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DIFFUSION APPROXIMATION FOR OPEN STATE-DEPENDENT QUEUEING NETWORKS IN THE HEAVY TRAFFIC SITUATION

BY KEIGO YAMADA

Kanagawa University

We consider open queueing networks in which arrival and service rates are dependent on the state (i.e., queue length) of the network. They are modeled as multidimensional birth and death processes. If a heavy traffic condition is satisfied on the behavior of arrival and service rates when the queue length becomes very large, it is shown that a properly normalized sequence of queue length converges in law to a reflecting diffusion process.

1. Introduction. We consider in this paper open queueing networks in which arrival and service rates depend on the state (i.e., queue length) of the network. They are modeled as multidimensional birth and death processes. It is shown that a properly normalized sequence of queue lengths converges to a reflecting diffusion process under a heavy traffic condition. For a K-station queueing network, this diffusion takes values in the K-dimensional nonnegative orthant with a fixed direction of reflection for each boundary hyperplane and has nonsingular drift and diffusion coefficients. The fact that the limit process is a diffusion and, hence, that drift and diffusion coefficients generally depend on the state of the process, is a reflection of the dependence of arrival and service rates on the state (i.e., queue length) of the queueing systems. This contrasts with the result obtained by Reiman [13]. In the networks considered there, arrival streams are renewal processes and service times have general probability distributions, but they do not depend on the state of the network; that is, arrival and service rates are constant. As a result, the limit process is a reflecting Brownian motion with a drift. We will see that to specify how arrival and service rates behave when the queue length becomes very large plays an important role in obtaining the diffusion limit, and this specification is called the heavy traffic condition if, roughly speaking, the traffic intensity defined appropriately using arrival and service rates is nearly unity when the queue length is very large. The limit diffusions obtained from our result are not of a special type like reflecting Brownian motions or Ornstein–Uhlenbeck processes but are rather general, and vari-

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ous different types of reflecting diffusions are obtained by specifying the state dependence differently in different ways. The limit diffusions are also characterized by the fact that their drift and diffusion coefficients are not singular. Hence, for example, Bessel diffusions, which have singular drift generally, cannot be obtained from our results. (See Example 3 in Section 6.) To see how our result is applied, some examples are given. In Example 1 in Section 6, it is shown that, roughly speaking, a very large class of reflecting diffusions with nonsingular drifts and diffusion coefficients can be in the class of limit processes for queueing models. In Example 2, a multidimensional reflecting Ornstein–Uhlenbeck process appears as a limit process.

As mentioned before, the queue length at each station is modeled by multidimensional birth and death processes. They represent the network which consists of K single server stations. At least one station has an arrival stream from outside the network. Customers are served in order of arrival and they are randomly routed, after service, to either another station or out of the network entirely. Interarrival and service times have exponential distributions whose parameters (i.e., arrival and service rates) are dependent on the state (i.e., queue length) of the network. In setting our models, we have tried to accommodate models with arrival and service rates which are as general as possible. For that purpose, we have introduced the concept "c.c. convergence" for functions (see Definition 1 in Section 4), and this takes care of many discontinuous arrival and service rates which appear in application.

In view of the Markov property of our model, to show our result we use a stochastic calculus approach such as developed in [8]. The K-dimensional vector queue length process is represented as the solution of a Skorohod equation defined on the K-dimensional nonnegative orthant. Properties of such an equation were studied by Harrison and Reiman [5], and we make use of their result to establish our theorem. The limit process, which is a reflecting diffusion, is the solution of a stochastic differential equation with reflecting boundary, which is also represented as the Skorohod equation of the same type as above. We were unable to show the uniqueness of the solution of this equation under our general setting where no continuity of coefficients of the equation is assumed. See, however, Remark 5 and Proposition 1, where the uniqueness is shown for the equation with measurable drifts (but with Lipschitz continuous diffusion coefficients).

If the Markov property does not hold for the queueing systems, the modeling itself, which takes into account the state dependence of arrival and service rates, seems to be difficult in general. We point out, however, that for some specific models the analysis is possible using our approach. Such an example is considered in Section 7.

We denote by $D([0,\infty), R^d)$ the space of functions $f\colon [0,\infty) \to R^d$ which are right continuous and admit left limits, and we endow this space with Skorohod topology ([8], Chapter VI, Section 1). All the processes appearing in the sequel are assumed to be realized in $D([0,\infty), R^d)$, for some d. We also denote by $\to_P$ and $\to_{\mathscr{L}}$ convergence in probability and in law, respectively, and $\int_0^t$ stands for $\int_{(0,t]}$. Sometimes, we simply refer to the space $D([0,\infty), E)$ as D

when the range space E is obvious but cumbersome to write down. When we refer to vectors, it should be understood that we mean row vectors.

2. The queueing network model. The queueing network model in this paper consists of K service stations. At each station, customers arrive from outside the network or from other stations. Let $Q_i(t)$ be the number of customers at station i at time t. Then we have the following identity:

$$Q_i(t) = Q_i(0) + A_i(t) + \sum_{j=1}^{K} N_{ji}(t) - N_i(t), \qquad 1 \le i \le K, \tag{2.1}$$

where $A_i(t)$ is the number of customers who arrive at station i from outside the network by time t, $N_{ji}(t)$ is the number of customers who move from station j to station i after completion of service by time t and $N_i(t)$ is the number of customers who complete service at station i by time t. If we let $N_{j0}(t)$ be the number of customers who leave the network from station j, we have the relation

$$N_i(t) = \sum_{j=0}^{K} N_{ij}(t).$$

Our basic assumption is as follows: The processes $A_i(t)$, $N_{ij}(t)$ and $N_i(t)$, $1 \le i \le K$, $0 \le j \le K$, are counting processes defined on a stochastic basis $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$ satisfying the usual conditions (see [8], page 2), and they have intensities which depend on the state of the network in the following way: There exist nonnegative functions $\lambda_i(\cdot), \mu_i(\cdot)$, $1 \le i \le K$, defined on $\{0, 1, 2, \cdot, \cdot, \cdot\}$ such that

$$\tilde{A}_i(t) := A_i(t) - \int_0^t \lambda_i(Q_i(s))\, ds,$$

$$\tilde{N}_i(t) := N_i(t) - \int_0^t \mu_i(Q_i(s))\, ds,$$

$$\tilde{N}_{ji}(t) := N_{ji}(t) - \int_0^t p_{ji}\, \mu_j(Q_j(s))\, ds, \qquad 1 \le i, j \le K,$$

are locally square integrable martingales, where p_{ij}, $1 \le i \le K$, $0 \le j \le K$, are real numbers such that $p_{ij} \ge 0$ and $\sum_{j=0}^{K} p_{ij} = 1$. Moreover, $A_i(t)$ and $N_{ij}(t)$, $1 \le i \le K$, $0 \le j \le K$, have no common jumps.

In the above queueing network model, p_{ij} represents the probability that a customer goes to station j or leaves the network ($j = 0$) after completion of service at station i; $P = (p_{ij})$, $1 \le i, j \le K$, is called a routing matrix. Note that we can construct the network with the above properties by using the method given in [2] (Chapter V, Section 2). We also note that the predictable quadratic variational processes are given by

$$\begin{aligned} \langle \tilde{A}_i \rangle(t) &= \int_0^t \lambda_i(Q_i(s))\, ds, \\ \langle \tilde{N}_i \rangle(t) &= \int_0^t \mu_i(Q_i(s))\, ds, \qquad (2.2) \\ \langle \tilde{N}_{ij} \rangle(t) &= \int_0^t p_{ij}\, \mu_i(Q_i(s))\, ds, \end{aligned}$$

respectively.

REMARK 1. The manner of dependence of arrival and service rates on the state of queues, which is considered here, is rather restricted in the sense that the rates λ_i and μ_i are dependent only on Q_i, but not on $Q = (Q_1, \dots, Q_K)$. This is due to the technical reason that we use an argument for a local time for semimartingales defined only for one-dimensional processes (see Lemma 3). This restriction can be removed for some queueing models, but we will not discuss it here.

3. An example. To see how the dependence of arrival and service rates on the state of the queue affects the limit processes of the queueing network, we consider the following simple example of a single station. (As long as we are interested in the influence of the state dependence, it suffices to consider a single station model.) This example will motivate some of the assumptions for our main result and also show the outline of our approach.

We consider a sequence of queues of a single station. Let $Q_n(t)$ be the number of customers at the station at time t for the nth queue. Then we have

$$Q_n(t) = Q_n(0) + A_n(t) - N_n(t), \qquad n = 1, 2, \dots,$$

where $A_n(t)$ and $N_n(t)$ are arrival and departure processes. They are assumed to have intensities $\lambda_n(Q_n(t))$ and $\mu_n(Q_n(t))$, respectively, given by

$$\lambda_n(x) = \begin{cases} n\lambda_n^1, & x \geq a_n, \\ n\lambda_n^2, & x < a_n, \end{cases}$$

$$\mu_n(x) = \begin{cases} n\mu_n^1, & x \geq a_n, \\ n\mu_n^2, & 0 < x < a_n. \end{cases}$$

Let $\{\varphi_n\}$ be a sequence of real numbers defined by $\varphi_n = \sqrt{n}$, and assume the following:

(i) $a_n / \varphi_n \to b$;
(ii) $\sqrt{n}(\lambda_n^1 - \mu_n^1) \to a_1, \qquad \sqrt{n}(\lambda_n^2 - \mu_n^2) \to a_2$;
(iii) $\lim \lambda_n^1 = \lim \mu_n^1 = \lambda_1 > 0, \qquad \lim \lambda_n^2 = \lim \mu_n^2 = \lambda_2 > 0.$

Let us define processes $\{X_n(t)\}$ by $X_n(t) = Q_n(t)/\varphi_n$ and consider the limit process of $\{X_n(t)\}$. Now $Q_n(t)$ can be written as

$$Q_n(t) = Q_n(0) + \int_0^t [\lambda_n(Q_n(s)) - \mu_n(Q_n(s))]\, ds + m_n(t),$$

$$m_n(t) := \tilde{A}_n(t) - \tilde{N}_n(t),$$

where

$$\tilde{A}_n(t) = A_n(t) - \int_0^t \lambda_n(Q_n(s))\, ds,$$

$$\tilde{N}_n(t) = N_n(t) - \int_0^t \mu_n(Q_n(s))\, ds$$

are orthogonal square integrable martingales with predictable quadratic variation processes given by

$$\langle \tilde{A}_n \rangle(t) = \int_0^t \lambda_n(Q_n(s))\, ds,$$

$$\langle \tilde{N}_n \rangle(t) = \int_0^t \mu_n(Q_n(s))\, ds,$$

respectively. [Note that the orthogonality of $\tilde{A}_n(t)$ and $\tilde{N}_n(t)$ comes from the fact that $A_n(t)$ and $N_n(t)$ have no common jumps.] Defining $\tilde{\mu}_n(x)$ by $\tilde{\mu}_n(x) = \mu_n(x)$ $(x > 0)$ and $\tilde{\mu}_n(0) = n\mu_n^2$, $Q_n(t)$ can be written as

$$Q_n(t) = Q_n(0) + \int_0^t [\lambda_n(Q_n(s)) - \tilde{\mu}_n(Q_n(s))]\, ds + m_n(t)$$

$$+ \tilde{\mu}_n(0) \int_0^t 1(Q_n(s) = 0)\, ds.$$

Thus $x_n(t)$ can be written as

$$X_n(t) = X_n(0) + A_n(t) + M_n(t) + \xi_n(t), \tag{3.1}$$

where

$$A_n(t) = \int_0^t \tilde{\alpha}_n(X_n(s))\, ds, \qquad \tilde{\alpha}_n(x) := \frac{1}{\varphi_n}[\lambda_n(\varphi_n x) - \tilde{\mu}_n(\varphi_n x)],$$

$$\xi_n(t) = \frac{1}{\varphi_n} \tilde{\mu}_n(0) \int_0^t 1(X_n(s) = 0)\, ds;$$

$$M_n(t) = \frac{1}{\varphi_n} m_n(t).$$

We note that $\xi_n(t)$ increases only when $X_n(t) = 0$; that is,

$$\xi_n(t) = \int_0^t 1(X_n(s) = 0)\, d\xi_n(s). \tag{3.2}$$

Thus (3.1) with (3.2) represents a Skorohod equation for $X_n(t)$. The predictable quadratic variation process for $M_n(t)$ can be calculated, using the orthogonality of $\tilde{A}_n(t)$ and $\tilde{N}_n(t)$, as

$$\langle M_n\rangle(t) = (1/\varphi_n^2)\langle m_n\rangle(t)$$
$$= \int_0^t (1/\varphi_n^2)\lambda_n(\varphi_n X_n(s))\,ds + \int_0^t (1/\varphi_n^2)\mu_n(\varphi_n X_n(s))\,ds.$$

Since $\tilde{a}_n(x)$, $(1/\varphi_n^2)\lambda_n(\varphi_n x)$ and $(1/\varphi_n^2)\mu_n(\varphi_n x)$ are bounded [see assumptions (A1) and (A3) in the next section], $\{A_n(t), M_n(t)\}$ is tight in $D([0,\infty), R^2)$. Then by the continuity property of solutions of the Skorohod equation (3.1) and (3.2) (see Lemma 1 in the next section), we conclude that $(X_n(t), \xi_n(t))$ is also tight. Now let $(X(t), \xi(t), A(t), M(t))$ be any weak limit of $(X_n(t), \xi_n(t), A_n(t), M_n(t))$. Then we have

$$X(t) = X(0) + A(t) + M(t) + \xi(t)$$

and

$$\xi(t) = \int_0^t 1(X(s) = 0)\,d\xi(s).$$

To identify the processes $A(t)$ and $M(t)$, we note that the following facts hold:

(a) Let $\alpha_n(x) = (1/\varphi_n)(\lambda_n(\varphi_n x) - \mu_n(\varphi_n x))$. Then $\alpha_n(x_n) \to a(x)$ if $x_n \to x \neq b$ and $x > 0$, where $a(x)$ is defined by $a(x) = a_1$, for $x > b$, and by $a(x) = a_2$, for $0 \le x \le b$. [In the terminology defined in the next section, this is equivalent to $\alpha_n(\cdot) \to_{\text{c.c.}} a(\cdot)$ a.e. on $D = \{x > 0\}$.]

(b) Let $\lambda(x) = \lambda_1$, if $x > b$, and $\lambda(x) = \lambda_2$, if $0 \le x \le b$. Then

$$(1/\varphi_n^2)\lambda_n(\varphi_n x_n) \to \lambda(x) \quad \text{if } x_n \to x \neq b,$$

and

$$(1/\varphi_n^2)\mu_n(\varphi_n x_n) \to \lambda(x) \quad \text{if } x_n \to x \neq b \text{ and } x > 0.$$

[See (A4) in the next section.]

(c) For any Borel set A in R_+ satisfying $\mathscr{L}(A) = 0$ ($\mathscr{L}$ denotes the Lebesgue measure), $\int_0^t 1(X(s) \in A)\,ds = 0$, for all $t \ge 0$. (See Lemma 2 in the next section.)

Combining these facts, we can identify $A(t)$ and $M(t)$ as

$$A(t) = \int_0^t a(X(s))\,ds,$$
$$M(t) = \int_0^t \sqrt{\lambda(X(s))}\,dB_1(s) + \int_0^t \sqrt{\lambda(X(s))}\,dB_2(s),$$

where $B_1(t)$ and $B_2(t)$ are independent Brownian motions. We note that independence of $B_1(t)$ and $B_2(t)$ comes from the fact that $\tilde{A}_n(t)$ and $\tilde{N}_n(t)$ are orthogonal. We can now write $X(t)$ as a solution of the following Skorohod equation:

$$\begin{aligned} X(t) &= X(0) + \int_0^t a(x(s))\,ds + \int_0^t \sqrt{\lambda(X(s))}\,dB_1(s) \\ &\quad + \int_0^t \sqrt{\lambda(X(s))}\,dB_2(s) + \xi(t), \qquad (3.3) \\ & X(t) \geq 0, \qquad \xi(t) = \int_0^t 1(X(s) = 0)\,d\xi(s), \end{aligned}$$

where $\xi(t)$ is an increasing process with $\xi(0) = 0$. Thus, if the above equation (3.3) has a unique solution, we conclude that $X_n(t) \to_{\mathscr{L}} X(t)$ in $D([0,\infty), R^1)$.

Suppose that $a_n = 0$, for all n. Then arrival and service rates do not depend on the state (i.e., queue length) of the queue. In this case, $a(x) = a_1$ and $\lambda(x) = \lambda_1$. Thus the limit process $X(t)$ is a reflecting Brownian motion with a drift. Assumption (ii) in this section is called a heavy traffic condition, and note that this condition is equivalent to fact (a) [see assumption (A2) in the next section].

4. Basic result. From now on we consider a sequence of queueing networks of the type described in Section 2, indexed by $n \geq 1$. Let $(\Omega_n, \mathscr{F}_n, P_n; \mathscr{F}_t^n)$ be the stochastic basis on which the nth such network is defined. All the notation is carried forward, except that we append an n in a convenient place to denote a quantity which depends on n. We assume that K and the routing matrix P are independent of n.

Choosing an appropriate sequence $\{\varphi_n\}$, we give conditions under which $X_n(t)$ converges weakly to a process, where

$$X_n(t) = (X_n^1(t), \dots, X_n^K(t)), \qquad X_n^i(t) = (1/\varphi_n) Q_n^i(t).$$

To this end, as in Section 3, we write $Q_n(t) = (Q_n^1(t), \dots, Q_n^K(t))$ as follows:

$$Q_n^i(t) = Q_n^i(0) + \int_0^t a_n^i(Q_n(s))\,ds + m_n^i(t), \qquad 1 \leq i \leq K,$$

where

$$m_n^i(t) = \tilde{A}_n^i(t) - \sum_{j=1}^K \tilde{N}_n^{ji}(t) - \tilde{N}_n^i(t),$$

$$a_n^i(x) = \lambda_n^i(x_i) + \sum_{j=1}^K p_{ji}\,\mu_n^j(x_j) - \mu_n^i(x_i), \qquad x = (x_1, \dots, X_K) \in R^K.$$

Choosing $\tilde{\mu}_n^i$, $1 \leq i \leq K$, appropriately, we define $\tilde{\mu}_n^i(x)$ by $\tilde{\mu}_n^i(x) = \mu_n^i(x)$, if $x > 0$, and by $\tilde{\mu}_n^i(x) = \tilde{\mu}_n^i$, if $x = 0$, and define $\tilde{a}_n(x) = (\tilde{a}_n^1(x), \dots, \tilde{a}_n^K(x))$ by

$$\tilde{a}_n^i(x) = \lambda_n^i(x_i) + \sum_{j=1}^K p_{ji}\,\tilde{\mu}_n^j(x_j) - \tilde{\mu}_n^i(x_i), \qquad 1 \leq i \leq K.$$

Then $X_n(t)$ can be written as

$$X_n^i(t) = X_n^i(0) + A_n^i(t) + M_n^i(t) + \xi_n^i(t) - \sum_{j=1}^{K} p_{ji}\xi_n^j(t), \qquad 1 \le i \le K, \tag{4.1}$$

where

$$\tilde{\alpha}_n^i(x) = (1/\varphi_n)\tilde{a}_n^i(\varphi_n x), \qquad x \in R_+^K \left[= (R_+)^K\right],$$

$$A_n^i(t) = \int_0^t \tilde{\alpha}_n^i(X_n(s))\,ds,$$

$$\xi_n^i(t) = (1/\varphi_n)\tilde{\mu}_n^i \int_0^t 1\left(X_n^i(s) = 0\right) ds,$$

$$M_n^i(t) = (1/\varphi_n)m_n^i(t).$$

Note that $\xi_n(t) = (\xi_n^1(t), \dots, \xi_n^K(t))$ satisfies

$$\xi_n^i(t) = \int_0^t 1\left(X_n^i(s) = 0\right) d\xi_n^i(s), \qquad 1 \le i \le K.$$

In vector form, $X_n(t)$ can be written as

$$X_n(t) = X_n(0) + A_n(t) + M_n(t) + \xi_n(t)(I - P), \tag{4.2}$$

where $\tilde{\alpha}_n(x) = (\tilde{\alpha}_n^1(x), \dots, \tilde{\alpha}_n^K(x))$ and $M_n(t) = (M_n^1(t), \dots, M_n^K(t))$.

We will make the following assumptions:

(A1) For any $R > 0$, there exists a constant C such that

$$\sup_{\substack{|x| \le R \\ x \in R^K}} |\tilde{\alpha}_n(x)| \le C.$$

(A2) There exists a measurable function $\alpha(x)$, $x \in R^K$, such that for $\alpha_n(x) := (1/\varphi_n)a_n(\varphi_n x)$, $x \in R_+^K (= R_+ \times R_+ \times \cdots \times R_+)$,

$$\alpha_n(\cdot) \to_{\text{c.c.}} \alpha(\cdot) \quad \text{a.e. on } D,$$

where $D = \{x \in R^K,\ x_i > 0,\ 1 \le i \le K\}$.

The definition of the above convergence in (A2) is as follows:

Definition 1. For measurable functions $f_n(x)$, $n \ge 1$, and $f(x)$ defined on R^K and a subset D in R^K, we write $f_n(\cdot) \to_{\text{c.c.}} f(\cdot)$ a.e. on D if, for a Borel set A in D with $\mathscr{L}(A) = 0$ ($\mathscr{L}$ denotes Lebesgue measure), $f_n(x_n) \to f(x)$ whenever $x_n \to x$, $x_n \in D$ and $x \in A^c$ (c.c. implies continuous convergence.)

(A3) For any $R > 0$, there exists a constant C such that

$$\sup_{0 \le x \le R} (1/\varphi_n^2)\lambda_n^i(\varphi_n x) \le C,$$

$$\sup_{0 \le x \le R} (1/\varphi_n^2)\mu_n^i(\varphi_n x) \le C, \qquad 1 \le i \le K.$$

(A4) There exist measurable functions $\lambda_i(\cdot)$ and $\mu_i(\cdot)$ defined on R_+ such that, for $1 \le i \le K$,

$$(1/\varphi_n^2)\lambda_n^i(\varphi_n \cdot) \to_{\text{c.c.}} \lambda_i(\cdot) \quad \text{a.e. on } R_+,$$
$$(1/\varphi_n^2)\mu_n^i(\varphi_n \cdot) \to_{\text{c.c.}} \mu_i(\cdot) \quad \text{a.e. on } \{x;\, x > 0\}.$$

(A5) (i) The routing matrix P has spectral radius strictly less than unity.
(ii) Let

$$\hat{\lambda}_i = \inf_{\substack{x \ge 0 \\ n \ge 1}} (1/\varphi_n^2)\lambda_n^i(\varphi_n x), \qquad 1 \le i \le K.$$

Then all the elements of the vector $\hat{\lambda}(I - P)^{-1}$ are strictly positive, where $\hat{\lambda} = (\hat{\lambda}_1, \dots, \hat{\lambda}_K)$.
(iii) We have

$$\inf_{x > 0} \mu_i(x) > 0, \qquad 1 \le i \le K.$$

[Note that $\mu_i(x)$ appears in (A4).]

We now state our result:

THEOREM 1. *Assume* (A1)–(A5) *and* $X_n(0) \to_{\mathscr{L}} X(0)$ *in* R^K, *where* $X(0)$ *is a random vector. Then* $X_n(t) \to_{\mathscr{L}} X(t)$ *in* $D([0,\infty), R^K)$. *Here* $X(t)$ *is assumed to be the unique solution in law of the following Skorohod equation*:

$$\begin{aligned} X_i(t) &= X_i(0) + \int_0^t a_i(X(s))\, ds + M_i(t) \\ &\quad + \xi_i(t) - \sum_{j=1}^K p_{ji}\xi_j(t), \\ M_i(t) &:= \int_0^t \sqrt{\lambda_i(X_i(s))}\, d\tilde{W}_i(s) + \sum_{j=1}^K \int_0^t \sqrt{p_{ji}\mu_j(X_j(s))}\, d\tilde{B}_{ji}(s) \\ &\quad - \sum_{j=0}^K \int_0^t \sqrt{p_{ij}\mu_i(X_i(s))}\, d\tilde{B}_{ij}(s), \qquad 1 \le i \le K, \end{aligned} \tag{4.3}$$

where $\xi_i(t)$, $1 \le i \le K$, *are increasing with* $\xi_i(0) = 0$ *and satisfy*

$$\xi_i(t) = \int_0^t 1(X_i(s) = 0)\, d\xi_i(s).$$

Moreover, $\tilde{W}_i(t)$ *and* $\tilde{B}_{ij}(t)$, $1 \le i \le K$, $0 \le j \le K$, *are independent standard Brownian motions*.

REMARK 2. In view of the discussion in Section 3, assumption (A2) may be called a heavy traffic condition.

REMARK 3. Let π_i be the ith row vector of $I - P$. Then π_i represents the direction of reflection when the process $X(t)$ hits the boundary $\partial D_i = \{x \in R_+^K,\ x_i = 0\}$.

REMARK 4. Assumption (A1) is given for $\tilde{\alpha}_n(x)$, which is defined using constants $\tilde{\mu}_n^i$, $1 \le i \le K$, not known a priori. Hence, it is desirable to give a condition which uses $\alpha_n(x)$ instead of $\tilde{\alpha}_n(x)$ and implies (A1). The following is such an example.

(A1′) For any $R > 0$, there exists a constant C such that

$$\sup_{\substack{x \in D \\ |x| \le R}} |\alpha_n(x)| \le C.$$

Moreover, for each i, $1 \le i \le K$, and n, there exists a positive integer z_n^i such that $\lambda_n^i(0) = \lambda_n^i(z_n^i)$.

Under (A1′), we set $\tilde{\mu}_n^i(0) = \mu_n^i(z_n^i)$. Then it is obvious that (A1) is satisfied since

$$\sup_{\substack{x \in R_+^K \\ |x| \le R}} |\tilde{\alpha}_n(x)| = \sup_{\substack{x \in D \\ |x| \le R}} |\alpha_n(x)|.$$

REMARK 5. As for the uniqueness (in law) of the solution of (4.3), it seems that there exist no general results in our setting under which $a(x)$, $\lambda(x)$ and $\mu(x)$ are generally discontinuous. However, for the following cases the uniqueness holds.

Case 1. Feedforward networks for which we can label the stations so that $p_{ij} = 0$, if $i > j$. A typical case is tandem queues. In this case, the uniqueness problem reduces to that of a one-dimensional Skorohod equation, and various results are available (e.g., [10] and [1]) for general drift and diffusion coefficients.

Case 2. The drift coefficients of (4.3) are measurable, but the diffusion coefficients are Lipschitz continuous. In this case, we can reduce the uniqueness problem to the case where (4.3) has no drift. Then the uniqueness is guaranteed by the Lipschitz continuity of the reflection mapping (see Lemma 1), and this is shown in the following proposition.

PROPOSITION 1. *In* (4.3), *we assume the following*:

(i) $a(x)$, $\lambda(x)$ $[= (\lambda_1(x_1), \lambda_2(x_2), \dots, \lambda_K(x_K))]$ *and* $\mu(x)$ $[= (\mu_1(x_1), \mu_2(x_2), \dots, \mu_K(x_K))]$ *satisfy a linear growth condition*; *that is*,

$$|a(x)|^2 + |\lambda(x)|^2 + |\mu(x)|^2 \le C(1 + |x|^2), \qquad x \in R_+^K.$$

(ii) $\lambda_i(x)$ *and* $\mu_i(x)$, $1 \le i \le k$, *are twice continuously differentiable and satisfy*

$$\inf_{0 \le x \in R^1} \min(\lambda_i(x), \mu_i(x)) > 0.$$

(iii) *For any* $x \in R_+^K$, $\lambda(x)(I - P)^{-1} > 0$ *and* $\mu(x) > 0$; *that is, each element of the vectors is strictly positive.*

Then (4.3) *has a unique solution in law.*

If $a(x)$ is Lipschitz continuous, we can weaken the above conditions and the following result holds:

PROPOSITION 2. *Assume condition* (i) *in Proposition* 1 *and that* $a(x)$, $x \in R^K$, $\sqrt{\lambda_i(x)}$ *and* $\sqrt{\mu_i(x)}$, $x \in R^1$, $1 \le i \le K$, *are locally Lipschitz continuous. Then* (4.3) *has a unique solution.*

5. The proof. For the proof of our theorem, we need several lemmas. To this end, we define the reflection map

$$(\Psi, \Phi)\colon D([0,T], R^d) \to D([0,T], R^{2d})$$

associated with the routing matrix P as a function which maps $w \in D([0,T], R^d)$ into a unique solution $(x, \xi) = (\Psi(w), \Phi(w)) \in D([0,T], R^{2d})$ of the following Skorohod equation:

$$\text{(SE)}\qquad \begin{aligned} &x(t) = w(t) + \xi(t)(I - P) \ge 0,\\ &\xi_i(t) \text{ is nondecreasing with } \xi_i(0) = 0,\\ &\xi_i(t) = \int_0^t 1(x_i(s) = 0)\, d\xi_i(s), \qquad 1 \le i \le d. \end{aligned}$$

For any $a \in R^d$, let $\|a\| = \Sigma_{i=1}^d |a_i|$ and, for any $x \in D$, let

$$\|x\| = \sup_{0 \le t \le T} \|x(t)\|,$$

for an arbitrarily fixed T.

LEMMA 1. *The reflection map* (Ψ, Φ) *is Lipschitz continuous. That is, there exists a constant C such that, for any* $x_1, x_2 \in D$,

$$\max(\|\Psi(x_1) - \Psi(x_2)\|, \|\Phi(x_1) - \Phi(x_2)\|) \le C\|x_1 - x_2\|.$$

COROLLARY 1. *For a given sequence of processes* $\{W_n\}_{n \ge 1}$ *in D, let* $X_n = \Psi(W_n)$ *and* $\xi_n = \Phi(W_n)$, $n = 1, 2, \dots$. *Suppose* $\{W_n\}$ *is tight in* $D([0,\infty), R^d)$. *Then* $\{X_n, W_n, \xi_n\}$ *is tight in* $D([0,\infty), E)$ *with* $E = R_+^d \times R^d \times R_+^d$. *Moreover, if* $\{X, W, \xi\}$ *is any weak limit of* $\{X_n, W_n, \xi_n\}$, $\{X, \xi\}$ *is the unique solution of the Skorohod equation (SE) with* $w(t)$ *being replaced by* $W(t)$, *that is,* $X = \Psi(W)$ *and* $\xi = \Phi(W)$.

REMARK 6. The definition of the reflection map (Ψ, Φ) and its Lipschitz continuity is due to Harrison and Reiman ([5], Theorem 1).

REMARK 7. The (Lipschitz) continuity of the map (Ψ, Φ) assures the result of Corollary 1. The Lipschitz continuity is used for establishing the uniqueness of the solution of a stochastic differential equation with reflection and with Lipschitz continuous diffusion coefficients (see Propositions 1 and 2).

LEMMA 2. *Assume* (A1) *and* (A3) *with* $R = \infty$ *and that* $\{X_n(0)\}$ *is tight in* R^K. *Then* $\{X_n(t), A_n(t), M_n(t), \xi_n(t)\}$ [*see* (4.2)] *is tight in* $D([0,\infty), F)$ *with* $F = R_+^K \times R^K \times R^K \times R_+^K$.

PROOF. Let $W_n(t) = A_n(t) + M_n(t)$. Then $(X_n(t), W_n(t), \xi_n(t))$ satisfies (4.1). Thus, in view of Lemma 1, it suffices to show that $\{A_n(t), M_n(t)\}$ is tight in $D([0,\infty), R^K \times R^K)$. For this purpose, it suffices to show that $\{M_n(t)\}$ is tight in $D([0,\infty), R^1)$ and $\{A_n(t)\}$ is C-tight in $D([0,\infty), R^1)$ ([8], Chapter VI, Corollary 3.3). However, since $\{A_n(t)\}$ is obviously C-tight by (A1), we will show that $\{M_n(t)\}$ is tight. Since $\{A_n^i(t), N_n^{ij}(t), 1 \le i \le K, 0 \le j \le K\}$ have no common jumps by our assumption (see Section 2), any pair of $\{\tilde{A}_n^i(t), \tilde{N}_n^{ij}(t), 1 \le i \le K, 0 \le j \le K\}$ is orthogonal (see [8], Chapter I, Section 4). Thus we have

$$\begin{aligned}\langle M_n^i\rangle(t) &= (1/\varphi_n^2)\langle \tilde{A}_n^i\rangle(t) + (1/\varphi_n^2)\sum_{\substack{l=1\\ l\neq i}}^{K}\langle \tilde{N}_n^{li}\rangle(t)\\ &\quad + (1/\varphi_n^2)\sum_{\substack{m=0\\ m\neq i}}^{K}\langle \tilde{N}_n^{im}\rangle(t).\end{aligned} \tag{5.1}$$

We note that

$$\begin{aligned}(1/\varphi_n^2)\langle \tilde{A}_n^i\rangle(t) &= \int_0^t (1/\varphi_n^2)\lambda_n^i(\varphi_n X_n^i(s))\,ds,\\ (1/\varphi_n^2)\langle \tilde{N}_n^{ij}\rangle(t) &= \int_0^t (1/\varphi_n^2)p_{ij}\mu_n^i(\varphi_n X_n^i(s))\,ds\end{aligned} \tag{5.2}$$

[see (2.2) in Section 2]. Hence, by (A3), $\langle M_n^i\rangle(t)$, $1 \le i \le K$, is C-tight in $D([0,\infty), R^1)$, and so is $\Sigma_{1\le i\le K}\langle M_n^i\rangle(t)$, which means the tightness of $\{M_n(t)\}$ in $D([0,\infty), R^K)$ ([8], Chapter VI, Theorem 4.13). □

LEMMA 3. *Assume* (A1) *and* (A3) *with* $R = \infty$ *and that* $\{X_n(0)\}$ *is tight in* R^1. *We also assume* (A5). *Then any weak limit* $X(t)$ *of* $X_n(t)$ *satisfies* $\mathscr{L}\{t, X_i(t) \in A\} = 0$, $1 \le i \le K$, *with probability* 1 *for any Borel set* A *in* R_+ *with* $\mathscr{L}(A) = 0$. *Here, as before,* $\mathscr{L}$ *denotes the Lebesgue measure in* R^1.

PROOF. *Step* 1. By assumption (A5)(ii), there exists an i, $1 \le i \le K$, such that $\hat{\lambda}_i > 0$. We will show that for such i the assertion is true. Let $\{X(t), A(t), M(t), \xi(t)\}$ be any weak limit of $\{X_n(t), A_n(t), M_n(t), \xi_n(t)\}$ in $D([0,\infty), F)$ (see Lemma 2). Then $X_i(t)$ can be written as

$$\begin{aligned}X_i(t) &= X_i(0) + A_i(t) + M_i(t) + \xi_i(t)\\ &\quad - \sum_{1\le j\le K} p_{ji}\xi_j(t).\end{aligned}$$

We can easily show that $M_i(t)$ is a continuous martingale since it is a weak limit of the martingale $M_n^i(t)$, and $A_i(t)$ is a continuous process of bounded variation. Hence, by Lemma 1, $X_i(t)$ is continuous. Moreover, since

$$\beta_n(t) := (1/\varphi_n)\tilde{A}_n^i(t),$$
$$\gamma_n(t) := (1/\varphi_n)\sum_{1\le j\le K}\tilde{N}_n^{ji}(t) - (1/\varphi_n)\tilde{N}_n^i(t)$$

are orthogonal martingales, $M_i(t)$ can be written as

$$M_i(t) = \beta(t) + \gamma(t),$$

where $\{\beta(t), \gamma(t)\}$ is a weak limit of $\{\beta_n(t), \gamma_n(t)\}$ in $D([0,\infty), R^2)$ and $\beta(t)$ and $\gamma(t)$ are orthogonal martingales. Since $X_i(t)$ is a continuous semimartingale, it has a local time and the density formula for local times ([7], page 188) implies

$$\int_0^t 1_A(X_i(s))d\langle M_i\rangle(s) = 0.$$

Thus we have

$$0 = \int_0^t 1_A(X_i(s))d\langle\beta\rangle(s).$$

We will show that there exists a measurable process $\hat{\beta}(t)$ such that $\langle\beta\rangle(t) = \int_0^t\hat{\beta}(s)\,ds$ with $\hat{\beta}(t)\ge\hat{\beta}>0$ for a constant $\hat{\beta}$. We note that this at once shows $\int_0^t 1_A(X_i(s))\,ds = 0$, a desired assertion in Step 1. By (5.2) and assumption (A3), we have

$$\left\langle(1/\varphi_n)\tilde{A}_n^i\right\rangle(t) - \left\langle(1/\varphi_n)\tilde{A}_n^i\right\rangle(s) \le C(t-s).$$

Noting that $\beta(t)$ is the weak limit of $(1/\varphi_n)\tilde{A}_n^i(t)$ and letting n tend to infinity in the above inequality, we have, by a standard argument, that

$$\langle\beta\rangle(t) - \langle\beta\rangle(s) \le C(t-s).$$

Hence, there exists a measurable process $\hat{\beta}(t)$ such that $\langle\beta\rangle(t) = \int_0^t\hat{\beta}(s)\,ds$. Moreover, by (5.2) and (A5)(ii), we can take $\hat{\beta}(t)$ satisfying $\hat{\beta}(t)\ge\hat{\lambda}_i$ (>0). This completes Step 1.

Step 2. Take an arbitrary station k. Then by (A5)(i), (ii), there exists a sequence of stations $i_1 = i$, $i_2 = j, \dots, i_n = k$ such that $\hat{\lambda}_i > 0$, $p_{i_1 i_2} > 0$, $p_{i_2 i_3} > 0, \dots, p_{i_{n-1}k} > 0$. Then, in view of the result of Step 1, it holds that $\mathscr{L}(t;\ X_i(t)\in A) = 0$ for any Borel set A in R_+ with $\mathscr{L}(A) = 0$. Using this fact, we will show that the assertion of the lemma also holds for station $i_2 = j$, that is, $\mathscr{L}(t;\ X_j(t)\in A) = 0$ with probability 1. Then by induction we reach the conclusion of the lemma. As in Step 1, we have

$$\int_0^t 1_A\big(X_j(s)\big)d\langle M_j\rangle(s) = 0.$$

We can write the process $M_j(t)$ as

$$M_j(t) = \tilde{A}_j(t) + \sum_{1\le l\le K}\tilde{N}_{lj}(t) - \tilde{N}_j(t), \qquad \tilde{N}_j(t) = \sum_{0\le l\le K}\tilde{N}_{jl}(t),$$

where $(\tilde{A}_i(t), \tilde{N}_{ij}(t),\ 1\le i\le K,\ 0\le j\le K)$ is the weak limit of $\{\mathscr{A}_n(t)\} = \{(1/\varphi_n)\tilde{A}_n^i(t), (1/\varphi_n)\tilde{N}_n^{ij}(t),\ 1\le i\le K,\ 0\le j\le K\}$. As in Step 1, using the orthogonality of the processes in $\mathscr{A}_n(t)$, by a standard argument we have

$$\langle M_j\rangle(t) = \langle \tilde{A}_j\rangle(t) + \sum_{\substack{1\le l\le K\\ l\ne j}} \langle \tilde{N}_{lj}\rangle(t) + \sum_{\substack{0\le m\le K\\ m\ne j}} \langle \tilde{N}_{jm}\rangle(t).$$

Thus, we have

$$0 = \int_0^t 1_A\big(X_j(s)\big)\,d\langle \tilde{N}_{ij}\rangle(s). \tag{5.3}$$

Here we note that

$$\langle \tilde{N}_{ij}\rangle(t) = \int_0^t p_{ij}\,\mu_i(X_i(s))\,ds. \tag{5.4}$$

Indeed, we consider the limit of

$$\Big\langle (1/\varphi_n)\tilde{N}_n^{ij}\Big\rangle(t) = \int_0^t (1/\varphi_n^2)\,p_{ij}\,\mu_n^i\big(\varphi_n X_n^i(s)\big)\,ds.$$

In doing this, we may assume by the Skorohod imbedding theorem that $X_n^i(t) \to X_i(t)$ uniformly on any compact t-set with probability 1 [recall that $X(t)$ is continuous]. Then, in view of assumption (A4) and the result of Step 1, we have, as n tends to infinity,

$$\Big\langle (1/\varphi_n)\tilde{N}_n^{ij}\Big\rangle(t) \to \int_0^t p_{ij}\,\mu_i(X_i(s))\,ds.$$

Thus, we obtain (5.4) again by a standard argument. Then, from (5.3) and (5.4) and the result of Step 1, we have

$$0 = \int_0^t 1_A\big(X_j(s)\big)p_{ij}\,\mu_i(X_i(s))1(X_i(s) > 0)\,ds$$

$$\ge p_{ij}\inf_{x>0}\mu_i(x)\int_0^t 1_A\big(X_j(s)\big)\,ds.$$

Then our desired result, $\mathscr{L}(t;\ X_j(t)\in A) = 0$, follows from $p_{ij} > 0$ and assumption (A5)(iii). □

PROOF OF THEOREM 1. In view of the standard cutoff argument ([14], Section 11.1), we may and do assume that assumptions (A1) and (A3) hold with $R = \infty$. Then by Lemma 2, $\{\mathscr{B}_n(t)\} := \{(X_n(t), A_n(t), M_n(t), \xi_n(t))\}$ is tight in $D([0,\infty), F)$ and we let $\mathscr{B}(t) := (X(t), A(t), \tilde{M}(t), \xi(t))$ be any weak limit of $\{\mathscr{B}_n(t)\}$. We have, by Lemma 1,

$$X(t) = X(0) + A(t) + \tilde{M}(t) + \xi(t)(I - P),$$

where $\xi_i(t)$ is increasing with $\xi_i(0) = 0$, and

$$\xi_i(t) = \int_0^t 1(X_i(s) = 0)\,d\xi_i(t) \quad \text{for each } i\ (1\le i\le K).$$

We recall that $X(t)$ is continuous as mentioned in the proof of Lemma 3. It remains to identify $A(t)$ as

$$A(t) = \int_0^t a(X(s))\,ds$$

and $\tilde{M}(t)$ as $M(t)$ given in (4.3). In doing this, we may assume, as in Lemma 3, that $\mathscr{B}_n(t) \to \mathscr{B}(t)$ uniformly on any compact t-set with probability 1. By Lemma 3, it holds that $\mathscr{L}(t; X(t) \in \partial D) = 0$, where ∂D is the boundary of D [D was defined in (A2) in Section 3]. Hence,

$$A_n(t) = \int_0^t \tilde{\alpha}_n(X_n(s)) 1_D(X(s))\, ds.$$

Note that if $X(s) \in D$, then, for sufficiently large n, $X_n(s) \in D$ and hence $\tilde{\alpha}_n(X_n(s)) = \alpha_n(X_n(s))$. Thus letting n go to infinity in the above equation, we have, by (A2) and Lemma 3,

$$A(t) = \int_0^t a(X(s))\, ds.$$

The identification of $\tilde{M}(t)$ as $M(t)$ in (4.3) is almost contained in the proof of Lemma 3. We note that each pair of the weak limit $\{\tilde{A}_i(t), \tilde{N}_{ij}(t), 1 \le i \le K, 0 \le j \le K\}$ of $\{\tilde{A}_n^i(t), \tilde{N}_n^{ij}(t), 1 \le i \le K, 0 \le j \le K\}$ is orthogonal. Moreover, we can show, using (A4) and Lemma 3, that

$$\langle \tilde{A}_i \rangle(t) = \int_0^t \lambda_i(X_i(s))\, ds, \qquad 1 \le i \le K.$$

Taking into account these facts and that (5.4) holds for $1 \le i, j \le K$, due to the representation theorem for martingales ([6], Chapter 2, Theorem 7.1′) we have the desired result for the identification of $\tilde{M}(t)$. □

PROOF OF PROPOSITION 2. This follows directly from the Lipschitz continuity of the reflection map Ψ defined in Lemma 1 by a standard calculation using Doob's inequality for martingales (see [4]). □

PROOF OF PROPOSITION 1. We write (4.3) in vector form. For this, we define a matrix function $\Sigma(x)$, $x \in R^K$, as follows: $\Sigma(x)$ is written as $\Sigma(x) = (A(x), B_1(x), \dots, B_K(x))$, where A and B_i, $1 \le i \le K$, are $K \times K$ and $K \times (K + 1)$ matrices and the components of each matrix are as follows:

$$A(x) = \begin{bmatrix} \sqrt{\lambda_1(x_1)} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \sqrt{\lambda_k(x_K)} \end{bmatrix},$$

$$B_i(x) = \left(B_i^0(x), B_i^1(x), \dots, B_i^K(x)\right)^t,$$

where $B_i^j(x)$, $0 \le j \le K$, are K-dimensional column vectors,

$$B_i^0(x) = \left(0,\dots,0,-\sqrt{\mu_i(x_i)},0,\dots,0\right)^t$$

$$\left[\text{the } i\text{th element is } -\sqrt{\mu_i(x_i)}\,\right],$$

$$B_i^j(x) = \left(0,\dots,0,-\sqrt{p_{ij}\mu_i(x_i)},0,\dots,0,\sqrt{p_{ij}\mu_i(x_i)},0,\dots,0\right)^t$$

$$\left(\text{the } i\text{th element is } -\sqrt{} \text{ and the } j\text{th element is } \sqrt{}\right),$$

$$B_i^i(x) = (0,\dots,0)^t.$$

Then (4.3) can be written as

$$X(t) = X(0) + \int_0^t a(X(s))\,ds + \int_0^t \Sigma(X(s))\,d\tilde{B}(s),$$

where $\tilde{B}(t) = (\tilde{W}_i(t), \tilde{B}_{ij}(t),\ 1 \le i,j \le K)$. We note that $\langle M_i, M_j\rangle(t) = \int_0^t \alpha_{ij}(X(s))\,ds$, where $\alpha_{ij}(x)$ is the (i,j) element of $\Sigma(x)\Sigma(x)^t$. [Recall that $M(t) = (M_1(t), M_2(t),\dots, M_K(t))$ was defined in Theorem 1.] We will show that $\Sigma(x)\Sigma(x)^t$ is positive definite for each $x \in R^d$. For that purpose, it suffices to show that, for each $x \in R^d$,

$$u\Sigma(x) = 0 \quad\to\quad u = 0, \qquad u \in R^K.$$

Let $x \in R^d$ be arbitrarily fixed. By assumption (iii) of Proposition 1, there exists a station i with $\lambda_i(x) > 0$. On the other hand, since $uA(x) = 0$ [note $u\Sigma(x) = 0$], $u_i\sqrt{\lambda_i(x_i)} = 0$. Hence, $u_i = 0$. Next, suppose that $p_{ij} > 0$ for $j \ne i$. For such j, we have

$$0 = uB_i^j(x) = \sqrt{p_{ij}\mu_i(x_i)}\,u_j - \sqrt{p_{ij}\mu_i(x_i)}\,u_i = \sqrt{p_{ij}\mu_i(x_i)}\,u_j.$$

Then, since $p_{ij} > 0$ and $\mu_i(x_i) > 0$, we have $u_j = 0$. We now take an arbitrary station k. Then, as in the proof of Lemma (iii), in view of assumption (iii), there exists a sequence of stations $i, i_1,\dots, i_{n-1}, k$ such that $p_{ii_1} > 0$, $p_{i_1 i_2} > 0,\dots, p_{i_{n-1}k} > 0$. Then repeating the above discussion, it follows that $u_k = 0$. Hence $u = 0$ and we have shown that $\Sigma(x)\Sigma(x)^t$ is positive definite. Now, since $\alpha_{ij}(x)$ is twice continuously differentiable, there exists a nonsingular $K \times K$ matrix function $\sigma(x)$ such that $\Sigma(x)\Sigma(x)^t = \sigma(x)\sigma(x)^t$ and $\sigma(x)$ is locally Lipschitz continuous (see [6] Chapter 6, Proposition 6.2). It follows, by a representation theorem for martingales (see [6], Chapter 2, Theorem 7.1), that there exists a K-dimensional standard Brownian motion $B(t)$ such that

$$M_i(t) = \sum_{1\le j\le K}\int_0^t \sigma_{ji}(X(s))\,dB_j(s), \qquad 1 \le i \le K.$$

Thus (4.3) becomes

$$(4.3^*)\quad X(t) = X(0) + \int_0^t a(X(s))\,ds + \int_0^t \sigma(X(s))\,dB(s) + \xi(t)(I-P),$$

and, to show Proposition 1, it suffices to show the uniqueness result for (4.3*). To this end, first we note that any solution of (4.3*) [hence of (4.3)] is not explosive. (We can show this easily by using the Lipschitz continuity of the reflection map Ψ in Lemma 1.) For any solution $X(t)$, let $\tau_R = \inf\{t; |X(t)| \geq R\}$ and let $\tilde{X}(t) = X(t \wedge \tau_R)$ and $\tilde{\xi}(t) = \xi(t \wedge \tau_R)$. Then $\tilde{X}(t)$ satisfies the following Skorohod equation:

$$\begin{aligned} \tilde{X}(t) &= \tilde{X}(0) + \int_0^t a(X(s))1\big(|\tilde{X}(s)| < R\big)\,ds \\ &\quad + \int_0^t 1\big(|\tilde{X}(s)| < R\big)\sigma\big(\tilde{X}(s)\big)\,dB(s) + \tilde{\xi}(t)(I-P), \end{aligned} \tag{5.5}$$

where $\tilde{\xi}_i(t)$ is nondecreasing and satisfies

$$\tilde{\xi}_i(t) = \int_0^t 1\big(\tilde{X}_i(s) = 0\big)\,d\tilde{\xi}_i(s).$$

Thus, since $X(t)$ is not explosive as noted above, it suffices to show the uniqueness result for the solution of (5.5). For this, obviously it is sufficient to show the uniqueness of the solution of the following Skorohod equation:

$$\begin{aligned} Y(t) &= X(0) + \int_0^t a(Y(s))1(|Y(s)| < R)\,ds \\ &\quad + \int_0^t \sigma(Y(s))\,dB(s) + \varphi(t)(I-P), \end{aligned}$$

where $\varphi_i(t)$ is nondecreasing with $\varphi_i(0) = 0$ and

$$\varphi_i(t) = \int_0^t 1(Y_i(s) = 0)\,d\varphi_i(s), \qquad 1 \leq i \leq K.$$

Since $1(|Y(s)| < R)a(Y(s))\sigma^{-1}(Y(s))$ is bounded (see assumptions 1 and 2), it is well known that by a Girsanov measure transformation technique, the uniqueness problem of the above equation reduces to that of the following driftless Skorohod equation:

$$Y(t) = X(0) + \int_0^t \sigma(Y(s))\,dB(s) + \varphi(t)(I-P),$$

where $\varphi(t)$ satisfies the same relation as above. (See Section 4 of Chapter 4 in [6].) Now the uniqueness of the solution of the above equation is assured by Proposition 1, since $\sigma(x)$ is locally Lipschitz continuous. □

6. Examples for queueing networks.

EXAMPLE 1. In this example, we treat a case which is an extension of the model considered in Section 3 to network models. Let $\lambda_n^i(x)$ and $\mu_n^i(x)$ be given as follows:

$$\begin{aligned} \lambda_n^i(x) &= n\hat{\lambda}_n^i\big(x/\sqrt{n}\big), && x \in R^1, \\ \mu_n^i(x) &= n\hat{\mu}_n^i\big(x/\sqrt{n}\big), && 0 < x \in R^1, 1 \leq i \leq K. \end{aligned}$$

We take $\varphi_n = \sqrt{n}$ and assume the following:

(E1) There exists a measurable function $a(x)$ such that

$$\alpha_n^i(x) = \sqrt{n}\left(\hat{\lambda}_n^i(x_i) + \sum_{1\le j\le K} p_{ji}\,\hat{\mu}_n^j(x_j) - \hat{\mu}_n^i(x_i)\right) \to_{\text{c.c.}} a_i(x)$$

$$\text{a.e. on } D\ (x \in R_+^K),\ 1 \le i \le K.$$

(E2) There exist measurable functions $\lambda_i(x)$ and $\mu_i(x)$ on R_+^1, $1 \le i \le K$, such that

$$(1/\varphi_n^2)\lambda_n^i(\varphi_n x) = \hat{\lambda}_n^i(x) \to_{\text{c.c.}} \lambda_i(x) \quad \text{a.e. on } \{x \in R^1;\ x \ge 0\},$$

$$(1/\varphi_n^2)\mu_n^i(\varphi_n x) = \hat{\mu}_n^i(x) \to_{\text{c.c.}} \mu_i(x) \quad \text{a.e. on } \{x \in R^1;\ x > 0\}.$$

If we assume (A1), (A3) and (A5), then $X_n(t) \to_{\mathscr{L}} X(t)$, where $X(t)$ is the unique solution of (4.3). As a special case, let

$$\hat{\lambda}_n^i(x) = \hat{\lambda}_i(x)/\sqrt{n} + \lambda_i(x), \qquad x \in R^1,$$

$$\hat{\mu}_n^i(x) = \hat{\mu}_i(x)/\sqrt{n} + \mu_i(x), \qquad x \in R^1,$$

where $\hat{\lambda}_i(x)$, $\lambda_i(x)$, $\hat{\mu}_i(x)$ and $\mu_i(x)$ are arbitrary nonnegative measurable functions. Then (E1) implies

$$a_i(x) = \hat{\lambda}_i(x_i) + \sum_{1\le j\le K} p_{ji}\,\hat{\mu}_j(x_j) - \hat{\mu}_i(x_i), \tag{6.1}$$

$$\lambda_i(x_i) + \sum_{1\le j\le K} p_{ji}\,\mu_j(x_j) - \mu_i(x_i) = 0 \quad \text{for all } x \in R_+^K. \tag{6.2}$$

Thus the following fact follows: Given an arbitrary equation (4.3) with $a(x)$ being given by (6.1) and assuming the uniqueness of the solution of (4.3) for which (6.2) holds, then we can construct a sequence of queueing networks $\{Q_n(t)\}_{n\ge 1}$ such that the normalized processes $\{X_n(t)\}_{n\ge 1}$ converge in law to the process $X(t)$ which is the unique solution of the arbitrarily given equation (4.3). That is, quite roughly speaking, a very large class of reflecting diffusions with nonsingular drift and diffusion coefficients can be approximated by an appropriate sequence of queueing networks with suitable normalization.

EXAMPLE 2. Let

$$\lambda_n^i(x) = n\hat{\lambda}_i(x/n) \quad \text{and} \quad \mu_n^i(x) = n\hat{\mu}_i(x/n)1(x > 0), \qquad x \in R^1.$$

[Note that we do not assume $\hat{\mu}_i(0) = 0$, $1 \le i \le K$, but $\hat{\mu}_i(0)$ are defined so that they satisfy the conditions to be stated soon.] We assume the following conditions:

(E3) $\hat{\lambda}_i(x)$, $\hat{\mu}_i(x)$, $1 \le i \le K$, $x \in R^1$, are differentiable at zero and satisfy

$$|\hat{\lambda}_i(x) - \hat{\lambda}_i(0)| + |\hat{\mu}_i(x) - \hat{\mu}_i(0)| \le Cx,$$

for any $x \ge 0$.

(E4) (Heavy traffic condition)

$$\hat{\lambda}_i(0) + \sum_{1\le j\le K} p_{ji}\,\hat{\mu}_j(0) - \hat{\mu}_i(0) = 0, \qquad 1 \le i \le K.$$

(E5) $\hat{\mu}_i(0) > 0$, $1 \le i \le K$.

In view of (E3) and (E4), $\alpha_n^i(x_n) \to a_i(x)$, if $x_n \to x \neq 0$, $x_n, x \in R^K$, where

$$\alpha_n^i(x) = \sqrt{n}\left(\hat{\lambda}_i\left(x_i/\sqrt{n}\right) + \sum_{1\le j\le K} p_{ji}\,\hat{\mu}_j\left(x_j/\sqrt{n}\right) - \hat{\mu}_i\left(x_i/\sqrt{n}\right)\right),$$

$$a_i(x) = \hat{\lambda}_i'(0)\,x_i + \sum_{1\le j\le K} p_{ji}\,\hat{\mu}_j'(0)\,x_j - \hat{\mu}_i'(0)\,x_i, \qquad 1 \le i \le K$$

[$\hat{\lambda}_i'(0)$, $\hat{\mu}_i'(0)$, $1 \le i \le K$, denote the derivatives at 0 of $\hat{\lambda}_i(x)$ and $\hat{\mu}_i(x)$]. Thus assumption (A2) is satisfied. Clearly, assumption (A4) is also satisfied if we let $\lambda_i(x) = \hat{\lambda}_i(0)$ and $\mu_i(x) = \hat{\mu}_i(0)$, for $1 \le i \le K$. Thus if assumption (A5)(ii) is satisfied [i.e., if $\inf_{x\ge 0} \hat{\lambda}_i(x) > 0$, for $1 \le i \le K$], all the assumptions (A1)–(A5) are satisfied. Equation (4.3) becomes

$$X_i(t) = X_i(0) + \int_0^t a_i(X(s))\,ds + M_i(t) + \xi_i(t) - \sum_{1\le j\le K} p_{ji}\,\xi_j(t),$$

$$M_i(t) = \sqrt{\lambda_i(0)}\,\tilde{W}_i(t) + \sum_{1\le j\le K} \sqrt{p_{ji}\,\mu_j(0)}\,\tilde{B}_{ji}(t)$$

$$- \sum_{0\le j\le K} \sqrt{p_{ij}\,\mu_i(0)}\,\tilde{B}_{ij}(t), \qquad 1\le i\le K.$$

Since $a(x)$ is Lipschitz continuous, by Proposition 2, (4.3) (i.e., the equation just above) has a unique solution. [$X(t)$ may be called a reflecting Ornstein–Uhlenbeck process.] Thus, taking $\varphi_n = \sqrt{n}$, we have $X_n(t) \to_{\mathscr{L}} X(t)$ by Theorem 1.

REMARK 8. The problem of Example 2 was considered for a single station model without feedback by Liptser and Shiryaev ([11], Chapter 10, Section 4), where $\hat{\lambda}_1(x)$ and $\hat{\mu}_1(x)$ are assumed to be differentiable with bounded and continuous derivatives. Pats [12] considered the same problem more extensively in the framework of M_1 convergence. (See [15] for M_1 convergence.)

EXAMPLE 3. This example is given to show that there exists a case which is the outside of the application of our result. We consider a fixed state-dependent single station model $Q(t)$ with arrival rate $\lambda(x)$ and service rate $\mu(x)$. We let $Q_n(t) = Q(nt)$, $n \ge 1$, and $X_n(t) = (1/\sqrt{n})Q_n(t)$. Then, for a sequence of queueing processes $\{Q_n(t)\}_{n\ge 1}$, we have $\lambda_n^1(x) = n\lambda(x)$ and $\mu_n^1(x) = n\mu(x)$. Hence,

$$\alpha_n(x) = (1/\sqrt{n})\left(\lambda_n(\sqrt{n}\,x) - \mu_n(\sqrt{n}\,x)\right)$$
$$= \sqrt{n}\left(\lambda(\sqrt{n}\,x) - \mu(\sqrt{n}\,x)\right).$$

Hence, $\alpha_n(x) \to_{\text{c.c.}} \alpha(x) = c/x$ a.e. on $D = \{x;\ x > 0\}$ under the following heavy traffic condition (E6):

$$\text{(E6)} \qquad \lim_{x\to\infty} x(\lambda(x) - \mu(x)) = c.$$

Thus, the drift coefficient $\alpha(x)$ is singular and we cannot have (A1) since $\alpha(x) = c/x \to +\infty$ or $-\infty$ as $x \to 0+$ except $c = 0$. [For this example, we can show that the limit process of $\{X_n(t)\}$ is a Bessel process (see [16]).] As this example shows, reflecting diffusion processes with singular drift and/or diffusion coefficients cannot be obtained from queueing processes considered in this paper. More comments on the comparison of models considered in Examples 2 and 3 can be found in [12].

7. A non-Markovian case. In this section, we consider a non-Markovian case which can be treated by our approach. The model is a mixture of our model and that considered by Reiman [13].

For each i, $1 \le i \le K$, there is a sequence of positive iid random variables $\{v_i(l),\ l \ge 1\}$ and a sequence of iid K-dimensional random vectors $\{\eta_i(l),\ l \ge 1\}$, which take values $e_0, e_1, \dots, e_K$, where e_j, $j \ge 1$, is the K-vector whose jth component is 1 and other components are 0, and e_0 is the K-vector all of whose components are 0. We let $p_{ij} = P(\eta_i(l) = e_j)$. We also assume that all these sequences are mutually independent. Let

$$S_i(t) = \max\left\{k;\ \sum_{1\le l\le K} v_i(l) \le t\right\}.$$

$v_i(l)$ represents the service time for lth customer at station i and $\eta_i(l)$ is a routine vector; $\eta_i(l) = e_j$ means that the lth customer at station i goes to station j after the completion of service. Following Reiman [13], $Q_i(t)$ is assumed to satisfy

$$Q_i(t) = Q_i(0) + A_i(t) + \sum_{j=1}^{K} \sum_{l=1}^{S_j(B_j(t))} \eta_j^i(l) - S_i(B_i(t)),$$

$$B_i(t) = \int_0^t 1(Q_i(s) > 0)\,ds, \qquad 1 \le i \le K$$

[$\eta_j^i(l)$ is the ith component of $\eta_j(l)$]. $N_i(t)$ in (2.1) is obviously $N_i(t) = S_i(B_i(t))$ and $N_{ji}(t) = \sum_{1\le l\le L}\eta_j^i(l)$ with $L = S_j(B_j(t))$. We assume that all the processes above are defined on a stochastic basis and $A_i(t)$ has an intensity as in Section 2. Moreover, we assume that $\{A_i(t), N_i(t),\ 1 \le i \le K\}$ have no common jumps.

Hereafter, we consider a sequence of the queueing networks of the above type, and let $X_n^i(t) = Q_n^i(t)/\varphi_n$, $1 \le i \le K$. The process $X_n(t) =$

$(X_n^1(t), \dots, X_n(t))$ can be written as in (4.1) or (4.2), where

$$\tilde{\alpha}_n^i(x) = (1/\varphi_n)\left(\lambda_n^i(\varphi_n x_i) + \sum_{1 \le j \le K} p_{ji}\,\mu_n^j - \mu_n^i\right),$$

$$\mu_n^i = 1/Ev_n^i(l),$$

$$M_n^i(t) = (1/\varphi_n)\left[\tilde{A}_n^i(t) + \sum_{j=1}^{K} \sum_{l=1}^{S_n^j(B_n^j(t))} \left(\eta_{n,j}^i(l) - p_{ji}\right) + \sum_{j=1}^{K} p_{ji}\hat{S}_n^j\left(B_n^j(t)\right) - \hat{S}_n^i\left(B_n^i(t)\right)\right],$$

$$\hat{S}_n^i(t) := S_n^i(t) - \mu_n^i t,$$

$$\xi_n^i(t) = \left(\mu_n^i/\varphi_n\right)\int_0^t 1\left(X_n^i(s) = 0\right) ds, \qquad 1 \le i \le K.$$

For each $1 \le i \le K$, we define

$$\delta_n^i(t) := (1/\varphi_n) \sum_{1 \le l \le [\varphi_n^2 t]} (\eta_i(l) - P_i),$$

where P_i is the ith row of the routing matrix P,

$$\tilde{S}_n^i(t) := (1/\varphi_n)\left(S_n^i(t) - \mu_n^i t\right),$$

$$\theta_n^i(t) := \left(1/\varphi_n^2\right) S_n^i(t).$$

We impose the following assumptions, most of which are the same as those in Section 3:

(B1) For any $R > 0$, there exists a constant C such that

$$\sup_{\substack{|x| \le R \\ x \in R^K}} |\tilde{\alpha}_n(x)| \le C \quad \text{and} \quad \sup_{0 \le x \le R} (1/\varphi_n)\lambda_n^i(\varphi_n x) \le C, \qquad 1 \le i \le K.$$

(B2) There exists a measurable function $\alpha(x)$, $x \in R^K$, such that $\tilde{\alpha}_n(x) \to_{\text{c.c.}} \alpha(x)$ a.e. on R^K. We also have, for $1 \le i \le K$, $(1/\varphi_n^2)\lambda_n^i(\varphi_n \,\cdot\,) \to_{\text{c.c.}} \lambda_i(\cdot)$ a.e. on R_+ for a measurable function $\lambda_i(\cdot)$ on R_+.

(B3) The sequence $(\delta_n^1(t), \dots, \delta_n^K(t), \tilde{S}_n(t), \theta_n(t)) \to_{\mathscr{L}} (\delta_1(t), \dots, \delta_K(t), \tilde{S}(t), \theta(t))$ in D, where $\delta_1(t), \dots, \delta_K(t), \tilde{S}(t)$ are independent K-dimensional Brownian motions. Moreover, $\tilde{S}_1, \dots, \tilde{S}_K$ are independent one-dimensional Brownian motions and $\theta_i(t) = \theta_i t$ with $\theta_i > 0$, for $1 \le i \le K$.

(B4) For $1 \le i \le K$, $\sup_{l \le \varphi_n^2}(1/\varphi_n)\mu_n^i v_n^i(l) \to_P 0$.

(B5) The routing matrix P has spectral radius strictly less than unity.

(B6) For each $t \geq 0$, $\sup_n E(\tilde{S}_n^i(t))^2 < \infty$, $1 \leq i \leq K$.

We have the following result similar to Theorem 1:

THEOREM 2. *Assume* (B1)–(B6) *and* $X_n(0) \to_{\mathscr{L}} X(0)$ *in* R^K. *Then* $X_n(t) \to_{\mathscr{L}} X(t)$ *in* $D([0,\infty), R^K)$. *Here* $X(t)$ *is assumed to be the unique (in law) solution of the Skorohod equation given in Theorem* 1 *with* $M(t)$ *being replaced by*

$$M_i(t) = \int_0^t \sqrt{\lambda_i(X_i(s))}\, d\tilde{W}_i + \sum_{j=1}^K \sqrt{\theta_j}\, \delta_j^i(t) + \sum_{j=1}^K \hat{p}_{ji} \tilde{S}_j(t),$$

$$\hat{p}_{ij} = \begin{cases} p_{ij}, & \text{if } j \neq i, \\ 1 - p_{ii}, & \text{if } j = i, \end{cases}$$

where $\delta_j(t)$, $1 \leq j \leq K$, *and* $\tilde{S}(t)$ *are K-dimensional processes given in* (B3), *and* $\tilde{W}_1(t), \dots, \tilde{W}_K(t)$ *are independent standard Brownian motions which are also independent of* $\delta_j(t)$, $1 \leq j \leq K$, *and* $\tilde{S}(t)$.

This theorem can be proved in almost the same way as Theorem 1. We will only roughly indicate the points where the proof differs. The tightness of $\{A_n(t), M_n(t)\}$ follows easily from the discussion in Lemma 2 and (B3). Then by Lemma 1, $\{X_n(t), A_n(t), M_n(t), \xi_n(t)\}$ is tight. We note that this implies

$$\begin{aligned} &\left\{\delta_n^1(t), \dots, \delta_n^K(t), \tilde{S}_n(t), \theta_n(t), B_n(t)\right\} \\ &\qquad \to_{\mathscr{L}} \left\{\delta_1(t), \dots, \delta_K(t), \tilde{S}(t), \theta(t), t\right\} \quad \text{in } D. \end{aligned} \tag{7.1}$$

Indeed, by (B3) we have $\mu_n^i/\varphi_n \to \infty$. On the other hand, $\xi_n^i(t) = (\mu_n^i/\varphi_n)(t - B_n^i(t))$, $1 \leq i \leq K$. Hence, the tightness of $\{\xi_n(t)\}$ implies the above fact. Now let, for each $1 \leq i \leq K$,

$$\begin{aligned} a_n^i(t) &:= (1/\varphi_n) \tilde{A}_n^i(t), \\ b_n^i(t) &:= (1/\varphi_n) \sum_{j=1}^K \sum_{1 \leq l \leq S_n^j(B_n^j(t))} \left(\eta_j^i(l) - P_i\right), \\ c_n^i(t) &:= (1/\varphi_n) \sum_{j=1}^K \hat{p}_{ji} \hat{S}_n^j\left(B_n^j(t)\right). \end{aligned}$$

Let $\{a(t), b(t), c(t), M(t)\}$ be any weak limit of $\{a_n(t), b_n(t), c_n(t), M_n(t)\}$. Then we have

$$M(t) = a(t) + b(t) + c(t).$$

Using (7.1), we can easily show that

$$\begin{aligned} b(t) &= \left(\delta_1(\theta_1 t), \dots, \delta_K(\theta_K t)\right), \\ c(t) &= \left(\sum_{j=1}^K \hat{p}_{j1} \tilde{S}_j(t), \dots, \sum_{j=1}^K \hat{p}_{jK} \tilde{S}_j(t)\right). \end{aligned}$$

We must show that $a_i(t)$ is orthogonal to $b_j(t)$ and $c_j(t)$; that is, $\langle a_i, b_j\rangle(t) = 0$ and $\langle a_i, c_j\rangle(t) = 0$. Let us, for example, show that $\langle a_i, c_j\rangle(t) = 0$. To see this, it suffices to show that $\langle a_i, \tilde{S}_j\rangle(t) = 0$. We note that

$$\tilde{S}_n^j(t) = (1/\varphi_n) \sum_{1\le l\le S_n^j(t)} \left(1 - \mu_n^j v_n^j(l)\right) - \left(\mu_n^j/\varphi_n\right)\left(t - \sum_{1\le l\le S_n^j(t)} v_n^j(l)\right)$$
$$:= d_n^j(t) - e_n^j(t).$$

By (B4), $e_n^j(t) \to_{\mathscr{L}} 0$. Thus, if $(\tilde{S}_n^j(t), d_n^j(t)) \to_{\mathscr{L}} (\tilde{S}_j(t), d_j(t))$, then $\tilde{S}_j(t) = d_j(t)$. Hence it suffices to show that $\langle a_i, d_j\rangle(t) = 0$ since $d_j(t) = \tilde{S}_j(t)$. However, $d_n^j(t)$ is a locally square integrable martingale [note that $\tilde{S}_n^j(t)$ is not a martingale] and has no common jumps with $a_n^i(t)$ since $A_n^i(t)$ and $S_n^j(t)$ have no common jumps by our assumption. Hence, we have $\langle a_n^i, d_n^j\rangle(t) = 0$. By letting n go to infinity and taking into consideration assumption (B6), we reach the conclusion $\langle a_i, d_j\rangle(t) = 0$. Thus, we have shown the orthogonality of $a_i(t)$ and $c_j(t)$. That $a_i(t)$ is also orthogonal to $b_j(t)$ can be shown similarly. We can then show the result of Lemma 3 by the same (but easier) discussion as in the proof of Lemma 3. Then the identification of the process $a(t)$ can be carried out exactly in the same way as the proof of Theorem 1. Now, we obtain the conclusion of Theorem 2.

As a final remark, we point out that it is also possible to obtain a result similar to that above for the model where the arrival streams are renewal processes not dependent on the state of the network, but the service streams are the same as in the model in Theorem 1.

8. Concluding remarks. This section lists some problems related directly or indirectly to our result.

1. We have treated queueing networks for which the heavy traffic condition (A2) holds for each station. That is, if we use the terminology of Chen and Mandelbaum [3], balanced networks were considered, and hence, it remains to investigate unbalanced networks.
2. Although some non-Markovian models were treated in Section 7, more general models of this type should be considered. It would be interesting to see whether our stochastic calculus approach is applicable to such models.
3. This problem is not directly related to our result, but is interesting from the viewpoint of application. Given queueing network processes $\{Q_n(t)\}_{n\ge 1}$, we often need to investigate the asymptotic behavior of functionals of jumps of $\{Q_n(t)\}_{n\ge 1}$ of the form $\sum_{Q_n(s)\ne Q_n(s-)} f(Q_n(s), Q_n(s-))$. Such a problem was considered in [17] for single station queues under a heavy traffic situation. The approach there depends on the basic fact that the limit process for such queues is reflecting Brownian motion. In the problem for our network queues, it is also expected that our result will play a basic role. However, the direct application of the method in [17] seems difficult, and it is important to find the methodology to cope with our problem.

4. As is mentioned in the Introduction, we have introduced the concept "c.c. convergence" for functions, which is weaker than uniform convergence, to take care of some discontinuous arrival and service rates, that is, to deal with the convergence of stochastic integrals

$$\int_0^t f_n(X_n(s-))\, dM_n(s) \to \int_0^t f(X(s))\, dM(s),$$

where $(X_n, M_n) \to (X, M)$ in the J_1-topology and $f_n \to f$ in an appropriate sense. Note that in our case X is a diffusion and M is a Brownian motion or a deterministic process t. (A similar problem is treated by Kurtz [9], but it seems his result is not applicable to our problem.) With respect to the above convergence problem, a referee suggested the use of M_1-convergence [15]. Although this seems to be an interesting approach, at present we are not able to clarify this point.

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REFERENCES

[1] BARLOW, M. T. and PERKINS, E. (1984). One-dimensional stochastic differential equations involving a singular increasing process. *Stochastics* **12** 229–249.

[2] BREMAUD, P. (1981). *Point Processes and Queues: Martingale Dynamics*. Springer, Berlin.

[3] CHEN, H. and MANDELBAUM, A. (1991). Discrete flow networks: Diffusion approximations and bottlenecks. *Ann. Probab.* **19** 1463–1519.

[4] DUPUIS, P. and ISHII, H. (1993). SDEs with oblique reflection on nonsmooth domains. *Ann. Probab.* **21** 554–580.

[5] HARRISON, M. and REIMAN, M. I. (1981). Reflected Brownian motion on an orthant. *Ann. Probab.* **9** 302–308.

[6] IKEDA, N. and WATANABE, S. (1981). *Stochastic Differential Equations and Diffusions*, 2nd ed. North-Holland, Amsterdam.

[7] JACOD, J. (1979). *Calcule Stochastique et Problèmes de Martingales. Lecture Notes in Math.* **714**. Springer, Berlin.

[8] JACOD, J. and SHIRYAEV, A. N. (1987). *Limit Theorems for Stochastic Processes*. Springer, Berlin.

[9] KURTZ, T. G. (1991). Weak limit theorems for stochastic integrals and stochastic differential equations. *Ann. Probab.* **19** 1035–1070.

[10] LE GALL, J.-F. (1984). One-dimensional stochastic differential equations involving the local times of the unknown process. *Stochastic Analysis and Applications. Lecture Notes in Math.* **1095** 51–82. Springer, Berlin.

[11] LIPTSER, R. SH. and SHIRYAYEV, A. N. (1989). *Theory of Martingales*. Kluwer, Dordrecht.

[12] PATS, G. (1993). State-dependent queueing networks: approximations and applications. Ph.D. dissertation.

[13] REIMAN, M. I. (1981). Open queueing networks in heavy traffic. *Math. Oper. Res.* **9** 441–458.

[14] STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multidimensional Diffusion Processes*. Springer, Berlin.

[15] WHITT, W. (1980). Some useful functions for functional limit theorems. *Math. Oper. Res.* **5** 67–85.

[16] YAMADA, K. (1986). Multi-dimensional Bessel processes as heavy traffic limits of certain tandem queues. *Stochastic Process. Appl.* **23** 35–56.

[17] YAMADA, K. (1993). A limit theorem for functionals of jumps of birth and death processes under heavy traffic condition. *Ann. Appl. Probab.* **3** 840–862.

DEPARTMENT OF MATHEMATICS
KANAGAWA UNIVERSITY
ROKKAKUBASHI, KANAGAWA-KU
YOKOHAMA 221
JAPAN