The Annals of Applied Probability
1995, Vol. 5, No. 3, 666–680

DYNAMIC ASYMPTOTIC RESULTS FOR A GENERALIZED STAR-SHAPED LOSS NETWORK

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We consider a network in which a call holds a given number of uniformly chosen links and releases them simultaneously. We show pathwise propagation of chaos and convergence of the process of empirical fluctuations to a Gaussian Ornstein–Uhlenbeck process. The limiting martingale problem is obtained by closing a hierarchy. The drift term is given by a simple factorization technique related to mean-field interaction, but the Doob–Meyer bracket contains special terms coming from the strong interaction due to simultaneous release. This is treated by closing another hierarchy pertaining to a measure-valued process related to calls routed through couples of links, and the factorization is again related to mean-field interaction. Fine estimates obtained by pathwise interaction graph constructions are used for tightness purposes and are thus shown to be of optimal order.

0. Introduction. Generalized star-shaped networks are symmetric networks in which each call involves a fixed number K of links. This may model many situations of simultaneous service, for instance, telecommunication or computer networks, locking of items in databases, parallel computing or job processing in factories. In the case $K = 2$ we can imagine that the network ensures connections through a central hub, hence the term "star-shaped."

The network consists of n links, all having the same capacity of C channels. Calls involving subsets of K links arrive according to independent Poisson processes with rate ν_n. If each link in the subset has spare capacity, the call is accepted and lasts an exponential time with mean 1, at the expiration of which it releases all K channels simultaneously. The call is lost if at least one of the K links is full.

The global attempt rate for calls seen by a link is assumed to be constant and equal to ν, and thus $\nu_n = \nu/\binom{n-1}{K-1} = O(1/n^{K-1})$. We could as well treat a situation where the above equality is replaced by an asymptotic equivalence.

The process of the occupancies of the links is not Markovian because of the simultaneous releases. We shall introduce the Markov process of the number of calls on the routes. The birth of calls gives mean-field interaction, while their death introduces strong interaction, giving rise to intricate hierarchies as in the realistic Boltzmann equation in Cercigniani [1] and Uchiyama [14],

Received January 1994; revised January 1995.
AMS 1991 *subject classifications.* Primary 60K35, 60F05, 60F17; secondary 68M10, 90B12.
Key words and phrases. Networks, interaction graphs, hierarchies, propagation of chaos, fluctuations.

Smoluchowski's theory of coagulation in Lang and Nguyen [10] or annihilating Brownian spheres in Sznitman [12, 13].

These network models were introduced by Whitt [16] and studied by, among others, Ziedins and Kelly [17], Hunt [6, 7] and Kelly [9]. All these authors were principally interested in the properties at equilibrium, for instance, the stationary blocking probability.

We are interested in the dynamic behavior of the network as the number of links grows to infinity. The laws of the processes of occupancies satisfy a propagation-of-chaos result in total variation norm, converging to a limit law solving a nonlinear martingale problem on path space. The model is a particular case of a general model studied in Graham and Méléard [5], in which it is given as an example.

We now investigate the speed of convergence. The main result of the paper is the convergence of the fluctuations of the empirical processes around the limit law to a Gaussian Ornstein–Uhlenbeck process which we characterize. We thus get a precise asymptotic view of the behavior of a fixed subnetwork as the size of the global network increases.

Some results have been obtained previously. Whitt [16] proved that the process of the fraction of links with occupancies $0, 1, \dots, C$ converges to the deterministic process of the masses at $0, 1, \dots, C$, of the limit law. This limit process solves a nonlinear differential equation on the simplex. Whitt [16] studies its long-time behavior and shows it converges to its unique fixed point. He conjectures a fluctuation result for a process started at its stationary distribution. Hunt [7] proves this result for a network with an initial state converging to the limit equilibrium point. Our motivation is not the same and we start far from equilibrium.

The proof uses the precise pathwise estimates obtained with the random graphs to show tightness for the processes of empirical fluctuations. This shows the estimates are of optimal order. The martingale problem for the empirical fluctuations gives rise to a hierarchy involving high-order empirical processes corresponding to sequences of links brought into play by the interaction. We must obtain a single limit martingale problem with unique solution in order to characterize the accumulation points and thus prove the convergence, which means we must close the hierarchy. The term corresponding to the drift of the accumulation points is easily closed since only the mean-field interaction appears in it: the corresponding high-order empirical processes are easily factored in terms of the basic empirical fluctuations, and a simple linearization technique takes care of the multiple interaction.

The Doob–Meyer bracket involves a complex term due to the simultaneous releases. We introduce a measure-valued process to control the releases, which gives rise to its own hierarchy. The mean-field nature of the interaction helps us close this hierarchy by the factorization of high-order measures in terms of the low-order ones and thus characterize the limit Ornstein–Uhlenbeck process.

We then obtain a centered Gaussian convergence result for the empirical fluctuations of the sample paths somewhat stronger than the finite-dimensional distribution sense. We thus succeed in proving for this model a functional central limit theorem by closing the hierarchies in all their terms and then a fluctuation field result giving insight into the temporal correlations of the processes.

Graham and Méléard [4] investigate fluctuations for a loss network with alternate routing. The lack of exchangeability and above all the strong interaction in the birth of alternate calls prevents the closing of the hierarchy in the drift. The Doob–Meyer bracket involving the simultaneous releases was successfully closed as above with additional difficulties and precise estimates because of the local interaction, thus characterizing the Gaussian martingale part of the fluctuations.

The factorization and linearization in the drift term are easy in any mean-field model and are purely algebraic. The method of treatment of simultaneous release is fairly general, is straightforward for mean-field interaction and can be adapted to different kinds of interaction with a precise study of the speed of propagation of chaos for each model.

Notation. For integers p and q, $(p)_q$ denotes the number $p(p-1)\cdots(p-q+1)$ of ordered subsets of size q chosen from a set of size p, $\binom{p}{q}$ denotes the number $(p)_q/q!$ of subsets, n is the number of links in the network, C is the capacity of each link and K is the number of links involved in a call.

The symbol R^n denotes the set of routes $\{r = i_1 \cdots i_K: 1 \le i_1 < \cdots < i_K \le n\}$ and $R^n_{j_1 \cdots j_p}$ denotes the set $\{i_1 \cdots i_K \in R^n: j_1, \dots, j_p \in \{i_1, \dots, i_K\}\}$ of routes involving the links $j_1, \dots, j_p$. On routes $r = i_1 \cdots i_K$ we have independent Poisson processes $N_r^n = N^n_{i_1 \cdots i_K}$ of call arrivals, of rate $\nu_n = \nu / \binom{n-1}{K-1}$. The global rate of call attempts seen by a link is ν.

The symbol $\langle \ , \ \rangle$ denotes either martingale Doob–Meyer brackets or duality brackets between measures and functions, and if needed we mention the integration variable: $\langle f(x), m(dx) \rangle = \int f(x) m(dx)$. The symbol $| \ |$ denotes either the total variation norm or the cardinality of a set, and $| \ |_T$ denotes the total variation norm on the set $\Pi(D([0,T],\{0,1,\dots,C\}))$ of probability measures on the Skorokhod space.

We study the process of occupancies of the links: for $1 \le i \le n$, X_i^n is the process of the number of occupied channels on link i. The process $(X_i^n)_{1 \le i \le n}$ is an exchangeable process if the initial values are, but it is not a Markov process. We introduce for distinct $1 \le i_1, \dots, i_K \le n$ the process $Y^n_{i_1 \cdots i_K}$ of calls involving the set $\{i_1, \dots, i_K\}$, and thus for any permutation σ of $\{1, \dots, K\}$, $Y^n_{i_1 \cdots i_K} = Y^n_{i_{\sigma(1)} \cdots i_{\sigma(K)}}$. This convention is practical for notational purposes. All these processes belong to the Skorokhod space $D(\mathbb{R}^+,\{0,1,\dots,C\})$.

The process $(Y_r^n)_{r \in R^n}$ is Markovian and, for $1 \le i \le n$,

$$X_i^n = \sum_{r \in R_i^n} Y_r^n = \sum_{\substack{i_2 < \cdots < i_K \\ i \notin \{i_2, \dots, i_K\}}} Y^n_{i i_2 \cdots i_K}. \tag{0.1}$$

It is easy to write down the generator of the Markov process and the martingale problem satisfied by $(X_i^n)_{1 \le i \le n}$: For any function ϕ,

$$(0.2)\quad \begin{aligned} \phi(X_i^n(t)) - \phi(X_i^n(0)) - \int_0^t \Bigg[& \big(\phi(X_i^n(s)+1) - \phi(X_i^n(s))\big) \mathbb{1}_{X_i^n(s)<C} \\ & \times \left(\frac{\nu}{\binom{n-1}{K-1}} \sum_{\substack{i_2 < \cdots < i_K \\ i \notin \{i_2, \dots, i_K\}}} \mathbb{1}_{X_{i_2}^n(s)<C, \dots, X_{i_K}^n(s)<C} \right) \\ & + \big(\phi(X_i^n(s)-1) - \phi(X_i^n(s))\big) X_i^n(s) \Bigg] ds \end{aligned}$$

is a martingale. Notice that the release rate for one particle depends only on its own occupancy and not on which other channels must be released simultaneously, and the Y do not appear at this stage. We define $\phi^+(x) = \phi(x+1) - \phi(x)$ and $\phi^-(x) = \phi(x-1) - \phi(x)$, and the empirical measure

$$(0.3)\quad \mu_i^{K-1,n} = \frac{1}{(n-1)_{K-1}} \sum_{\substack{|\{i_2, \dots, i_K\}| = K-1 \\ i \notin \{i_2, \dots, i_K\}}} \delta_{X_{i_2}^n, \dots, X_{i_K}^n}$$

of the $(K-1)$-tuples not containing i. Using this notation, (0.2) can be written

$$(0.4)\quad \begin{aligned} & \phi(X_i^n(t)) - \phi(X_i^n(0)) \\ & - \int_0^t \Big[\phi^+(X_i^n(s)) \nu \mathbb{1}_{X_i^n(s)<C} \langle \mathbb{1}_{x_2<C, \dots, x_K<C}, \mu_{i,s}^{K-1,n}(dx_2, \dots, dx_K) \rangle \\ & \qquad + \phi^-(X_i^n(s)) X_i^n(s) \Big] ds. \end{aligned}$$

1. Propagation of chaos and the hierarchy for fluctuations. Propagation of chaos means that a fixed set of links will behave like independent links following a limit law, as the size of the network grows to infinity. A fixed subnetwork will then act at the limit as if the links were independent, allowing various approximations often called fixed-point approximations. If the limit law is Q, we speak of Q-chaoticity, and for exchangeable random variables this is equivalent to the convergence in law of the empirical measures to Q.

In Graham and Méléard [5] this network was an illustrative example and we considered an initially empty network. Use of the interaction graphs in Graham and Méléard [3–5] necessitates independent initial conditions, and the K links initially connected by a call are somewhat related. This constraint vanishes as the number n of links goes to infinity, as seen in the pathwise propagation-of-chaos result, and we could have a propagation-of-chaos result for general chaotic initial conditions.

We shall need precise speeds of convergence and thus assume the network is asymptotically empty for the sake of simplicity. We consider an initial law for the network which is invariant under permutations of the routes, and this exchangeability property is preserved by the time evolution. The $X_i^n(0)$ take values in a finite set for which convergence in law or in total variation coincide.

THEOREM 1.1. *Assume that the initial network is exchangeable and* $|\mathscr{L}(X_1^n(0),\dots,X_n^n(0)) - \delta_0^{\otimes n}| = O(1/n)$. *Then for given* T *and* q, *uniformly for distinct* $i_1,\dots,i_q$, $|\mathscr{L}(X_{i_1}^n,\dots,X_{i_q}^n) - \tilde{\mathrm{P}}^{\otimes q}|_T = O(1/n)$, *where* $\tilde{\mathrm{P}}$ *on* $D(\mathbb{R}^+,\{0,1,\dots,C\})$ *is the unique solution to a nonlinear martingale problem starting at* 0: *For any function* ϕ,

$$(1.1)\quad \phi(X_t) - \phi(0) - \int_0^t \Big[\phi^+(X_s)\nu \mathbb{1}_{X_s<C}\big(1-\tilde{\mathrm{P}}_s^C\big)^{K-1} - \phi^-(X_s)X_s\Big]\,ds$$

is a $\tilde{\mathrm{P}}$-*martingale, where* X *is the canonical process and* $\tilde{\mathrm{P}}_s^C = \mathrm{P}(X_s = C)$ *is the nonlinear term. Moreover,* $\mathscr{L}(X_{i_1}^n,\dots,X_{i_q}^n)$ *converges weakly to* $\tilde{\mathrm{P}}^{\otimes q}$ *for the Skorokhod topology on* $D(\mathbb{R}^+,\{0,1,\dots,C\})$.

PROOF. As in Graham and Méléard [3] we can find a set of probability $1 - O(1/n)$ on which the intial values are zero. We use this to couple the network with an initially empty network. Then the result follows from Graham and Méléard [5].

We first prove the chaos hypothesis under which a finite fixed number of links behaves independently at the limit. Given a finite subset of links, we construct a random graph describing its past history with the least necessary knowledge. We devise a coupling between independent graphs that each describe the past history of a different link of the subset and the global random graph. We then consider the event that the graphs differ, which is called a chain interaction and is accurately described in Graham and Méléard [5]. This pathwise construction uses the Poisson processes $N_{i_1\cdots i_K}^n$. The existence of a chain interaction is expressed in terms of these processes, and its probability is evaluated and is shown to vanish at the limit. This proves the chaos hypothesis according to which particles become independent at the limit.

We then couple the interaction graph for one link with a Boltzmann tree corresponding to a graph without self-interactions, in which the intervening links are taken from an infinite supply of independent links. We evaluate the probability that the tree differs from the graph, and we show the convergence of the law of the occupancy of a link to that of the same computed on the tree, which does not depend on n. The constructions are recursive and based on the notion of direct interaction, which happens when two links are used by the same call.

This shows $\tilde{\mathrm{P}}$-chaoticity in variation norm with speed of convergence to the law of the process constructed on the Boltzmann tree. We only need now to identify $\tilde{\mathrm{P}}$ as the solution to the nonlinear martingale problem (1.1), which has a unique solution thanks to classical contraction methods. Classical martingale characterization techniques will enable us to deduce this from a convergence result on the terms of the martingale problem (0.4). The only problem comes from $\mu_i^{K-1,n}$, which converges to $\tilde{\mathrm{P}}^{\otimes K-1}$ by usual considerations on propagation of chaos for exchangeable systems or by the next theorem. The uniformity result on the choice of the links is very strong and Graham and Méléard [5] give results for empirical measures without symmetry assumptions. □

THEOREM 1.2. *We assume the initial conditions as in Theorem* 1.1. *Let* J^n *be a set of q-tuples of distinct indices with empty intersection, such that* $\lim_{n\to\infty}|J^n| = \infty$. *Then for measurable bounded* ϕ,

$$E\left(\left\langle \phi, \frac{1}{|J^n|} \sum_{i_1\cdots i_q \in J^n} \delta_{X^n_{i_1},\dots,X^n_{i_q}} - \tilde{P}^{\otimes q}\right\rangle^2\right) = O\left(\max\left\{\frac{1}{n}, \frac{1}{|J^n|}\right\}\right)$$

uniformly over T and $\|\phi\|_\infty$, *and the empirical measure converges to* $\tilde{P}^{\otimes q}$ *in law and in probability, for the total variation norm on* $\Pi(D([0,T],\{0,1,\dots,C\}))$ *or for convergence in law for the Skorokhod topology on* $D(\mathbb{R}^+, \{0,1,\dots,C\})$.

PROOF.

$$\begin{aligned}
&E\left(\left\langle \phi, \frac{1}{|J^n|} \sum_{i_1\cdots i_q \in J^n} \delta_{X^n_{i_1},\dots,X^n_{i_q}} - \tilde{P}^{\otimes q}\right\rangle^2\right)\\
&\quad = \frac{1}{|J^n|^2}\Bigg(\sum_{i_1\cdots i_q \in J^n} E\left(\left(\phi\left(X^n_{i_1},\dots,X^n_{i_q}\right) - \langle \phi, \tilde{P}^{\otimes q}\rangle\right)^2\right)\\
&\qquad + \sum_{i_1\cdots i_q \neq j_1\cdots j_q} E\Big(\left(\phi\left(X^n_{i_1},\dots,X^n_{i_q}\right) - \langle \phi, \tilde{P}^{\otimes q}\rangle\right)\\
&\qquad\qquad \times \left(\phi\left(X^n_{j_1},\dots,X^n_{j_q}\right) - \langle \phi, \tilde{P}^{\otimes q}\rangle\right)\Big)\Bigg)\\
&\quad = O\left(\max\left\{\frac{1}{n}, \frac{1}{|J^n|}\right\}\right)
\end{aligned}$$

using the uniformity in Theorem 1.1. We do not need exchangeability. □

REMARK. We give in Graham and Méléard [5] an explicit value for $O(1/n)$ for an initially empty network given as $q(q-1)/n$ times a function of νT and K. We can state a propagation-of-chaos result for P_0-chaotic initial values as in Graham and Méléard [3], for a general P_0 on $\{0,1,\dots,C\}$, but fail to achieve the $O(1/n)$ we need for tightness of the fluctuation processes.

We have closed the hierarchy at the first level as if the interaction were mean-field, which was intuitively expected since the simultaneous releases do not appear in (0.4). We shall see that at the fluctuation level the strong interaction due to simultaneous releases will introduce troublesome terms. Notice that the simultaneous releases appear in the Doob–Meyer brackets of the martingales given by (0.4). We define the empirical measure and empirical fluctuation over path space as

$$(1.2)\qquad \mu^n = \frac{1}{n}\sum_{i=1}^{n}\delta_{X_i^n}, \qquad \eta^n = \sqrt{n}\left(\mu^n - \tilde{\mathrm{P}}\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\delta_{X_i^n} - \tilde{\mathrm{P}}\right).$$

Theorem 1.2 shows that $\lim_{n\to\infty}\mu^n = \tilde{\mathrm{P}}$. Recall that in the case of exchangeability this property is equivalent to $\tilde{\mathrm{P}}$-chaoticity. We now wish to study the fluctuations associated with this convergence. For this we follow the ideas in Graham and Méléard [4]:

1. We define the martingale problems satisfied by the processes $(\mu_t^n)_{t\geq 0}$ and $(\eta_t^n)_{t\geq 0}$. This starts a hierarchy much simpler than in Graham and Méléard [4] because of symmetry and the mean-field interaction for births, but it contains difficult terms due to simultaneous release.
2. We prove bounds which give tightness.
3. We study the limit of the terms in the martingale problem for $(\eta_t^n)_{t\geq 0}$, the mean-field terms being easy, but not the simultaneous release ones.

We consider the empirical measure and fluctuation over K-tuples of distinct indices

$$(1.3)\quad \mu^{K,n} = \frac{1}{(n)_K}\sum_{|\{i_1,\dots,i_K\}|=K}\delta_{X_{i_1}^n,\dots,X_{i_K}^n}, \qquad \eta^{K,n} = \sqrt{n}\left(\mu^{K,n} - \tilde{\mathrm{P}}^{\otimes K}\right)$$

and deduce from (0.2) or (0.4) that, for any function ϕ,

$$(1.4)\quad \begin{aligned} &\langle \phi,\mu_t^N\rangle - \langle \phi,\mu_0^N\rangle \\ &\quad - \int_0^t \Big[\langle \phi^+(x_1)\nu\,\mathbb{1}_{x_1<C,\dots,x_K<C},\mu_s^{K,n}(dx_1\cdots dx_K)\rangle \\ &\qquad\qquad + \langle \phi^-(x)x,\mu_s^n(dx)\rangle\Big]\,ds \end{aligned}$$

is a martingale. By taking limits or using (1.1), the deterministic limit law $(\tilde{\mathrm{P}}_t)_{t\geq 0}$ satisfies the evolution equation obtained by replacing μ^n by $\tilde{\mathrm{P}}$, $\mu^{K,N}$ by $\tilde{\mathrm{P}}^{\otimes K}$ and the martingale by 0 in (1.4).

PROPOSITION 1.3. *For all functions ϕ on $\{0,1,\dots,C\}$,*

$$(1.5)\quad \begin{aligned} &\langle \phi,\eta_t^n\rangle - \langle \phi,\eta_0^N\rangle \\ &\quad - \int_0^t \Big[\langle \phi^+(x_1)\nu\,\mathbb{1}_{x_1<C,\dots,x_K<C},\eta_s^{K,n}(dx_1\cdots dx_K)\rangle \\ &\qquad\qquad + \langle \phi^-(x)x,\eta_s^n(dx)\rangle\Big]\,ds \end{aligned}$$

is a martingale with Doob–Meyer bracket

$$(1.6)\quad \begin{aligned} &\int_0^t \Bigg[\left\langle \left(\phi^+(x_1)+\cdots+\phi^+(x_K)\right)^2\frac{\nu}{K}\mathbb{1}_{x_1<C,\dots,x_K<C},\mu_s^{K,n}(dx_1\cdots dx_K)\right\rangle \\ &\qquad + \frac{1}{n}\sum_{i_1<\cdots<i_K}\left(\phi^-(X_{i_1}^n)+\cdots+\phi^-(X_{i_K}^n)\right)^2 Y_{i_1\cdots i_K}^n(s)\Bigg]\,ds. \end{aligned}$$

We remark that the propagation of chaos gives the asymptotics of the first term of (1.6), but the second is specific to the strong interaction due to simultaneous release.

These martingale problems are each the start of a hierarchy, since the martingale problem for links involves K-tuples of links, and the one for K-tuples would involve more complex empirical measures and fluctuations over $(K+1)$-tuples, $\dots, (2K-1)$-tuples and so on. Furthermore, the Y processes in (1.6) will necessitate the use of a positive bounded measure process, giving rise to its own hierarchy. In order to obtain a limit martingale problem, we must close the hierarchy by factoring the high-order measures (involving long multiples) in terms of the low-order ones.

REMARK. The process $(\eta_t^n)_{t\ge 0}$ takes its values in the bounded signed measures on the finite set $\{0,1,\dots,C\}$ and is canonically isomorphic to the $\mathbb{R}^{C+1}$-valued process of its densities $(\eta_t^n(\{0\}),\dots,\eta_t^n(\{C\}))_{t\ge 0}$. The techniques and compactness criteria we use are thus those in $\mathbb{R}^{C+1}$. The test functions for the martingale problems and the evolution equations can be limited to $\mathbb{1}_{\{0\}}, \mathbb{1}_{\{1\}},\dots, \mathbb{1}_{\{C\}}$. Using these, the evolution equations can be easily rewritten as ordinary differential equations on $\mathbb{R}^{C+1}$.

2. Bounds and compactness.

We now use our pathwise estimates to obtain bounds for tightness results.

THEOREM 2.1. *The quantity* $\mathrm{E}(\langle\phi,\eta^n\rangle^2)$ *is bounded for* ϕ *in* $\mathscr{L}^\infty(D[0,T], \{0,1,\dots,C\}))$ *uniformly in* $\|\phi\|_\infty$, *and* $\mathrm{E}(\langle\psi,\eta^{K,n}\rangle^2)$ *is bounded for* ψ *in* $\mathscr{L}^\infty(D([0,T], \{0,1,\dots,C\})^K)$ *uniformly in* $\|\psi\|_\infty$.

PROOF. This is a special case of Theorem 1.2. □

THEOREM 2.2. *The empirical fluctuation processes* $(\eta_t^n)_{t\ge 0}$ *are tight in* $\mathrm{D}(\mathbb{R}+, \mathbb{R}^{C+1})$ *and any accumulation point is continuous.*

PROOF. It is classical to prove tightness using pathwise bounds and martingale problems as in Proposition 3.2.3 in Joffe and Métivier [8]. The boundedness assumptions needed are given by Theorem 2.1 and $\sum_{r\in R_i^n} Y_r^n = X_i^n \le C$.

Let η denote the canonical process, Q^n the law of η^n and $\tilde{\mathrm{Q}}$ an accumulation point. There are K simultaneous jumps of amplitude $1/\sqrt{n}$ and thus $|\eta_t^n - \eta_{t-}^n| \ge 2K/\sqrt{n}$, and for any $\varepsilon > 0$, $\lim_{n\to\infty} \mathrm{Q}^n(\sup_{t\ge 0}|\eta_t - \eta_{t-}| > \varepsilon) = 0$. The set $\{\sup_{t\ge 0}|\eta_t - \eta_{t-}| > \varepsilon\}$ is open; thus $\tilde{\mathrm{Q}}(\sup_{t\ge 0}|\eta_t - \eta_{t-}| \le \varepsilon) = 0$ and accumulation points are continuous. □

REMARK. We likewise see that the empirical fluctuation processes $(\eta_t^{K,n})_{t\ge 0}$ and those following in the hierarchy are tight and have continuous limit points.

3. Convergence of the fluctuation process.

After a tightness result, it is classical to prove convergence by uniqueness of the accumulation points.

We have introduced in Section 1 the martingale problem satisfied by $(\eta_t^n)_{t\geq 0}$ and it is natural to show that any accumulation point must satisfy a limiting martingale problem with unique solution.

A process governing simultaneous release. We need a new notion for the limit of the product terms, first introduced in Graham and Méléard [4]. We denote by Y_{ij}^n the process of the number of calls in progress that involve both i and j. Naturally $Y_{ij}^n = \sum_{r\in R_{ij}^n} Y_r^n$ and since each call on a route involving i contributes to X_i and to $K-1$ distinct Y_{ij}^n, we have $(K-1)X_i = \sum_{j:j\neq i} Y_{ij}^n$. We introduce the process

$$\lambda^n = \frac{1}{n}\sum_{i\neq j} Y_{ij}^n \delta_{X_i^n, X_j^n} \tag{3.1}$$

on the set of positive measures on $\{0,1,\dots,C\}^2$, and

$$\begin{aligned}|\lambda^n| &= \frac{1}{n}\sum_{i\neq j} Y_{ij}^n = \frac{1}{n}\sum_{i=1}^n \sum_{j:j\neq i} Y_{ij}^n\\ &= \frac{1}{n}\sum_{i=1}^n (K-1)X_i^n = (K-1)\langle x,\mu^n(dx)\rangle \leq (K-1)C.\end{aligned} \tag{3.2}$$

Note that $|\lambda^n|$ converges to $(K-1)$ times the mean value of $\tilde{\mathrm{P}}$ and λ^n can be considered as a process in the simplex $\{0\leq x_{ij},\ \sum_{i,j}x_{ij}\leq (K-1)C\}$ of $\mathbb{R}^{\{0,1,\dots,C\}^2}$. Moreover, λ_0^n converges to zero if $X_i^n(0)$ does.

THEOREM 3.1. *We take the initial conditions of Theorem* 1.1 *and assume that* η_0^n *converges to* η_0. *Let* $\tilde{\mathrm{P}}$ *be the solution to the nonlinear martingale problem* (1.1) *starting at* 0. *Then* $(\eta_t^n)_{t\geq 0}$ *converges in law to the unique Gaussian Ornstein–Uhlenbeck process such that, for each function* ϕ,

$$\begin{aligned}&\langle\phi,\eta_t\rangle - \langle\phi,\eta_0\rangle\\ &\quad -\int_0^t\Big[\nu\big(1-\tilde{\mathrm{P}}_s^C\big)^{K-1}\langle\phi^+(x)\mathbb{1}_{x<C},\eta_s(dx)\rangle\\ &\qquad +\nu(K-1)\big(1-\tilde{\mathrm{P}}_s^C\big)^{K-2}\langle\phi^+(x)\mathbb{1}_{x<C},\tilde{\mathrm{P}}_s(dx)\rangle\eta_s(\{0,\dots,C-1\})\\ &\qquad +\langle\phi^-(x)x,\eta_s(dx)\rangle\Big]\,ds\end{aligned} \tag{3.3}$$

is a Gaussian continuous martingale with deterministic Doob–Meyer bracket

$$\begin{aligned}&\int_0^t\Big[\nu\big(1-\tilde{\mathrm{P}}_s^C\big)^{K-1}\langle\phi^+(x)^2\mathbb{1}_{x<C},\tilde{\mathrm{P}}_s(dx)\rangle\\ &\qquad +\nu(K-1)\big(1-\tilde{\mathrm{P}}_s^C\big)^{K-2}\langle\phi^+(x)\mathbb{1}_{x<C},\tilde{\mathrm{P}}_s(dx)\rangle^2\\ &\qquad +\langle\phi^-(x)^2x,\tilde{\mathrm{P}}_s(dx)\rangle + \langle\phi^-\otimes\phi^-,\lambda_s\rangle\Big]\,ds,\end{aligned} \tag{3.4}$$

where λ is the deterministic unique solution to the affine evolution equation that holds for all α and β on $\{0,1,\dots,C\}$:

$$\begin{aligned}\langle \alpha \otimes \beta, \lambda_t \rangle = \int_0^t \Big[\Big\langle \nu\big(1 - \tilde{\mathrm{P}}_s\big)^{K-1} \big(\alpha^+(x)\beta(y)\mathbb{1}_{x<C} + \alpha(x)\beta^+(y)\mathbb{1}_{y<C}\big) \\ + \alpha^-(x)\beta(y)(x-1) + \alpha(x)\beta^-(y)(y-1) \\ -\alpha(x)\beta(y), \lambda_s(dx,dy) \Big\rangle + \nu(K-1)\big(1-\tilde{\mathrm{P}}_s\big)^{K-2} \\ \times \langle \alpha(x+1)\mathbb{1}_{x<C}, \tilde{\mathrm{P}}_s(dx)\rangle \langle \beta(x+1)\mathbb{1}_{x<C}, \tilde{\mathrm{P}}_s(dx)\rangle \Big] \, ds \end{aligned} \tag{3.5}$$

and that describes the limit behavior of the simultaneous releases.

PROOF. Classical martingale characterization techniques show that, for any accumulation point of $(\eta_t^n)_{t\geq 0}$, the limit of (1.5) should be a martingale with Doob–Meyer bracket given by the limit of (1.6), if these limits exist. There are two steps in the study of these limits.

The first step consists of closing the hierarchy for the drift term. In Graham and Méléard [4] we were not able to do this because of the strong interaction in the alternate routings. In the present situation, the drift is just mean field, as for spatially homogeneous Boltzmann equations, and can be readily linearized at the limit. The main point is that the empirical measure over K-tuples differs very little from the Kth power of the empirical measure.

The second step studies the convergence of the martingale term. We have seen in Section 1 that an unusual term appears in the Doob–Meyer bracket, due to the strong interaction introduced by simultaneous releases, which we must study accurately.

Step 1. A simple factorization is the key for linearizing the fluctuations and thus expressing the limit of $\eta_s^{K,n}$ in terms of η_s, which closes this hierarchy. This is always true for mean-field interaction. The computations are algebraic:

$$\begin{aligned} \mu^{K,n} - (\mu^n)^{\otimes K} &= \frac{1}{(n)_K} \sum_{|\{i_1,\dots,i_K\}|=K} \delta_{X_{i_1}^n,\dots,X_{i_K}^n} - \frac{1}{n^K} \sum_{i_1,\dots,i_K} \delta_{X_{i_1}^n,\dots,X_{i_K}^n} \\ &= \left(\frac{1}{(n)_K} - \frac{1}{n^K} \right) \sum_{|\{i_1,\dots,i_K\}|=K} \delta_{X_{i_1}^n,\dots,X_{i_K}^n} \\ &\quad - \frac{1}{n^K} \sum_{|\{i_1,\dots,i_K\}|<K} \delta_{X_{i_1}^n,\dots,X_{i_K}^n}, \end{aligned} \tag{3.6}$$

and it is obvious that in variation norm, uniformly on the randomness,

$$|\mu^{K,n} - (\mu^n)^{\otimes K}| \leq 2\left(1 - \frac{(n)_K}{n^K}\right) = O\left(\frac{1}{n}\right)$$

and $|\eta^{K,n} - \sqrt{n}\,((\mu^n)^{\otimes K} - \tilde{\mathrm{P}}^{\otimes K})| = O(1/\sqrt{n})$. We are thus required only to consider the limit behavior of the factored expression $\sqrt{n}\,((\mu^n)^{\otimes K} - \tilde{\mathrm{P}}^{\otimes K})$. A simple linearization technique does the job:

$$\begin{aligned}\sqrt{n}\left((\mu^n)^{\otimes K} - \tilde{\mathrm{P}}^{\otimes K}\right) &= \sqrt{n} \sum_{i=1}^{K} (\mu^n)^{\otimes K-i} \otimes \left(\mu^n - \tilde{\mathrm{P}}\right) \otimes \tilde{\mathrm{P}}^{\otimes i-1} \\ &= \sum_{i=1}^{K} (\mu^n)^{\otimes K-i} \otimes \eta^n \otimes \tilde{\mathrm{P}}^{\otimes i-1}\end{aligned} \tag{3.7}$$

and propagation of chaos then shows that any accumulation point of the laws of $(\eta_t^n)_{t\ge 0}$ satisfies the limit (3.3) of (1.5).

Step 2. We study now the martingale term. The first term in the Doob–Meyer process (1.6) converges by propagation of chaos to

$$\int_0^t \left\langle (\phi^+(x_1) + \cdots + \phi^+(x_K))^2 \frac{\nu}{K} \mathbb{1}_{x_1 < C, \dots, x_K < C}, \tilde{\mathrm{P}}_s^{\otimes K}(dx_1 \cdots dx_K) \right\rangle ds \tag{3.8}$$

in which we develop the square and use symmetry to get the first term in (3.4). For simple mean-field interaction, this would be all that is needed, but we must consider the second troublesome term in (1.6) coming from the strong interaction due to simultaneous release,

$$\frac{1}{n} \sum_{i_1 < \cdots < i_K} \left(\phi^-(X_{i_1}^n) + \cdots + \phi^-(X_{i_K}^n)\right)^2 Y_{i_1 \cdots i_K}^n(s). \tag{3.9}$$

We could study directly the measure-valued process

$$\frac{1}{nK!} \sum_{|\{i_1, \dots, i_K\}| = K} Y_{i_1 \cdots i_K}^n \delta_{X_{i_1}^n, \dots, X_{i_K}^n}, \tag{3.10}$$

but at the fluctuation level we only need knowledge of the correlations, and thus of its marginals λ^n given in (3.1). We develop the square in (3.9). The diagonal terms give

$$\begin{aligned}&\frac{1}{n} \sum_{i_1 < \cdots < i_K} \sum_{i \in \{i_1, \dots, i_K\}} \phi^-(X_i^n)^2 Y_{i_1 \cdots i_K}^n \\ &\quad = \frac{1}{n} \sum_{i=1}^{n} \phi^-(X_i^n)^2 \sum_{r \in R_i^n} Y_r^n \\ &\quad = \frac{1}{n} \sum_{i=1}^{n} \phi^-(X_i^n)^2 X_i^n = \langle \phi^-(x)^2 x, \mu^n(dx) \rangle\end{aligned} \tag{3.11}$$

and converge to $\langle \phi^-(x)^2 x, \tilde{\mathrm{P}}(dx) \rangle$ using propagation of chaos, which would correspond to independent release. The product terms give

$$
\begin{aligned}
&\frac{1}{n}\sum_{i_1<\cdots<i_K}\ \sum_{\substack{i\neq j\\ i,j\in\{i_1,\ldots,i_K\}}} \phi^-(X_i^n)\phi^-(X_j^n)Y_{i_1\cdots i_K}^n\\
&\quad=\frac{1}{n}\sum_{i\neq j}\phi^-(X_i^n)\phi^-(X_j^n)\sum_{r\in R_{ij}^n}Y_r^n \qquad (3.12)\\
&\quad=\frac{1}{n}\sum_{i\neq j}\phi^-(X_i^n)\phi^-(X_j^n)Y_{ij}^n=\langle\phi^-\otimes\phi^-,\lambda^n\rangle
\end{aligned}
$$

and we only need to know the limit behavior of λ^n. For functions α and β on $\{0,1,\ldots,C\}$,

$$
\begin{aligned}
&Y_{ij}^n(t)\,\alpha(X_i^n(t))\beta(X_j^n(t))-Y_{ij}^n(0)\,\alpha(X_i^n(0))\beta(X_j^n(0))\\
&\quad-\int_0^t\Bigg[\frac{\nu}{\binom{n-1}{K-1}}\Bigg(Y_{ij}^n(s)\,\alpha^+(X_i^n(s))\beta(X_j^n(s))\mathbb{1}_{X_i^n(s)<C}\\
&\qquad\times\sum_{\substack{i_2<\cdots<i_K\\ i,j\notin\{i_2,\ldots,i_K\}}}\mathbb{1}_{X_{i_2}^n(s)<C,\ldots,X_{i_K}^n(s)<C}\\
&\qquad+Y_{ij}^n(s)\,\alpha(X_i^n(s))\beta^+(X_j^n(s))\mathbb{1}_{X_j^n(s)<C}\\
&\qquad\times\sum_{\substack{i_2<\cdots<i_K\\ i,j\notin\{i_2,\ldots,i_K\}}}\mathbb{1}_{X_{i_2}^n(s)<C,\ldots,X_{i_K}^n(s)<C}\\
&\qquad+\Big(\big(Y_{ij}^n(s)+1\big)\alpha(X_i^n(s)+1)\beta(X_j^n(s)+1) \qquad (3.13)\\
&\qquad\qquad-Y_{ij}^n(s)\,\alpha(X_i^n(s))\beta(X_j^n(s))\Big)\\
&\qquad\times\mathbb{1}_{X_i^n(s)<C,X_j^n(s)<C}\sum_{\substack{i_3<\cdots<i_K\\ i,j\notin\{i_3,\ldots,i_K\}}}\mathbb{1}_{X_{i_3}^n(s)<C,\ldots,X_{i_K}^n(s)<C}\Bigg)\\
&\qquad+Y_{ij}^n(s)\,\alpha^-(X_i^n(s))\beta(X_j^n(s))(X_i^n(s)-Y_{ij}^n(s))\\
&\qquad+Y_{ij}^n(s)\,\alpha(X_i^n(s))\beta^-(X_j^n(s))(X_j^n(s)-Y_{ij}^n(s))\\
&\qquad+\Big(\big(Y_{ij}^n(s)-1\big)\alpha(X_i^n(s)-1)\beta(X_j^n(s)-1)\\
&\qquad\qquad-Y_{ij}^n(s)\,\alpha(X_i^n(s))\beta(X_j^n(s))\Big)Y_{ij}^n(s)\Bigg]\,ds
\end{aligned}
$$

is a martingale M_{ij}^n. Moreover, $(\lambda_t^n)_{t\geq 0}$ solves a martingale problem: If M^n is the martingale corresponding to $\langle \alpha \otimes \beta, \lambda_t^n \rangle$, then using (3.1), $M^n = (1/n)\Sigma_{i \neq j} M_{ij}^n$.

LEMMA 3.2. *$E(\sup_{0 \le t \le T} M^n(t)^2) = O(1/n)$, uniformly in T, $\|\alpha\|_\infty$ and $\|\beta\|_\infty$.*

PROOF. Because of the Doob inequality we only need to prove $E(\langle M^n \rangle_T) = O(1/n)$. Naturally $E(\langle M^n \rangle_T) = E([M^n]_T)$, the sum of the squares of the jumps of M. A jump is either the arrival or the end of a call, and there are $K(K-1)$ couples taken from the K links on the route on which this happens; thus, the size of the squares of the jumps is $O(1/n^2)$. The call ends come from either the calls present at time zero or from calls that arrived afterward. There is a global capacity of at most nC circuits in the network, and each call necessitates K circuits; thus, there are at most nC/K calls present at the start, and their contribution to $E([M^n]_T)$ is$O(1/n)$ [actually $o(1/n)$ since the network is asymptotically initially empty]. The ends of the calls that arrived afterward contribute at most as much as their arrival, so we now only need to show that the contribution from call arrivals is $O(n)$, which is the case since there are $\binom{n}{K}$ routes each with a Poisson stream of arrivals of rate $\nu/\binom{n-1}{K-1}$. This proves the lemma. □

The process λ^n jumps as $[M^n]$ except that its jumps are of size $O(1/n)$. Thus $(\lambda_t^n)_{t \ge 0}$ is tight and its accumulation points are absolutely continuous and have no martingale part. We only need to close the hierarchy implicit in (3.13) to obtain the deterministic affine evolution problem satisfied by the limit process $(\lambda_t)_{t \ge 0}$. This martingale problem involves the higher-order measure-valued processes

$$
\begin{aligned}
\lambda^{K,n} &= \frac{K-1}{(n)_K} \sum_{|\{i,j,i_3,\dots,i_K\}|=K} Y_{ij}^n \delta_{X_i^n, X_j^n, X_{i_3}^n, \dots, X_{i_K}^n}, \\
\lambda^{K+1,n} &= \frac{1}{(n)_K} \sum_{|\{i,j,i_2,\dots,i_K\}|=K+1} Y_{ij}^n \delta_{X_i^n, X_j^n, X_{i_2}^n, \dots, X_{i_K}^n},
\end{aligned} \tag{3.14}
$$

the first of which vanishes at the limit since its total mass is seen to be $O(1/n)$ by reasoning as in (3.2): there are not enough routes containing both i and j for them to be seen at the limit. Instead, $\lambda^{K+1,n}$ necessitates a delicate factorization.

LEMMA 3.3. *$E(\langle \alpha_0 \otimes \cdots \otimes \alpha_K, \lambda_t^{K+1,n} - \lambda_t^n \otimes \tilde{P}_t^{\otimes K-1} \rangle^2) = O(1/n)$ uniformly on $[0,T]$, $\|\alpha_0\|_\infty, \dots, \|\alpha_K\|_\infty$.*

PROOF. Let $\mu^{K-1,n}$ denote the empirical measure over distinct $(K-1)$-tuples

$$\mu^{K-1,n} = \frac{1}{(n)_{K-1}} \sum_{|\{i_2,\dots,i_K\}|=K-1} \delta_{X_{i_2}^n, \dots, X_{i_K}^n}.$$

By Theorem 1.2 and (3.2), $\mathrm{E}(\langle \alpha_0 \otimes \cdots \otimes \alpha_K, \lambda_t^n \otimes \tilde{\mathrm{P}}_t^{\otimes K-1} - \lambda_t^n \otimes \mu^{K-1,n}\rangle^2) = O(1/n)$,

$$\lambda^{K+1,n} - \lambda_t^n \otimes \frac{n}{n-K+1}\mu^{K-1,n}$$

$$(3.15) \qquad = \frac{1}{(n)_K}\Bigg(\sum_{|\{i,j,i_2,\dots,i_K\}|=K+1} Y_{ij}^n \delta_{X_i^n, X_j^n, X_{i_2}^n, \dots, X_{i_K}^n} - \sum_{i\neq j}\sum_{|\{i_2,\dots,i_K\}|=K-1} Y_{ij}^n \delta_{X_i^n, X_j^n, X_{i_2}^n, \dots, X_{i_K}^n}\Bigg)$$

is bounded in variation norm by

$$(3.16) \qquad \frac{1}{(n)_K}\sum_{i\neq j} Y_{ij}^n \Big|\{\{i_2,\dots,i_K\} : |\{i_2,\dots,i_K\}| = K-1, \quad |\{i,j,i_2,\dots,i_K\}| \le K\}\Big|,$$

which is $O(1/n)$ uniformly on the randomness using (3.2). This proves the lemma. □

We now consider (3.13) in light of these results. It is now simple to find the limiting deterministic evolution equation (3.5), which has a unique solution that can be made explicit by the method of variation of constants. The affine part comes from the term involving increases of Y_{ij}^n and propagation of chaos (it is the only part left in the limit of the term involving $\lambda^{K,n}$), and we use the following lemma.

LEMMA 3.4. *Uniformly for t in $[0,T]$, for $i \neq j$, $\mathrm{P}(Y_{ij}^n(t) > 0) = O(1/n)$, $\mathrm{E}(Y_{ij}^n(t)) = O(1/n)$ and $\mathrm{E}(|(Y_{ij}^n(t))^2 - Y_{ij}^n(t)|) = o(1/n)$, and if $k \neq l$ and $\{i,j\} \neq \{k,l\}$, then $\mathrm{P}(Y_{ij}^n(t)Y_{kl}^n(t) > 0) = O(1/n^2)$ and $\mathrm{E}(Y_{ij}^n(t)Y_{kl}^n(t)) = O(1/n^2)$.*

PROOF. All the following bounds are uniform for $t \in [0,T]$: $(K-1)\mathrm{E}(X_i^n(t)) = (n-1)\mathrm{E}(Y_{ij}^n(t)) \le (K-1)C$ and thus $\mathrm{P}(Y_{ij}^n(t) > 0) = O(1/n)$. By developing $(X_i^n)^2$ we see that for $|\{i,j,k\}| = 3$, $\mathrm{E}(Y_{ij}^n(t)Y_{ik}^n(t)) = O(1/n^2)$ and thus $\mathrm{P}(Y_{ij}^n(t)Y_{ik}^n(t) > 0) = O(1/n^2)$. By developing $X_i^n X_j^n$ for distinct i and j, we see that for $|\{i,j,k,l\}| = 4$, $\mathrm{E}(Y_{ij}^n(t)Y_{kl}^n(t)) = O(1/n^2)$ and thus $\mathrm{P}(Y_{ij}^n(t)Y_{kl}^n(t) \neq 0) = O(1/n^2)$. Since $\mathrm{E}(X_i^n(0)) = o(1)$, $\mathrm{P}(Y_{ij}^n(0) > 0) = o(1/n)$, which implies that $\mathrm{P}(Y_{ij}^n(t) \ge 2) = o(1/n)$ and if the network is started empty, $\mathrm{P}(Y_{ij}^n(t) \ge 2) = O(1/n^2)$. Births of Y_{ij}^n arrive at rate $O(1/n)$ independently of the past, and thus for Y_{ij}^n to reach 2 we must have $\mathrm{P}(Y_{ij}^n(0) \ge 2)$ or have both the independent events that $Y_{ij}^n(0) = 1$ and that one call has arrived before time T or the event of probability $O(1/n^2)$ of two arrivals or more. This proves the lemma. □

We finish the proof of Theorem 3.1 by remarking that a continuous martingale with deterministic Doob–Meyer bracket is Gaussian. □

REMARK. We can likewise find the limit deterministic affine evolution problem for the limit of the measure-valued process (3.10).

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