The Annals of Applied Probability
1995, Vol. 5, No. 4, 897–905

EFFICIENT MONTE CARLO SIMULATION OF SECURITY PRICES

BY DARRELL DUFFIE AND PETER GLYNN

Stanford University

This paper provides an asymptotically efficient algorithm for the allocation of computing resources to the problem of Monte Carlo integration of continuous-time security prices. The tradeoff between increasing the number of time intervals per unit of time and increasing the number of simulations, given a limited budget of computer time, is resolved for first-order discretization schemes (such as Euler) as well as second- and higher-order schemes (such as those of Milshtein or Talay).

1. Introduction. It sometimes occurs that one must numerically compute $E[f(X_T)]$, where $f\colon \mathbb{R}^N \to \mathbb{R}$ and X is the solution of a stochastic differential equation of the form

$$dX_t = a(X_t, t)\,dt + b(X_t, t)\,dB_t, \qquad X_0 = x \in \mathbb{R}^N, \tag{1}$$

where B is a standard Brownian motion in $\mathbb{R}^d$ on some given probability space and where $a\colon \mathbb{R}^N \times [0,\infty) \to \mathbb{R}^N$ and $b\colon \mathbb{R}^N \times [0,\infty) \to \mathbb{R}^{N\times d}$ are assumed to have sufficient regularity to ensure the existence of a unique solution to (1). A common application is the determination of the price of a financial security. (An example of this appears in subsequent text.) Depending on the application and technical conditions, the usual Kolmogorov backward equation gives, via one of many possible finite-difference algorithms, a good numerical approximation for $E[f(X_T)]$.

In some cases, however, it is convenient and simple to obtain a Monte Carlo approximation of $E[f(X_T)]$ by simulating a discrete-time approximation of (1). For example, the Euler approximation, with periods of length $h > 0$, takes the approximation $\hat{X}^h_{T/h}$ for X_T, where $\hat{X}^h$ is the discrete-time $\mathbb{R}^N$-valued process defined on some (possibly different) probability space $(\Omega, \mathcal{F}, P)$ by

$$\hat{X}^h_{k+1} - \hat{X}^h_k = a\big(\hat{X}^h_k, kh\big)h + b\big(\hat{X}^h_k, kh\big)\sqrt{h}\,\varepsilon_{k+1}, \qquad \hat{X}^h_0 = x, \tag{2}$$

where $\varepsilon_1, \varepsilon_2, \ldots$ is an i.i.d. sequence of standard normal vectors valued in $\mathbb{R}^d$.

This paper shows how to make a computationally efficient tradeoff between reducing the length h of a time period and increasing the number of Monte Carlo simulations of the sample path of $\hat{X}^h$. Extensions to path-dependent and other cases are discussed in the concluding remarks.

Received April 1993; revised May 1995.
AMS 1991 *subject classifications.* 65C05, 90A09
Key words and phrases. Monte Carlo simulation, stochastic differential equations, option pricing, finance.

As a first step, we can ask: How good is $f_h \equiv E[f(\hat{X}^h_{T/h})]$ as an approximation for $E[f(X_T)]$? Let $e_h = E[f(X_T)] - f_h$ denote the approximation error. It can be shown, under mild technical conditions, that $\lim_{h \downarrow 0} e_h = 0$. Even better, we can given an order of convergence. A sequence $\{y_h\}$ has order-k convergence if $y_h h^{-k}$ is bounded in h. The Euler approximation (2) is said to be a first-order discretization scheme in that e_h has order-1 convergence. More precisely, under additional regularity conditions, there is a constant C such that $e_h + Ch$ has order-2 convergence. The error coefficient C may be positive or negative, and gives a notion of bias in the approximation. Although C is usually unknown, it turns out that C can itself be approximated to first order by $C_h = 2(f_{2h} - f_h)/h$. As shown by Talay and Tubaro (1990), these properties are satisfied under the following regularity condition:

CONDITION A. The function f is C^∞ and satisfies a polynomial growth condition: for some positive integer s,

$$|f(x)| \leq C(+1\|x\|^s), \qquad x \in \mathbb{R}^N.$$

Both a and b are C^∞ and have bounded derivatives of any order.

The assumption that f is C^∞ is not essential, and even continuity of f can be sacrificed, as shown by Bally and Talay (1995).

Talay and Tubaro (1990) also provide references to more complicated discretization schemes with order-2 error. For instance, with $N = 1$ and under the same technical conditions, an order-2 scheme is given by the Milshtein (1978) approximation. Given μ and σ in $C^2(\mathbb{R})$ such that, for all t, we have $a(x,t) = \mu(x)$ and $b(x,t) = \sigma(x)$, the Milshtein approximation is given by

$$\begin{aligned}\hat{X}^h_{k+1} - \hat{X}^h_k = {} & h\left[\mu\left(\hat{X}^h_k\right) - \frac{1}{2}\sigma\left(\hat{X}^h_k\right)\sigma'\left(\hat{X}^h_k\right)\right] + \sqrt{h}\,\sigma\left(\hat{X}^h_k\right)\varepsilon_{k+1} \\ & + \frac{h}{2}\sigma\left(\hat{X}^h_k\right)\sigma'\left(\hat{X}^h_k\right)\varepsilon^2_{k+1} + h^{3/2}\nu\left(\hat{X}^h_k\right)\varepsilon_{k+1} + h^2\eta\left(\hat{X}^h_k\right),\end{aligned} \tag{3}$$

where

$$\nu(x) = \tfrac{1}{2}\mu(x)\sigma'(x) + \tfrac{1}{2}\mu'(x)\sigma(x) + \tfrac{1}{4}\sigma(x)^2\sigma''(x)$$

and

$$\eta(x) = \tfrac{1}{2}\mu(x)\mu'(x) + \tfrac{1}{4}\mu''(x)\sigma(x)^2.$$

For $N > 1$, Talay (1984, 1986) provides second-order discretization schemes. Kloeden and Platen (1992) review other approximation schemes, some of even higher order, under conditions.

Of course, we cannot generally calculate even the approximation $E[f(\hat{X}^h_{T/h})]$, but we can in turn estimate this quantity by Monte Carlo simulation. Let $Y_1 = f(\hat{X}^h_{T/h})$ be defined as before. This is the first simulation of the random variable whose mean is to be computed. Let Y_2 be defined in the same manner, with the exception that, for each j, we substitute $\varepsilon_{j+T/h}$

for ε_j. Let Y_3 be defined likewise, substituting $\varepsilon_{j+2T/h}$ everywhere for ε_j and so on. Since $\varepsilon_1, \varepsilon_2, \ldots$ is i.i.d., the sequence $Y_1, Y_2, \ldots$ qualifies for an application of the law of large numbers, leaving

$$f(h,k) \equiv \frac{Y_1 + \cdots + Y_k}{k} \to E\left[f\left(\hat{X}^h_{T/h}\right)\right] \quad \text{a.s.} \tag{4}$$

In practice, one often substitutes pseudorandom numbers for $\{\varepsilon_j\}$, using some deterministic scheme. Variance reduction techniques can improve the convergence properties of Monte Carlo simulation. For example, our previous first approach was to simulate Y, with the disturbance sequence

$$D_i \equiv (\varepsilon_{1+T(i-1)/h}, \varepsilon_{2+T(i-1)/h}, \varepsilon_{3+T(i-1)/h}, \ldots, \varepsilon_{Ti/h}).$$

Instead, one can typically improve the convergence properties of the simulation by substituting, for even-numbered i, the disturbance sequence $-D_{i-1}$. Barraquand (1993) reviews other variance reduction techniques in this setting.

At this point, (4) gives us an approximation $f(h,k)$ of $E[f(X_T)]$ based on a discrete-time approximation $\hat{X}^h$ of the Itô process X with periods of length h, and with k simulations of the process $\hat{X}^h$. The number of additions required to compute $f(h,k)$ is roughly proportional to N^2Tk/h. Since N and T are presumably fixed for a given problem, we are concerned about the size of k/h, given limited computation time. We are also concerned with the error size $e(h,k) = |f(h,k) - E[f(X_T)]|$.

2. Main result. This section presents the optimal tradeoff between h and k in a general setting. We fix a probability space and a random variable Z. Suppose that we want to compute $\alpha \equiv E(Z)$ by Monte Carlo simulation, but we cannot generate the random variable Z directly. However, for every $h > 0$, we can generate a corresponding random variable $Z(h)$. In our principal application, $Z = f(X_T)$ and $Z(h) = f(\hat{X}^h_{T/h})$, where $\hat{X}^h$ is a discrete approximation of the solution X of a given stochastic differential equation, using periods of length h. As with this example, we assume that the family $\{Z(h): h > 0\}$ can be chosen so that $\alpha(h) \triangleq E[Z(h)] \to \alpha$ as $h \downarrow 0$.

ASSUMPTIONS.

(i) $Z(h) \Rightarrow Z$ as $h \downarrow 0$.

(ii) $E[Z^2(h)] \to E(Z^2) < \infty$ as $h \downarrow 0$ (i.e., $\{Z^2(h): h > 0\}$ is uniformly integrable).

(iii) $\alpha(h) = \alpha + \beta h^p + o(h^p)$ as $h \downarrow 0$, where $\beta \neq 0$ and $p > 0$.

(iv) The (computer) time required to generate $Z(h)$ is given by $\tau(h)$, where $\tau(h)$ is deterministic and satisfies $\tau(h) = \gamma h^{-q} + o(h^{-q})$ as $h \downarrow 0$, where $\gamma > 0$ and $q > 0$.

Given t units of computer time, consider the estimator

$$\alpha(t,h) = \frac{1}{n(t,h)} \sum_{i=1}^{n(t,h)} Z_i(h), \tag{5}$$

where $Z_1(h), Z_2(h), \ldots$ is a sequence of i.i.d. copies of $Z(h)$ and $n(t,h) = \lfloor t/\tau(h) \rfloor$. For our main example of simulation of $f(X_T)$, these four assumptions are satisfied for $q = 1$ and $p = 1$ (for first-order schemes such as Euler) or $p = 2$ [for second-order schemes, such as those of Milshtein (1978) or Talay (1984, 1986, 1990)] under Condition A, or under other sets of technical conditions given, for example, in Talay and Tubaro (1990) or Kloeden and Platen (1992). The following theorem describes the convergence characteristics of $\alpha(t, h_t)$.

THEOREM 1. *Suppose Assumptions* (i)–(iv) *hold.*

(a) *If* $h_t\, t^{1/(q+2p)} \to +\infty$ *or if* $h_t\, t^{1/(q+2p)} \to 0$ *as* $t \to \infty$, *then, as* $t \to \infty$,

$$t^{p/(q+2p)} |\alpha(t, h_t) - \alpha| \Rightarrow +\infty. \tag{6}$$

(b) *If* $h_t\, t^{1/(q+2p)} \to c$, *where* $0 < c < \infty$, *then, as* $t \to \infty$,

$$t^{p/(q+2p)} \big(\alpha(t, h_t) - \alpha\big) \Rightarrow \sigma(\gamma/c^q)^{1/2} W + \beta c^p, \tag{7}$$

where W *is standard normal and* $\sigma^2 = \operatorname{var}(Z)$.

PROOF. Note that

$$\alpha(t, h_t) - \alpha = \frac{1}{n(t, h_t)} \sum_{i=1}^{n(t, h_t)} \hat{Z}_i(h_t) + \alpha(h_t) - \alpha,$$

where $\hat{Z}_i(h) = Z_i(h) - \alpha(h)$. Then,

$$t^{p/(q+2p)} \big(\alpha(t, h_t) - \alpha\big) = t^{p/(q+2p)} \big(n(t, h_t)\big)^{-1/2} \left(\sum_{i=1}^{n(t, h_t)} \frac{\hat{Z}_i(h_t)}{\sqrt{n(t, h_t)}} \right) + t^{p/(q+2p)} \big(\alpha(h_t) - \alpha\big).$$

The uniform integrability conditions (i) and (ii) guarantee that the Lindberg–Feller theorem applies here. That is, as $t \to \infty$,

$$\sum_{i=1}^{n(t, h_t)} \frac{\hat{Z}_i(h_t)}{\sqrt{n(t, h_t)}} \Rightarrow \sigma W,$$

where W is standard normal. We use the facts that, as $t \to \infty$,

$$t^{p/(q+2p)} \big(\alpha(h_\tau) - \alpha\big) = \beta \big(h_t\, t^{1/(q+2p)}\big)^p + o\big(h_t\, t^{1/(1+2p)}\big),$$

$$t^{p/(q+2p)} n(t, h_t)^{-1/2} = \gamma^{1/2} \big(h_t\, t^{1/(1+2p)}\big)^{-q/2} + o\big(h_t^{-q/2} t^{-q/(2q+4p)}\big).$$

Combining these results completes the proof. □

We can interpret Theorem 1 as follows. As the total budget t of computer time gets large, the length h_t of a time interval should converge to zero with order $1/(q + 2p)$. If this rule is followed, then (7) implies that the estimation error has convergence with t of order $p/(q + 2p)$ (in distribution) to zero. If

this rule is not followed, then (6) implies that the estimation error does not converge in distribution to zero "as quickly" (i.e., the error does not converge at this order). It follows that this rule is "optimal," in the sense of asymptotic distribution. We therefore call a scheme $\{h_t\}$ for reducing h (the length of a time interval, in our main application) *asymptotically optimal* if $h_t\, t^{1/(q+2p)} \to c$, with $0 < c < \infty$.

Just to formalize the application to SDE's, we can record the following proposition, which follows from Theorem 1 of Talay and Tubaro (1990).

PROPOSITION 1. *Under Condition* A, *let* $Z(h) = f(\hat{X}^h_{T/h})$ *and* $Z = f(X_T)$, *where* X *is the solution of* (1). *For* $p = 1$ *or* $p = 2$, *let* X^h *be the solution of the first-order (Euler) scheme* (2) *or of one of the second-order schemes of Talay* (1984, 1986), *respectively. Then the conclusions of Theorem* 1 *apply for* $p = 1$ *or* $p = 2$, *respectively.*

We can also view the result in terms of the asymptotic relation between the number k of Monte Carlo simulations and the size h of a time interval. The theorem implies that, asymptotically, it is optimal to have k increasing at the order of h^{-2p}. For instance, with the Euler scheme ($p = 1$), the number of simulations should quadruple with each doubling of the number of time intervals. With a second-order scheme, the number of simulations should be on the order of the number of time intervals to the fourth power and so on. Similarly, with an optimal Euler scheme, asymptotically speaking, for each doubling of the number of time intervals the root-mean-squared estimation error is halved. For an asymptotically optimal second-order scheme such as the Milshtein (1978) scheme (for $N = 1$) or Talay (1984, 1986) scheme (for $N \geq 1$), for each doubling of the number of time intervals, the root-mean-squared estimation error is reduced by a factor of 4.

The option pricing example shown in the next section has estimation errors for finite samples that are consistent with this asymptotic error behavior. Numerical examples given by Kloeden and Platen (1992) are also consistent with this predicted behavior.

3. An option pricing example. We illustrate the results with a simple option pricing example. We assume a constant continuously compounding interest rate r. Under a given probability measure Q, an asset price process X is assumed to satisfy a stochastic differential equation of the form

$$dX_t = -rX_t\, dt + \sigma X_t^{\gamma}\, dB_t, \tag{8}$$

where $\sigma \neq 0$ and $\gamma \in [0.5, 1]$ are constant parameters. [In order to formally map into the setting of the previous sections, we actually take the diffusion function b to be given by $b(x) = \sigma x^{\gamma}$ for $x \geq 0$ and $b(x) = 0$ otherwise.] This is the so-called constant-elasticity-of-variance model of Cox (1975) and Cox and Ross (1976), which specializes to the Black and Scholes (1973) model for $\gamma = 1$. As explained by Harrison and Kreps (1979) [following on Black and Scholes (1973) and Cox and Ross (1976)], the absence of arbitrage implies

that a financial security, whose payoff at time T is a finite-variance random variable Y measurable with respect to the sample path $\{X_t\colon t \in [0,T]\}$, has the initial price $E(e^{-rT}Y)$, where E denotes expectation under Q. We could take, for example, the case examined by Black and Scholes of a European call option on the asset with strike price K and expiration date T, in which case $Y = (X_T - K)^+$. For simplicity, however, we take the rescaling $Y = e^{rT}(X_T - K)^+$, so that in our main application we can take $f(x) = (x - K)^+$.

For the Black–Scholes special case ($\gamma = 1$), all of the regularity required by Condition A is satisfied, except for the single point K at which f is not differentiable. In this particular case, however, f can be uniformly well approximated from above by a C^∞ family of functions $\{f_\delta\colon \delta \in (0,1)\}$, where

$$f_\delta(x) = \frac{x - K + \sqrt{(x-K)^2 + \delta}}{2}$$

Of course, f_δ satisfies the polynomial growth condition of Condition A. If we take δ to be of order h^{-2p}, we therefore preserve the quality of the approximation given in the previous proposition. Even this smoothing of f is unnecessary, as shown in an as yet unpublished work by Bally and Talay (1995).

For $\gamma \in [0.5, 1)$, however, little can be done to get into a situation covered formally by Condition A. We nevertheless have convergence in distribution of $Z(h) = (\hat{X}^h_{T/h} - K)^+$ to $(X_T - K)^+$ by the results of Yamada (1976) (who in fact shows convergence in mean of $X^h_{T/h}$ to X_T) and by the dominated convergence theorem.

Tables 1 and 2 show the results of an experiment for the particular case $\sigma = 0.2$, $r = 0.1$, $K = 40$, $X_0 = 42$ and $T = 0.5$. Various cases are considered:

1. Euler and Milshtein discretization schemes;
2. $\gamma = 0.5$ and $\gamma = 1$;
3. ε_t normally distributed and uniformly distributed.

TABLE 1
Option price estimation: Euler scheme, sample root mean squared error (10 *trials*)

		$\gamma = 1.0$		$\gamma = 0.5$	
n	***k***	**Gaussian**	**Uniform**	**Gaussian**	**Uniform**
8	64	0.33386	0.59804	0.13023	0.11984
16	256	0.31877	0.36335	0.04152	0.06275
32	1,024	0.14807	0.11988	0.01742	0.03525
64	4,096	0.10105	0.09688	0.01356	0.01568
128	16,384	0.02606	0.04554	0.00854	0.00428
256	65,536	0.01732	0.01246	0.00402	0.00341
512	262,144	0.00636	0.01040	0.00160	0.00171
1024	1,048,576	0.00616	0.00468	0.00064	0.00065
Avg. reduction		1.97	2.16	2.22	2.23

TABLE 2
Option price estimation: Milshtein scheme, sample root mean squared error (10 trials)

		$\gamma = 1.0$		$\gamma = 0.5$	
n	k	**Gaussian**	**Uniform**	**Gaussian**	**Uniform**
32	15	1.19471	1.22362	0.00041	0.00030
64	240	0.21429	0.41149	0.00009	0.00006
128	3,840	0.04765	0.08138	0.00001	0.000005
256	61,440	0.02003	0.02069	0.000002	0.000001
Avg. reduction		4.15	3.99	6.11	7.43

As shown by Talay (1984, 1986), under technical conditions, Assumptions (i)–(iii) are not affected by substitution of normally distributed increments with i.i.d. increments of other distributions with zero mean and unit variance, under technical conditions.

In each case, as the number $n = T/h$ of time intervals is doubled, the number k of simulations of sample paths of X is increased by the factor prescribed by Theorem 1 (a factor of 2 for the Euler scheme and a factor of 4 for the Milshtein scheme). Despite the fact that the assumptions of Theorem 1 are not satisfied in every case and that the conclusions of the theorem are only guaranteed to apply asymptotically, Tables 1 and 2 show results that conform well to the conclusions of the theorem. In particular, for the Euler scheme, the sample root-mean-squared error of the estimated option price are reduced at each time by a factor of roughly 2 (the average reduction factor is shown in each case) and, for the Milshtein scheme, the sample root-mean-squared errors are reduced at each stage by a factor of roughly 4.

4. Concluding remarks. For applications of Monte Carlo methods to asset pricing problems in continuous time settings, see, for example, Boyle (1977, 1988, 1990), Jones and Jacobs (1986) and Boyle, Evnine and Gibbs (1989). For an alternative (large deviations) perspective on the tradeoff between number of time periods and number of Monte Carlo simulations, see Chapter 10 of Duffie (1992).

Extensions to several cases are possible. For example, with path-dependent security payoffs, one can sometimes augment the state space so as to reduce the problem to the setting considered here. Consider, for instance, with $N = 1$, an "Asian" option paying $(X_T - \int_0^T X_t\,dt)^+$ at time T. Let $Y_0 = 0$ and $dY_t = X_t\,dt$. Then (X, Y) is a diffusion in $\mathbb{R}^2$ whose drift and diffusion have bounded derivatives of all orders, provided the same is true of X. Extensions of this example are obvious. Next consider the case of stochastic short-term interest rates of the form $r_t = R(X_t)$ for some measurable R: $\mathbb{R}^N \to \mathbb{R}$. The price of a claim to $f(X_T)$ at time T can be reduced, under technical conditions and possibly after a change of measure, to the form $E[Y_T f(X_T)]$, where $dY_t = -R(X_t)Y_t\,dt$, with $Y_0 = 1$. [For a survey, see Duffie

(1992).] By augmenting the state process from X to (X, Y), we have a Monte Carlo problem of the same form, although strong conditions will be required on R in order to apply the standard results on weak orders of convergence. For a path-dependent option of the "lookback" payoff form $X_T - \inf_t X_t$, we can once again augment the state process by including $Y_t = \inf_{s \le t} X_s$, and reduce the problem to the computation of $E(X_T - Y_T)$. Although (X, Y) is not the solution of a standard stochastic differential equation, at least low orders of convergence can be obtained from the strong convergence of X^h to X. See, for example, the same arguments used in Talay and Tubaro (1990). In order to obtain high orders of convergence using the "Feynman–Kac" style of approach for weak convergence used by Talay and Tubaro, however, one would want to establish the smoothness of $E(Y_T)$ as a function of X_0.

For more general path-dependent functionals, one would want to extend the general theory of weak convergence to apply to functions on the sample path space, with suitable growth conditions. Our theorem is of course applicable whenever one can obtain an order of convergence of $E[f(X^h)]$ to $E[f(X)]$.

For the case of X which is a diffusion with reflection in a well-behaved domain, we can immediately apply Theorem 1 by using the weak convergence results for the discretizations of Slominski (1993, 1994) and Liu (1993). For the case of diffusions that are stopped at a well-behaved boundary, we can apply our theorem with the aid of the weak convergence results for discretizations given by Milshtein (1993).

We have allowed for a resource requirement that is of some general order q in the discretization step h. While a time constraint would generally require $q = 1$, one can imagine a setting in which parallel processing and the need to consider a path-dependent payoff would call for a memory constraint that is of an order q possibly higher than 1.

Acknowledgments. We are grateful for many conversations with Don Iglehart and Costis Skiadas, and we thank Ekhard Platen, Philip Protter, Denis Talay and others at the Colloque en Probabilités Numériques of INRIA, Paris, March 1993, for references and comments. We are also grateful for computational assistance from Robert Ashcroft, Kingston Duffie and Ping Jiang.

REFERENCES

BALLY, V. and TALAY, D. (1995). The law of the Euler scheme for stochastic differential equations: I. Convergence rate of the distribution function. *Probab. Theory Related Fields.* To appear.

BARRAQUAND, J. (1993). Numerical valuation of high dimensional multivariate European securities. Unpublished manuscript, Digital Research Laboratory, Paris.

BLACK, F. and SCHOLES, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy* **81** 637–654.

BOYLE, P. (1977). Options: a Monte Carlo approach. *Journal of Financial Economics* **4** 323–338.

BOYLE, P. (1988). A lattice framework for option pricing with two state variables. *Journal of Financial and Quantitative Analysis* **23** 1–12.

BOYLE, P. (1990). Valuation of derivative securities involving several assets using discrete time methods. Unpublished manuscript, Accounting Group, Univ. Waterloo.

BOYLE, P., EVNINE, J. and GIBBS, S. (1989). Numerical evaluation of multivariate contingent claims. *Review of Financial Studies* **2** 241–250.

COX, J. (1975). Notes on option pricing I: constant elasticity of variance diffusions. Working paper, Stanford Univ.

COX, J. and ROSS, S. (1976). The valuation of options for alternative stochastic processes. *Journal of Financial Economics* **7** 229–263.

DUFFIE, D. (1992). *Dynamic Asset Pricing Theory*. Princeton Univ. Press.

HARRISON, M. and KREPS, D. (1979). Martingales and arbitrage in multiperiod security markets. *J. Econom. Theory* **20** 381–408.

JONES, R. and JACOBS, R. (1986). History dependent financial claims: Monte Carlo valuation. Unpublished manuscript, Simon Fraser Univ.

KLOEDEN, P. and PLATEN, E. (1992). *Numerical Solutions of Stochastic Differential Equations*. Springer, New York.

LIU, Y.-J. (1993). Numerical approaches to reflected diffusion processes. Unpublished manuscript, Purdue Univ.

MILSHTEIN, G. (1978). A method of second order accuracy integration of stochastic differential equations. *Theory Probab. Appl.* **3** 557–562.

MILSHTEIN, G. (1993). The solving of boundary value problems by the numerical integration of stochastic equations. Unpublished manuscript, Ural State Univ., Ekaterinburg, Russia.

SLOMINSKI, L. (1993). On existence and stability of solutions of multidimensional S.D.E.'s with reflecting boundary conditions. *Ann. Inst. H. Poincaré* **29** 1–36.

SLOMINSKI, L. (1994). On approximation of solutions of S.D.E.'s with reflecting boundary conditions. *Stochastic Process. Appl.* **50** 197–219.

TALAY, D. (1984). Efficient numerical schemes for the approximation of expectations of functionals of S.D.E. *Filtering and Control of Random Processes. Lecture Notes in Control and Inform. Sci.* **61**. Springer, Berlin.

TALAY, D. (1986). Discrétisation d'une equation différentielle stochastique et calcul approché d'espérances de fonctionelles de la solution. *Mathematical Modeling and Numerical Analysis* **20** 141–179.

TALAY, D. (1990). Second order discretization schemes of stochastic differential systems for the computation of the invariant laws. *Stochastics Stochastics Rep.* **29** 13–36.

TALAY, D. and TUBARO, L. (1990). Expansion of the global error for numerical schemes solving stochastic differential equations. *Stochastic Anal. Appl.* **8** 483-509.

YAMADA, T. (1976). Sur l'approximation des solutions d'équations différentielles stochastiques. *Z. Wahrsch. Verw. Gebiete* **36** 153–164.

GRADUATE SCHOOL OF BUSINESS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305-5015

DEPARTMENT OF OPERATIONS RESEARCH
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305-4022