

THERE IS NO NONTRIVIAL HEDGING PORTFOLIO FOR OPTION PRICING WITH TRANSACTION COSTS

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Conventional wisdom holds that since continuous-time, Black–Scholes hedging is infinitely expensive in a model with proportional transaction costs, there is no continuous-time strategy which hedges a European call option perfectly. Of course, if one is attempting to dominate the European call rather than replicate it, then one can use the trivial strategy of buying one share of the underlying stock and holding to maturity. In this paper we prove that this is, in fact, the least expensive method of dominating a European call in a Black–Scholes model with proportional transaction costs.

1. Introduction. In a complete, continuous-time financial market without transaction costs, every contingent claim (i.e., integrable random variable whose value will be revealed at time T in the future) can be replicated by starting with a certain initial capital at time $t = 0$, and investing thereafter according to the Black–Scholes hedging portfolio. This, and related arbitrage considerations, justify the definition of the value of a claim to be the amount of initial capital required for this hedging [see, e.g., Karatzas and Shreve (1991) for the presentation of these ideas in a Brownian motion context; see Black and Scholes (1973), Harrison and Kreps (1979), Harrison and Pliska (1981, 1983) for their original derivations]. In particular, consider the example of the European call option, that is, the option of buying one share of stock at time T for an a priori specified amount $q > 0$. If $P(\cdot)$ denotes the price process for a share of the stock, then the value of this option at time T is $(P(T) - q)^+$. In a frictionless market, its value at time 0 is given by the celebrated Black–Scholes option pricing formula, and is a number strictly between $(P(0) - q)^+$ and $P(0)$.

However, the Black–Scholes hedging portfolio requires trading at all time instants, and the total turnover of stock in the time interval $[0, T]$ is infinite. Therefore, in a model with transaction costs proportional to the amount of trading, the Black–Scholes hedging portfolio is prohibitively expensive. Indeed, through Monte Carlo simulation, Figlewski (1989) demonstrates that even in discrete-time models, transaction costs are a substantial factor in hedging. This has led to a general belief that in the continuous-time model

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with proportional transaction costs, there is no portfolio which replicates the European call option and incurs finite transaction costs.

In this paper, we relax slightly the hedging condition, requiring only that the hedging portfolio almost surely dominate rather than replicate the value of the European call at maturity. With this relaxation, there is always the trivial hedging strategy of buying and holding one share of the stock on which the call is written. We show that this is in fact the cheapest dominating hedging strategy; this portfolio is, of course, of no interest to practitioners. As a corollary to this result, we see that any strategy which replicates the European call is also uninteresting. The problem solved in this paper was brought to our attention by Davis and Clark (1994), who cast as a formal conjecture the “conventional wisdom” concerning Black–Scholes hedging in the presence of transaction costs. We prove the conjecture of Davis and Clark. Their paper obtains a necessary and sufficient condition for the existence of a nontrivial hedging portfolio dominating the European call and offers arguments in support of their contention that this condition cannot hold.

Our model (with proportional transaction costs) is the same as that of Davis and Norman (1990), except that it uses the notation of Shreve and Soner (1994). We describe it in Section 1, and define what we mean by hedging a European call option in this market. The rest of the paper consists of a series of theorems, propositions and lemmas, which eventually lead to the proof of the main result. Here is a brief account of the proof: The method is based on a careful analysis of the function $\varphi(t, p, x)$, which is defined to be the minimal amount of money which must be invested in stock at time $t \in [0, T)$ in order to dominate the call at time T , when at time t the amount of money in the bond is x and the price of one share of the stock is p . The function φ is convex in (p, x) for each fixed t , and this enables us to treat it as if it were differentiable in these variables. One discovers then that the partial derivative of $\varphi(t, p, \cdot)$ with respect to x is constant in each of the two half-lines which comprise the complement of a certain interval $(a(t, p), b(t, p))$ [defined by (3.22) and (3.23)]. A maximum principle argument using Itô’s rule shows that $\varphi(t, \cdot, x)$ is a linear function in p , when $x \in (a(t, p), b(t, p))$ (Theorem 3.11). This linear growth, bounds on φ analogous to the above-mentioned bounds on the Black–Scholes price, and the (not so easy to prove) fact that $a(\cdot)$ and $b(\cdot)$ are nondecreasing in p imply $a(t, p) = b(t, p) \equiv 0$. Once this is established, the function φ is known and the result follows.

To overcome the negative result confirmed in this paper, various relaxations of perfect hedging have been proposed. Leland (1985) considers a model in which trades occur at discrete times, and thus a certain “hedge slippage” occurs at the time of each trade. However, the total cost of transaction remains finite, and the Black–Scholes partial differential equation for the value of the European call is obtained, albeit with an adjusted volatility. As the time between trades approaches zero, the hedge slippage disappears, the adjusted volatility approaches infinity and the value of the European call approaches the value we obtain here. Hoggard, Whalley and Wilmott (1994) extend Leland’s analysis to other derivatives securities. Using the notion of domination

rather than replication, Avellaneda and Parás (1994) extend Leland's approach even further.

A different approach to transaction costs is to introduce preferences. Hodges and Neuberger (1989) maximize the utility of the difference between a desired cash flow and the cash flow realized by a portfolio which must account for transaction costs. Davis, Panas and Zariphopoulou (1993) continue this approach, defining the value of an option in a market with transaction costs to be the minimal price under which an agent with preferences will choose to hold it.

In a discrete, binomial model with transaction costs, there is a replicating portfolio, as well as dominating portfolios which may have cheaper initial value than the replicating portfolio. We refer the reader to Bensaid, Lesne, Pagès and Scheinkman (1992), Boyle and Vorst (1992), Edirisinghe, Naik and Uppal (1993) and Boyle and Tan (1994).

Related works are Davis and Zariphopoulou (1994), Flesaker and Hughston (1994), Hodges and Clewlow (1993), Gilster and Lee (1984), Henrotte (1993), Merton (1989), Morton and Pliska (1993), Panas (1993), Pliska and Selby (1993), Shen (1990), Toft (1993) and Dewynne, Whalley and Wilmott (1994).

2. Model and definition of hedging. We consider a financial market which consists of one *bond* and one *stock*, the price of which evolves according to

$$(2.1) \quad dP(t) = \sigma P(t) dW(t), \quad P(t_0) = p \in (0, \infty),$$

for $t \in [t_0, T]$. Here t_0 is the initial and T the terminal time, W is a standard Brownian motion, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and we shall denote by $\{\mathcal{F}_t\}$ the \mathbf{P} -augmentation of the filtration $\mathcal{F}_t^W = \sigma(W(s); 0 \leq s \leq t)$ generated by W . The volatility σ of the stock is assumed to be a positive constant. We assume that the return rate of the stock, as well as the interest rate of the bond, are equal to zero. This represents no loss of generality, as explained in Remark 2.2(i) below. Notice that (2.1) implies

$$(2.2) \quad P(t) = P^{t_0, p}(t) = p \exp\{\sigma(W(t) - W(t_0)) - \sigma^2(t - t_0)/2\}, \quad t \in [t_0, T].$$

Let $X(\cdot)$ and $Y(\cdot)$ be the processes of dollar holdings in the bond and stock, respectively. A *trading strategy* is a pair (L, M) of $\{\mathcal{F}_t\}$ -adapted, left-continuous nondecreasing processes, satisfying $L(t_0) = M(t_0) = 0$, which are to be interpreted as the cumulative amounts transferred from bond to stock and stock to bond, respectively. Given fractional transaction costs $\lambda, \mu \in (0, 1)$ and the initial holdings x, y , the corresponding portfolio holdings $X = X_{L, M}^{t_0, x, y}$ and $Y = Y_{L, M}^{t_0, x, y}$ evolve according to

$$(2.3) \quad X(s) = x - L(s) + (1 - \mu)M(s),$$

$$(2.4) \quad Y(s) = y - M(s) + (1 - \lambda)L(s) + \int_{t_0}^s \sigma Y(u) dW(u),$$

for $s \in [t_0, T]$. Introduce the set $\mathcal{S} \triangleq \{(x, y) \mid (1-\lambda)x+y \geq 0, x+(1-\mu)y \geq 0\}$. We shall say that a trading strategy (L, M) is *admissible* if the corresponding holdings satisfy the *solvency constraint* (see Figure 1)

$$(2.5) \quad (X_{L,M}^{t_0,x,y}(t), Y_{L,M}^{t_0,x,y}(t)) \in \mathcal{S} \quad \text{a.s.,}$$

for every $t \in [t_0, T]$ [see Remark 2.2(ii) below for a weaker condition of solvency]. For $t = t_0$, (2.5) means that we require $(x, y) \in \mathcal{S}$ [strictly speaking, a trading strategy is a quadruple (L, M, x, y)]. We denote the set of admissible trading strategies by \mathcal{A} .

We shall say that the European call option with the strike price $q > 0$ is hedged by an admissible trading strategy (L, M) if the corresponding holdings satisfy

$$(2.6) \quad (1-\lambda)X(T) + Y(T) \geq [-(1-\lambda)q + P(T)]\mathbf{1}_{\{P(T) > (1-\lambda)q\}},$$

$$(2.7) \quad X(T) + (1-\mu)Y(T) \geq [-q + (1-\mu)P(T)]\mathbf{1}_{\{P(T) > (1-\lambda)q\}}.$$

The interpretation here is hedging in the sense of being able to transact at time T so as to deliver a share of the stock, and to cover the remaining position in the bond with the strike price q , if the option is exercised. Also, we assume that the buyer of the option is eager to own the stock, so she exercises the option already if $P(T) > (1-\lambda)q$ (rather than $P(T) > q$), since, to buy a share at time T one needs $P(T)/(1-\lambda)$ dollars. This is one of several ways conditions at the final time could be specified. We have chosen this particular specification for mathematical convenience. Let us mention that the results of this paper still hold with the less natural definition of hedging in which the right-hand sides of (2.6) and (2.7) are replaced by $(P(T) - q)^+$ and $(1-\mu)(P(T) - q)^+$, respectively. This definition is perhaps closer to the case of hedging with no transaction costs. Of course, we may rewrite (2.6) as

$$(2.6') \quad (1-\lambda)X(T) + Y(T) \geq [-(1-\lambda)q + P(T)]^+.$$

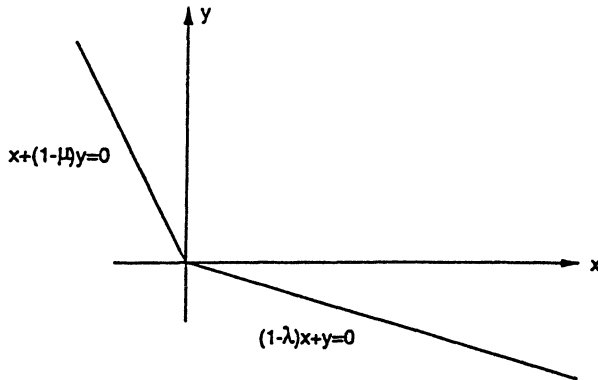


FIG. 1.

Moreover, the solvency condition $X(T) + (1 - \mu)Y(T) \geq 0$ combined with (2.7) yields

$$(2.7') \quad X(T) + (1 - \mu)Y(T) \geq [-q + (1 - \mu)P(T)]^+.$$

We define the *set of hedgeable initial conditions*

$$(2.8) \quad H \triangleq \{(t_0, p, x, y) \in [0, T) \times (0, \infty) \times \mathcal{S} \mid \text{there exists } (L, M) \in \mathcal{A} \text{ such that (2.6) and (2.7) are satisfied by } X_{L, M}^{t_0, x, y}(T), Y_{L, M}^{t_0, x, y}(T), P^{t_0, p}(T)\}.$$

For each $t_0 \in (0, T)$, we define

$$(2.9) \quad H(t_0) \triangleq \{(p, x, y) \mid (t_0, p, x, y) \in H\}.$$

For each $(t_0, p) \in [0, T) \times (0, \infty)$, we define

$$(2.10) \quad H(t_0, p) \triangleq \{(x, y) \mid (t_0, p, x, y) \in H\}.$$

Define a function $\varphi: [0, T) \times (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$(2.11) \quad \varphi(t_0, p, x) = \inf\{y \in \mathbb{R} \mid (x, y) \in H(t_0, p)\}.$$

The following theorem is the main result of the paper. It shows that the least costly way of hedging the European call option in a market with transaction costs is the trivial one—to buy a share of the stock and hold it.

THEOREM 2.1. *For a given $q > 0$, we have for every $p \in (0, \infty)$ and $t_0 \in [0, T)$,*

$$\varphi(t_0, p, x) = \begin{cases} p - (1 - \lambda)x, & \forall x \geq 0, \\ p - \frac{1}{1 - \mu}x, & \forall x \leq 0. \end{cases}$$

REMARK 2.2. (i) We want to show that there is no loss of generality in assuming that the interest rate r of the bond and the return rate b of the stock are equal to zero. Indeed, suppose they are not, so that the stock price obeys

$$dP(t) = P(t)[b dt + \sigma dW(t)],$$

and the holdings satisfy

$$\begin{aligned} X(s) &= x + \int_{t_0}^s r X(s) ds - L(s) + (1 - \mu)M(s), \\ Y(s) &= y + \int_{t_0}^s b Y(s) ds - M(s) + (1 - \lambda)L(s) + \int_{t_0}^s \sigma Y(u) dW(u). \end{aligned}$$

Suppose also that the analogue of Theorem 2.1 for this case does not hold, for example, that $(L, M) \in \mathcal{A}$ are such that (2.6) and (2.7) hold with $t_0 = 0$, $(x, y) \in \mathcal{S}$, $x \geq 0$ and $(1 - \lambda)x + y < p$. Consider the discounted processes $\tilde{X}(t) = e^{-rt} X(t)$, and similarly for \tilde{Y}, \tilde{P} ; set $\tilde{L}(t) = \int_{[0, t)} e^{-rs} dL(s)$

and similarly for \tilde{M} , and $\tilde{q} = e^{-rT}q$. Introduce also the process $\tilde{W}(t) = W(t) + (b - r)\sigma^{-1}t$, a Brownian motion under a probability measure $\tilde{\mathbf{P}}$, equivalent to the original measure \mathbf{P} (Girsanov theorem). It is then easy to see that \tilde{P} is a price process satisfying (2.1) with W replaced by \tilde{W} , that $\tilde{L}, \tilde{M}, \tilde{X}$ and \tilde{Y} satisfy corresponding versions of (2.3) and (2.4), and that (\tilde{L}, \tilde{M}) is an admissible trading strategy which hedges the European call option on the stock with price $\tilde{P}(\cdot)$ and strike price \tilde{q} , starting with the initial capital $(x, y) \in \mathcal{S}$, $x \geq 0, (1 - \lambda)x + y < p$. This is in contradiction to Theorem 2.1.

(ii) The solvency constraint (2.5) is a standard, no-arbitrage type assumption. It could be relaxed by requiring that $(1 - \lambda)X(T) + Y(T) \geq 0, (1 - \lambda)X(t) + Y(t) > -B$ for $t < T$, and similarly for $X(\cdot) + (1 - \mu)Y(\cdot)$, where $B \geq 0$ is a square-integrable random variable.

The rest of the paper is devoted to the preparations for the proof of Theorem 2.1.

3. The proof. Let $X = X_{L,M}^{t_0,x,y}$ and $Y = Y_{L,M}^{t_0,x,y}$ be the portfolio holdings corresponding to the admissible trading strategy (L, M) and initial holdings x, y . Extend the processes to $[0, T]$ by left continuity, that is,

$$(3.1) \quad L(s) \equiv M(s) \equiv 0, \quad X(s) = x, \quad Y(s) = y, \quad s \in [0, t_0].$$

Define the process

$$(3.2) \quad I(s) \triangleq \int_{t_0}^{s \vee t_0} Y(u) dW(u), \quad s \in [0, T].$$

We have the following lemma.

LEMMA 3.1. *There is a positive constant C such that*

$$(3.3) \quad EI^2(t) \leq C,$$

for every $t \in [0, T]$.

PROOF. Define $\gamma \triangleq 1 - (1 - \mu)(1 - \lambda)$. Then the solvency conditions $X + (1 - \mu)Y \geq 0, (1 - \lambda)X + Y \geq 0$ imply

$$(3.4) \quad \begin{aligned} \gamma L(s) &\leq x + (1 - \mu)y + (1 - \mu)\sigma I(s), \\ \gamma M(s) &\leq (1 - \lambda)x + y + \sigma I(s), \quad s \in [0, T]. \end{aligned}$$

Therefore, we have from (2.4) (suppressing s in the notation), $Y \leq y + \sigma I + (1 - \lambda)L \leq A + BI, Y \geq y + \sigma I - M \geq -A - BI$, for some positive constants A, B , which do not depend on s . Hence

$$(3.5) \quad |Y(s)| \leq A + B|I(s)|.$$

Next, fix $t \in [t_0, T]$ and define

$$(3.6) \quad \begin{aligned} \tau_k &\triangleq t \wedge \inf\{s \leq t; |Y(s)| \geq k\}, \\ Y^{(k)}(s) &\triangleq Y(s) \quad \text{if } s \leq \tau_k, \\ &= 0 \quad \text{if } s > \tau_k, \end{aligned}$$

for all $k \in \mathcal{N}$. Let $I^{(k)}$ be the corresponding integral of $Y^{(k)}$, that is, $I^{(k)}(s) = I(s)$ for $s \leq \tau_k$ and $= I(\tau_k)$ for $s \geq \tau_k$. It is easy to see from (3.5) that $|Y^{(k)}(s)| \leq A + B|I^{(k)}(s)|$. We have then, for $s \in [t_0, t]$,

$$(3.7) \quad E[I^{(k)}(s)]^2 = \int_{t_0}^s E[Y^{(k)}(u)]^2 du \leq 2A^2T + 2B^2T \int_{t_0}^s E[I^{(k)}(u)]^2 du.$$

Therefore, by Gronwall's inequality, $E[I^{(k)}(t)]^2 \leq C$ (C independent of t). When $k \rightarrow \infty$, then $\tau_k \rightarrow t$, $I^{(k)}(t) = I(\tau_k) \rightarrow I(t)$ and (3.3) follows by Fatou's lemma. \square

THEOREM 3.2. *The set H of (2.8) is relatively closed in $[0, T] \times (0, \infty) \times \mathcal{S}$, that is, if $(t_n, p_n, x_n, y_n) \in [0, T] \times (0, \infty) \times \mathcal{S}$ converges to $(t_0, p, x, y) \in [0, T] \times (0, \infty) \times \mathcal{S}$ and satisfies $(t_n, p_n, x_n, y_n) \in H$ for every $n \in \mathbb{N}$, then $(t_0, p, x, y) \in H$.*

The proof is given in the Appendix.

COROLLARY 3.3. *The function φ of (2.11) is lower semicontinuous.*

PROPOSITION 3.4. *For each $t_0 \in [0, T]$, the set $H(t_0)$ is convex.*

PROOF. Let (p_1, x_1, y_1) and (p_2, x_2, y_2) be in $H(t_0)$ and let L_1, M_1 and L_2, M_2 be such that $X_{L_i, M_i}^{t_0, x_i, y_i}(T)$, $Y_{L_i, M_i}^{t_0, x_i, y_i}(T)$ and $P^{t_0, p_i}(T)$ satisfy (2.6) and (2.7) for $i = 1, 2$. For $\alpha_1, \alpha_2 \in [0, 1]$ with $\alpha_1 + \alpha_2 = 1$, define

$$\begin{aligned} p &= \alpha_1 p_1 + \alpha_2 p_2, & x &= \alpha_1 x_1 + \alpha_2 x_2, & y &= \alpha_1 y_1 + \alpha_2 y_2, \\ L &= \alpha_1 L_1 + \alpha_2 L_2, & M &= \alpha_1 M_1 + \alpha_2 M_2. \end{aligned}$$

Then

$$\begin{aligned} X_{L, M}^{t_0, x, y}(T) &= \alpha_1 X_{L_1, M_1}^{t_0, x_1, y_1}(T) + \alpha_2 X_{L_2, M_2}^{t_0, x_2, y_2}(T), \\ Y_{L, M}^{t_0, x, y}(T) &= \alpha_1 Y_{L_1, M_1}^{t_0, x_1, y_1}(T) + \alpha_2 Y_{L_2, M_2}^{t_0, x_2, y_2}(T), \\ P^{t_0, p}(T) &= \alpha_1 P^{t_0, p_1}(T) + \alpha_2 P^{t_0, p_2}(T). \end{aligned}$$

The solvency condition is clearly satisfied by $X_{L, M}^{t_0, x, y}$ and $Y_{L, M}^{t_0, x, y}$. To check the hedging property, it suffices to consider the set

$$(3.8) \quad \{P^{t_0, p}(T) > (1 - \lambda)q, P^{t_0, p_1}(T) > (1 - \lambda)q, P^{t_0, p_2}(T) \leq (1 - \lambda)q\},$$

other cases being either similar or trivial. On this set, we have

$$\begin{aligned}
 (1 - \lambda)X_{L,M}^{t_0,x,y}(T) + Y_{L,M}^{t_0,x,y}(T) &\geq \alpha_1[-(1 - \lambda)q + P^{t_0,p_1}(T)] \\
 &\quad + \alpha_2[P^{t_0,p_2}(T) - (1 - \lambda)q] \\
 &= -(1 - \lambda)q + P^{t_0,p}(T), \\
 X_{L,M}^{t_0,x,y}(T) + (1 - \mu)Y_{L,M}^{t_0,x,y}(T) &\geq \alpha_1[-q + (1 - \mu)P^{t_0,p_1}(T)] \\
 &\quad + \alpha_2(1 - \mu)[P^{t_0,p_2}(T) - (1 - \lambda)q] \\
 &\geq -q + (1 - \mu)P^{t_0,p}(T).
 \end{aligned}$$

Therefore, $(p, x, y) \in H(t_0)$. \square

Notice that we are entitled to write minimum rather than infimum in (2.11) because of Theorem 3.2. Proposition 3.4 implies $\varphi(t, \cdot, \cdot)$ is convex, and hence continuous for each t , and so the subdifferentials

$$\begin{aligned}
 \partial_p \varphi(t, p, x) \\
 (3.9) \quad &\triangleq \{\delta_p \in \mathbb{R} \mid \varphi(t, p', x) \geq \varphi(t, p, x) + \delta_p(p' - p), \forall p' \in (0, \infty)\},
 \end{aligned}$$

$$\begin{aligned}
 \partial_x \varphi(t, p, x) \\
 (3.10) \quad &\triangleq \{\delta_x \in \mathbb{R} \mid \varphi(t, p, x') \geq \varphi(t, p, x) + \delta_x(x' - x), \forall x' \in \mathbb{R}\}
 \end{aligned}$$

are nonempty, compact sets for each $(t, p, x) \in [0, T] \times (0, \infty) \times \mathbb{R}$. In particular, the left and right derivatives $D_x^- \varphi(t, p, x)$ and $D_x^+ \varphi(t, p, x)$ with respect to x exist, are left and right continuous, respectively, and satisfy

$$(3.11) \quad D_x^- \varphi(t, p, x) \leq D_x^+ \varphi(t, p, x),$$

with equality holding whenever one of them is continuous in x (with t and p held fixed). We have

$$\begin{aligned}
 \partial_x \varphi(t, p, x) &= [D_x^- \varphi(t, p, x), D_x^+ \varphi(t, p, x)] \\
 (3.12) \quad &\forall (t, p, x) \in [0, T] \times (0, \infty) \times \mathbb{R}.
 \end{aligned}$$

PROPOSITION 3.5. *Let $v_1 = (-(1 - \mu), 1)$ and $v_2 = (1, -(1 - \lambda))$. If, for some $(t_0, p) \in [0, T] \times (0, \infty)$, we have $(x, y) \in H(t_0, p)$, then*

$$(x, y) + \alpha v_1 \in H(t_0, p), \quad (x, y) + \alpha v_2 \in H(t_0, p) \quad \forall \alpha > 0.$$

In particular,

$$\begin{aligned}
 -\frac{1}{1 - \mu} &\leq D_x^- \varphi(t, p, x) \leq D_x^+ \varphi(t, p, x) \leq -(1 - \lambda) \\
 (3.13) \quad &\forall (t, p, x) \in [0, T] \times (0, \infty) \times \mathbb{R}.
 \end{aligned}$$

PROOF. From $(x, y) + \alpha v_1$ and $(x, y) + \alpha v_2$, one can jump to (x, y) by a transaction. \square

PROPOSITION 3.6. For every $(t_0, p) \in [0, T] \times (0, \infty)$, we have

$$(3.14) \quad (p - q)^+ - (1 - \lambda)x \leq \varphi(t_0, p, x) \leq p - (1 - \lambda)x \quad \forall x \geq 0,$$

$$(3.15) \quad \begin{aligned} & ((1 - \mu)p - q)^+ - \left(\frac{1}{1 - \mu}\right)x \\ & \leq \varphi(t_0, p, x) \leq p - \left(\frac{1}{1 - \mu}\right)x \quad \forall x \leq 0. \end{aligned}$$

PROOF. Let $\gamma = 1 - (1 - \mu)(1 - \lambda)$. For $(t_0, p, x) \in [0, T] \times (0, \infty) \times \mathbb{R}$, define $y = \varphi(t_0, p, x)$. Let L, M be such that $X_{L,M}^{t_0,x,y}(T), Y_{L,M}^{t_0,x,y}(T)$ and $P^{t_0,p}(T)$ satisfy (2.6) and (2.7). We have

$$\begin{aligned} E(P^{t_0,p}(T) - q)^+ & \leq E(P^{t_0,p}(T) - (1 - \lambda)q)^+ \\ & \leq E[(1 - \lambda)X_{L,M}^{t_0,x,y}(T) + Y_{L,M}^{t_0,x,y}(T)] \\ & = (1 - \lambda)x + y - \gamma EM(T) + E \int_{t_0}^T \sigma Y_{L,M}^{t_0,x,y}(s) dW(s) \\ & \leq (1 - \lambda)x + y, \end{aligned}$$

because $E \int_{t_0}^T (Y_{L,M}^{t_0,x,y}(s))^2 ds < \infty$, according to (3.3) and (3.5). However, $(P^{t_0,p}(s) - q)^+, t_0 \leq s \leq T$, is a submartingale, so

$$(p - q)^+ \leq E(P^{t_0,p}(T) - q)^+ \leq (1 - \lambda)x + y,$$

and the first inequality in (3.14) follows. On the other hand, if $X(t_0) = x, Y(t_0) = p - (1 - \lambda)x$ and $x \geq 0$, we can jump to $(0, p)$, and hold one share of stock until the final time, so that $X(T) = 0, Y(T) = P(T)$ and (2.6) and (2.7) are satisfied. This establishes the second inequality in (3.14).

For (3.15), we observe from (2.7') that

$$\begin{aligned} ((1 - \mu)p - q)^+ & \leq E((1 - \mu)P^{t_0,p}(T) - q)^+ \\ & \leq \frac{1}{1 - \mu} E[-q + (1 - \mu)P^{t_0,p}(T)]^+ \\ & \leq \frac{1}{1 - \mu} E[X_{L,M}^{t_0,x,y}(T) + (1 - \mu)Y_{L,M}^{t_0,x,y}(T)] \\ & = \frac{1}{1 - \mu} \left[x + (1 - \mu)y - \gamma EL(T) \right. \\ & \quad \left. - (1 - \mu)E \int_{t_0}^T \sigma Y_{L,M}^{t_0,x,y}(s) dW(s) \right] \\ & \leq \frac{x}{1 - \mu} + y. \end{aligned}$$

If $X(t_0) = x, Y(t_0) = p - (1/(1 - \mu))x$, we can again jump to $(0, p)$. \square

LEMMA 3.7. For every $(t, p, x) \in [0, T) \times (0, \infty) \times \mathbb{R}$, we have

$$\partial_p \varphi(t, p, x) \subset \left[0, \frac{1}{p} \left(\varphi(t, p, x) + \frac{x^+ + q}{(1 - \lambda)(1 - \mu)} \right) \right].$$

PROOF. It is clear from the hedging conditions (2.6') and (2.7') that φ is nondecreasing in p , so $\partial_p \varphi(t, p, x) \subset [0, \infty)$. It remains to find an upper bound for $D_p^+ \varphi(t, p, x)$.

Let $y = \varphi(t, p, x)$ and let L, M be such that $X_{L,M}^{t,x,y}(T), Y_{L,M}^{t,x,y}(T)$ and $P^{t,p}(T)$ satisfy (2.6') and (2.7'). Let $\alpha > 1$ be given and set $\bar{x} = \alpha x + (\alpha - 1)q/(1 - \lambda)$, $\bar{y} = \alpha y$, $\bar{p} = \alpha p$, $\bar{L} = \alpha L$ and $\bar{M} = \alpha M$. Then

$$X_{\bar{L},\bar{M}}^{t,\bar{x},\bar{y}}(T) = \alpha X_{L,M}^{t,x,y}(T) + \frac{(\alpha - 1)q}{1 - \lambda},$$

$$Y_{\bar{L},\bar{M}}^{t,\bar{x},\bar{y}}(T) = \alpha Y_{L,M}^{t,x,y}(T),$$

$$P^{t,\bar{p}}(T) = \alpha P^{t,p}(T)$$

and

$$\begin{aligned} (1 - \lambda)X_{\bar{L},\bar{M}}^{t,\bar{x},\bar{y}}(T) + Y_{\bar{L},\bar{M}}^{t,\bar{x},\bar{y}}(T) &= \alpha[(1 - \lambda)X_{L,M}^{t,x,y}(T) + Y_{L,M}^{t,x,y}(T)] + (\alpha - 1)q \\ &\geq \alpha(P^{t,p}(T) - (1 - \lambda)q)^+ + (1 - \lambda)(\alpha - 1)q. \end{aligned}$$

However,

$$\begin{aligned} \alpha(P^{t,p}(T) - (1 - \lambda)q)^+ + (1 - \lambda)(\alpha - 1)q \\ &\geq \alpha(P^{t,p}(T) - (1 - \lambda)q) + (1 - \lambda)(\alpha - 1)q \\ &= P^{t,\bar{p}}(T) - (1 - \lambda)q, \end{aligned}$$

and since the left-hand side is nonnegative, we in fact have

$$\alpha(P^{t,p}(T) - (1 - \lambda)q)^+ + (1 - \lambda)(\alpha - 1)q \geq (P^{t,\bar{p}}(T) - (1 - \lambda)q)^+.$$

Similarly,

$$\begin{aligned} X_{\bar{L},\bar{M}}^{t,\bar{x},\bar{y}}(T) + (1 - \mu)Y_{\bar{L},\bar{M}}^{t,\bar{x},\bar{y}}(T) &\geq \alpha[X_{L,M}^{t,x,y}(T) + (1 - \mu)Y_{L,M}^{t,x,y}(T)] + (\alpha - 1)q \\ &\geq \alpha[-q + (1 - \mu)P^{t,p}(T)]^+ + (\alpha - 1)q \\ &\geq [-q + (1 - \mu)P^{t,\bar{p}}(T)]^+. \end{aligned}$$

Therefore, $\bar{y} \in H(t, \bar{p}, \bar{x})$ or, equivalently,

$$\begin{aligned} \alpha \varphi(t, p, x) &\geq \varphi\left(t, \alpha p, \alpha x + (\alpha - 1)\frac{q}{(1 - \lambda)}\right) \\ &\geq \varphi(t, \alpha p, x) - \frac{(\alpha - 1)(x^+ + q)}{(1 - \lambda)(1 - \mu)}, \end{aligned}$$

where we have used (3.13). From this inequality, we have

$$\varphi(t, \alpha p, x) - \varphi(t, p, x) \leq (\alpha - 1) \left[\varphi(t, p, x) + \frac{x^+ + q}{(1 - \lambda)(1 - \mu)} \right].$$

Dividing by $(\alpha - 1)p$ and letting $\alpha \downarrow 1$, we obtain

$$D_p^+ \varphi(t, p, x) \leq \frac{1}{p} \left[\varphi(t, p, x) + \frac{x^+ + q}{(1 - \lambda)(1 - \mu)} \right]. \quad \square$$

PROPOSITION 3.8. *The mapping $\varphi(t, \cdot, \cdot)$ is locally Lipschitz on $(0, \infty) \times \mathbb{R}$, uniformly in $t \in [0, T]$.*

PROOF. The bounds in Proposition 3.6 are independent of t and so, because of Lemma 3.7, $\partial_p \varphi(t, p, x)$ is locally bounded in p and x , independently of t . According to (3.13), $\partial_x \varphi(t, p, x)$ is bounded, independently of t, p and x . \square

PROPOSITION 3.9. *We have*

$$\lim_{\substack{t' \downarrow t \\ p' \rightarrow p \\ x' \rightarrow x}} \varphi(t', p', x') = \varphi(t, p, x) \quad \forall (t, p, x) \in [0, T] \times (0, \infty) \times \mathbb{R}.$$

PROOF. In light of Proposition 3.8, it suffices to show that for some function $f: [t, T] \rightarrow (0, \infty)$ with f continuous at t and $f(t) = p$, that

$$(3.16) \quad \lim_{t' \downarrow t} \varphi(t', f(t'), x) = \varphi(t, p, x).$$

Let $y = \varphi(t, p, x)$ and let L, M be such that $X_{L,M}^{t,x,y}(T), Y_{L,M}^{t,x,y}(T)$ and $P^{t,p}(T)$ satisfy (2.6) and (2.7). Then, for $t' \in (t, T)$,

$$Y_{L,M}^{t,x,y}(t') \geq \varphi(t', P^{t,p}(t'), X_{L,M}^{t,x,y}(t')).$$

From (2.3) and (3.13), we have

$$\begin{aligned} \varphi(t', P^{t,p}(t'), X_{L,M}^{t,x,y}(t')) &= \varphi(t', P^{t,p}(t'), x - L(t') + (1 - \mu)M(t')) \\ &\geq \varphi(t', P^{t,p}(t'), x + (1 - \mu)M(t')) + (1 - \lambda)L(t') \\ &\geq \varphi(t', P^{t,p}(t'), x) - M(t') + (1 - \lambda)L(t'). \end{aligned}$$

From (2.4) we have

$$Y_{L,M}^{t,x,y}(t') = \varphi(t, p, x) - M(t') + (1 - \lambda)L(t') + \int_t^{t'} \sigma Y_{L,M}^{t,x,y}(s) dW(s).$$

These three relations imply

$$\varphi(t, p, x) + \int_t^{t'} \sigma Y_{L,M}^{t,x,y}(s) dW(s) \geq \varphi(t', P^{t,p}(t'), x),$$

and thus

$$\varphi(t, p, x) \geq \limsup_{t' \downarrow t} \varphi(t', P^{t,p}(t'), x).$$

Corollary 3.3 implies

$$\varphi(t, p, x) \leq \liminf_{t' \downarrow t} \varphi(t', P^{t,p}(t'), x)$$

and we have (3.16). \square

LEMMA 3.10. *Let $(t_0, p_0, x_0) \in [0, T) \times (0, \infty) \times \mathbb{R}$ be given, define $y_0 = \varphi(t_0, p_0, x_0)$ and let L, M be such that $X_{L,M}^{t_0, x_0, y_0}(T), Y_{L,M}^{t_0, x_0, y_0}(T)$ and $P^{t_0, p_0}(T)$ satisfy (2.6) and (2.7). If*

$$(3.17) \quad \partial_x \varphi(t_0, p_0, x_0) \cap \left(-\frac{1}{1-\mu}, -(1-\lambda) \right) \neq \emptyset,$$

then L and M are continuous at t_0 .

PROOF. Define $\ell = L(t_0+)$ and $m = M(t_0+)$, so

$$X(t_0+) = x - \ell + (1 - \mu)m, \quad Y(t_0+) = y - m + (1 - \lambda)\ell.$$

Choose $\delta_x \in \partial_x \varphi(t_0, p_0, x_0) \cap (-1/(1 - \mu), -(1 - \lambda))$. We have

$$\begin{aligned} \varphi(t_0, p_0, x_0) - m + (1 - \lambda)\ell &= Y(t_0+) \\ &\geq \varphi(t_0, p_0, X(t_0+)) \\ &= \varphi(t_0, p_0, x_0 - \ell + (1 - \mu)m) \\ &\geq \varphi(t_0, p_0, x_0) + \delta_x(-\ell + (1 - \mu)m), \end{aligned}$$

which implies

$$(1 - \lambda + \delta_x)\ell \geq (1 + \delta_x(1 - \mu))m.$$

However, $1 - \lambda + \delta_x < 0$ and $1 + \delta_x(1 - \mu) > 0$, so we must have $\ell = m = 0$. \square

THEOREM 3.11. *Let $(t_0, p_0, x_0) \in [0, T) \times (0, \infty) \times \mathbb{R}$ be given. If (3.17) holds, then $(\partial/\partial p)\varphi(t_0, p_0, x_0)$ is defined and*

$$\frac{\partial}{\partial p} \varphi(t_0, p_0, x_0) = \frac{1}{p_0} \varphi(t_0, p_0, x_0).$$

PROOF. Choose $\delta_x \in \partial_x \varphi(t_0, p_0, x_0) \cap (-1/(1 - \mu), -(1 - \lambda))$ and $\delta_p \in \partial_p \varphi(t_0, p_0, x_0)$. Define

$$\psi(p, x) = \varphi(t_0, p_0, x_0) + \delta_p(p - p_0) + \delta_x(x - x_0) - (p - p_0)^2 - (x - x_0)^2,$$

so $\psi(p_0, x_0) = \varphi(t_0, p_0, x_0)$ and

$$(3.18) \quad \psi(p, x) < \varphi(t_0, p, x) \quad \text{for } (p, x) \neq (p_0, x_0).$$

For $\varepsilon \in (0, (T - t_0) \wedge p_0)$, define

$$B_\varepsilon = [t_0, t_0 + \delta_\varepsilon) \times (p_0 - \varepsilon, p_0 + \varepsilon) \times (x_0 - \varepsilon, x_0 + \varepsilon),$$

where $\delta_\varepsilon \in (0, \varepsilon)$ will be determined below. For ε sufficiently small, we have

$$(3.19) \quad \frac{\partial}{\partial x} \psi(p, x) \in \left(-\frac{1}{1 - \mu}, -(1 - \lambda) \right), \quad \forall (t, p, x) \in \overline{B}_\varepsilon.$$

Let us consider the face $F_0 \triangleq \{t_0\} \times [p_0 - \varepsilon, p_0 + \varepsilon] \times [x_0 - \varepsilon, x_0 + \varepsilon]$ of the cube \overline{B}_ε . This face has four edges, $\{t_0\} \times \{p_0 \pm \varepsilon\} \times [x_0 - \varepsilon, x_0 + \varepsilon]$ and $\{t_0\} \times [p_0 - \varepsilon, p_0 + \varepsilon] \times \{x_0 \pm \varepsilon\}$, and on these edges, $\varphi - \psi$ is strictly positive because of (3.18). In fact, because $\varphi - \psi$ is lower semicontinuous, $\varphi - \psi$ is bounded away from zero on these edges and, by Proposition 3.9, for some $\delta_\varepsilon \in (0, \varepsilon)$, $\varphi - \psi$ is also strictly positive on the four faces

$$F_1^\pm \triangleq [t_0, t_0 + \delta_\varepsilon] \times \{p_0 \pm \varepsilon\} \times [x_0 - \varepsilon, x_0 + \varepsilon],$$

$$F_2^\pm \triangleq [t_0, t_0 + \delta_\varepsilon] \times [p_0 - \varepsilon, p_0 + \varepsilon] \times \{x_0 \pm \varepsilon\}.$$

On the face $F_3 \triangleq \{t_0 + \delta_\varepsilon\} \times [p_0 - \varepsilon, p_0 + \varepsilon] \times [x_0 - \varepsilon, x_0 + \varepsilon]$, $\varphi - \psi$ may fail to be positive, but it is bounded from below, and for some k_ε sufficiently large and

$$\psi_\varepsilon(t, p, x) \triangleq \psi(p, x) - k_\varepsilon(t - t_0),$$

we have $\varphi - \psi_\varepsilon$ positive on the five faces F_1^\pm, F_2^\pm, F_3 .

At (t_0, p_0, x_0) , $\varphi - \psi_\varepsilon$ is zero, and so the lower semicontinuous function $\varphi - \psi_\varepsilon$ attains a nonpositive minimum over \overline{B}_ε at some point $(t_\varepsilon, p_\varepsilon, x_\varepsilon) \in B_\varepsilon$. Because $(\varphi - \psi_\varepsilon)(t_\varepsilon, p_\varepsilon, x)$ is minimized over $x \in [x_0 - \varepsilon, x_0 + \varepsilon]$ at $x_\varepsilon \in (x_0 - \varepsilon, x_0 + \varepsilon)$, we have

$$\frac{\partial}{\partial x} \psi(p_\varepsilon, x_\varepsilon) = \frac{\partial}{\partial x} \psi_\varepsilon(t_\varepsilon, p_\varepsilon, x_\varepsilon) \in \partial_x \varphi(t_\varepsilon, p_\varepsilon, x_\varepsilon).$$

From (3.19), we conclude that

$$(3.20) \quad \partial_x \varphi(t_\varepsilon, p_\varepsilon, x_\varepsilon) \cap \left(-\frac{1}{1 - \mu}, -(1 - \lambda) \right) \neq \emptyset.$$

Set $y_\varepsilon = \varphi(t_\varepsilon, p_\varepsilon, x_\varepsilon)$ and let $L_\varepsilon, M_\varepsilon$ be such that $X_{L_\varepsilon, M_\varepsilon}^{t_\varepsilon, x_\varepsilon, y_\varepsilon}(T), Y_{L_\varepsilon, M_\varepsilon}^{t_\varepsilon, x_\varepsilon, y_\varepsilon}(T)$ and $P^{t_\varepsilon, p_\varepsilon}(T)$ satisfy (2.6) and (2.7). To simplify notation, we suppress superscripts and subscripts on the processes. According to Lemma 3.10, $L = L_\varepsilon$ and $M = M_\varepsilon$ are continuous at t_ε . Thus,

$$\tau_\varepsilon \triangleq \min\{t \geq t_\varepsilon \mid (t, P(t), X(t)) \notin \overline{B}_\varepsilon\}$$

is strictly greater than t_ε , almost surely.

Itô's rule for semimartingales [Meyer (1976), page 301, or Protter (1990), page 74] implies

$$\begin{aligned} &\psi_\varepsilon(t \wedge \tau_\varepsilon, P(t \wedge \tau_\varepsilon), X(t \wedge \tau_\varepsilon)) \\ &= \psi_\varepsilon(t_\varepsilon, p_\varepsilon, x_\varepsilon) + \int_{t_\varepsilon}^{t \wedge \tau_\varepsilon} \sigma P \frac{\partial}{\partial p} \psi_\varepsilon dW \\ &\quad + \int_{t_\varepsilon}^{t \wedge \tau_\varepsilon} \left(\frac{\partial}{\partial t} \psi_\varepsilon + \frac{1}{2} \sigma^2 P^2 \frac{\partial^2}{\partial p^2} \psi_\varepsilon \right) ds \\ &\quad + \int_{t_\varepsilon}^{t \wedge \tau_\varepsilon} \frac{\partial}{\partial x} \psi_\varepsilon (-dL^c + (1 - \mu) dM^c) \\ &\quad + \sum_{0 \leq s < t \wedge \tau_\varepsilon} [\psi_\varepsilon(s, P(s), X(s+)) - \psi_\varepsilon(s, P(s), X(s))], \end{aligned}$$

where L^c and M^c denote the continuous parts of L and M . Unlike the references cited, we have adopted the convention of left-continuous processes, so any jump in X at time $t \wedge \tau_\varepsilon$ does not appear on the left-hand side of the above equation; hence the sum on the right-hand side is over $s < t \wedge \tau_\varepsilon$ rather than $s \leq t \wedge \tau_\varepsilon$. In \bar{B}_ε , the absolute value of the integrand $|(\partial/\partial t)\psi_\varepsilon + \frac{1}{2}\sigma^2 P^2(\partial^2/\partial p^2)\psi_\varepsilon|$ is bounded by some constant C_ε , so

$$\int_{t_\varepsilon}^{t \wedge \tau_\varepsilon} \left(\frac{\partial}{\partial t} \psi_\varepsilon + \frac{1}{2} \sigma^2 P^2 \frac{\partial^2}{\partial p^2} \psi_\varepsilon \right) ds \geq -C_\varepsilon(t - t_\varepsilon).$$

With ε chosen small enough to satisfy (3.19), with $0 \leq s < \tau_\varepsilon$, and with $\ell = L(s+) - L(s)$ and $m = M(s+) - M(s)$, the segment connecting $(s, P(s), X(s+))$ and $(s, P(s), X(s))$ lies in \bar{B}_ε and so $-1/(1 - \mu) \leq (\partial/\partial x)\psi_\varepsilon \leq -(1 - \lambda)$ at every point on the segment. Hence

$$\begin{aligned} &\psi_\varepsilon(s, P(s), X(s+)) - \psi_\varepsilon(s, P(s), X(s)) \\ &= \psi_\varepsilon(s, P(s), X(s) - \ell + (1 - \mu)m) - \psi_\varepsilon(s, P(s), X(s)) \\ &= \theta(-\ell + (1 - \mu)m), \end{aligned}$$

where $-1/(1 - \mu) \leq \theta \leq -(1 - \lambda)$. If $-\ell + (1 - \mu)m \geq 0$, then

$$\begin{aligned} \theta(-\ell + (1 - \mu)m) &\geq -\frac{1}{1 - \mu}(-\ell + (1 - \mu)m) \\ &\geq -m + (1 - \lambda)\ell, \end{aligned}$$

whereas if $-\ell + (1 - \mu)m < 0$, then

$$\begin{aligned} \theta(-\ell + (1 - \mu)m) &\geq -(1 - \lambda)(-\ell + (1 - \mu)m) \\ &\geq -m + (1 - \lambda)\ell. \end{aligned}$$

In either case, we have

$$\begin{aligned} & \psi_\varepsilon(s, P(s), X(s+)) - \psi_\varepsilon(s, P(s), X(s)) \\ & \geq -(M(s+) - M(s)) + (1 - \lambda)(L(s+) - L(s)). \end{aligned}$$

Therefore,

$$\begin{aligned} & \psi_\varepsilon(t \wedge \tau_\varepsilon, P(t \wedge \tau_\varepsilon), X(t \wedge \tau_\varepsilon)) - \psi_\varepsilon(t_\varepsilon, P_\varepsilon, x_\varepsilon) \\ & \geq \int_{t_\varepsilon}^{t \wedge \tau_\varepsilon} \sigma P \frac{\partial}{\partial p} \psi_\varepsilon dW - C_\varepsilon(t - t_\varepsilon) + \int_{t_\varepsilon}^{t \wedge \tau_\varepsilon} \frac{\partial}{\partial x} \psi_\varepsilon (-dL^c + (1 - \mu) dM^c) \\ & \quad + \sum_{t_\varepsilon \leq s < t \wedge \tau_\varepsilon} [-(M(s+) - M(s)) + (1 - \lambda)(L(s+) - L(s))]. \end{aligned}$$

On the other hand,

$$Y(t \wedge \tau_\varepsilon) - \varphi(t_\varepsilon, p_\varepsilon, x_\varepsilon) = \int_{t_\varepsilon}^{t \wedge \tau_\varepsilon} [-dM(s) + (1 - \lambda) dL(s)] + \int_{t_\varepsilon}^{t \wedge \tau_\varepsilon} \sigma Y(s) dW(s).$$

Subtracting these relations, and using (3.19) and the fact that

$$Y(t \wedge \tau_\varepsilon) \geq \varphi(t \wedge \tau_\varepsilon, P(t \wedge \tau_\varepsilon), X(t \wedge \tau_\varepsilon)),$$

we arrive at the inequality

$$\begin{aligned} & [\varphi(t \wedge \tau_\varepsilon, P(t \wedge \tau_\varepsilon), X(t \wedge \tau_\varepsilon)) - \psi_\varepsilon(t \wedge \tau_\varepsilon, P(t \wedge \tau_\varepsilon), X(t \wedge \tau_\varepsilon))] \\ & \quad - [\varphi(t_\varepsilon, p_\varepsilon, x_\varepsilon) - \psi_\varepsilon(t_\varepsilon, p_\varepsilon, x_\varepsilon)] \\ & \leq \int_{t_\varepsilon}^{t \wedge \tau_\varepsilon} \sigma [Y(s) - P(s)] \frac{\partial}{\partial p} \psi_\varepsilon(s, P(s), X(s)) dW(s) + C_\varepsilon(t - t_\varepsilon). \end{aligned}$$

However, $\varphi - \psi$ attains its minimum over \bar{B}_ε at $(t_\varepsilon, p_\varepsilon, x_\varepsilon)$, so the left side of this inequality is nonnegative. We conclude that

$$(3.21) \quad -C_\varepsilon(t - t_\varepsilon) \leq \int_{t_\varepsilon}^t \sigma \mathbf{1}_{\{s \leq \tau_\varepsilon\}} \left[Y(s) - P(s) \frac{\partial}{\partial p} \psi_\varepsilon(s, P(s), X(s)) \right] dW(s) \quad \forall t \in [t_\varepsilon, T].$$

According to the following lemma, (3.21) implies

$$\varphi(t_\varepsilon, p_\varepsilon, x_\varepsilon) = p_\varepsilon \frac{\partial}{\partial p} \psi_\varepsilon(t_\varepsilon, p_\varepsilon, x_\varepsilon) = p_\varepsilon [\delta_p - 2(p_\varepsilon - p_0)].$$

Letting $\varepsilon \downarrow 0$ and using Proposition 3.9, we obtain

$$\varphi(t_0, p_0, x_0) = p_0 \delta_p, \quad \forall \delta_p \in \partial_p \varphi(t_0, p_0, x_0). \quad \square$$

LEMMA 3.12. *Let $X(t)$ be an adapted, left-continuous, right-limited process which is continuous at $t = 0$, with $X(0)$ nonrandom. Let C be a constant and*

assume

$$-Ct \leq \int_0^t X(s) dW(s) \quad \forall t \in [0, 1].$$

Then $X(0) = 0$.

PROOF. We first consider the case $C = 0$. Let $x = X(0)$ and assume $x \neq 0$. We take $x > 0$ with no loss of generality. Define

$$\tau = 1 \wedge \inf\{t \in [0, 1] \mid X(t) < \frac{1}{2}x\},$$

$$M(t) = \int_0^t X(s) dW(s),$$

$$\hat{M}(t) = \int_0^t (X(s) \vee \frac{1}{2}x) dW(s),$$

so that

$$M(t) = \hat{M}(t), \quad 0 \leq t \leq \tau.$$

Now $\langle \hat{M} \rangle(t) = \int_0^t (X(s) \vee \frac{1}{2}x)^2 ds$ is strictly increasing and $\hat{M}(\langle \hat{M} \rangle^{-1}(s))$ is a standard Brownian motion. Therefore, for P -almost every ω , there is a sequence $s_1 > s_2 > \dots > 0$ with $s_n \downarrow 0$ such that $\hat{M}(\langle \hat{M} \rangle^{-1}(s_n)) < 0$ for every n . Set $t_n = \langle \hat{M} \rangle^{-1}(s_n)$, so $t_n \downarrow 0$ and $\hat{M}(t_n) < 0$ for every n . It follows that $M(t_n) < 0$ for n sufficiently large, and we have a contradiction.

If $C \neq 0$, use Girsanov's theorem to change the probability measure so that $\tilde{W}(t) \triangleq W(t) + \int_0^t C ds / ((X(s) \vee \frac{1}{2}x))$ is a Brownian motion. Then $0 \leq Ct + \int_0^t X(s) dW(s) = \int_0^t X(s) d\tilde{W}(s)$ for $0 \leq t \leq \tau$. Proceed as before. \square

Henceforth, we hold $t_0 \in [0, T)$ fixed and suppress t_0 in the notation. We define

$$(3.22) \quad a(p) = \max\left\{x \in \mathbb{R}; D_x^- \varphi(p, x) = -\frac{1}{1-\mu}\right\},$$

$$(3.23) \quad b(p) = \min\{x \in \mathbb{R}; D_x^+ \varphi(p, x) = -(1-\lambda)\},$$

where $a(p)$ is $-\infty$ and $b(p)$ is $+\infty$ if the sets in (3.22) and (3.23), respectively, are empty. We want to show that $a(\cdot)$ and $b(\cdot)$ are finite and nondecreasing, with $a(0) = b(0) = 0$.

LEMMA 3.13. For every $p > 0$, we have

$$(3.24) \quad a(p) \leq a^* \triangleq \frac{1-\mu}{\gamma} (p - ((1-\mu)p - q)^+),$$

$$(3.25) \quad b(p) \geq b^* \triangleq -\frac{1-\mu}{\gamma} (p - ((1-\mu)p - q)^+),$$

where $\gamma \triangleq 1 - (1-\lambda)(1-\mu)$.

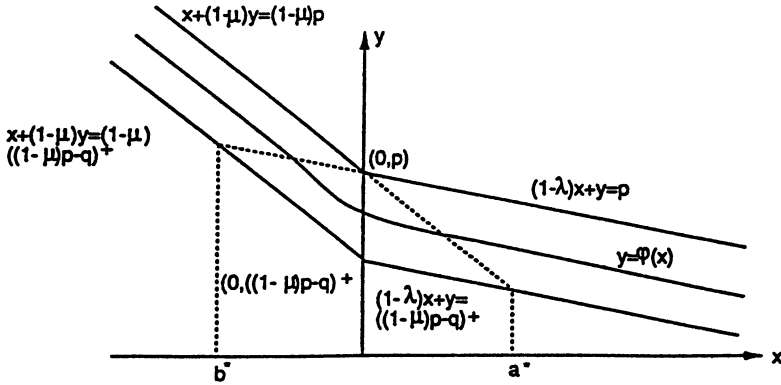


FIG. 2.

PROOF. From Proposition 3.6 [and $((1 - \mu)p - q)^+ \leq (p - q)^+$] we have the bounds on φ as in Figure 2. The figure shows that $a(p) \leq a^*$ and $b(p) \geq b^*$. We find a^* by solving simultaneously the equations

$$\begin{aligned} a^* + (1 - \mu)y &= (1 - \mu)p, \\ (1 - \lambda)a^* + y &= ((1 - \mu)p - q)^+. \end{aligned}$$

The solution is $a^* = [(1 - \mu)/\gamma](p - ((1 - \mu)p - q)^+)$.

We find b^* by solving simultaneously the equations

$$\begin{aligned} b^* + (1 - \mu)y &= (1 - \mu)((1 - \mu)p - q)^+, \\ (1 - \lambda)b^* + y &= p. \end{aligned}$$

The solution is $b^* = -[(1 - \mu)/\gamma](p - ((1 - \mu)p - q)^+)$. \square

LEMMA 3.14. *The quantity $a(\cdot)$ is upper semicontinuous and $b(\cdot)$ is lower semicontinuous.*

PROOF. Suppose $p_n \rightarrow p$. If $a(p_n) > -\infty$, then $D_x^- \varphi(p_n, a(p_n)) = -1/(1 - \mu)$, that is,

$$\varphi(p_n, x') \geq \varphi(p_n, a(p_n)) - \frac{1}{1 - \mu}(x' - a(p_n)) \quad \forall x' \in \mathbb{R}.$$

If $a(p_n)$ converges to $a_0 \in \mathbb{R}$ along some subsequence, then taking the limit along this subsequence, we obtain

$$\varphi(p, x') \geq \varphi(p, a_0) - \frac{1}{1 - \mu}(x' - a_0) \quad \forall x' \in \mathbb{R},$$

so $-1/(1 - \mu) \in \partial_p \varphi(p, a_0)$ and $a(p) \geq a_0$. Hence,

$$\limsup_{n \rightarrow \infty} a(p_n) \leq a(p).$$

A similar argument proves the lower semicontinuity of $b(\cdot)$. \square

For $\beta \in (-1/(1-\mu), -(1-\lambda))$, we set

$$(3.26) \quad a(p, \beta) = \max\{x; D_x^- \varphi(p, x) \leq \beta\},$$

$$(3.27) \quad b(p, \beta) = \min\{x; D_x^+ \varphi(p, x) \geq \beta\}.$$

Proposition 3.6 together with the monotonicity of $\varphi(p, \cdot)$ shows that

$$\begin{aligned} \lim_{x \rightarrow -\infty} D_x^- \varphi(p, x) &= \lim_{x \rightarrow -\infty} D_x^+ \varphi(p, x) = -\frac{1}{1-\mu}, \\ \lim_{x \rightarrow \infty} D_x^- \varphi(p, x) &= \lim_{x \rightarrow \infty} D_x^+ \varphi(p, x) = -(1-\lambda), \end{aligned}$$

so $a(p, \beta)$ and $b(p, \beta)$ are finite for all $p > 0$ and $\beta \in (-1/(1-\mu), -(1-\lambda))$.

The following lemma is proved just as we proved Lemma 3.14.

LEMMA 3.15. *For each $\beta \in (-1/(1-\mu), -(1-\lambda))$, $a(\cdot, \beta)$ is upper semicontinuous and $b(\cdot, \beta)$ is lower semicontinuous.*

From the inequalities

$$D_x^- \varphi(p, x) \leq D_x^+ \varphi(p, x) \leq D_x^- \varphi(p, \xi) \quad \text{if } x < \xi,$$

we have the following lemma.

LEMMA 3.16. *If $\hat{\beta}, \beta \in (-1/(1-\mu), -(1-\lambda))$ and $\hat{\beta} < \beta$, then*

$$a(p, \hat{\beta}) \leq b(p, \beta) \leq a(p, \beta).$$

Furthermore,

$$\lim_{\beta \downarrow -1/(1-\mu)} a(p, \beta) = a(p), \quad \lim_{\beta \uparrow -(1-\lambda)} b(p, \beta) = b(p).$$

LEMMA 3.17. *If $\beta \in (-1/(1-\mu), -(1-\lambda))$ and $x \in [b(p, \beta), a(p, \beta)]$, then*

$$\beta \in \partial_x \varphi(p, x), \quad \frac{\partial}{\partial p} \varphi(p, x) = \frac{1}{p} \varphi(p, x).$$

PROOF. Because $D_x^- \varphi(p, \cdot)$ is left continuous and $D_x^+ \varphi(p, \cdot)$ is right continuous, we have

$$D_x^- \varphi(p, x) \leq D_x^- \varphi(p, a(p, \beta)) \leq \beta \leq D_x^+ \varphi(p, b(p, \beta)) \leq D_x^+ \varphi(p, x),$$

which implies $\beta \in \partial_x \varphi(p, x)$. Theorem 3.11 now implies $(\partial/\partial p)\varphi(p, x) = (1/p)\varphi(p, x)$. \square

PROPOSITION 3.18. *Let $0 < p_1 < p_2$ and $-1/(1 - \mu) < \beta < -(1 - \lambda)$ be given. Then*

$$(3.28) \quad \begin{aligned} & \varphi(p_2, a(p_2, \beta)) - \varphi(p_1, a(p_1, \beta)) \\ &= \int_{p_1}^{p_2} \frac{1}{p} \varphi(p, a(p, \beta)) dp + \beta[a(p_2, \beta) - a(p_1, \beta)], \end{aligned}$$

$$(3.29) \quad \begin{aligned} & \varphi(p_2, b(p_2, \beta)) - \varphi(p_1, b(p_1, \beta)) \\ &= \int_{p_1}^{p_2} \frac{1}{p} \varphi(p, b(p, \beta)) dp + \beta[b(p_2, \beta) - b(p_1, \beta)]. \end{aligned}$$

REMARK. Proposition 3.18 gives validity to the formal computation

$$\begin{aligned} \frac{d}{dp} \varphi(p, a(p, \beta)) &= \frac{\partial}{\partial p} \varphi(p, a(p, \beta)) + \frac{\partial}{\partial x} \varphi(p, a(p, \beta)) \frac{\partial}{\partial p} a(p, \beta) \\ &= \frac{1}{p} \varphi(p, a(p, \beta)) + \beta \frac{\partial}{\partial p} a(p, \beta). \end{aligned}$$

PROOF OF PROPOSITION 3.18. We set

$$F(p) \triangleq \varphi(p, a(p, \beta)) - \beta a(p, \beta), \quad p \in [p_1, p_2],$$

and show that F is convex. Indeed, for $p_1 \leq p' < p'' \leq p_2$ and with $\bar{p} = \frac{1}{2}(p' + p'')$, $\alpha' = a(p', \beta)$, $\alpha'' = a(p'', \beta)$ and $\bar{\alpha} = \frac{1}{2}(\alpha' + \alpha'')$, we have

$$\begin{aligned} & \frac{1}{2} F(p') + \frac{1}{2} F(p'') - F(\bar{p}) \\ &= \frac{1}{2} \varphi(p', \alpha') + \frac{1}{2} \varphi(p'', \alpha'') - [\varphi(\bar{p}, a(\bar{p}, \beta)) + \beta(\bar{\alpha} - a(\bar{p}, \beta))] \\ &\geq \frac{1}{2} \varphi(p', \alpha') + \frac{1}{2} \varphi(p'', \alpha'') - \varphi(\bar{p}, \bar{\alpha}) \\ &\geq 0, \end{aligned}$$

where we have used the fact that $\beta \in \partial_x \varphi(\bar{p}, a(\bar{p}, \beta))$ and the joint convexity of φ . Lemma 3.17 implies

$$\beta \in \partial_x \varphi(p, a(p, \beta)), \quad \frac{\partial}{\partial p} \varphi(p, a(p, \beta)) = \frac{1}{p} \varphi(p, a(p, \beta)) \quad \forall p > 0.$$

Therefore, for any $p, p' \in [p_1, p_2]$, we have

$$\begin{aligned} F(p') &= \varphi(p', a(p', \beta)) - \beta a(p', \beta) \\ &\geq \varphi(p, a(p, \beta)) + \frac{1}{p} \varphi(p, a(p, \beta))(p' - p) \\ &\quad + \beta[a(p', \beta) - a(p, \beta)] - \beta a(p', \beta) \\ &= F(p) + \frac{1}{p} \varphi(p, a(p, \beta))(p' - p). \end{aligned}$$

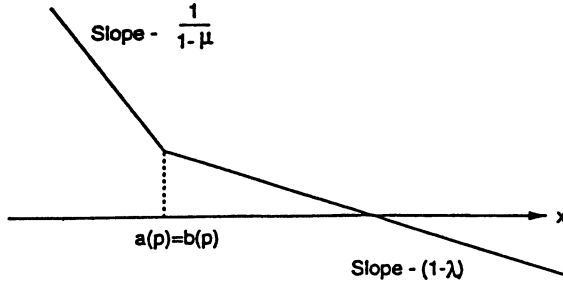


FIG. 3.

This shows that

$$\frac{1}{p}\varphi(p, a(p, \beta)) \in \partial F(p), \quad \forall p > 0.$$

Consequently,

$$F(p_2) - F(p_1) = \int_{p_1}^{p_2} \frac{1}{p}\varphi(p, a(p, \beta)) dp,$$

which is (3.28). The proof of (3.29) is completely analogous. \square

We need to understand how $a(p)$ and $b(p)$ vary with p in regions where $a(p) = b(p)$. This is the case that φ has either slope $-1/(1 - \mu)$ or $-(1 - \lambda)$, and the two regimes are separated by a “corner” (see Figure 3). If $a(p) = b(p)$, then for any $\hat{\beta}, \beta \in (-1/(1 - \mu), -(1 - \lambda))$,

$$a(p) = a(p, \hat{\beta}) = b(p, \beta) = b(p).$$

We show now that when $\hat{\beta} < \beta$ and the middle equality holds for all p in an interval $[p_1, p_2]$, then $a(p, \hat{\beta}) = b(p, \beta)$ is nondecreasing in p .

COROLLARY 3.19. *If $0 < p_1 < p_2$ and $-1/(1 - \mu) < \hat{\beta} < \beta < -(1 - \lambda)$, and if*

$$a(p_1, \hat{\beta}) = b(p_1, \beta), \quad a(p_2, \hat{\beta}) = b(p_2, \beta),$$

then $a(p_1, \hat{\beta}) \leq a(p_2, \hat{\beta})$.

PROOF. From Proposition 3.18 we have

$$\begin{aligned} & \int_{p_1}^{p_2} \frac{1}{p}\varphi(p, a(p, \hat{\beta})) dp + \hat{\beta}[a(p_2, \hat{\beta}) - a(p_1, \hat{\beta})] \\ &= \varphi(p_2, a(p_2, \hat{\beta})) - \varphi(p_1, a(p_1, \hat{\beta})) \\ &= \varphi(p_2, b(p_2, \beta)) - \varphi(p_1, b(p_2, \beta)) \end{aligned}$$

$$\begin{aligned}
 &= \int_{p_1}^{p_2} \frac{1}{p} \varphi(b(p, \beta)) dp + \beta[b(p_2, \beta) - b(p_1, \beta)] \\
 &= \int_{p_1}^{p_2} \frac{1}{p} \varphi(p, b(p, \beta)) dp + \beta[a(p_2, \hat{\beta}) - a(p_1, \hat{\beta})].
 \end{aligned}$$

Since $a(p, \hat{\beta}) \leq b(p, \beta)$ (Lemma 3.16) and $\varphi(p, \cdot)$ is nonincreasing, we must have

$$(\beta - \hat{\beta})[a(p_2, \hat{\beta}) - a(p_1, \hat{\beta})] \geq 0. \quad \square$$

PROPOSITION 3.20. *For all $p > 0$, $a(p)$ and $b(p)$ are finite.*

PROOF. It suffices to show $a(p) > -\infty$ and $b(p) < \infty$ for all $p > 0$. We consider only $a(p)$; the argument for $b(p)$ is completely analogous.

Suppose for some $p^* > 0$, we have $a(p^*) = -\infty$. Define for $0 < p \leq p^*$,

$$\begin{aligned}
 A(p) &\triangleq \max\{a(p'); p' \in [p, p^*]\}, \\
 P(p) &\triangleq \max\{p' \in [p, p^*]; a(p') = A(p')\}.
 \end{aligned}$$

[Recall that $a(\cdot)$ is upper semicontinuous, so $A(p)$ and $P(p)$ are well defined, although we have not yet ruled out the possibility $A(p) = -\infty$.]

Suppose for some $p_0 \in (0, p^*)$, we had $A(p_0) = -\infty$. Then $a(p) = -\infty$ for all $p \in [p_0, p^*]$. We have $b(p) \geq b^*$ given by (3.25), and according to Theorem 3.11,

$$\frac{\partial}{\partial p} \varphi(p, x) = \frac{1}{p} \varphi(p, x) \quad \forall p \in [p_0, p^*], x \in (-\infty, b^*).$$

This implies that

$$\varphi(p, x) = \frac{p}{p_0} \varphi(p_0, x) \quad \forall p \in [p_0, p^*], x \in (-\infty, b^*),$$

and thus

$$-\frac{1}{1 - \mu} = \lim_{x \rightarrow -\infty} D_x^- \varphi(p^*, x) = \frac{p^*}{p_0} \lim_{x \rightarrow -\infty} D_x^- \varphi(p_0, x) = -\left(\frac{p^*}{p_0}\right) \left(\frac{1}{1 - \mu}\right).$$

This is a contradiction, which shows that

$$A(p) > -\infty, \quad P(p) < p^* \quad \forall p \in (0, p^*).$$

Now let $0 < q_1 < q_2 < \dots$ be a sequence with $q_n \uparrow p^*$ and set $p_n = P(q_n)$. Then $p_n \uparrow p^*$, but $p_n < p^*$ for every n , $a(p_n) = A(p_n)$, and $\{a(p_n)\}_{n=1}^\infty$ is a nonincreasing sequence. Because $a(\cdot)$ is upper semicontinuous,

$$(3.30) \quad \lim_{n \rightarrow \infty} a(p_n) = a(p^*) = -\infty.$$

Therefore, there is some n_0 such that $a(p_n) < b^*$, for $n \geq n_0$. Theorem 3.11 implies for $n \geq n_0$ that

$$\frac{\partial}{\partial p} \varphi(p, x) = \frac{1}{p} \varphi(p, x) \quad \forall p \in [p_n, p^*], x \in (a(p_n), b^*),$$

and so

$$\varphi(p^*, x) = \frac{p^*}{p_n} \varphi(p_n, x) \quad \forall x \in (a(p_n), b^*).$$

In particular,

$$\begin{aligned} D_x^+ \varphi(p_n, a(p_n)) &= \lim_{x \downarrow a(p_n)} D_x^+ \varphi(p_n, x) \\ &= \frac{p_n}{p^*} \lim_{x \downarrow a(p_n)} D_x^+ \varphi(p^*, x) \\ &\geq -\left(\frac{p_n}{p^*}\right) \left(\frac{1}{1-\mu}\right). \end{aligned}$$

On the other hand, $D_x^- \varphi(p_n, a(p_n)) = -1/(1-\mu)$. We conclude that

$$\partial_x \varphi(p_n, a(p_n)) = \left[-\frac{1}{1-\mu}, \beta_n\right] \quad \forall n \geq n_0,$$

where β_n is some number satisfying $\beta_n \geq -(p_n/p^*)(1/(1-\mu))$.

Consider $n \geq n_0$ and choose $\hat{\beta}, \beta \in (-1/(1-\mu), \beta_n \wedge \beta_{n+1})$ with $\hat{\beta} < \beta$. We have $a(p_n, \hat{\beta}) = a(p_n) = b(p_n, \beta)$ and $a(p_{n+1}, \hat{\beta}) = a(p_{n+1}) = b(p_{n+1}, \beta)$. Corollary 3.19 implies $a(p_n) \leq a(p_{n+1})$, and this contradicts (3.30).

We conclude that there can be no $p^* > 0$ with $a(p^*) = -\infty$. \square

THEOREM 3.21. *The quantities $a(\cdot)$ and $b(\cdot)$ are nondecreasing.*

PROOF. Suppose there are $0 < p_0 < p^*$ such that $a(p_0) > a(p^*)$. For $p_0 \leq p \leq p^*$, define

$$\begin{aligned} A(p) &= \max\{a(p'); p' \in [p, p^*]\}, \\ P(p) &= \max\{p' \in [p, p^*]; a(p') = A(p')\}. \end{aligned}$$

Because $a(\cdot)$ is upper semicontinuous, these maxima are attained. Set $p_1 = P(p_0)$, so $p_0 \leq p_1 < p^*$ and

$$(3.31) \quad a(p) < a(p_1) \quad \forall p \in (p_1, p^*].$$

If $D_x^+ \varphi(p_1, a(p_1)) = -1/(1-\mu)$, then $b(p_1) > a(p_1)$, and because of (3.31) and the lower semicontinuity of $b(\cdot)$, there is a $\delta > 0$ such that

$$[a(p_1), a(p_1) + \delta] \subset [a(p), b(p)] \quad \forall p \in [p_1, p_1 + \delta].$$

Since $(\partial/\partial p)\varphi(p, x) = (1/p)\varphi(p, x)$ for $x \in (a(p), b(p))$, we have $\varphi(p, x) = (p/p_1)\varphi(p_1, x)$, $\forall x \in (a(p_1), a(p_1) + \delta)$, $p \in [p_1, p_1 + \delta]$. Therefore,

$$\lim_{x \downarrow a(p_1)} D_x^+ \varphi(p, x) = \frac{p}{p_1} \lim_{x \downarrow a(p_1)} D_x^+ \varphi(p_1, x) = \frac{p}{p_1} D_x^+ \varphi(p_1, a(p_1)) < -\frac{1}{1 - \mu},$$

which contradicts (3.13). We conclude that

$$(3.32) \quad D_x^+ \varphi(p_1, a(p_1)) > -\frac{1}{1 - \mu}.$$

We next show that

$$(3.33) \quad D_x^+ \varphi(p, a(p)) = -\frac{1}{1 - \mu} \quad \forall p \in (p_1, p^*].$$

If not, then for some $p_2 \in (p_1, p^*]$, we would have $D_x^+ \varphi(p_2, a(p_2)) > -1/(1 - \mu)$. We could then choose $\hat{\beta}, \beta$ satisfying $-1/(1 - \mu) < \hat{\beta} < \beta < \min\{D_x^+ \varphi(p_1, a(p_1)), D_x^+ \varphi(p_2, a(p_2))\}$, and would have

$$a(p_1, \hat{\beta}) = b(p_1, \beta), \quad a(p_2, \hat{\beta}) = b(p_2, \beta).$$

According to Corollary 3.19,

$$a(p_1) = a(p_1, \hat{\beta}) \leq a(p_2, \hat{\beta}) = a(p_2)$$

and (3.31) is contradicted.

We now claim that $a(\cdot)$ is nondecreasing on $(p_1, p^*]$. If it were not, we could find \tilde{p}_0, \tilde{p}^* with $p_1 < \tilde{p}_0 < \tilde{p}^* \leq p^*$ and $a(\tilde{p}_0) > a(\tilde{p}^*)$. Replacing p_0 by \tilde{p}_0 and p^* by \tilde{p}^* in the preceding argument, we would obtain [cf. (3.32)]

$$D_x^+ \varphi(\tilde{p}_1, a(\tilde{p}_1)) > -\frac{1}{1 - \mu}$$

for some $\tilde{p}_1 \in [\tilde{p}_0, \tilde{p}^*)$. This would contradict (3.33).

Because $a(\cdot)$ is nondecreasing on $(p_1, p^*]$, we have

$$\lim_{p \downarrow p_1} a(p) \leq a(p^*) < a(p_1) \leq b(p_1).$$

By the lower semicontinuity of $b(\cdot)$, there is a $\delta > 0$ satisfying

$$b(p) \geq a(p_1 + \delta) + \delta \quad \forall p \in (p_1, p_1 + \delta],$$

which implies that

$$[a(p), b(p)] \subset [a(p_1) + \delta, a(p_1 + \delta) + \delta] \quad \forall p \in (p_1, p_1 + \delta].$$

We have

$$\begin{aligned} \frac{\partial}{\partial p} \varphi(p, x) &= \frac{1}{p} \varphi(p, x) \\ \forall x &\in (a(p_1 + \delta), a(p_1 + \delta) + \delta), \quad p \in (p_1, p_1 + \delta], \end{aligned}$$

and so,

$$\begin{aligned}\varphi(p, x) &= \frac{p}{p_1 + \delta} \varphi(p_1 + \delta, x) \\ \forall x &\in (a(p_1 + \delta), a(p_1 + \delta) + \delta), \quad p \in (p_1, p_1 + \delta].\end{aligned}$$

Thus, for $p \in (p_1, p_1 + \delta)$, we have

$$\begin{aligned}D_x^+ \varphi(p, a(p_1 + \delta)) &= \lim_{x \downarrow a(p_1 + \delta)} D_x^+ \varphi(p, x) \\ &= \frac{p}{p_1 + \delta} \lim_{x \downarrow a(p_1 + \delta)} D_x^+ \varphi(p_1 + \delta, x) \\ &= \frac{p}{p_1 + \delta} D_x^+(p_1 + \delta, a(p_1 + \delta)) \\ &= -\left(\frac{p}{p_1 + \delta}\right) \left(\frac{1}{1 - \mu}\right),\end{aligned}$$

where we have used (3.33). Letting $p \downarrow p_1$, we obtain

$$-\left(\frac{p_1}{p_1 + \delta}\right) \left(\frac{1}{1 - \mu}\right) \in \partial_x \varphi(p_1, a(p_1 + \delta)).$$

From (3.31) we know that $a(p_1 + \delta) < a(p_1)$, which implies

$$\partial_x \varphi(p_1, a(p_1 + \delta)) = \left\{ -\frac{1}{1 - \mu} \right\}.$$

We have again obtained a contradiction, and this time we conclude that there can be no pair p_0, p^* with $0 < p_0 < p^*$ and $a(p_0) > a(p^*)$. In other words, $a(\cdot)$ is nondecreasing.

Reversing the p -variable in the above proof, we can use it to show that $b(\cdot)$ also is nondecreasing. More specifically, suppose there are $0 < p^* < p_0$ such that $b(p_0) < b(p^*)$. For $p^* \leq p \leq p_0$, define

$$\begin{aligned}B(p) &\triangleq \min\{b(p'); p' \in [p^*, p]\}, \\ Q(p) &\triangleq \min\{p' \in [p^*, p]; b(p) = B(p)\}.\end{aligned}$$

Now proceed as before. \square

COROLLARY 3.22. $0 \leq a(p) \leq b(p), \forall p > 0.$

PROOF. Lemma 3.13 implies $b(0) \triangleq \lim_{p \downarrow 0} b(p) \geq 0$. Suppose we had $a(p^*) < 0$ for some p^* . Then

$$a(p) \leq a(p^*) < 0 \leq b(p) \quad \forall p \in (0, p^*].$$

It follows from Theorem 3.11 that

$$\frac{\partial}{\partial x} \varphi(x, p) = \frac{1}{p} \varphi(x, p) \quad \forall x \in (a(p^*), 0), \quad p \in (0, p^*],$$

and so

$$D_x^+ \varphi(x, p) = \frac{p}{p^*} D_x^+ \varphi(x, p^*) \quad \forall x \in (\alpha(p^*), 0), p \in (0, p^*].$$

Consequently,

$$\lim_{p \downarrow 0} D_x^+ \varphi(x, p) = 0 \quad \forall x \in (\alpha(p^*), 0),$$

which contradicts (3.13). \square

COROLLARY 3.23. $\alpha(0) \triangleq \lim_{p \downarrow 0} \alpha(p) = 0$ and $b(0) \triangleq \lim_{p \downarrow 0} b(p) = 0$.

PROOF. From Corollary 3.22 and Lemma 3.13, we have $\alpha(0) = 0$. If $b(0) > 0$, we can find $p^* > 0$ such that

$$\alpha(p) \leq \alpha(p^*) < b(0) \leq b(p) \quad \forall p \in (0, p^*].$$

Arguing as in Corollary 3.22, we conclude that

$$\lim_{p \downarrow 0} D_x^+ \varphi(x, p) = \lim_{p \downarrow 0} \frac{p}{p^*} D_x^+ \varphi(x, p^*) = 0 \quad \forall x \in (\alpha(p^*), b(0)),$$

and (3.13) is contradicted. \square

THEOREM 3.24. We have $\alpha(p) = b(p) = 0$ for every $p > 0$.

PROOF. Letting $\beta \uparrow -(1 - \lambda)$ in (3.29), using Lemma 3.16 and Proposition 3.8, we obtain for $0 \leq p_1 \leq p_2$,

$$\begin{aligned} & \varphi(p_2, b(p_2)) + (1 - \lambda)b(p_2) \\ (3.34) \quad & = \varphi(p_1, b(p_1)) + (1 - \lambda)b(p_1) + \int_{p_1}^{p_2} \frac{1}{p} \varphi(p, b(p)) dp. \end{aligned}$$

Using (3.14) twice in this equality, we see that

$$\begin{aligned} (p_2 - q)^+ & \leq \varphi(p_2, b(p_2)) + (1 - \lambda)b(p_2) \\ & \leq \varphi(p_1, b(p_1)) + (1 - \lambda)b(p_1) + \int_{p_1}^{p_2} \left[1 - \frac{1}{p}(1 - \lambda)b(p) \right] dp \\ & \leq \varphi(p_1, b(p_1)) + (1 - \lambda)b(p_1) + p_2 - p_1 - (1 - \lambda)b(p_1) \log \frac{p_2}{p_1}. \end{aligned}$$

This can hold for all $p_2 \geq p_1$ only if $b(p_1) = 0$. Thus, $b(p_1) = 0$ for all $p_1 > 0$. Because of Corollary 3.22, $\alpha(p_1) = 0$ for all $p_1 > 0$ also. \square

Finally, we prove the main result.

PROOF OF THEOREM 2.1. According to (3.34),

$$\varphi(p_2, 0) = \varphi(p_1, 0) + \int_{p_1}^{p_2} \frac{1}{p} \varphi(p, 0) dp, \quad 0 < p_1 \leq p_2,$$

which shows that $(\partial/\partial p)\varphi(p, 0) = (1/p)\varphi(p, 0)$ for all $p > 0$. This implies $\varphi(p, 0) = \varphi(1, 0)p$ for all $p > 0$, and (3.14) shows that $\varphi(1, 0)$ must be 1. Thus, $\varphi(p, 0) = p$ for all $p > 0$, and since $D_x^-\varphi(p, x) = -1/(1 - \mu)$ for $x \leq a(p) = 0$ and $D_x^+\varphi(p, x) = -(1 - \lambda)$ for $x \geq b(p) = 0$, the theorem holds. \square

APPENDIX

PROOF OF THEOREM 3.2. To prove the theorem we need to produce a strategy (L, M) such that the corresponding pair (X, Y) , starting at (x, y) at time t_0 , hedges the option. We first find a weak limit of subsequences of the admissible trading strategies L_n, M_n , corresponding to the initial investments x_n, y_n .

Denote by X_n, Y_n the associated holdings in bond and stock and extend all our processes to $[0, T]$ by left continuity, as in (3.1). Following the proof of Lemma 3.1 and taking into account that x_n, y_n are bounded, it is seen that there is a constant C such that

$$(A.1) \quad EI_n^2(t) \leq C,$$

for every n, t , where $I_n(s) := \int_{t_n}^{s \vee t_n} Y_n(u) dW(u)$, $s \in [0, T]$.

Now, (3.4), (3.5) and (A.1) imply boundedness of the sequences $\{M_n(\cdot)\}$, $\{L_n(\cdot)\}$, $\{Y_n(\cdot)\}$ in L_2 , the Hilbert space of progressively measurable, meas \times \mathbf{P} square-integrable processes. Here, meas stands for Lebesgue measure on $[0, T]$. Consequently, there exist weakly convergent (relabelled) subsequences in L_2 . Denote the respective limits by $M(\cdot), L(\cdot)$, and $Y(\cdot)$.

Following Karatzas and Shreve (1984), we say that two jointly measurable processes N and \tilde{N} on $[0, T] \times \Omega$ are *modifications* of each other if

$$(A.2) \quad \mathbf{P}[\omega \in \Omega: \text{meas}\{0 \leq s \leq T; N_t(\omega) \neq \tilde{N}_t(\omega)\} = 0] = 1.$$

Using arguments of Karatzas and Shreve (1984), Lemmas 4.5 and 4.6, we can show that M and L admit modifications (again denoted by M , and L) which satisfy $M(t_0) = L(t_0) = 0$, a.s, which are left-continuous, nondecreasing and $\{\mathcal{F}_t\}$ -adapted, and are still weak limits of M_n , and L_n . Moreover, their Lemma 4.7 implies that $L_n(t)$, and $M_n(t)$ converge weakly in $L_1 = L_1(\Omega, \mathcal{F}_T, \mathbf{P})$ (i.e., as L_1 -random variables) to $L(t)$, and $M(t)$, for almost every $t \in [0, T]$. Also, following the proof of that lemma, we have by weak convergence, left continuity and monotonicity of M, M_n , for every $A \in \mathcal{F}_T$,

$$\begin{aligned} \liminf_{n \rightarrow \infty} EM_n(T)\mathbf{1}_A &\geq \frac{1}{h} \lim_{n \rightarrow \infty} \int_{T-h}^T EM_n(s)\mathbf{1}_A ds \\ &= \frac{1}{h} \int_{T-h}^T EM(s)\mathbf{1}_A ds \geq EM(T-h)\mathbf{1}_A. \end{aligned}$$

Letting $h \downarrow 0$, we get

$$(A.3) \quad \liminf_{n \rightarrow \infty} EM_n(T)\mathbf{1}_A \geq EM(T)\mathbf{1}_A \quad \forall A \in \mathcal{F}_T.$$

Similarly for $L_n(T), L(T)$.

Let $X(\cdot)$ be the solution to (2.3), that is, the process of holdings in bond corresponding to the strategy (L, M) , starting with x at time t_0 . Clearly, $X(\cdot)$ is the weak limit of $X_n(\cdot)$ in L_2 .

Recall that Y is the weak limit of Y_n in L_2 . Fix $r < t_0$ and let $\eta \in L_2$ be supported on $[0, r] \times \Omega$. Then

$$E \int_0^T Y_n(s)\eta(s) ds = y_n E \int_0^r \eta(s) ds \longrightarrow y E \int_0^r \eta(s) ds E \int_0^r y\eta(s) ds.$$

Hence, a modification of Y satisfies $Y(s) = y$ for all $s \leq t_0$.

We next claim that

$$(A.4) \quad I_n(t) \text{ converges weakly in } L_2(\Omega, \mathcal{F}_T, \mathbf{P}) \text{ to } I(t) := \int_{t_0}^t Y(s) dW(s) \quad \forall t \in [t_0, T].$$

Indeed, fix $t \in [t_0, T]$ and notice that given an $L_2(\Omega, \mathcal{F}_T, \mathbf{P})$ -random variable Z , we have the representation $Z = EZ + \int_0^T z(s) dW(s)$, with $z(\cdot) \in L_2$. Assume, without loss of generality, that $EZ = 0$. Denote by $z_0(\cdot)$ the process equal to zero for $s > t$, but otherwise equal to $z(\cdot)$. We have then

$$(A.5) \quad \begin{aligned} E[ZI_n(t)] &= E \int_0^{t \vee t_n} Y_n(s) dW(s) \int_0^T z(s) dW(s) \\ &\quad - E \int_0^{t_n} Y_n(s) dW(s) \int_0^T z(s) dW(s) \\ &= E \int_0^T Y_n(s) z_0(s) ds - y_n E \int_0^{t_n} z(s) ds \\ &\rightarrow E \int_0^T Y(s) z_0(s) ds - y E \int_0^{t_0} z(s) ds \\ &= E \int_{t_0}^T Y(s) z_0(s) ds = \dots = E[ZI(t)]. \end{aligned}$$

Hence the claim (A.4).

This implies that $I_n(\cdot)$ converges weakly in L_2 to $I(\cdot)$. Indeed, for a process $\eta(\cdot) \in L_2$ we have $E \int_0^T I_n(s)\eta(s) ds = \int_0^T E[I_n(s)\eta(s)] ds$, which converges to $\int_0^T E(I(s)\eta(s)) ds$. Consequently, taking weak limits in (2.4) for $Y_n(\cdot)$, we see that $Y(\cdot)$ satisfies equation (2.4).

We show now that the trading strategy (M, L) is admissible, that is, that $X + (1 - \mu)Y$ and $(1 - \lambda)X + Y$ are nonnegative processes. We know that $X_n(t)$ and $Y_n(t)$ converge weakly in L_1 to $X(t)$ and $Y(t)$ for almost every t . The solvency condition is then satisfied for almost every t , and actually for every t , because of the left continuity.

Finally, we prove the hedging property. Let $S \triangleq \{\omega \in \Omega; (1 - \lambda)X(T) + Y(T) < [-(1 - \lambda)q + P^{t_0, P}(T)]\mathbf{1}_A\}$, where $A \triangleq \{P^{t_0, P}(T) > (1 - \lambda)q\}$. First

observe that since

$$R_n \triangleq (1 - \lambda)X_n(T) + Y_n(T) + \gamma M_n(T) = (1 - \lambda)x_n + y_n + I_n(T),$$

$$R \triangleq (1 - \lambda)X(T) + Y(T) + \gamma M(T) = (1 - \lambda)x + y + I(T),$$

from (A.4) we conclude that R_n converges to R weakly. Then by (A.3), Fatou's lemma and (2.6') used twice, we obtain

$$\begin{aligned} \gamma E\mathbf{1}_S M(T) &\leq \liminf_{n \rightarrow \infty} \gamma E\mathbf{1}_S M_n(T) \\ &= \lim_{n \rightarrow \infty} E\mathbf{1}_S \{R_n - ((1 - \lambda)X_n(T) + Y_n(T))\} \\ &\leq E\mathbf{1}_S R - \limsup_{n \rightarrow \infty} E\mathbf{1}_S [- (1 - \lambda)q + P^{t_n, P_n}(T)]^+ \\ &\leq E\mathbf{1}_S R - E\mathbf{1}_S [- (1 - \lambda)q + P^{t_0, P}(T)]^+ \\ &\leq \gamma E\mathbf{1}_S M(T). \end{aligned}$$

It follows that we have equalities everywhere, hence $\mathbf{P}(S) = 0$. Similarly for $X(T) + (1 - \mu)Y(T)$, so that the strategy (L, M) hedges the European option and the proof has been completed. \square

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