}_k\}_{k=0}^{\infty}$ adapted to the filtration $\mathscr{F}_t := \sigma\{\mathbf{x}_s;\ s \le t\}$, where by convention $\tau_0 := 0$, and a sequence of impulse control actions $\{u_\delta(\mathbf{x}_{\tau_k})\}_{k=0}^{\infty}$, where u_δ: $E \to U_\delta$ is measurable [cf. Costa and Davis (1989)].

An *admissible* (*full*) *control* policy u involves both a dynamic control and an impulse control. We denote the set of admissible control policies by $\mathscr{C}$.

We also make the following *assumptions* in the sequel:

(A5) For any admissible (full) control u, $P_x^u[\lim_n \mathbf{T}_n = \infty] = 1$ for all $x \in E$.

(A6) The *running cost* l_0: $E^0 \times U_0 \to \mathbb{R}_+$ is nonnegative, bounded, continuous and Lipschitz continuous in $x \in E^0$ uniformly in $u \in U_0$. The *boundary (jump) cost* l_∂: $\partial E \times U_\partial \to \mathbb{R}_+$ is nonnegative, bounded, continuous and Lipschitz continuous in $x \in \partial E$ uniformly in $v \in U_\partial$. The *intervention cost* l_δ: $E \times U_\delta \to \mathbb{R}_+$ is bounded away from 0 below, bounded above, continuous and Lipschitz continuous in $x \in E$ uniformly in $v \in U_\delta$.

The PDP *optimal (full) control problem* is to find an admissible (full) control u such that the expected cost

$$J_x(u) = E_x^u\Bigg[\int_0^\infty \exp(-\delta t) l_0(\mathbf{x}_t, u_0(\boldsymbol{\tau}_t, \mathbf{z}_t))\,dt + \sum_{\mathbf{T}_k \neq \boldsymbol{\tau}_k} \exp(-\delta \mathbf{T}_k) l_\partial\big(\mathbf{x}_{\mathbf{T}_k^-}, u_\partial(\mathbf{x}_{\mathbf{T}_k^-})\big) I_{\{\mathbf{x}_{\mathbf{T}_k^-} \in \partial E\}} + \sum_{k=0}^{\infty} \exp(-\delta \boldsymbol{\tau}_k) l_\delta\big(\mathbf{x}_{\boldsymbol{\tau}_k^-}, u_\delta(\mathbf{x}_{\boldsymbol{\tau}_k^-})\big)\Bigg] \tag{6}$$

is minimized, where $\boldsymbol{\tau}_t$ represents the time elapsed since the last jump, $\mathbf{z}_t$ represents the post-jump state, $\delta > 0$ is the *discount factor*, $I_{\{\cdot\}}$ denotes the indicator function of the event $\{\cdot\}$, $\mathbf{T}_k$ is a process jump epoch and $\boldsymbol{\tau}_k$ is an intervention epoch.

Finally, we define the *value function* V: $E \to \mathbb{R}_+$ of the PDP optimal control problem by

$$V(x) := \inf_{u \in \mathscr{C}} J_x(u)$$

for all $x \in E$.

3. Reduction to a new problem with only dynamic control. First we shall reformulate the PDP optimal full control problem in terms of a different implementation of impulse controls.

Due to the (strong) Markov nature of PDPs and by the definition of stopping times [cf. Davis (1976)], for any stopping time $\boldsymbol{\tau}$ there exists a sequence of nonnegative random variables $\mathbf{r}_n$ such that:

1. $\mathbf{r}_n$ is $\mathscr{F}_{\mathbf{T}_n}$ measurable for $n = 0, 1, 2, \dots$.
2. $\boldsymbol{\tau} = \sum 1_{\{\mathbf{T}_n < \boldsymbol{\tau} < \mathbf{T}_{n+1}\}}(\mathbf{T}_n + \mathbf{r}_n) \wedge \mathbf{T}_{n+1}$, where $\mathbf{T}_1, \mathbf{T}_2 \dots$ is the sequence of jump times of the (controlled) PDP.

Consequently, by specifying after each jump (either a process jump or a jump caused by an intervention) a *time remaining to intervene* $t' > 0$ ($\mathbf{r}_n$), which diminishes at unit rate as process time evolves, any realized stopping time $\tau \in (T_n, T_{n+1}]$ is either the time when $t' = 0$, provided no process jump has occurred (this corresponds to the case where $T_n + r_n < T_{n+1}$), or the jump epoch, if a process jump has occurred with $t' > 0$ (this corresponds to the case where $T_n + r_n > T_{n+1}$).

Therefore, impulse control strategies can be implemented as follows. For each possible *pre-jump state* $x \in E^0$ of the process, a post-jump time remaining to intervene $t'(x) > 0$ (in the absence of a process jump) is specified which subsequently diminishes at unit rate with the evolution of (process) time. Provided no process jump has occurred previously, an impulse control action is applied whenever $t' = 0$, and at each jump epoch a decision is made as to whether or not to intervene. Having implemented interventions in the way we have just described, we can reformulate the PDP optimal full control problem as follows.

The PDP optimal full control problem is to find an admissible full control u which involves both a dynamic control and an *impulse control* (*policy*) (u_δ, t'), which specifies for each (pre-jump) state $x \in E$ a (post-jump) time remaining to intervene $t'(x) > 0$ (i.e., a measurable function t': $E \to (0, \infty]$) and an *intervention control action* $u_\delta(x)$ [i.e., a measurable *intervention control* function u_δ: $E \to U_\delta$) which influences the (given) *intervention transition measure* Q_δ: $E \times U_\delta \to \mathbb{P}(E^0)$] so as to minimize the expected cost (6), where $\tau_0, \tau_1, \tau_2, \dots$ now denotes the sequence of stopping times corresponding to the impulse control (u_δ, t') (and $\tau_0 := 0$ by convention).

If we now compare a boundary control with an impulse control, we find that they both move a process instantaneously to a new state chosen according to the transition measures Q_∂ and Q_δ, respectively. The difference is only in the timing. A boundary control action is applied whenever the process hits the boundary of the state space, while an impulse control action is applied at intervention epochs. To reduce impulse controls to boundary controls, it is sufficient to embed the original process in a new process in such a way that at intervention epochs of the original process the new process will hit the boundary of the new state space.

It is obvious that if we let t' be one of the coordinates of the state of the new process, the new process will hit the boundary of the new space when $t' = 0$ since 0 is an end point of the interval $(0, \infty)$.

However, in the case when the process jumps while $t' > 0$, that is, an ordinary interior jump, a natural question to ask is how to embed the original process so that the new process will hit a piece of its state space boundary at this time. The idea here is to use a *fictitious time* construction, following Yushkevich (1983, 1987) and Dempster and Solel (1987). We consider an ordinary interior jump to be an interior jump of the new process. In this event, the new process jumps to a state where all the coordinates are kept constant except for t', which is set equal to -5, an interior point of the fictitious time interval $(-6, -4)$. Fictitious time then runs backward until it hits the boundary at $t' = -6$, at which time we decide whether or not to intervene.

To be consistent, we let fictitious time run (backward) after both jump epochs *and* interventions. Thus we can distinguish two kinds of boundary states for the new process: boundary states at which we can *decide* whether or not to intervene and ones at which we *always* intervene. Thus we define the state space for fictitious time as a union of two disjoint time intervals $(-6, -4) \cup (-3, -1)$. In the case when $t' = 0$, the new process will jump to a state where all the coordinates are kept constant except t', which is set equal to -2, an interior point of the fictitious time interval $(-3, -1)$. When the new process hits the boundary $t' = -3$, an impulse control action is taken.

Due to the use of fictitious time, the *new* process time increases one *unit* for each intervention and each process jump. To calculate the *original* process time, we must therefore keep track both of the number of original process jumps and the number of interventions. We shall also find it convenient to use these state variables to index the countable partition of the state space of the new process referred to in Section 2.

We must also keep track of both the post-jump state and the time elapsed since the last jump for the original process because interior control depends upon them.

We now give the precise formulation. Define from the given controlled process $\mathbf{x}_t$ a *new controlled process* $\hat{\mathbf{x}}_s$ with *state*

$$\hat{x} := (x, z, \tau, t', m, n), \tag{7}$$

where x is the *state* of the *original* process, z is the *post-jump* state of the *original* process, τ is the *time elapsed since the last jump* of the *original* process, t' is the *time remaining to intervene* or *fictitious time* and m_s and n_s are, respectively, the number of interventions and the number of original process jumps up to the present *process time s*.

If a strategy under consideration does not specify a next intervention decision time, we need to take t' to be ∞, but we will instead take $t' := -8$. Therefore, the new process $\hat{\mathbf{x}}_s$ evolves in a new *state space* defined as follows:

$$\begin{aligned}\hat{E}^0 = &\bigcup_{m,n=0}^{\infty} (E^0 \times E^0 \times T \times T' \times \{m\} \times \{n\})\\ &\cup \bigcup_{m,n=0}^{\infty} (\partial E \times E^0 \times T \times [(-6,-4) \cup (-3,-1)] \times \{m\} \times \{n\}),\end{aligned}$$

where

$$\begin{aligned}T &= (0,\infty),\\ T' &= (-\infty,-7) \cup (-6,-4) \cup (-3,-1) \cup (0,\infty).\end{aligned}$$

The *active boundary* of the new state space is

$$\begin{aligned}\hat{\Gamma}^* = &\bigcup_{m,n=0}^{\infty} (E^0 \times E^0 \times T \times [\{-6\} \cup \{-3\} \cup \{0\}] \times \{m\} \times \{n\})\\ &\cup \bigcup_{m,n=0}^{\infty} (\Gamma^* \times E^0 \times T \times [\{-6\} \cup \{-3\} \cup \{0\}] \times \{m\} \times \{n\}),\end{aligned}$$

where Γ^* is the active boundary of the original state space E as defined by (5).

Denote by $\{\mathbf{x}_t \in E^0: t \in T\}$ the *original* process and by $\{\hat{\mathbf{x}}_s \in \hat{E}^0: s \in S\}$ the *new* process, where the original time set T is called the *real time set* and S is termed the (new) *process time set*. Then real time t in the new process is represented implicitly in terms of (new) process time s as $t(s) := s - (m_s + n_s + 1)$ due to the fact that the new process time increases one unit for each intervention and each process jump and the new process must be run for one unit of fictitious time to allow the possibility of intervention in the original process at real time $t = 0$.

Since the time remaining to intervene and fictitious time t' runs backward at unit speed and all coordinates but fictitious time are kept constant while fictitious time is running, the dynamics of the new process are as follows:

In $E^0 \times E^0 \times T \times [(-\infty, -7) \cup (0, \infty)] \times \{m\} \times \{n\}$,

$$\dot{x} = f(x_s, u_0(\tau_s, z_s)), \qquad \dot{z}_s = 0, \qquad \dot{\tau}_s = 1, \qquad \dot{t}'_s = -1.$$

In $[E^0 \cup \partial E] \times E^0 \times T \times [(-6, -4) \cup (-3, -1)] \times \{m\} \times \{n\}$

$$\dot{x}_s = 0, \qquad \dot{z}_s = 0, \qquad \dot{\tau}_s = 0, \qquad \dot{t}'_s = -1.$$

While a trajectory of the original controlled process $\mathbf{x}_t$ starting at x_0 proceeds with time t, the corresponding trajectory of the new controlled process $\hat{\mathbf{x}}_s$ taking values in the state space $\hat{E}$ with dynamics defined as above proceeds with time s in the following way.

The new process $\hat{\mathbf{x}}_s$ starts at the initial point $(x_0, x_0, 0, -2, 0, 0)$ at time $s := 0$ and goes in fictitious time to $(x_0, x_0, 0, -3, 0, 0)$, which is a boundary point of $E^0 \times E^0 \times T \times (-3, -1) \times \{0\} \times \{0\}$ at $s = 1$.

If an impulse control action u_{δ_0} is applied, the original process jumps to x_0^+ chosen randomly by the transition measure $Q_\delta(\cdot; x_0, u_{\delta_0})$ and t'_0 is set. Otherwise, $Q_\delta(\cdot; x_0, u_{\delta_0}) := 1_{\{x_0\}}$ and t'_0 is set. This formulation thus allows impulse control action to be taken even at time $t = 0$. The new process jumps to either $(x_0^+, x_0^+, 0, t'_0, 1, 0)$ or $(x_0, x_0, 0, t'_0, 1, 0)$, which are interior points of $E^0 \times E^0 \times T \times T' \times \{1\} \times \{0\}$ and in the first case an intervention cost of $l_\delta(x_0, u_{\delta_0}(x_0)) > 0$ is incurred, while otherwise 0 cost is incurred. After the first jump the new process continues its motion described by the integral curves until one of two possible cases occurs at real time t or process time $s = t + 1$:

1. $t' = 0$.
2. $t' > 0$ or $t' < -8$ and t is a jump epoch (either an interior jump or a boundary jump).

In case 1, the new process hits the boundary. It jumps to $(x_{t^-}, x_0^+, \tau_{t^-}, -2, 1, 0) \in E^0 \times E^0 \times T \times (-3, -1) \times \{1\} \times \{0\}$ or $\partial E \times E^0 \times T \times (-3, -1) \times \{1\} \times \{0\}$ depending on whether $x_{t^-} \in E^0$ or $x_{t^-} \in \partial E$.

In case 2, if $x_{t^-} \in E^0$, the new process has an interior jump to $(x_{t^-}, x_0^+, \tau_{t^-}, -5, 1, 0) \in E^0 \times E^0 \times T \times (-6, -4) \times \{1\} \times \{0\}$. If $x_{t^-} \in \partial E$, the new process hits the boundary. It jumps to $(x_{t^-}, x_0^+, \tau_{t^-}, -5, 1, 0)$ which is an interior point of $\partial E \times E^0 \times T \times (-6, -4) \times \{1\} \times \{0\}$.

In both cases, the new process will continue along the appropriate integral curve until $t' = -3$ in case 1 or $t' = -6$ in case 2 at which point it will jump using the given control strategy to a new state in which $t' \in (0, \infty)$ or $t' := -8 \in (-\infty, -7)$.

In case 1, the original process jumps under an impulse control action u_δ from x_{t^-} to x_t according to the transition measure $Q_\delta(\cdot; x_{t^-}, u_\delta)$. In case 2, the original process jumps optimally under either an impulse control (as in case 2) or under an ordinary control, according to the appropriate transition measure $Q_0(\cdot; x_{t^-}, u_0(\tau_{t^-}, x_0^+))$ or $Q_\partial(\cdot; x_{t^-}, u_\partial(x_{t^-}))$, so as to minimize the relevant remaining expected total cost. In the first instance, a cost $e^{-\delta t} l_\delta(x_{t^-}, u_\delta(x_{t^+}))$ is incurred, while in the second instance, a cost 0 or $e^{-\delta t} l_\partial(x_{t^-}, u_\partial(x_{t^-}))$ is incurred according as the process enjoyed an interior or a boundary jump.

Note that in all cases, whether or not an intervention is dictated by the control policy, the state variable of the original process jumps to a point in E^0. In case 1, the process restarts again from the interior point $(x_t, x_t, 0, t', 2, 0)$. In case 2, the process restarts again from the interior point $(x_t, x_t, 0, t', 2, 0)$ or $(x_t, x_t, 0, t', 1, 1)$ depending on which action (impulse or not) takes place.

It remains to define the control sets $\hat{U}_0$ and $\hat{U}_\partial$, admissible controls $\hat{u} = (\hat{u}_0, \hat{u}_\partial)$, the jump rate $\hat{\lambda}$ and the transition measures $\hat{Q}_0, \hat{Q}_\partial$ so as to ensure that the new controlled process proceeds in the way described above. The new control sets $\hat{U}_0$ and $\hat{U}_\partial$ to be defined below will also be compact.

Since the new process undergoes an interior jump only when it is an interior jump epoch of the original process and $t' > 0$, the *interior control set* of the new process can be taken to be that of the original process, that is, $\hat{U}_0 := U_0$. The new *jump rate* is

$$\hat{\lambda}(\hat{x}, u) := \begin{cases} \lambda(x, u), & \text{if } \hat{x} \in E^0 \times E^0 \times T \\ & \quad \times [(0, \infty) \cup (-\infty, -7)] \times \{m\} \times \{n\}, \\ 0, & \text{otherwise.} \end{cases}$$

When the new process has an interior jump, we want it to jump to the state with all coordinates kept the same except that fictitious time is set to -5. Therefore, the new *interior jump transition measure* is given by

$$\hat{Q}_0(\cdot; \hat{x}) := \begin{cases} 1_{\{x, z, \tau, -5, m, n\}}(\cdot), & \text{if } \hat{x} \in E^0 \times E^0 \times T \\ & \quad \times [(-\infty, -7) \cup (0, \infty)] \times \{m\} \times \{n\}, \\ 1_{\{\hat{x}\}}(\cdot), & \text{otherwise.} \end{cases}$$

The new *boundary control set* is defined as

$$\hat{U}_\partial := (U_0 \cup U_\partial \cup U_\delta) \times U_{t'},$$

where $U_{t'} := [0,\infty) \cup \{-8\}$ is a one point compactification of $[0,\infty)$. It is thus a compact separable metric space.

An admissible *boundary control* is a feedback function $\hat{u}_\partial \colon \partial\hat{E} \to \hat{U}_\partial$ such that

$$\hat{u}_\partial(\hat{x}) \in \begin{cases} U_\delta \times U_{t'}, & \text{if } \hat{x} \in E^0 \times E^0 \times T \times \{-3\} \times \{m\} \times \{n\}, \\ [U_\delta \cup U_\partial] \times U_{t'}, & \text{if } \hat{x} \in \partial E \times E^0 \times T \times \{-6\} \times \{m\} \times \{n\}, \\ [U_\delta \cup \{u_0(\tau, z)\}] \times U_{t'}, & \text{if } \hat{x} \in E^0 \times E^0 \times T \times \{-6\} \times \{m\} \times \{n\}. \end{cases}$$

The *boundary jump transition measure* $\hat{Q}_\partial$ is defined as follows:

$$\hat{Q}_\partial(\cdot\,; \hat{x}, \hat{u}_\partial)$$
$$:= \begin{cases} 1_{\{x,z,\tau,-2,m,n\}}(\cdot), & \text{if } \hat{x} \in E \times E^0 \times T \times \{0\} \times \{m\} \times \{n\}, \\ 1_{\{x,z,\tau,-5,m,n\}}(\cdot), & \text{if } \hat{x} \in \partial E \times E^0 \times T \times [(0,\infty) \cup (-\infty,-7)] \times \{m\} \times \{n\}, \\ 1_{\{y\}}(dz) Q_\delta(dy; x, u_\delta) 1_{\{0,t',m+1,n\}}(\cdot), & \text{if } \hat{x} \in E \times E^0 \times T \times \{-3\} \times \{m\} \times \{n\} \text{ and } \hat{u}_\partial := (u_\delta, t'), \\ 1_{\{y\}}(dz) Q_0(dy; x, u_0(\tau, z)) 1_{\{0,t',m,n+1\}}(\cdot), & \text{if } \hat{x} \in E^0 \times E^0 \times T \times \{-6\} \times \{m\} \times \{n\} \text{ and } \hat{u}_\partial := (u_0(\tau,z), t'), \\ 1_{\{y\}}(dz) Q_\delta(dy; x, u_\delta) 1_{\{0,t',m+1,n\}}(\cdot), & \text{if } \hat{x} \in E^0 \times E^0 \times T \times \{-6\} \times \{m\} \times \{n\} \text{ and } \hat{u}_\partial := (u_\delta, t'), \\ 1_{\{y\}}(dz) Q_\partial(dy; x, u_\partial) 1_{\{0,t',m,n+1\}}(\cdot), & \text{if } \hat{x} \in \partial E \times E^0 \times T \times \{-6\} \times \{m\} \times \{n\} \text{ and } \hat{u}_\partial := (u_\partial, t'), \\ 1_{\{y\}}(dz) Q_\delta(dy; x, u_\delta) 1_{\{0,t',m+1,n\}}(\cdot), & \text{if } \hat{x} \in \partial E \times E^0 \times T \times \{-6\} \times \{m\} \times \{n\} \text{ and } \hat{u}_\partial := (u_\delta, t'). \end{cases}$$

We have now finished the construction embedding the original process in the new process.

Next we identify cost functions for the new problem so that it has the same expected total cost as the original problem.

Arrange the jump epochs $\mathbf{T}_i$ and the intervention epochs $\boldsymbol{\tau}_i$ of the original process in increasing order (on each of its sample paths) and denote the resulting sequence of (combined) epochs by $\{\mathbf{T}_i^*\}$. Since an intervention (perhaps trivial) must occur at $t = 0$, this sequence of epochs will begin with $T_0^* = 0$.

From the construction of the new process, we can see that the new process jumps twice as often as the original process. Due to the unit increase in new process time for each original process jump, the jth jump epoch $\hat{T}_j$ of the new process can be obtained from the (combined) epochs of the original process as follows:

$$\hat{T}_j := \begin{cases} T_i^* + i, & \text{if } j = 2i \text{ and } i = 0,1,2,\dots, \\ T_i^* + i + 1, & \text{if } j = 2i+1 \text{ and } i = 0,1,2,\dots. \end{cases} \tag{8}$$

In other words, the $2i$th jump epoch of the new process corresponds to a *pre*-jump or intervention epoch of the original process and the $(2i+1)$st jump epoch of the new process corresponds to a *post*-jump epoch of the original process, $i = 0,1,2,\dots$. These epochs are of course only *infinitesimally* different in real (original process) time.

Now rewrite the expected total cost of the original problem as follows:

$$J_x(u) = E_x\left[\sum_{i=0}^{\infty}\int_{\mathbf{T}_i^*}^{\mathbf{T}_{i+1}^*} e^{-\delta t} l_0(\mathbf{x}_t, u_0(\boldsymbol{\tau}_t, \mathbf{z}_t))\,dt + \sum_{i=0}^{\infty} e^{-\delta T_i^*}\tilde{l}(\mathbf{x}_{T_i^*-}, \tilde{u}(\mathbf{x}_{T_i^*-}))\right], \tag{9}$$

where

$$\tilde{l}(\mathbf{x}_{T_i^*-}, \tilde{u}(\mathbf{x}_{T_i^*-})) := \begin{cases} l_\partial(\mathbf{x}_{T_i^-}, u_\partial(\mathbf{x}_{T_i^-}))I_{\{\mathbf{x}_{T_i^-} \in \partial E\}}, & \text{if } T_i^* = T_j \text{ for some } j \in \mathbb{N}, \\ l_\delta(\mathbf{x}_{\tau_i^-}, u_\delta(x_{\tau_i^-})), & \text{if } T_i^* = \tau_j \text{ for some } j \in \mathbb{N}. \end{cases}$$

For $T_i^* < t < T_{i+1}^*$, setting $t := s - (i+1)$, we have

$$\begin{aligned} &\int_{T_i^*}^{T_{i+1}^*} \exp(-\delta t) l_0(x_t, u_0(\tau_t, z_t))\,dt \\ &= \int_{T_i^* + (i+1)}^{T_{i+1}^* + (i+1)} \exp(-\delta(s-(i+1)) l_0(x_{s-(i+1)}), u_0(\tau_{s-(i+1)}, z_{s-(i+1)}))\,ds \\ &= \int_{\hat{T}_{2i+1}}^{\hat{T}_{2i+2}} \exp(-\delta s)\exp(\delta(i+1)) l_0(x_{s-(i+1)}, u_0(\tau_{s-(i+1)}, z_{s-(i+1)}))\,ds. \end{aligned}$$

Since T_i^* is the $(i+1)$st jump or intervention epoch of the original fully controlled process (due to the fact that a, perhaps trivial, intervention necessarily occurs at $t = 0$), the original PDP $\mathbf{x}_t$ has $i+1$ combined epochs before $t \in [T_i^*, T_{i+1}^*]$. Therefore, if we define the new *running cost* function $\hat{l}_0$ for $v \in U_0$ as

$$\hat{l}_0(\hat{x}, v) := \begin{cases} e^{\delta(m+n)} l_0(x, v), & \text{if } \hat{x} \in E^0 \times E^0 \times T \times [(-\infty, -7) \cup (0,\infty)] \\ & \qquad \times \{m\} \times \{n\}, \\ 0, & \text{otherwise}, \end{cases}$$

we have

$$\int_{\hat{T}_{2i}}^{\hat{T}_{2i+1}} e^{-\delta s}\hat{l}_0(\hat{x}_s, \hat{u}_0(\hat{\tau}_s, \hat{z}_s))\,ds = 0$$

and

$$\int_{\hat{T}_{2i+1}}^{\hat{T}_{2i+2}} e^{-\delta s}\hat{l}_0(\hat{x}_x, \hat{u}_0(\hat{\tau}_s, \hat{z}_s))\, ds = \int_{T_i^*}^{T_{i+1}^*} e^{-\delta t} l_0(x_t, u_0(\tau_t, z_t))\, dt.$$

Consequently, we have

$$\begin{aligned}\sum_{i=0}^{\infty} & \int_{T_i^*}^{T_{i+1}^*} e^{-\delta t} l_0(x_t, u_0(\tau_t, z_t))\, dt \\ &= \sum_{i=0}^{\infty} \int_{\hat{T}_{2i}}^{\hat{T}_{2i+1}} e^{-\delta s}\hat{l}_0(\hat{x}_s, \hat{u}_0(\hat{\tau}_s, \hat{z}_s))\, ds \\ &\quad + \sum_{i=0}^{\infty} \int_{\hat{T}_{2i+1}}^{\hat{T}_{2i+2}} e^{-\delta s}\hat{l}_0(\hat{x}_s, \hat{u}_0(\hat{\tau}_s, \hat{z}_s))\, ds \\ &= \int_0^{\infty} e^{-\delta s}\hat{l}_0(\hat{x}_s, \hat{u}_0(\hat{\tau}_s, \hat{z}_s))\, ds. \end{aligned} \tag{10}$$

Moreover, $\hat{l}_0$ as defined remains bounded by virtue of (A6) on each of the countable number of components of the state space of the new process indexed by (m, n). Similarly, from (8), we have for (9),

$$\begin{aligned}\exp(-\delta T_i^*)&\tilde{l}(x_{T_i^*-}, \tilde{u}(x_{T_i^*-})) \\ &= \exp(-\delta \hat{T}_{2i+1})\exp(\delta(i+1))\tilde{l}(x_{T_i^*-}, \tilde{u}(x_{T_i^*-}))\end{aligned}$$

and $\mathbf{x}_t$ has i combined epochs before T_i^{*-}. Therefore, if we define the new *boundary cost* function $\hat{l}_\partial$ as

$$\hat{l}_\partial(\hat{x}, \hat{u}_\partial) := \begin{cases} \exp(\delta(m+n+1)) l_\partial(x, u_\partial), & \text{if } \hat{x} \in \partial E \times E^0 \times T \times \{-6\} \\ & \quad \times \{m\} \times \{n\} \text{ and } \hat{u}_\partial := (u_\partial, t'), \\ \exp(\delta(m+n+1)) l_\delta(x, u_\delta), & \text{if } \hat{x} \in \partial E \times E^0 \times T \\ & \quad \times [\{-3\} \cup \{-6\}] \times \{m\} \times \{n\} \\ & \quad \text{and } \hat{u}_\partial := (u_\delta, t'), \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$\exp(-\delta \hat{T}_{2i})\hat{l}_\partial(\hat{x}_{\hat{T}_{2i}^-}, \hat{u}_\partial(\hat{x}_{\hat{T}_{2i}^-}))1_{(\hat{x}_{\hat{T}^-{}_{2i}} \in \partial\hat{E})} = 0$$

and

$$\begin{aligned}\exp(-\delta \hat{T}_{2i+1})&\hat{l}_\partial(\hat{x}_{\hat{T}_{2i+1}^-}, \hat{u}_\partial(\hat{x}_{\hat{T}_{2i+1}^-}))1_{(\hat{x}_{\hat{T}_{2i+1}^-} \in \partial\hat{E})} \\ &= \exp(-\delta T_i^*)\tilde{l}(x_{T_i^*-}, \tilde{u}(x_{T_i^*-})).\end{aligned}$$

Consequently,

$$\sum_{i=0}^{\infty} \exp(-\delta T_i^*)\tilde{l}\left(x_{T_i^*-}, \tilde{u}(x_{T_i^*-})\right) \tag{11}$$
$$= \sum_{j=1}^{\infty} \exp\left(-\delta \hat{T}_j\right)\hat{l}_\partial\left(\hat{x}_{\hat{T}_j^-}, \hat{u}_\partial\left(\hat{x}_{\hat{T}_j^-}\right)\right)1_{(\hat{x}_{\hat{T}j^-} \in \partial\hat{E})}.$$

Let $\hat{C}$ denote the set of all admissible dynamic control policies for the new dynamic control problem. Then the new PDP optimal control problem is to find an optimal control among all admissible dynamic controls $\hat{u} := (\hat{u}_0, \hat{u}_\partial) \in \hat{C}$ such that the *expected total cost*

$$\hat{J}_{\hat{x}}(\hat{u}) := E_{\hat{x}}\left[\int_0^{\infty} \exp(-\delta s)\hat{l}_0(\hat{\mathbf{x}}_s, \hat{u}_0(\hat{\boldsymbol{\tau}}_s, \hat{\mathbf{z}}_s))\, ds \right.$$
$$\left. + \sum_{i=1}^{\infty} \exp\left(-\delta \hat{\mathbf{T}}_i\right)\hat{l}_\partial\left(\hat{\mathbf{x}}_{\hat{T}_i^-}, \hat{u}_\partial\left(\hat{\mathbf{x}}_{\hat{T}_i^-}\right)\right)I_{(\hat{\mathbf{x}}_{\hat{T}_i^-} \in \partial\hat{E})}\right]$$

is minimized. Here $\hat{x}_0 := (x_0, x_0, 0, -2, 0, 0)$.

We conclude from the equalities (9), (10) and (11) that the expected cost of the new problem is the same as that of the original problem.

We end this section with an example illustrating the construction of the new boundary controlled process from an original (only) *impulse controlled* process.

EXAMPLE. A repair/maintenance model [cf. Costa and Davis (1989)]. Suppose x_t represents the cumulative *degree of damage* to a machine at time t. This increases at rate $f(x)$ when the degree of damage is x, and also discontinuously due to independent random shocks which occur at Poisson times and have some known distribution function G.

The *intervention strategy* is to replace the machine (i.e., set $\mathbf{x}_t$ to 0) when the cumulative damage first exceeds some fixed level $x_{\max}$. (Of course, this could happen either at a shock time or between shocks; see Figure 1). There may or may not be some delay in machine replacement.

Since there is no dynamic control in this case, we can take the new state space to be $\hat{E} := \bigcup_{m,n=0}^{\infty}(E^0 \times T' \times \{m\} \times \{n\})$, where T' is defined as above.

While the trajectory of the original impulse controlled process $\mathbf{x}_t$ starting at x_0 proceeds with (real) time t, the corresponding trajectory of the new process $\hat{\mathbf{x}}_s$ taking values in the new state space $\hat{E}$ proceeds with (process) time s in the following way.

The new process starts at the initial state $(x_0, -2, 0, 0)$ at time $s := 0$ and goes in fictitious time to $(x_0, -3, 0, 0)$ which is a boundary point of $E^0 \times (-3, -1) \times \{0\} \times \{0\}$ at $s = 1$.

Set the time remaining to intervene (i.e., the time remaining to replace the machine provided no random shocks have occurred) $t' := t'_0$, the time at which, starting from the initial damage level x_0, the cumulative damage will first exceed $x_{\max}$ at $t = t'_0$, that is, such that $\int_0^{t'_0} f(x)\, dx + x_0 = x_{\max}$, providing no random shocks occur before time t'_0 and let the impulse control action

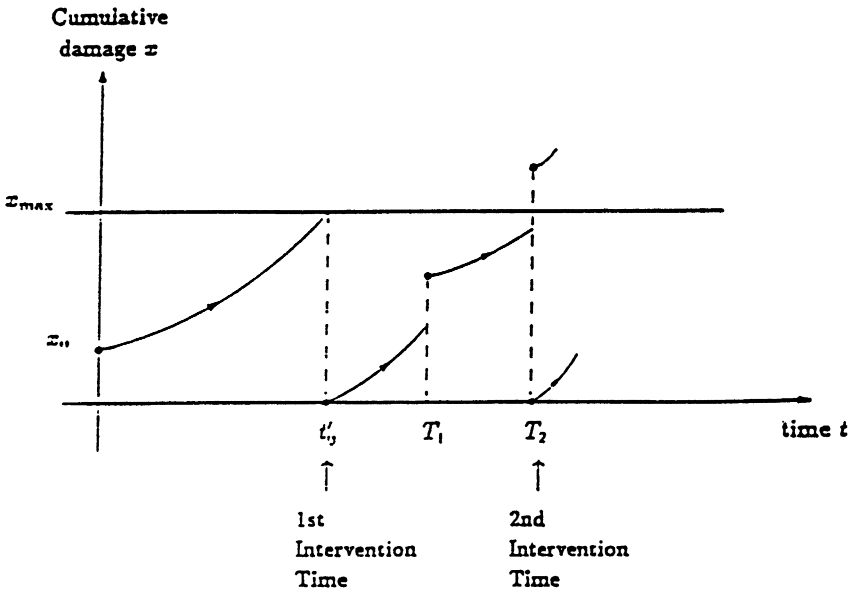

FIG. 1.

be equal to zero. This is equivalent to taking a new boundary control $\hat{u}_\partial := (u_\delta, t') := (0, t'_0)$. Under this boundary control action, the new process jumps to $(x_0, t'_0, 1, 0) \in E^0 \times T' \times \{1\} \times \{0\}$ and, if (as shown in the figure) there are no shocks before $t = t'_0$, continues its motion until it reaches the state $(x_{\max}, 0, 1, 0)$ which is a boundary point of $E^0 \times T' \times \{1\} \times \{0\}$. At this boundary point, the new process has an uncontrolled boundary jump to $(x_{\max}, -2, 1, 0) \in E^0 \times (-3, -1) \times \{1\} \times \{0\}$ and goes in fictitious time to $(x_{\max}, -3, 1, 0)$ which is a boundary point of $E^0 \times (-3, -1) \times \{1\} \times \{0\}$. Applying the boundary control action $\hat{u}_\partial := (-x_{\max}, t'_1)$, where t'_1 satisfies $x_{\max} = \int_0^{t'_1} f(x)\,dx$, the new process $\hat{x}_t$ jumps to $(0, t'_1, 2, 0) \in E^0 \times T' \times \{2\} \times \{0\}$ and continues its motion until it reaches the state $(x_{T_1^-}, t'_1 - (T_1 - t'_0), 2, 0)$ at the first jump time $t = T_1$ or $s = T_1 + 2$. The process $\hat{x}_s$ then takes an interior jump to $(x_{T_1^-}, -5, 2, 0) \in E^0 \times (-6, -4) \times \{2\} \times \{0\}$ and runs in fictitious time to $(x_{T_1^-}, -6, 2, 0)$, which is a boundary point of $E^0 \times (-6, -4) \times \{2\} \times \{0\}$. Applying the boundary control action $\hat{u}_\partial := t'_2$ (i.e., do not intervene), where $x_{\max} = \int_0^{t'_2} f(x)\,dx + x_{T_1}$, the new process jumps to $(x_{T_1}, t'_2, 2, 1) \in E^0 \times T' \times \{2\} \times \{1\}$, where x_{T_1} is determined by the distribution function G. The new process $\hat{x}_s$ again continues its motion until it reaches the state $(x_{T_2^-}, t'_2 - (T_2 - T_1), 2, 1)$ at the second jump time $t = T_2$ or $s = T_2 + 3$. It then takes an interior jump to $(x_{T_2^-}, -5, 2, 1)$ and proceeds in fictitious time to $(x_{T_2^-}, -6, 2, 1)$, a boundary point. Applying the boundary control action $\hat{u}_\partial := (-x_{T_2^-}, t'_3)$ (i.e., intervene to replace the machine), where $t'_3 := t'_1$ [i.e., $x_{\max} = \int_0^{t'_3} f(x)\,dx$], the new process jumps to $(0, t'_3, 1)$ and restarts again from this interior point.

This example shows that three possible cases can occur (see Figure 1):

1. At time $t_0', t' = 0$ and we intervene to replace the machine.
2. At time T_1, a jump epoch, a decision is made not to intervene.
3. At time T_2, a (second) jump epoch, a decision is made to intervene and replace the machine.

A machine replacement *delay*—possibly independent random with a common known distribution function—together with a *penalty* for cumulative wear $\mathbf{x}_t$ exceeding $x_{\max}$ is an extension easily incorporated within the framework of our theory, as is the optimal setting of $x_{\max}$ itself.

4. Necessary and sufficient optimality conditions for the PDP full optimal control problem. The purpose of this section is to give generalized BHJ necessary and sufficient optimality conditions for the PDP full control problem. To this end, make the following further *assumptions*.

(A7) There exists $\alpha > 0$ such that for all $x \in \partial E$ and all $v \in U_0$,

$$f(x,v) \cdot n(x) \geq \alpha > 0, \tag{12}$$

where $n(x)$ is the unit outward normal to $\partial E \in \mathbb{R}^n$ at the point $x \in \partial E$ and $\cdot$ denotes inner product.

Assumption (A7) postulates that when the deterministic controlled flows get sufficiently close to the boundary, they *must hit* the boundary in finite time by virtue of requirement (12) that on the boundary the corresponding field element makes an acute angle with the unit outward normal. (Any other similar condition implying this finite boundary hitting time property would suffice.)

(A8) The set

$$N_\theta(x) := \left\{ (f(x,u)', \lambda(x,u), l)' : l \geq l_0(x,u) + \lambda(x,u) \int_{E^0} \theta(y) Q_0(dy; x,u), u \in U_0 \right\}$$

is convex for all $x \in E^0$, $\theta \in C(E^0)$, where the prime denotes transpose.

This assumption is made only in the interests of clear presentation to obviate the necessity for considering relaxed or generalized control policies in cumbersome detail.

(A9) The jump rate satisfies

$$\inf_{\substack{x \in E^0 \\ u \in U_0}} \lambda(x,u) + \delta > \lambda_+^0, \tag{13}$$

where $\lambda^0 := \sup_{x,y \in E^0, u \in U_0} (x-y)'(f(x,v) - f(y,v))/\|x-y\|^2$.

In Section 3, we have transformed an original fully controlled PDP process starting at $x := x_0$ to a new dynamically controlled PDP starting at $\hat{x} := (x_0, x_0, 0, -2, 0, 0)$. In order to use the dynamic programming approach, we embed the new optimal control problem with initial state $(x_0, x_0, 0, -2, 0, 0)$ in a family of optimal control problems obtained by varying the initial condition $\hat{x} \in \hat{E}$. For any $\hat{x} = (x, z, \tau, t', m, n) \in \hat{E}$, the new process starting at $\hat{x}$ is constructed from the original process in exactly same way as we did for $\hat{x} := (x_0, x_0, 0, -2, 0, 0)$ except that at the new process time $s = 0$, the original process has m interventions and n jumps, the post-jump state is z, the time elapsed is τ and the fictitious time is t'. We can then define the *value function* for the new dynamic control problem by

$$\hat{V}(\hat{x}) := \min_{\hat{u} \in \hat{C}} \hat{J}_{\hat{x}}(\hat{u}) \quad \text{for any } \hat{x} \in \hat{E}.$$

Since value functions are in general not smooth even for deterministic processes, $\hat{V}$ does not satisfy the Bellman–Hamilton–Jacobi equation in the conventional sense. By replacing the conventional gradient in the BHJ equation with an appropriate minimum element in the Clarke generalized gradient, we have given a necessary and sufficient optimality condition for the PDP dynamic control problem in terms of the resulting *generalized* BHJ equation in Dempster and Ye (1992).

To apply results in Dempster and Ye (1992), we next introduce some definitions of nonsmooth analysis which we will need. The reader is referred to Clarke (1983) for more details.

Let Y be a subset of a Banach space X. Let $\phi: Y \to \mathbb{R}$ be Lipschitz near a given point x and let d be any vector in X. The *generalized directional derivative of* ϕ at x in the *direction* d, denoted $\phi^0(x, d)$, is defined as follows:

$$\phi^0(x; d) := \limsup_{\substack{y \to x \\ t \downarrow 0}} [\phi(y + td) - \phi(y)]/t,$$

where y is a vector in X and t is a positive scalar.

Denote by X^* the dual space of X. The *generalized gradient* of ϕ at x, denoted by $\partial\phi(x)$, is the subset of X^* given by

$$\partial\phi(x) := \{\zeta' \in X^* : \phi^0(x; d) \geq \zeta' d \text{ for all } d \in X\}.$$

If ϕ is smooth, $\partial\phi(x)$ reduces to the conventional gradient.

If ϕ is continuous and convex, the generalized gradient coincides with the subgradient of convex analysis.

The function ϕ is said to be *regular at x* provided:

1. For all directions d, the usual one-sided directional derivative $\phi'(x; d)$ exists.
2. For all d, $\phi'(x; d) = \phi^0(x; d)$.

A function ϕ is called *regular* if it is regular at all $x \in Y$.

The following theorem gives the existence of an optimal full control characterization of the new value function $\hat{V}(\hat{x})$ and a necessary and sufficient condition for optimality.

THEOREM 1. *Under assumptions* (A1)–(A9), *there exists an optimal full control which solves the PDP optimal full control problem and the value function* $\hat{V}(\hat{z})$ *of the equivalent transformed process* $\hat{\mathbf{x}}_t$ *is a Lipschitz continuous solution of the generalized BHJ equation on* $\hat{E}^0$ *given by*

$$\begin{aligned}(14)\quad &\min_{\substack{\xi' \in \partial \hat{V}(\hat{x}) \\ v \in U_0}} \Big\{ \xi'(f(x,v),0,1,-1) \\ &\quad + \lambda(x,v)\big[\hat{V}(x,z,\tau,-5,m,n) - \hat{V}(\hat{x})\big] \\ &\qquad - \delta \hat{V}(\hat{x}) + e^{\delta(m+n)} l_0(x,v) \Big\} = 0, \\ &\forall\, \hat{x} \in E^0 \times E^0 \times T \times [(-\infty,-7) \cup (0,\infty)] \times \{m\} \times \{n\},\end{aligned}$$

$$\begin{aligned}(15)\quad &\min_{\xi' \in \partial \hat{V}(\hat{x})} \{\xi'(0,0,0,-1)\} - \delta \hat{V}(\hat{x}) = 0, \\ &\forall \hat{x} \in [E^0 \cup \partial E] \times E^0 \times T \times [(-6,-4) \cup (-3,-1)] \times \{m\} \times \{n\}\end{aligned}$$

(*corresponding, respectively, to real and fictitious time interior evolution*), *with boundary conditions*

$$\begin{aligned}(16)\quad &\hat{V}(\hat{x}) = \min_{\substack{v \in U_\delta \\ t' \in U_{t'}}} \left\{ e^{\delta(m+n+1)} l_\delta(x,v) + \int_{E^0} \hat{V}(y,y,0,t',m+1,n) Q_\delta(dy;x,v) \right\}, \\ &\forall\, \hat{x} \in [E^0 \cup \partial E] \times E^0 \times T \times \{-3\} \times \{m\} \times \{n\},\end{aligned}$$

$$\begin{aligned}(17)\quad &\hat{V}(\hat{x}) = \min_{\substack{v \in U_\delta \\ t' \in U_{t'}}} \left\{ e^{\delta(m+n+1)} l_\delta(x,v) - \int_{E^0} \hat{V}(y,y,0,t',m+1,n) Q_\delta(dy;x,v) \right\} \\ &\quad \wedge \min_{t \in U_{t'}} \left\{ \int_{E^0} \hat{V}(y,y,0,t',m,n+1) Q_0(dy;x,u_0(\tau,z)) \right\}, \\ &\forall\, \hat{x} \in E^0 \times E^0 \times T \times \{-6\} \times \{m\} \times \{n\},\end{aligned}$$

$$\begin{aligned}(18)\quad &\hat{V}(\hat{x}) = \min_{\substack{v \in U_\partial \\ t' \in U_{t'}}} \Big\{ e^{\delta(m+n+1)} l_\partial(x,v) \\ &\qquad + \int_{E^0} \hat{V}(y,y,0,t',m,n+1) Q_\partial(dy;x,v) \Big\} \\ &\quad \wedge \min_{\substack{v \in U_\delta \\ t' \in U_{t'}}} \left\{ e^{\delta(m+n+1)} l_\delta(x,v) + \int_{E^0} \hat{V}(y,y,0,t',m+1,n) Q_\delta(dy;x,v) \right\}, \\ &\forall\, \hat{x} \in \partial E \times E^0 \times T \times \{-6\} \times \{m\} \times \{n\}.\end{aligned}$$

[*These three boundary conditions for the Dirichlet problem for the generalized BHJ equation, characterizing optimality for the transformed dynamic control* *problem in terms of its value function, correspond to (potential) interventions, respectively, at nonjump epochs, interior jump epochs and boundary jump epochs of the original fully controlled process.*]

A full control u^* is optimal if and only if the interior control action u_0^* achieves the minimum in the generalized BHJ equation (14) *and* (15) *in the sense that for each possible post-jump state of the new process* $\hat{z} := (z, z, 0, t', m, n) \in E^0 \times E^0 \times T \times [(-\infty, 7) \cup (0, \infty)] \times \{m\} \times \{n\}$ *and for all* $\xi_\tau' \in \partial \hat{V}(\hat{\phi}_\tau^{\hat{u}^*}(\hat{z}))$ [*there exists* $\xi_\tau' \in \partial \hat{V}(\hat{\phi}_\tau^{\hat{u}^*}(\hat{z}))$ *achieving equality provided that* $\hat{V}$ *is regular*],

$$\begin{aligned}&\xi_\tau'(f(\phi_\tau^{u^*}(z), u_{0\tau}^*(z)), 0, 1, -1) + \lambda(\phi_\tau^{u^*}(z), u_{0\tau}^*(z))\\ &\quad\times\left[\hat{V}(\phi_\tau^{u^*}(z), z, \tau, -5, m, n) - \hat{V}(\phi_\tau^{u^*}(z), z, \tau, t' - \tau, m, n)\right]\\ &\quad - \delta\hat{V}(\phi_\tau^{u^*}(z), z, \tau, t' - \tau, m, n) - e^{\delta(m+n)} l_0(\phi_\tau^{u^*}(z), u_{0\tau}^*(z)) = 0\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad a.e\ \tau \in [0, t_*^{u^*}(z) \wedge t'],\end{aligned}$$

where $\phi_\tau^{u^*}(z)$ *is the flow of the original problem,* $\hat{\phi}_\tau^{\hat{u}^*}(\hat{z}) := (\phi_\tau^{u^*}(x), x, \tau, t' - \tau, m, n)$ *denotes the flow of the new process and for every* $\hat{x} \in \hat{\Gamma}^*$, *the boundary control and impulse control actions achieve the minimum in the boundary conditions* (16), (17) *and* (18). *In other words, whenever* $t' = 0$, *we intervene by choosing an impulse control action* $u_\delta^* \in U_\delta$ *and a time remaining to intervene* t'^* *such that the right-hand side of* (16) *is minimized. Whenever* $t' > 0$ (*or* $-\infty < t' < -8$) *and there is a process jump, we either let the process jump and choose the time remaining to intervene* t'^* *if the second term in the right-hand side of* (17) *is the minimal value or otherwise intervene by choosing the impulse control action* $u_\delta^* \in U_\delta$ *and a time remaining to intervene* t'^*. *Whenever* $t' > 0$ (*or* $-\infty < t' < -8$) *and the original PDP reaches the boundary of the state space, we either apply the boundary control action* $u_\partial^* \in U_\partial$ *and a time remaining to intervene* t'^* *if* (u_∂^*, t'^*) *achieves the minimum in the right-hand side of* (18) *or otherwise intervene by choosing the impulse control action* $u_\delta^* \in U_\delta$ *and the time remaining to intervene* t'^*.

REMARK. We do not have the uniqueness result for the solutions for the generalized BHJ equation, as in Dempster and Ye (1990, 1991, 1992), since the new state space $\hat{E}$ is not bounded.

PROOF. Notice that the new state space is a union of sets indexed by (m, n). Although the control theory for the PDP optimal dynamic control problem developed in Dempster and Ye (1992) is stated for the case when the state space is connected, it is easy to see that the result is also true for the general case when the state space is a union of sets with smooth boundary, provided the problem data satisfies the appropriate assumptions on each component of the state space. Under assumptions (A1)–(A9), it is straightforward to verify that the assumptions (A1)–(A9) of Dempster and Ye (1992) are satisfied on each component of the state space indexed by (m, n).

The assumption in (A6) that $l_\delta(x, v) > \varepsilon > 0$ for all $x \in E$ and $v \in U_\delta$ is necessary to ensure the existence of an optimal control policy. Indeed, otherwise the infimum of the original fully controlled problem cost functional (6) might be approached—but never achieved—by a sequence of impulse controls involving an increasingly dense sequence of intervention times at which the controlled process $\mathbf{x}_t$ is reset to $\arg\min_{z \in E_0} [\min_{v \in U_0} l_0(z, v)] \in E_0$ (assuming that U_δ allows such interventions).

Applying the main result of Dempster and Ye (1992) to the transformed dynamic control problem, we conclude that there exists an optimal control for the new problem and the value function of the new dynamic problem $\hat{V}$ is a Lipschitz continuous solution of the following generalized BHJ equation:

$$\min_{\substack{\xi' \in \partial \hat{V}(\hat{x}) \\ v \in U_0}} \left\{ \xi' \hat{f}(\hat{z}, v) + \hat{\lambda}(\hat{z}, v) \int_{\hat{E}^0} (\hat{V}(\hat{y}) - \hat{V}(\hat{z})) \hat{Q}_0(d\hat{y}; \hat{z}) - \delta \hat{V}(\hat{z}) + \hat{l}_0(\hat{z}, v) \right\} = 0, \quad \forall\, \hat{z} \in \hat{E}^0, \tag{19}$$

with *boundary condition*

$$\hat{V}(\hat{z}) = \min_{\hat{v} \in \hat{U}_\partial} \left\{ \hat{l}_\partial(\hat{z}, \hat{v}) + \int_{\hat{E}^0} \hat{V}(\hat{y}) \hat{Q}_\partial(d\hat{y}; \hat{z}, \hat{v}) \right\} \qquad \forall\, \hat{z} \in \partial \hat{E}. \tag{20}$$

We also conclude that an optimal control $\hat{u}^* = (\hat{u}_0^*, \hat{u}_\partial^*)$ is optimal if and only if for all $\hat{z} \in \hat{E}^0$ and for all $\xi_\tau' \in \partial \hat{V}(\phi_\tau^{\hat{u}^*}(\hat{z}))$ [there exists $\xi_\tau' \in \partial \hat{V}(\phi_\tau^{\hat{u}^*}(\hat{z}))$ provided $\hat{V}$ is regular], we have

$$\begin{aligned} &\xi_\tau' \hat{f}(\phi_\tau^{\hat{u}^*}(\hat{z}), u_{0\tau}^*(\hat{z})) + \hat{\lambda}(\phi_\tau^{\hat{u}^*}(\hat{z}), u_{0\tau}^*(\hat{z})) \\ &\quad \times \int_{\hat{E}^0} (\hat{V}(\hat{y}) - \hat{V}(\phi_\tau^{\hat{u}^*}(\hat{z}))) \hat{Q}_0(d\hat{y}; \phi_\tau^{\hat{u}^*}(\hat{z})) \\ &\quad - \delta \hat{V}(\phi_\tau^{\hat{u}^*}(\hat{z})) + \hat{l}_0(\phi_\tau^{\hat{u}^*}(\hat{z}), u_{0\tau}^*(\hat{z})) = 0 \qquad \text{a.e. } \tau \in [0, t_*^{\hat{u}^*}(\hat{z})], \end{aligned}$$

where $t_*^{\hat{u}^*}(\hat{z})$ is the first time the flow $\phi_\tau^{\hat{u}^*}(\hat{z})$ hits the boundary of the state space $\hat{E}^0$ defined as in (3) and $\hat{u}_\partial^*$ achieves the minimum in (20).

Since the PDP full control problem is equivalent to the new dynamic control problem, we conclude from the existence of an optimal control for the new dynamic control problem the existence of an optimal full control.

Substituting the uncareted (non-hat) counterparts into equations (19) and (20), we obtain the BHJ equation with boundary conditions involving original problem data (14)–(18).

Similarly, interpreting the necessary and sufficient optimality condition for the new control problem in terms of the original problem data according to the construction of the new process, the necessary and sufficient optimality condition for the PDP full control problem follows. □

5. The generalized quasivariational inequality. The purpose of this final section is to state our results in terms of the generalized quasivariational inequality for the PDP full control problem, which yields the quasivariational inequality for the problem of impulse control (only) of PDPs studied by other authors.

To this end, we first give the relationship between the value function for the new dynamic control problem and that for the original full control problem.

PROPOSITION 3. *Prior to jumps of the transformed process,*

$$\hat{V}(x, z, \tau, -3, m, n) = \exp(\delta(m + n + 1))V(x) \quad (\textit{intervention}),$$
$$\hat{V}(x, z, \tau, -6, m, n) = \exp(\delta(m + n + 1))V(x) \quad (\textit{jump}),$$

while post-jump we have

$$\hat{V}(x, z, \tau, t', m, n) = \exp(\delta(m + n))V(x) \quad \textit{for } t' \in (-\infty, -7) \cup (0, \infty), \tag{21}$$

$$\min_{t' \in U_{t'}} \hat{V}(z, z, 0, t', m + 1, n) = \exp(\delta(m + n + 1))V(z) \quad (\textit{intervention}),$$

$$\min_{t' \in U_{t'}} \hat{V}(z, z, 0, t', m, n + 1) = \exp(\delta(m + n + 1))V(z) \quad (\textit{jump}).$$

At an interior jump epoch τ, where the interior control is optimal, we have

$$\hat{V}(x, z, \tau, -5, m, n) = e^{\delta(m+n)} \int_{E^0} V(y) Q_0(dy; x, u_0(z, \tau)).$$

PROOF. By definition of the value function for the new dynamic control problem, $\hat{V}(x, z, \tau, t', m, n)$ is the optimal cost for this transformed problem with initial point (x, z, τ, t', m, n). For any initial point $\hat{x} = (x, z, \tau, t', m, n)$, the new process starting from (x, z, τ, t', m, n) is constructed from the original process in exactly the same way as we did for $\hat{x}_0 := (x_0, x_0, 0, -2, 0, 0)$ except that at the new process time $s = 0$, the original process has m interventions and n jumps, the post-jump state is z, the time elapsed since the last jump is τ and the fictitious time is t'. Therefore, interpreting the optimal cost in terms of the original problem data, using the definitions of $\hat{l}_0$ and $\hat{l}_\partial$, it is easy to see that for all $t' \in (-\infty, -7) \cup (0, \infty)$, we have

$$\hat{V}(x, z, \tau, t', m, n) = e^{\delta(m+n)} V(x).$$

Similarly, we have

$$\begin{aligned} \min_{t' \in U_{t'}} \hat{V}(z, z, 0, t', m + 1, n) &= \inf_{\hat{u}} J_{\hat{x}}(\hat{u}) \\ &= \exp(\delta(m + n + 1))V(x) \end{aligned}$$

and

$$\begin{aligned} \min_{t' \in U_{t'}} \hat{V}(x, x, 0, t', m, n + 1) &= \inf_{\hat{u}} J_{\hat{x}}(\hat{u}) \\ &= \exp(\delta(m + n + 1))V(x). \end{aligned}$$

By virtue of (15), we have

$$\hat{V}(x,z,\tau,-2,m,n) = e^{-\delta}\hat{V}(x,z,\tau,-3,m,n),$$
$$\hat{V}(x,z,\tau,-5,m,n) = e^{-\delta}\hat{V}(x,z,\tau,-6,m,n).$$

Therefore, we have

$$\begin{aligned}\hat{V}(x,z,\tau,-3,m,n) &= e^{\delta}\hat{V}(x,z,\tau,-2,m,n)\\ &= e^{\delta}e^{\delta(m+n)}V(x,z,\tau,-2,0,0)\\ &= e^{\delta(m+n+1)}V(x)\end{aligned}$$

and

$$\begin{aligned}\hat{V}(x,z,\tau,-6,m,n) &= e^{\delta}\hat{V}(x,z,\tau,-5,m,n)\\ &= e^{\delta}e^{\delta(m+n)}V(x,z,\tau,-5,0,0)\\ &= e^{\delta(m+n+1)}V(x).\end{aligned}$$

At post-jump elapsed time points τ, where the interior control $u_0(z,\tau)$ is optimal, we have by virtue of (17),

$$\begin{aligned}&\hat{V}(x,z,\tau,-5,m,n)\\ &\quad= \exp(-\delta)\hat{V}(x,z,\tau,-6,m,n)\\ &\quad= \exp(-\delta)\min_{t'\in U_{t'}}\int_{E^0}\hat{V}(y,y,0,t',m,n+1)Q_0(dy;x,u_0(z,\tau))\\ &\quad= \exp(-\delta)\int_{E^0}\exp(\delta(m+n+1))V(y)Q_0(dy;x,u_0(z,\tau))\\ &\quad= \exp(\delta(m+n))\int_{E^0}V(y)Q_0(dy;x,u_0(z,\tau)).\end{aligned}$$ □

The following theorem characterizes the value function $V(z)$ of the original PDP control problem as a Lipschitz continuous solution of the generalized quasivariational inequality.

THEOREM 4. *Under assumptions* (A1)–(A9), *the value function $V(z)$ of the original PDP optimal full control problem is a Lipschitz continuous solution of the generalized quasivariational inequality*

$$\begin{aligned}&\min_{\substack{\xi'\in\partial V(x)\\ v\in U_0}}\left[\xi' f(x,v) + \lambda(x,v)\int_{E^0}(V(y)-V(x))Q_0(dy;x,v)\right.\\ &\qquad\qquad\left. -\delta V(x) + l_0(x,v)\right]\\ &\wedge\left[\min_{v\in U_\delta}\left\{l_\delta(x,v) + \int_{E^0}V(y)Q_\delta(d;x,v)\right\} - V(x)\right] = 0 \quad \text{for } x\in E^0,\end{aligned}\tag{22}$$

with the boundary condition

$$V(x) = \min_{v \in U_\partial}\left[l_\partial(x,v) + \int_{E^0} V(y) Q_\partial(dy; x, v) \right] \tag{23}$$
$$\wedge \min_{v \in U_\delta}\left[l_\delta(x,v) + \int_{E^0} V(y) Q_\delta(dy; x, v) \right] \quad \textit{for } x \in \partial E.$$

A full control $u^* = \{\{\tau_k^*\}_{k=0}^\infty, (u_0^*, u_\partial^*, u_\delta^*)\}$ [*with corresponding trajectory* $x^*(t)$] *is optimal if and only if*

$$\tau_k^* = \inf\Big\{ t \geq \tau_{k-1}^* : V(x^*(t))$$
$$= \min_{v \in U_\delta}\left[l_\delta(x^*(t), v) + \int_{E^0} V(y) Q_\delta(dy; x^*(t), v) \right]\Big\}$$
$$\textit{for } k = 0, 1, 2, \dots,$$

and for all $\xi_t' \in \partial V(\phi_t^*(z))$ [*or there exists* $\xi_t' \in \partial V(\phi_t^*(z))$ *provided* V *is regular*] *one has*

$$\xi_t' V(\phi_t^*(z)) f(\phi_t^*(z), u_{0t}^*(z)) + \lambda(\phi_t^*(z), u_{0t}^*(z))$$
$$\times \int_{E^0} (V(y) - V(\phi_t^*(z))) Q_0(dy; \phi_t^*(z), u_{0t}^*(z))$$
$$- \delta V(\phi_t^*(z)) + l_0(\phi_t^*(z), u_{0t}^*(z)) = 0$$
$$\textit{for all post-jump states } z \textit{ and a.e. } t \in [0, t_*(z)],$$

where $\phi_t^*(z)$ *is the flow corresponding to the optimal control* u^* *starting at the post-jump state* z *and*

$$u_\partial^*(z) = \arg\min_{v \in U_\partial} \left\{ l_\partial(z, v) + \int_{E^0} V(y) Q_\partial(dy; z, v) \right\}$$
$$\textit{for all } z \in \partial E \textit{ such that } \min_{v \in U_\partial}\left\{ l_\partial(z,v) + \int_{E^0} V(y) Q_\partial(dy; z, v) \right\} = V(z),$$
$$u_\delta^*(x'(\tau_k)) = \arg\min_{v \in U_\delta} \left\{ l_\delta(x^*(\tau_k), v) + \int_{E_0} V(y) Q_\delta(dy; x^*(\tau_k), v) \right\}$$
$$\textit{for } k = 0, 1, 2, \dots .$$

PROOF. Since, by Theorem 1, the value function $\hat{V}$ of the transformed problem is Lipschitz continuous on $\hat{E}$, it follows easily from Proposition 3 that the value function V of the original problem is Lipschitz continuous on E. By virtue of (21), $\hat{V}$ is independent of (z, τ, t'). Therefore, by Proposition 1.8 of Ye (1990), we have

$$\partial \hat{V}(\hat{x}) = \exp(\delta(m + n))\, \partial V(x) \times \{0\} \times \{0\} \times \{0\}. \tag{24}$$

At $x \in E^0$, where the second term on the quasivariational inequality (22) is strictly greater than zero, that is,

$$V(x) < \min_{v \in U_\delta} \left\{ l_\delta(x, v) + \int_{E^0} V(y) Q_\delta(dy; x, v) \right\},$$

we have by virtue of Proposition 3 that

$$\hat{V}(x, z, \tau, -5, m, n) = e^{\delta(m+n)} \int_{E^0} V(y) Q_0(dy; x, u_0(z, \tau)). \tag{25}$$

Substituting (24), (25) and (21) into (14), expression (14) is reduced to the generalized BHJ equation for the original problem, namely,

$$\min_{\substack{\xi' \in \partial V(x) \\ v \in U_0}} \left\{ \xi' f(x, v) + \lambda(x, v) \int_{E^0} (V(y) - V(x)) Q_0(dy; x, v) - \delta V(x) + l_0(x, v) \right\} = 0.$$

It follows that the quasivariational inequality (22) holds. Similarly (23) follows from (18) using Proposition 3.

The remaining results follow from Proposition 3 and Theorem 1 in the same way. □

REMARK 5. To relate our approach to the quasivariational approach taken by other authors, assume that the value function $V(z)$ is C^1. Then in the case in which there is no dynamic control, the impulse control action does not take place at a jump epoch, $l_\partial := 0$ and $Q_\partial(\cdot; x, v) := 1_{x+v}(\cdot)$. (22) and (23) reduce, respectively, to the quasivariational inequality (1) with boundary condition (2).

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