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SOME FORMULAE FOR A NEW TYPE OF PATH-DEPENDENT OPTION

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In this paper we present an explicit form of the distribution function of the occupation time of a Brownian motion with a constant drift (if there is no drift, this is the well-known arc-sine law). We also define the α-percentile of the stock price and give an explicit form of the distribution function of this random variable. Using this explicit distribution, we calculate the price of a new type of path-dependent option, called the *α-percentile option*. This option was first introduced by Miura and is based on order statistics.

1. A generalized arc-sine law. In this paper, B_t denotes a standard Brownian motion starting at 0, $\mathscr{F}_t$ denotes its canonical filtration and P_0 denotes its probability measure. Let

$$A(t, x; \mu) = \frac{1}{t}\int_0^t 1_{\{B_s + \mu s < x\}}\, ds, \qquad \mu > 0, t > 0, x \in \mathbb{R}^1. \tag{1.1}$$

Then we have the following theorem.

THEOREM 1.1. (i) *We have*

$$\begin{aligned} &P_0(A(t,0;\mu) < y) \\ &\quad = \frac{1}{2}\int_0^{ty}\left(\sqrt{\frac{2}{\pi s}}\exp\left(-\frac{\mu^2}{2}s\right) - 2\mu\Phi(\mu\sqrt{s})\right) \\ &\qquad \times\left\{\left(2\mu + \sqrt{\frac{2}{\pi(t-s)}}\exp\left(-\frac{\mu^2}{2}(t-s)\right)\right) - 2\mu\Phi(\mu\sqrt{t-s})\right\} ds, \end{aligned}$$

where Φ denotes the tail of the distribution function of the normal distribution; that is,

$$\Phi(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right) dy.$$

(ii) *More generally, we have*

$$P_0(A(t, x; \mu) < y) = \int_0^{t\alpha} h(s, x; \mu)\varphi(t-s, y - s/t; \mu)\, ds, \qquad x \neq 0,$$

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where $h(s, x; \mu)$ denotes a density of the first hitting time of $B_t + \mu t$ to x; that is,

$$h(s, x; \mu) = \frac{|x|}{\sqrt{2\pi s}} \exp\left(\frac{(|x| - \mu s)^2}{2s}\right) \quad and \quad \varphi(t, x; \mu) = P_0(A(t, 0; \mu) < x).$$

PROOF. First we prove assertion (i). Let

$$u(x) = E_x \int_0^\infty \exp(-\zeta t) \exp(-\lambda t A(t, 0; \mu))\, dt, \qquad x \in \mathbb{R}^1. \tag{1.2}$$

Then the Feynman–Kac formula [cf. Kac (1951) and Itô and McKean (1965)] claims that $u(x)$ is the unique bounded solution of the equation

$$-\tfrac{1}{2}u'' - \mu u' + \zeta u + \lambda 1_{\{x>0\}} u = 1, \qquad \zeta > 0, \lambda > 0. \tag{1.3}$$

By solving (1.3),

$$\begin{aligned} u(0) = {} & \frac{\mu}{2}\frac{1}{\zeta}\frac{\sqrt{\mu^2 + 2(\zeta + \lambda)}}{\zeta + \lambda} - \frac{\mu}{2}\frac{1}{\zeta + \lambda}\frac{\sqrt{\mu^2 + 2\zeta}}{\zeta} \\ & + \frac{1}{2}\frac{\sqrt{\mu^2 + 2(\zeta + \lambda)}}{\zeta + \lambda}\frac{\sqrt{\mu^2 + 2\zeta}}{\zeta} - \frac{\mu^2}{2}\frac{1}{\zeta(\zeta + \lambda)}. \end{aligned} \tag{1.4}$$

Since we have, by inverting the Laplace transform [see, e.g., Widder (1946)],

$$\begin{aligned} \frac{\sqrt{\mu^2 + 2\zeta}}{\zeta} &= \frac{1}{\zeta}\int_0^\zeta \frac{d\lambda}{\sqrt{\mu^2 + 2\lambda}} + \frac{1}{\zeta}\mu \\ &= \int_0^\infty \exp(-\zeta t)\left(\int_t^\infty \frac{\exp(-(\mu^2/2)s)}{\sqrt{2\pi s^3}}\, ds + \mu\right) dt, \end{aligned} \tag{1.5}$$

we get

$$\begin{aligned} u(0) = {} & \int_0^\infty \exp(-\zeta t) \int_0^t \frac{\exp(-\lambda s)}{2} \\ & \times \left(2\mu + \int_{t-s}^\infty \frac{\exp(-(\mu^2/2)\tau)}{\sqrt{2\pi\tau^3}}\, d\tau\right) \\ & \times \left(\int_s^\infty \frac{\exp(-(\mu^2/2)\tau)}{\sqrt{2\pi\tau^3}}\, d\tau\right) ds\, dt. \end{aligned} \tag{1.6}$$

Comparing (1.6) with (1.2), we get

$$\begin{aligned} P_0(A(t, 0; \mu) < y) = {} & \int_0^{ty} \frac{1}{2}\left(2\mu + \int_{t-s}^\infty \frac{\exp(-(\mu^2/2)\tau)}{\sqrt{2\pi\tau^3}}\, d\tau\right) \\ & \times \left(\int_s^\infty \frac{\exp(-(\mu^2/2)\tau)}{\sqrt{2\pi\tau^3}}\, d\tau\right) ds. \end{aligned} \tag{1.7}$$

Integrating by parts, we get

$$\int_s^\infty \frac{\exp(-(\mu^2/2)\tau)}{\sqrt{2\pi\tau^3}}\,d\tau = \sqrt{\frac{2}{\pi s}}\exp\left(-\frac{\mu^2}{2}s\right) - 2\mu\int_{\mu\sqrt{s}}^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\tau^2}{2}\right)d\tau. \tag{1.8}$$

So we have the assertion (1).

Assertion (2) follows directly from the Markov property of $B_t + \mu t$. □

REMARK 1.2. Since $A(t, x; -\mu) =_{\text{law}} 1 - A(t, -x; \mu)$, we now obtain Theorem 1.1 for all $\mu \in \mathbb{R}^1$.

REMARK 1.3. Watanabe (1993) also obtained Theorem 1.1 (1) by studying the generalized arc-sine laws of some classes of one-dimensional diffusion processes.

2. The pricing formula for the α-percentile option. Let us consider the Black–Scholes model [cf. Black and Scholes (1973)]: The stock price X_t is a geometric Brownian motion and the bond price b_t is nonstochastic; that is,

$$X_t = X_0 \exp\left(\sigma B_t + \left(\mu - \tfrac{1}{2}\sigma^2\right)t\right), \qquad X_0 > 0,\ \sigma > 0,\ \mu \in \mathbb{R}^1, \tag{2.1}$$

$$b_t = b_0 \exp(rt), \qquad r \geq 0,\ b_0 > 0. \tag{2.2}$$

We define the α-percentile $m(T; \alpha)$ for $0 < \alpha < 1$, $T > 0$ of $\{X_t\}_{t\in[0,T]}$ as

$$m(T; \alpha) = \inf\left\{x \in \mathbb{R}^1;\ \frac{1}{T}\int_0^T 1_{\{X_t < x\}}\,dt > \alpha\right\}. \tag{2.3}$$

We introduce the α-percentile option $(m(T; \alpha) - c)^+$, $c > 0$, and present the pricing formulae for this.

Here we think of "pricing" as the stochastic integral representation of the option with respect to the discounted stock price under martingale measure [cf. Harrison and Pliska (1981) and Föllmer (1991)].

We define a discounted price process Z_t by setting

$$Z_t = b_t^{-1}X_t. \tag{2.4}$$

Let us introduce a probability measure P_0^* under which Z_t is a martingale and let E_0^* denote its expectation. Let π be the price of the option, ζ_t be the amount of stock and ν_t be the amount of bond.

Then we have a stochastic representation [see, e.g., Rogers and Williams (1987)] of the option as follows:

$$b_T^{-1}(m(T; \alpha) - c)^+ = \pi + \int_0^T \zeta_t\,dZ_t, \tag{2.5}$$

where

$$\pi = E_0^*\left((m(T; \alpha) - c)^+ b_T^{-1}\right), \tag{2.6}$$

and we have

$$\nu_t = E_0^*\left((m(T; \alpha) - c)^+ b_T^{-1} \mid \mathcal{F}_t\right) - \zeta_t Z_t. \tag{2.7}$$

We can give the following formulae for π, ζ_t, ν_t by virtue of Theorem 1.1.

THEOREM 2.1. *We have*

$$
\begin{aligned}
\pi &= b_T^{-1} \int_c^\infty G\left(T, \sigma^{-1} \log \frac{y}{X_0}; \alpha, \frac{r}{\sigma} - \frac{1}{2}\sigma\right) dy \\
&\quad + c b_T^{-1} G\left(T, \sigma^{-1} \log \frac{c}{X_0}; \alpha, \frac{r}{\sigma} - \frac{1}{2}\sigma\right),
\end{aligned} \tag{2.8}
$$

$$
\zeta_t = -\frac{b_T^{-1}}{\sigma Z_t} \int_0^\infty \frac{\partial G}{\partial x}\left(T - t, \sigma^{-1} \log \frac{b_0 y}{Z_t}; \frac{T\alpha - C_t}{T - t}, \frac{r}{\sigma} - \frac{1}{2}\sigma\right) dy, \tag{2.9}
$$

$$
\begin{aligned}
\nu_t &= \int_c^\infty b_t^{-1} G\left(T - t, \sigma^{-1} \log \frac{b_0 y}{Z_t}; \frac{T\alpha - C_t}{T - t}, \frac{r}{\sigma} - \frac{1}{2}\sigma\right) dy \\
&\quad - \zeta_t Z_t + c b_t^{-1} G\left(T - t, \sigma^{-1} \log \frac{b_0 c}{Z_t}; \frac{T\alpha - C_t}{T - t}, \frac{r}{\sigma} - \frac{1}{2}\sigma\right),
\end{aligned} \tag{2.10}
$$

where

$$
G(t, x; \alpha, \mu) = \int_0^{t\alpha} h(s, x; \mu) \varphi\left(t - s, \alpha - \frac{s}{t}; \mu\right) ds, \tag{2.11}
$$

$\partial G/\partial x$ denotes the derivative with respect to the second variable and

$$
C_t = A\left(t, \sigma^{-1} \log x; \frac{r}{\sigma} - \frac{1}{2}\sigma\right). \tag{2.12}
$$

REMARK 2.2. To calculate $\partial G/\partial x$, we must notice that $h(s, \cdot, \cdot)$ is not uniformly integrable. However, $h(s, x, \mu)\varphi(t, \alpha; \mu)$ is integrable in s, and so easy calculations reduce this problem to the reflection principle $P(\sup_{s \leq t} B_t \geq a) = P(|B_t| > a)$, that is,

$$
\int_0^t \frac{a}{\sqrt{2\pi s^3}} \exp\left(-\frac{a^2}{2s}\right) ds = 2\int_a^\infty \frac{\exp(-b^2/2t)}{\sqrt{2\pi t}} db. \tag{2.13}
$$

Therefore, we have

$$
\begin{aligned}
&\frac{\partial G}{\partial x}(t, x; \alpha, \mu) \\
&= \int_0^{t\alpha} \left(\mu + \frac{1}{x} - \frac{x}{s}\right) h(s, x; \mu) \varphi\left(t - s, \alpha - \frac{s}{t}; \mu\right) ds \\
&\quad - \varphi(t, \alpha; \mu)\left(\int_0^{t\alpha} \left(\frac{1}{2} - \frac{x}{s}\right) \exp\left(\frac{\mu x}{2}\right) h(s, x; \mu)\, dx \right. \\
&\qquad \left. + \exp(\mu x) \frac{\exp(-x^2/2t\alpha)}{\sqrt{2\pi t\alpha}}\right).
\end{aligned} \tag{2.14}
$$

PROOF OF THEOREM 2.1. Let $B_t^* = B_t + ((\mu - r)/\sigma)t$. Then B_t^* is a Brownian motion under P_0^* and we have $Z_t = X_0 b_0^{-1} \exp(\sigma B_t^* - \frac{1}{2}\sigma^2 t)$ and $X_t = X_0 \exp(\sigma B_t^* + (r - \frac{1}{2}\sigma^2)t)$.

Since

$$(2.15) \qquad P_0^*(m(T;\alpha) > x) = P_0^*\left(A\left(T, \frac{1}{\sigma}\log\frac{x}{X_0}; \frac{r}{\sigma} - \frac{1}{2}\sigma\right) < \alpha\right),$$

we have from Theorem 1.1 (ii),

$$(2.16) \qquad P_0^*(m(T;\alpha) > x) = G\left(T, \sigma^{-1}\log\frac{x}{X_0}; \alpha, \frac{r}{\sigma} - \frac{1}{2}\sigma\right).$$

Therefore, we get (2.8).

To obtain (2.9) and (2.10), we first observe

$$\begin{aligned}
&P_0^*(m(\alpha, T) > x \mid \mathscr{F}_t)\\
&= P_0^*\left(A\left(T, \frac{1}{\sigma}\log\frac{x}{X_0}; \frac{r}{\sigma} - \frac{1}{2}\sigma\right) < \alpha \mid \mathscr{F}_t\right)\\
(2.17) \qquad &= P_0\left(\int_t^T 1_{\{X_s \le x\}}\, ds < T\alpha - C_t \mid \mathscr{F}_t\right)\\
&= P_0\left(A\left(T - t, \frac{1}{\sigma}\log\frac{x}{X_0} - B_t^* + \frac{1}{2}\sigma t; \frac{r}{\sigma} - \frac{1}{2}\sigma\right) < \frac{T\alpha - C_t}{T - t}\right)\\
&= G\left(T - t, \sigma^{-1}\log\frac{b_0 x}{Z_t}; \frac{T\alpha - C_t}{T - t}, \frac{r}{\sigma} - \frac{1}{2}\sigma\right).
\end{aligned}$$

By integrating both sides of (2.17) with respect to x, we obtain $E^*((m(T;\alpha) - c)^+ \mid \mathscr{F}_t)$.

Itô's formula claims that the integrand $\tilde{\zeta}_t$ should be the partial derivative of (2.17) with respect to Z_t, so we get (2.9) and then (2.10). □

REMARK 2.3. Yor (1993) studied the relationship between the arc-sine law and the distribution of the α-percentiles, or as he called them, the quantiles.

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REFERENCES

BLACK, F. and SCHOLES, M. (1973). The pricing of options and corporate liabilities. *J. Political Econom.* **81** 637–659.

FÖLLMER, H. (1991). Probabilistic aspect of options. Discussion paper B202, SFB 303. Univ. Bonn.

HARRISON, J. M. and PLISKA, S. R. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process. Appl.* **15** 214–260.

ITÔ, K. and MCKEAN, H. P., JR. (1965). *Diffusion Processes and Their Sample Paths*. Springer, New York.

KAC, M. (1951). On some connections between probability theory and differential and integral equations. *Proc. Second Berkeley Symp. Math. Statist. Probab.* 189–215. Univ. California Press, Berkeley.

MIURA, R. (1992). A note on look-back options based on order statistics. *Hitotsubashi J. Commerce Manage.* **27** 15–28.

ROGERS, L. and WILLIAMS, D. (1987). *Diffusions, Markov Processes, and Martingales* **2**, Chap. 4. Wiley, New York.

WATANABE, S. (1993). Generalized arc-sine laws for one dimensional diffusion processes and random walks. Preprint, Kyoto Univ.

WIDDER, D. V. (1946). *The Laplace Transform*. Princeton Univ. Press.

YOR, M. (1993). The distribution of Brownian quantiles. *J. Appl. Probab.* To appear.

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