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ON CONVERGENCE OF THE UNIFORM NORMS FOR GAUSSIAN PROCESSES AND LINEAR APPROXIMATION PROBLEMS

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We consider the large values and the mean of the uniform norms for a sequence of Gaussian processes with continuous sample paths. The convergence of the normalized uniform norm to the standard Gumbel (or *double exponential*) law is derived for distributions and means. The results are obtained from the Poisson convergence of the associated point process of exceedances for a general class of Gaussian processes. As an application we study the piecewise linear interpolation of Gaussian processes whose local behavior is like fractional (integrated fractional) Brownian motion (or with *locally stationary increments*). The overall interpolation performance for the random process is measured by the pth moment of the approximation error in the uniform norm. The problem of constructing the optimal sets of observation locations (or interpolation knots) is done asymptotically, namely, when the number of observations tends to infinity. The developed limit technique for a sequence of Gaussian nonstationary processes can be applied to analysis of various linear approximation methods.

1. Introduction. Let $X_n(t)$, $t \in [0, T]$, $n \geq 1$, be a sequence of Gaussian zero-mean processes with continuous sample paths. In the following we deal with the uniform norm $M_n(T) = \max_{t\in[0,T]} |X_n(t)|$ and its asymptotic distribution and mean as $T = T(n) \to \infty$ and $n \to \infty$. The asymptotic tail distribution of $M_n(T)$ is derived from the Poisson convergence of the associated point process of exceedances under general conditions. The general class of Gaussian processes considered is motivated mainly by applications for linear interpolation of a random process.

For a Gaussian stationary process, the limit distribution of the maximum $\max_{t\in[0,T]} X(t)$ [and $\max_{t\in[0,T]} |X(t)|$] is known to be of Gumbel type [Cramér and Leadbetter (1967)]. Further developments are given in Pickands (1969), Berman (1971), Qualls and Watanabe (1972) and Lindgren, de Maré and Rootzén (1975) assuming that the correlation function $r(t)$ satisfies the condition

$$r(t) = 1 - c|t|^{\alpha} + o(|t|^{\alpha}) \qquad \text{as } t \to 0, \tag{1}$$

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with $c > 0, 0 < \alpha \leq 2$, together with Berman's condition

$$r(v) \log v \to 0 \qquad \text{as } v \to \infty. \tag{2}$$

The asymptotic distribution of the maxima for some classes of nonstationary Gaussian processes are studied in Piterbarg and Prisyazn'yuk (1978), Berman (1985), Hüsler (1990, 1995) and Bräker (1993); see Leadbetter, Lindgren and Rootzén (1983), Adler (1990) and Piterbarg (1996) for related results and further references. Limit results for the distribution of the maxima $M_n(T)$ of a *sequence* of Gaussian processes were obtained in Seleznjev (1991, 1993, 1996), Piterbarg and Seleznjev (1994), see also Hüsler (1999). They investigate the tail distribution for the uniform norms of a sequence of certain nonstationary Gaussian processes which are close to cyclo-stationary processes, that is, with a periodic covariance function and nonconstant variance [for Gaussian cyclostationary processes, see Konstant and Piterbarg (1993)]. The asymptotic behavior of the mean $\mathbf{E}M_n(T)$ is far less studied even in the stationary case. The only known result, to the best of our knowledge, is due to Pickands (1968) for the extremes of independent, identically distributed random variables; see also Resnick (1987) for a different proof.

On the other hand, these large extreme value results are related to various random process approximation problems where small random uniform deviations are of interest. A usual scheme is as follows: for $X(\tau), \tau \in [0, 1]$, a Gaussian process, let

$$L_n(\tau) = L_n(X, \tau) \quad \text{and} \quad \delta_n(\tau) = X(\tau) - L_n(\tau) \tag{3}$$

be a random process approximating X and the deviation process, respectively. Here n is the number of known functionals on sample paths of X [e.g., Fourier coefficients, values of the process (and/or its derivatives) at interpolation knots]. As a rule, the maximal quadratic mean (q.m.) error $\sigma_n := \max_{[0,1]}(\mathbf{E}\delta_n^2(\tau))^{1/2} \to 0$ as $n \to \infty$. Then after a time and scale transformation, large extreme value problems arise for a sequence of random processes $X_n(t) = \delta_n(\tau)/\sigma_n,\ \tau = g_n(t), t \in [0, T], \tau \in [0, 1], T = T(n) \to \infty$ [e.g., $g_n(t) = t/n$ for trigonometric polynomials and equidistant linear interpolation] and levels $u = \delta/\sigma_n \to \infty$ as $n \to \infty$. The overall approximation performance is measured by the maximal q.m. error σ_n, the distribution and the mean of the deviation in the uniform norm,

$$\mathbf{P}\left\{\max_{\tau\in[0,1]} |\delta_n(\tau)| \leq \delta\right\} = \mathbf{P}\left\{\max_{t\in[0,T]} |X_n(t)| \leq u\right\}$$

and

$$e_p(L_n) = \left(\mathbf{E} \max_{\tau\in[0,1]} |\delta_n(\tau)|^p\right)^{1/p} = \sigma_n\left(\mathbf{E} \max_{t\in[0,T]} |X_n(t)|^p\right)^{1/p}, \qquad p \geq 1,$$

respectively. This relationship between extreme value and approximation problems is considered in Belyaev and Simonyan (1979) (for regression broken lines), in Seleznjev (1991) (for trigonometrical polynomials), in Seleznjev (1993, 1996) and

Piterbarg and Seleznjev (1994) (for piecewise linear interpolation with equidistant knots). Hüsler (1999) shows that these results hold under less stringent conditions on the correlation function. The next important question for interpolation is how to set interpolation knots (or design points) in an "optimal" way [see Seleznjev (2000)]. A number of results are obtained about optimal designs for certain classes of stochastic processes satisfying Sacks–Ylvisaker conditions [Sacks and Ylvisaker (1966)]. These conditions imply that a process is locally like the Brownian motion. For processes satisfying Sacks–Ylvisaker conditions, Su and Cambanis (1993) and Müller-Gronbach (1996) study optimal designs for the best linear unbiased estimators, or BLUEs, and piecewise linear interpolators with respect to the integrated quadratic mean error, that is $(\int_{[0,1]} \mathbf{E}X^2(t)\,dt)^{1/2}$. The order of the minimal errors is $n^{-1/2}$. Müller-Gronbach and Ritter (1997) investigate the mean weighted uniform norm for Gaussian (Sacks–Ylvisaker) processes with the minimal errors of order $(\log n)^{1/2}n^{-1/2}$. Ritter (1999) contains the most recent results and a very detailed survey of the approximation problems for stochastic processes. Optimal designs for Hermite spline interpolation of random processes with locally stationary increments are studied in Seleznjev (2000) with respect to the integral q.m. p-norm, $(\int_{[0,1]}(\mathbf{E}X^2(t))^{p/2}\,dt)^{1/p}$. The addressed problems arise in various practical areas, namely, in numerical analysis with realizations of stochastic processes [see Wahba (1990) and Weba (1992)], in simulation studies [see Eplett (1986) and Sacks, Welch, Mitchell and Wynn (1989)], in earth sciences [see Cressie (1993)].

Our aim in this paper is to extend some extreme value and approximation results of Seleznjev (1991, 1996, 2000), Piterbarg and Seleznjev (1994) and Hüsler (1999) to the case of general Gaussian processes and to apply these results to linear interpolation and optimal design problems. In order to demonstrate this general approach, the piecewise linear interpolation of a Gaussian process with locally stationary increments is considered with respect to the mean uniform norm. For a piecewise linear interpolator L_n with interpolation knots τ_k, $k = 0, \dots, n$, we have,

$$L_n(\tau) = X(\tau_{k-1})\frac{\tau_k - \tau}{\tau_k - \tau_{k-1}} + X(\tau_k)\frac{\tau - \tau_{k-1}}{\tau_k - \tau_{k-1}}, \qquad \tau \in [\tau_{k-1}, \tau_k], k = 1, \dots, n. \tag{4}$$

This widely used method provides the best rate of approximation for some classes of continuous functions [see Seleznjev (1989), Buslaev and Seleznjev (1999) and Seleznjev (1999)]. While piecewise linear interpolation is studied, the proposed technique for a sequence of Gaussian nonstationary processes and the corresponding maxima behavior can also be applied to various linear interpolation methods (e.g., interpolation splines for random functions).

For the sake of simplicity we denote also by $M_n(A) = \max_{t\in A} |X_n(t)|$, $A \subseteq [0, T]$, which should not be confused with $M_n(T) = M_n([0, T])$. Let $|J|$ be the length of an interval J. The following function $\mu_a(u) := u^a \exp\{-u^2/2\}$ is frequently used in extreme value theory for Gaussian processes. Also Φ and ϕ denote the standard Gaussian distribution and its density function, respectively. For two sequences a_n and b_n, let $a_n \asymp b_n$ if there exist positive constants c_1 and c_2 such that $c_1 b_n \le a_n \le c_2 b_n$; similar notation will be used for two sequences of arrays $a_{k,n}$ and $b_{k,n}$, $k = 1, \dots, n$.

The paper is organized as follows. In Section 2, we consider the problem of asymptotic behavior of the associated point process of exceedances (Poisson limit theorem) and the tail distribution and the mean of the uniform norms for a sequence of nonstationary Gaussian processes. The convergence for a sequence of means is derived from the Poisson limit theorem and the bounds of Theorem 3. These limit results allow us to study the linear interpolation error in the uniform p-norm and to construct an asymptotically optimal design sequence for a continuous and square mean differentiable process in Section 3. Section 4 contains the proofs of the statements from Sections 2 and 3.

2. Limit theorems for the uniform norm. Let $X_n(t), t \in [0, T]$, be a Gaussian zero-mean process with variance and correlation functions $\sigma_n^2(t)$ and $r_n(t, s)$, respectively, for each $n \ge 1$. We consider a sequence of processes $\{X_n(t)\}_{n\ge 1}$ with continuous sample paths and assume that $T = T(n) \to \infty$ as $n \to \infty$. Assume that for every n there exists a partition of $[0, T]$ of successive intervals $J_{k,n} = [t_{k-1,n}, t_{k,n}]$ such that $0 < h_1^* \le |J_{k,n}| \le h_2^* < \infty$, $k = 1, \dots, n$, with $[0, T] = \bigcup_{1\le k\le n} J_{k,n}$, $t_{0,n} = 0$, $t_{n,n} = T$. Note that $nh_1^* \le T \le nh_2^*$.

First we consider the convergence of the point process of exceedances to a Poisson process. We introduce the following general conditions for a sequence of Gaussian processes.

The first condition describes the behavior of tail distribution of local maximum $M_n(J_{k,n})$. It is well known that in many cases the asymptotic behavior is mainly determined by the value of $\sigma_{k,n} := \max_{t\in J_{k,n}} \sigma_n(t)$. We denote by $\sigma_n = \max_k \sigma_{k,n}$ and $u_{k,n} = u/\sigma_{k,n}$. If $\sigma_n > 0$, we norm the processes $X_n(t)$ without loss of generality in such a way that $\sigma_n = 1$. Let $u_{\min,n} := \min_k u_{k,n} \to \infty$ as $n \to \infty$.

C1. Assume that there exists a positive constant a such that for all n,

$$\mathbf{P}\{M_n(J_{k,n}) > u\} = c_{k,n}\mu_a(u_{k,n})(1 + \epsilon_{k,n}) \tag{5}$$

for all $k \le n$, where $c_{k,n} \asymp 1$ and

$$\sup\{|\epsilon_{k,n}|, k = 1, \dots, n\} \to 0 \qquad \text{as } u \to \infty.$$

The second condition allows us to formulate our results for random processes with a general structure of a maximum variance set. It describes the distributional approximation of a maxima of $X_n(t)$ defined on a continuous time interval and on a discrete but dense enough lattice. For any sufficiently small $0 < \delta < h_1^*/2$, let $J_{k,n}^{\delta} = (t_{k-1,n} + \delta, t_{k,n} - \delta)$ with $1 \le k \le n$, $J_{k,n}^{\delta} \subset J_{k,n}$. For $\alpha, \gamma > 0$, define $q_{k,n} = \gamma u_{k,n}^{-2/\alpha}$. We select $\gamma = \gamma(n) \to 0$ as $n \to \infty$. For every $q > 0$ and $J \subseteq [0, T]$, denote by $\mathbf{P}_q\{M_n(J) \ge u\} := \mathbf{P}\{\max_{iq \in J} |X_n(iq)| \ge u\}$.

C2. Assume that for some positive β, $0 < \beta \le \infty$, such that $a \ge 2/\alpha - 2/\beta - 1$ ($2/\beta = 0$ for $\beta = \infty$), and for all sufficiently large n, there exist $S = S(u) > 0$ and subintervals $J_{k,n}^{\star} \subset J_{k,n}^{\delta}$ with $\#\{iq_{k,n} \in J_{k,n}^{\star}\} \le S u_{k,n}^{2/\alpha - 2/\beta}$ such that for all $k \le n$,

$$0 \le \mathbf{P}\{M_n(J_{k,n}^{\delta}) > u\} - \mathbf{P}_{q_{k,n}}\{M_n(J_{k,n}^{\star}) \ge u\} \le g_1(\gamma, S)\mu_a(u_{k,n}),$$

$$0 \le \mathbf{P}\{M_n(J_{k,n}) > u\} - \mathbf{P}\{M_n(J_{k,n}^{\delta}) > u\} \le g_2(\delta)\mu_a(u_{k,n}),$$

where $g_1(\gamma, S) \to 0$ and $S \to S_0 \le \infty$ as $u \to \infty$, $\gamma \to 0$, and $g_2(\delta) \to 0$ as $\delta \to 0$.

The next assumption is based on C1 and controls the stabilizing rate of levels $u_{k,n}$ so that the number of intervals with "large" deviations of X_n tends to be "regularly" distributed on $[0, T]$ [cf. with the standard Poisson assumption in Leadbetter, Lindgren and Rootzén (1983)].

C3. Assume that $u \to \infty$ as $n \to \infty$ so that for any Borel set $A \subseteq [0, 1]$,

$$\sum_{J_{k,n} \subset T \cdot A} c_{k,n}\mu_a(u_{k,n}) \to \nu(A) < \infty \qquad \text{as } n \to \infty. \tag{6}$$

Obviously $\nu(A)$ is a finite measure on Borel subsets of $[0, 1]$. Without loss of generality we can suppose that $\nu([0, 1]) = 1$. Finally we assume a mixing property of the processes X_n (Berman's type condition).

C4. Assume that for some $b \ge \max(1, 2/\alpha - 2/\beta - a)$ the function $\rho(v) := \sup\{|r_n(t, s)|, |t - s| \ge v, n \ge 1\}$ is such that $\rho(v) < 1$ for all $v > 0$ and

$$\rho(v)(\log v)^b \to 0 \qquad \text{as } v \to \infty.$$

In linear interpolation problems, where α, β are smoothness parameters, [see Seleznjev (1993, 1996), Piterbarg and Seleznjev (1994) and Hüsler (1999)], we can put $b = 1$ and $a = 2/\alpha - 2/\beta - 1$ in C4. Hence this assumption is the known Berman's condition for asymptotic results of extremes of Gaussian processes. Conditions C1, C2 have a general form and can be used both for stationary and nonstationary cases with various structures of maximum variance set. One can find verifications of C1 and C2 for stationary Gaussian processes in Piterbarg (1996); in fact the proof is a repetition of the original Pickands' proof of C1 [see also Leadbetter, Lindgren and Rootzén (1983)]. Taking into account further applications for linear interpolation problems we give also the following sufficient conditions A1–A3. In these problems, usually, the time interval $[0, T]$ can be divided into subintervals $J_{k,n}$ in such a way that the maximum variance set consists of a single point. In order to prove the implication A1–A3 $\Rightarrow$ C1, C2, one can use the arguments from Seleznjev (1993, 1996) and Piterbarg and Seleznjev (1994), where this statement has been proved for the most frequently used cases $\alpha < \beta$ and $\alpha = \beta$, respectively. For a fixed interval, conditions of such type have been introduced in Piterbarg and Prisyazhn'uk (1978).

Let $\sigma^*(t)$ be a continuous function, $0 \le \sigma^*(t) \le 1$, $t \in [0, \infty)$, with a sequence of distinct maximum points $\{t : \sigma^*(t) = 1\} = \{t_{\max,k} : t_{\max,k+1} - t_{\max,k} \asymp 1,\ k = 1, 2, \dots\}$. Let $n = n_T$ be the number of points $t_{\max,k}$ in $[0, T]$. For every k, we introduce also the intervals $I^{\delta}_{k,n} = \{t : |t - t_{\max,k}| < \delta\}$, $I^{\delta}_{k,n} \subset J_{k,n}$, and $I(\delta) := \bigcup_{k \le n} I^{\delta}_{k,n} = I(\delta, n)$ being dependent on n.

A1. For every k, $\sigma_n(t) = \sigma_{k,n}\sigma^*(t)(1 + \epsilon_n(t))$, $t \in J_{k,n}$, where $\sup\{|\epsilon_n(t)|, t \in [0, T]\} = o(1/\log n)$ as $n \to \infty$, and $\sigma^*(t)$ can be expanded near $t_{\max,k}$ in the following form:

$$\sigma^*(t) = 1 - \bigl(a_k + \gamma(t)\bigr)|t - t_{\max,k}|^{\beta}, \qquad a_k \asymp 1, \beta > 0,$$

with $\max\{|\gamma(t)|, t \in I^{\delta}_{k,n}, k = 1, 2, \dots\} = o(1)$ as $\delta \to 0$.

A2. For every k,

$$r_n(t, s) = 1 - \bigl(b_{k,n} + \gamma_n(t, s)\bigr)|t - s|^{\alpha}, \qquad t, s \in I^{\delta}_{k,n}, 0 < \alpha \le 2,$$

where $\sup_n\{|\gamma_n(t, s)|, t, s \in I(\delta), n = 1, 2, \dots\} = o(1)$ as $\delta \to 0$ and $b_{k,n} \to b_k > 0$ as $n \to \infty$ uniformly in $k = 1, \dots, n$.

A3. There exist positive r and C such that for any $t, s \in [0, T]$ and all n, $\mathbf{E}(X_n(t) - X_n(s))^2 \le C|t - s|^r$.

Now we formulate the first main result of the paper, a Poisson limit theorem for the number of large maxima. Let $N(\cdot)$ be a Poisson point process on $[0, 1]$ with the intensity measure $\nu(\cdot)$ and for any Borel set $A \subseteq [0, 1]$, the point process of exceedances $N_n(A) := \#\{k : M_n(J_{k,n}) > u,\ J_{k,n} \subset T \cdot A\}$.

THEOREM 1. *Suppose that a sequence of Gaussian processes X_n, $n \ge 1$, satisfies* C1–C4. *Then the point process of exceedances N_n converges in distribution to a Poisson point process N as $n \to \infty$.*

Consider now the related problem of asymptotic behavior of the tail distribution and the mean of the uniform norm for a sequence $X_n(t)$, $n \ge 1$, and an increasing time interval $[0, T]$, $T \to \infty$ as $n \to \infty$. Suppose a regular behavior of $\sigma_{i,n}$ and $c_{i,n}$, $i = 1, \dots, n$, as $n \to \infty$. Let $[r]$ denotes the integer part of r.

C5. There exist continuous positive functions $c(s)$ and $\sigma(s)$, $s \in [0, 1]$, with $\sup_{t \in [0,1]} \sigma(t) = 1$, such that

$$\sigma_{[ns],n} = \sigma(s) + o(1/\log n) \quad \text{and} \quad c_{[ns],n} = c(s) + o(1)$$

as $n \to \infty$ uniformly in $s \in [0, 1]$.

For a Borel set $A \subseteq [0,1]$, denote

$$\begin{aligned} I_A(u) &= I_A\big(u, c(t), \sigma(t)\big) \\ &:= \int_A c(t)\sigma(t)^{-a} \exp\{-u^2\big(1-\sigma^2(t)\big)/(2\sigma^2(t))\}\,dt. \end{aligned}$$

The following condition supposes a regular (Laplace-like) behavior of $\sigma(t)$ near the points of its maximum.

C6. Assume that $I_{[0,1]}(u) \sim cu^{-\kappa}$ as $u \to \infty$ for a positive c and some $\kappa \geq 0$.

Let $u = u_n = b_n + x/b_n$, $x \in \mathbb{R}$. The above conditions provide exact asymptotic behavior of the norming values b_n of this sequence u_n.

PROPOSITION 1. *Let* C1, C3 *with* $A = [0,1]$, *and* C5 *hold. Then*

$$b_n \sim (2\log n)^{1/2} \qquad \text{as } n \to \infty. \tag{7}$$

If in addition C6 *holds, then*

$$\begin{aligned} b_n = (2\log n)^{1/2} + (2\log n)^{-1/2}\left(\frac{a-\kappa}{2}\log\log n + \log\big(c2^{(a-\kappa)/2}\big)\right) \\ + o\big((2\log n)^{-1/2}\big). \end{aligned} \tag{8}$$

REMARK 1. (1) One can generalize C5 using permutations, namely, assuming that for any n there exists a permutation π_n of indexes $1,\dots,n$ such that

$$\sigma_{\pi_n[ns],n} = \sigma(s) + o(1/\log n) \quad \text{and} \quad c_{\pi_n[ns],n} = c(s) + o(1)$$

uniformly in $s \in [0,1]$ as $n \to \infty$.

(2) By Proposition 1 we may interpret C3 and C6 as a variant of a standard "stabilizing" condition for Poisson approximation [cf. Leadbetter, Lindgren and Rootzén (1983), Theorem 12.3.4].

(3) If $S_1 := \{s \in [0,1] : \sigma(s) = 1\}$ is a finite set of isolated points s_k, and if $\sigma(\cdot)$ can be approximated near the points s_k by $\sigma(t) = 1 - \text{const}\cdot|t-s_k|^{\alpha_k}(1+o(1))$ as $t \to s_k$, with positive α_k's, then the standard Laplace technique gives $I_{[0,1]}(u) \sim cu^{-\kappa}$ as $u \to \infty$, for some $\kappa \geq 0$, and C6 follows. If S_1 contains at least one interval, then $I_{[0,1]}(u) \sim c$ as $u \to \infty$, where $c = \int_{S_1} c(t)\,dt > 0$ and $\kappa = 0$.

Taking into account C5 and C6, the following theorem is a straightforward consequence of (Poisson) Theorem 1 and Proposition 1.

THEOREM 2. *Suppose that a sequence of Gaussian processes* X_n, $n \geq 1$, *satisfies* C1–C6. *Then for any* x,

$$\lim_{n\to\infty} \mathbf{P}\{M_n(T) \leq b_n + x/b_n\} = \exp(-e^{-x}) \qquad \text{as } n \to \infty.$$

The following auxiliary result, the domination theorem, gives us necessary bounds in the moment convergence theorem for maxima (Theorem 4). Instead of Berman's condition C4 with $b = 1$ we assume a weaker condition.

C4.1. For some $L > 0$ and all $v > 1$, $\rho(v)\log v \le L$.

THEOREM 3. *Suppose that a sequence of Gaussian processes $X_n, n \ge 1$, satisfies* C1–C3, C4.1 *and* C5. *Then there exists a function $M(x)$; $x \in \mathbb{R}$, such that for some $\varepsilon > 0$, $\int_{-\infty}^{\infty} e^{\varepsilon|x|} M(x)\,dx < \infty$ and for all $x \ge 0$,*

$$\mathbf{P}\{M_n(T) \le b_n - x/b_n\} \le M(-x), \qquad \mathbf{P}\{M_n(T) \ge b_n + x/b_n\} \le M(x). \tag{9}$$

COROLLARY 1. *Under the conditions of Theorem* 3, *the statement of Theorem* 3 *holds also for* $\max_{t\in[0,T]} X_n(t)$.

THEOREM 4. *Under the conditions of Theorem* 2, *for any $p \ge 1$,*

$$\big(\mathbf{E}M_n(T)^p\big)^{1/p} = b_n + c_e b_n^{-1} + o(b_n^{-1}) \qquad \text{as } n \to \infty,$$

where $c_e = \int_R x\,d\exp(-e^{-x})$ is the first moment of Gumbel distribution.

It is well known that $c_e = 0.5772\ldots$ (Euler's constant) [see, e.g., Johnson and Kotz (1970)]. Our derivation shows that $(\mathbf{E}M_n(T)^p)^{1/p} = b_n + c_n b_n^{-1} + O(b_n^{-2})$ where the constant c_n depends on n and converges to the Euler constant c_e.

3. Linear interpolation of a Gaussian process in the uniform norm. We apply Theorems 1, 2 and 4 to a sequence of random deviation processes when the number of interpolation knots tends to infinity. In interpolation problems, we assume that $X(t), t \in [0,1]$, is interpolated at distinct design points $\tau_k(n)$, $0 \le k \le n$, with $T_n = \{\tau_0 = 0 < \tau_1(n) < \cdots < \tau_n(n) = 1\}$, or *knots*. The set of all such designs T_n with $(n+1)$ design points is denoted by D_n. Usually we suppress the argument n for the design points $\tau_k = \tau_k(n)$ from T_n, $k = 0, 1, \ldots, n$. Let $h(t), t \in [0,1]$, be a positive continuous density function and a sequence of designs $\{T_n\}_{n\ge1}$ be such that

$$\int_0^{\tau_i} h(t)\,dt = i/n, \qquad i = 1, \ldots, n. \tag{10}$$

Such a sequence of designs $\{T_n(h^*)\}_{n\ge1}$ is called a *regular* sequence generated by $h(\cdot)$ and denoted by $\{T_n(h^*)\}_{n\ge1} = RS(h)$ [Sacks and Ylvisaker (1966) and Su and Cambanis (1993)]. We define asymptotic optimality of the sequence of sampling designs T_n^* by

$$\inf_{T\in D_n} e_p(L_n(X,T)) \sim e_p(L_n(X,T_n^*)) \qquad \text{as } n \to \infty.$$

Henceforth, we deal with regular sequences of designs and consider the problem of constructing an asymptotically optimal density for piecewise linear interpolation of a given random process.

Let $X(t), t \in [0, 1]$, be a Gaussian process with zero mean and almost sure continuous sample paths. Denote by $X^{(m)}$ the q.m. derivative of order m, $X^{(0)} = X$. We consider both continuous ($m = 0$) and q.m. differentiable ($m = 1$) cases. Introduce the following conditions for the incremental variance (or *structure*) function $d_{X^{(m)}}(t, s) := \mathbf{E}(X^{(m)}(t) - X^{(m)}(s))^2$ and its partial derivatives $d^{(p,q)}_{X^{(m)}}(t, s) := \partial^{p+q} d_{X^{(m)}}(t, s)/\partial^p t\, \partial^q s$. We give the conditions in a common form for both cases, $m = 0, 1$. The *continuity modulus* of a function $f(t)$, $t \in [0, T]$, is defined as

$$w\big(f(\cdot), x\big) = \sup_{t,s\in[0,T],|t-s|\le x} |f(t) - f(s)|.$$

B1. There exists a continuous positive function $C(t), t \in [0, 1]$, with continuity modulus $w(C(\cdot), x) = o(1/\log x)$ as $x \to 0$ such that for all $t, s \in [0, 1]$, $0 < \alpha < 2$,

$$d_{X^{(m)}}(t, s) = C(t)|t - s|^\alpha\big(1 + f(t, s)\big) \qquad \text{with } f(t, s) \log |t - s| = o(1)$$

as $t - s \to 0$.

B2. For all $t, s \in [0, 1]$, $t \neq s$, there exists a continuous derivative $d^{(1,1)}_{X^{(m)}}(t, s)$ with $d^{(1,1)}_{X^{(m)}}(t, s)|t - s|^{2-\alpha} = O(1)$ as $t - s \to 0$. If $m = 0$, then there exists $d^{(1,0)}_X(t, s)$ for all $t, s \in [0, 1]$, $t \neq s$, with $d^{(1,0)}_X(t, s)|t - s|^{1-\alpha} = O(1)$ as $t - s \to 0$.

REMARK 2. Condition B1 implies that the process $X^{(m)}$ has *locally stationary* increments, that is,

$$\lim_{s\to 0} d_{X^{(m)}}(t + s, t)/|s|^\alpha = C(t) \qquad \text{uniformly in } t \in [0, 1]. \tag{11}$$

This condition is similar to local stationarity introduced by Berman (1974) for a standardized random process.

EXAMPLE 1. Let $B_\alpha(t), t \in [0, 1], 0 < \alpha < 2$, be a zero-mean fractional Brownian motion with $\mathbf{E}(B_\alpha(t + s) - B_\alpha(t))^2 = |s|^\alpha$ where $C_{B_\alpha}(t) = 1$. A simple nonstationary example of a process with locally stationary increments is a time and/or scale transformation of $B_\alpha(\cdot) : Y(t) = b(t)B_\alpha(a(t)) + m(t)$, if $a'(\cdot) > 0$, $b(\cdot) > 0$ and $m(\cdot)$ are Hölder continuous with the exponent $\gamma > \alpha/2$, then $C_Y(t) = b(t)^2|a'(t)|^\alpha$. More examples can be found in Hüsler (1995) and Seleznjev (2000).

In order to apply Theorems 2 and 4, we introduce the following time transformation of the interval $\tau \in [0, 1]$ into $t \in [0, n]$:

$$\begin{aligned} t = k - 1 + (\tau - \tau_{k-1})/h_k, \\ \tau \in [\tau_{k-1}, \tau_k],\ h_k = \tau_k - \tau_{k-1}, k = 1, \dots, n, \end{aligned} \tag{12}$$

and a sequence of scaled deviation processes $X_n(t) := \delta_n(\tau)/\sigma_n$, $t \in [0, n]$, $\tau \in [0, 1]$. We assume that the points τ_k belong to a $RS(h)$ design $T_n = \{\tau_0 = 0 < \tau_1 < \cdots < \tau_n = 1\}$. This design generates the corresponding design for the interval $[0, n]$, namely, $\{t_0 = 0 < t_1 < \cdots < t_n = n\}$, where $t_k = k, k = 0, \dots, n$. Let the sequence of levels $u = u(n) = \delta/\sigma_n \to \infty$ as $n \to \infty$. For the sake of simplicity, we use the following common notation for the continuous ($m = 0$) case and differentiable ($m = 1$) one.

For the *continuous* case, let

$$S(t) := s_0^{-1}\big(t^\alpha(1-t) + t(1-t)^\alpha - t(1-t)\big), \qquad t \in [0, 1], \tag{13}$$

where s_0 is a normalizing constant such that $\max_{t\in[0,1]} S(t) = 1$. Denote by α_0 the unique solution of the equation $\alpha(3-\alpha)2^{1-\alpha} = 1$ over the interval $[0, 2]$ ($\alpha_0 \approx 0.2052$). It is shown in Seleznjev (1996) that for $2 > \alpha \ge \alpha_0$, $S(t)$ has a single absolute maximum at the point $t = 1/2$, so that $s_0 = 2^{-\alpha} - 2^{-2}$. If $0 < \alpha < \alpha_0$ then there exist two maximum points of $S(t)$, $t_1, t_2 = 1 - t_1, 0 < t_1 < 1/2$.

For the *differentiable* case, let

$$S(t) := s_0^{-1} t(1-t)\big(1 - t^{\alpha+1} - (1-t)^{\alpha+1}\big)/\big((\alpha+1)(\alpha+2)\big), \qquad t \in [0, 1], \tag{14}$$

with s_0 the normalizing constant such that again $\max_{t\in[0,1]} S(t) = 1$. It is shown in Piterbarg and Seleznjev (1994) that $S(t)$ has a single absolute maximum at the point $t = 1/2$ with $s_0 = (1 - 2^{-\alpha})/(4(\alpha+1)(\alpha+2))$.

We let, for $m = 0$ or $m = 1$,

$$\begin{aligned} \sigma(t) &= {\sigma_0}^{-1} C(t)^{1/2} h(t)^{-(m+\alpha/2)}, \quad \text{where} \\ \sigma_0 &= \max_{t\in[0,1]} \big(C(t)^{1/2} h(t)^{-(m+\alpha/2)}\big), \end{aligned} \tag{15}$$

so that $\max_{t\in[0,1]} \sigma(t) = 1$. In Section 4.5 we show that B1 and B2 imply conditions A1–A3 for the correponding sequence of Gaussian processes. Therefore there exist $c_{k,n}$, $\sigma_{k,n}$ and $a_\alpha > 0$ such that C1 holds, that is,

$$\begin{aligned} &\mathbf{P}\Big\{\max_{\tau\in[\tau_{k-1},\tau_k]} |\delta_n(\tau)| > \delta\Big\} \\ &\quad = \mathbf{P}\Big\{\max_{t\in[k-1,k]} |X_n(t)| > u\Big\} \\ &\quad = c_{k,n}\left(\frac{u}{\sigma_{k,n}}\right)^{a_\alpha} \exp\left\{-\frac{u^2}{2\sigma_{k,n}^2}\right\}\big(1 + o(1)\big) \end{aligned} \tag{16}$$

as $u \to \infty$. We also show that in both cases, $m = 0, 1$, $\sigma_{[ns],n} = \sigma(s) + o(1/\log n)$ and $c_{[ns],n} = A_\alpha + o(1)$ as $n \to \infty$. The constants A_α, a_α are defined for the continuous case, in Seleznjev (1996) and for the differentiable case, in Piterbarg and Seleznjev (1994), respectively. The integral I_A takes the form $I_A(u) = A_\alpha \int_A \sigma(t)^{-a_\alpha} \exp\{-u^2(1-\sigma^2(t))/(2\sigma^2(t))\}\,dt$ and the following condition is a special case of C6 for the function $C(\tau)$ in B1 and the design density $h(\tau)$.

B3. There exist $c > 0$ and γ such that for any Borel set $A \subseteq [0,1]$ and some nonnegative $\lambda(A)$, $u^\gamma I_A(u) \to c\lambda(A)$ as $u \to \infty$, with $\lambda([0,1]) = 1$.

Obviously λ is a finite measure on Borel subsets of [0, 1]. Denote

$$v_n = (2\log n)^{1/2} + (2\log n)^{-1/2}\big((a_\alpha - \gamma)\log\log n + \log(c2^{(a_\alpha-\gamma)/2})\big).$$

From B3 it follows after a straightforward calculation that

$$n v_n^{a_\alpha} I_A(v_n)\exp\{-v_n^2/2\} \to \lambda(A) \qquad \text{as } n \to \infty. \tag{17}$$

Note that under the assumptions of this section and due to the above mentioned regularity properties of $\sigma(s)$, (17) is equivalent to C3. This follows similarly to the proof of Proposition 1.

Again it is convenient to define the corresponding point process of exceedances. Let for any Borel set $A \subseteq [0,1]$, N_n^* be the time-normalized point process of δ-exceedances,

$$N_n^*(A) := N_n(n\cdot A) := \#\Big\{k : \max_{[\tau_{k-1},\tau_k]} |\delta_n(\tau)| > \delta \text{ and } [\tau_{k-1},\tau_k] \subset A\Big\}.$$

Let N be a Poisson point process on [0, 1] with the intensity measure being equal to $\lambda(\cdot)$. It has been shown in Seleznjev (2000) that σ_n is of order $n^{-(m+\alpha/2)}$ as $n \to \infty$, $m = 0, 1$. As an application of Theorems 1, 2 and 4 we get the following results.

THEOREM 5. *Let $X(\tau)$, $\tau \in [0,1]$, satisfy* B1–B3 *for the continuous case* ($m = 0$) *or for the differentiable case* ($m = 1$). *Let $p \geq 1$. Then*:

(i) $\sigma_n \sim s_0^{1/2}\sigma_0 n^{-(m+\alpha/2)}$ *as $n \to \infty$*;

(ii) $\mathbf{P}\{\max_{\tau\in[0,1]} |\delta_n(\tau)| \leq \sigma_n(v_n + x/v_n)\} \to e^{-e^{-x}}$ *as $n \to \infty$*;

(iii) *For any design sequence $T_n = T_n(h), n \geq 1$, such that $w(h(\cdot), x) = o(1/\log x)$ as $x \to 0$*,

$$e_p(L_n(X, T_n)) = \sigma_n\big(v_n + c_e/v_n + o(1/v_n)\big) \sim s_0^{1/2}\sigma_0(2\log n)^{1/2} n^{-(m+\alpha/2)}$$

as $n \to \infty$;

(iv) *If $\delta = \sigma_n(v_n + x/v_n), x \in \mathbb{R}$, then N_n^* converges in distribution to the Poisson point process N as $n \to \infty$.*

Note that the assertion (i) has been proved in Seleznjev (2000) and is stated here for completeness.

Theorem 5 provides the solution for the optimal regular sequence problem. It follows directly from Theorem 5(iii) that

$$e_p\big(L_n(X, T_n(h))\big) \sim \sigma_n(T_n(h))(2\log n)^{1/2},$$

where $\sigma_n(T_n(h))^2 = \max_{[0,1]} \mathbf{E}\delta_n(\tau)^2$ for a given generating density $h(\cdot)$. Therefore

$$\frac{e_p(L_n(X, T_n(h^*)))}{\inf_{T\in D_n} e_p(L_n(X, T))} \sim \frac{\sigma_n(T_n(h^*))}{\inf_{T\in D_n} \sigma_n(T(h))} \tag{18}$$

as $n \to \infty$.

Piecewise linear interpolation is the simplest case of Hermite spline approximation. The optimal sequence of designs for the L_2 error,

$$\sigma_n(T(h))^2 = \max_{[0,1]} \mathbf{E}\big(X(\tau) - L_n(\tau)\big)^2,$$

has been investigated in Seleznjev (2000). Applying the corresponding result [Seleznjev (2000), Theorem 2] to the right-hand side of (18) we obtain straightforward the following solution to the optimal design problem for the norm $e_p(L_n(X, T))$, $p \geq 1$. Let $C_{m,\alpha} := \int_0^1 C(\tau)^{1/(2m+\alpha)}\, d\tau$. Assume that $\sup_{T(h)\in D_n} w(h(\cdot), x) = o(1/\log x)$ as $x \to 0$.

THEOREM 6. *Let $X(\tau)$, $\tau \in [0, 1]$, satisfy* B1–B3 *for the continuous case* ($m = 0$) *or for the differentiable case* ($m = 1$). *Let $p \geq 1$. Then the regular sequence $\{T_n\}_{n\geq 1} = RS(h^*)$, where $h^*(\tau) = C(\tau)^{1/(2m+\alpha)}/C_{m,\alpha}$, is asymptotically optimal with*

$$e_p(L_n(X, T_n(h^*))) \sim s_0^{1/2} C_{m,\alpha}^{m+\alpha/2} \cdot (2\log n)^{1/2} n^{-(m+\alpha/2)} \qquad \text{as } n \to \infty.$$

EXAMPLE 2. Let $a(t)$, $t \in [0, 1]$, be an increasing continuously differentiable function, $a(0) = 0$, $a(1) = 1$, with positive derivative $a'(t)$ such that $w(a'(\cdot), x) = o(1/\log x)$ as $x \to 0$. Then for time-transformed fractional Brownian motion $B_\alpha(a(t))$, Theorem 6 yields $h^\star(t) = a'(t)$.

REMARK 3. Let us compare the above results for piecewise linear estimators (PLE), with those for best linear unbiased estimators (BLUE). Until recently, BLUEs have been investigated only for Gaussian processes satisfying Sacks–Ylvisaker conditions [see Müller-Gronbach and Ritter (1997) and Ritter (1999)]. The Sacks–Ylvisaker conditions imply that the process is q.m. Hölder continuous with exponent 1/2, that is, $\alpha = 1$. They hold for processes with covariance functions $R(t, s) = (|t|^\alpha + |s|^\alpha - |t - s|^\alpha)/2$ for $\alpha = 1$ (e.g., Brownian motion) or $R(t, s) = \exp\{-|t - s|^\alpha\}$ with $\alpha = 1$ and do *not* hold if $0 < \alpha < 2$, $\alpha \neq 1$. Müller-Gronbach and Ritter (1997) show that BLUEs and PLEs have the same optimal approximation performance and find that the minimal error for weighted Brownian motion $C(t)^{1/2}B_1(t), t \in [0,1]$, is $e_p(L_n) \sim 1/2(\int_0^1 C(t)\,dt)^{1/2}(2\log n)^{1/2}n^{-1/2}$ which corresponds with Theorem 6, for $\alpha = 1$, $m = 0$, and $s_0 = 1/4$. Moreover, BLUEs require the precise knowledge of the covariance function and tedious calculations with inversion of the covariance matrix. Thus, simple and nonparametric linear interpolators are helpful at least as for obtaining the upper bounds for approximation errors for BLUEs.

4. Proofs.

4.1. *Proof of Theorem* 1 (*Poisson limit theorem*). The following construction [similar to Hüsler (1990)] is used for the proof. Let $h \in \mathbb{N}$ and

$$a_h = 3^{h-1}u_{n,\min}$$

depending on n. Define for $h \geq 1$,

$$I_{h,n} = \{1 \leq k \leq n : a_h < u_{k,n} \leq a_{h+1}\}$$

and

$$f_h = \sum_{k \in I_{h,n}} c_{k,n} u_{k,n}^a \exp(-u_{k,n}^2/2).$$

Thus $\sum_{h\geq 1} f_h = O(1)$ by C3. If $I_{h,n} = \varnothing$ for some h, then $f_h = 0$. Let

$$G = \{h : f_h > \exp(-a_h^2/6)\}$$

be the sets of a certain weight in the last sum. In Berman's comparison lemma we want to consider only the discrete time points $iq_{k,n}$ which belong to some $J_{k,n}^\delta$ where $k \in I_{h,n}$ and $h \in G$. This is possible since

$$\begin{aligned}
0 &\leq \mathbf{P}\Big\{\max_{t \in J_{k,n}^\delta} |X_n(t)| \leq u, 1 \leq k \leq n, k \in \bigcup_{h \in G} I_{h,n}\Big\} \\
&\qquad - \mathbf{P}\Big\{\max_{t \in J_{k,n}^\delta} |X_n(t)| \leq u, 1 \leq k \leq n\Big\} \\
&\leq \sum_{h \notin G} \sum_{k \in I_{h,n}} \mathbf{P}\Big\{\max_{t \in J_{k,n}^\delta} |X_n(t)| > u\Big\} \\
&\leq C \sum_{h \notin G, f_h > 0} f_h \leq C \sum_{h \notin G, f_h > 0} \exp(-a_h^2/6) \\
&\leq C \sum_{h : h \notin G, f_h > 0} \exp(-9^{h-1}a_1^2/6) \\
&\leq C \sum_{h : h \geq 1} \exp(-9^{h-1}a_1^2/6) = o(1)
\end{aligned} \tag{19}$$

as $n \to \infty$, with C some positive generic constant (as convention), for any $a \in \mathbb{R}$, since we are using that $a_1 = u_{n,\min} \to \infty$.

The proof consists of the following approximations:

$$\mathbf{P}\Big\{\max_{t \le T(n)} |X_n(t)| \le u\Big\}$$

$$= \mathbf{P}\Big\{\max_{t \in J_{k,n}} |X_n(t)| \le u,\, 1 \le k \le n\Big\}$$

$$= \mathbf{P}\Big\{\max_{t \in J_{k,n}^{\delta}} |X_n(t)| \le u,\, k \in \bigcup_{h \in G} I_{h,n}\Big\} \tag{20}$$
$$+ \sum_{k \in \bigcup_{h \in G} I_{h,n}} g_2(\delta) c_{k,n} u_{k,n}^{a} \exp(-u_{k,n}^2/2) + o(1)$$

$$= \mathbf{P}\Big\{\max_{iq_{k,n} \in J_{k,n}^{*}} |X_n(iq_{k,n})| \le u,\, k \in \bigcup_{h \in G} I_{h,n}\Big\} \tag{21}$$
$$+ \sum_{k \in \bigcup_{h \in G} I_{h,n}} g_1(\gamma, S) c_{k,n} u_{k,n}^{a} \exp(-u_{k,n}^2/2) + O(g_2(\delta)) + o(1)$$

$$= \prod_{k \in \bigcup_{h \in G} I_{h,n}} \mathbf{P}\Big\{\max_{iq_{k,n} \in J_{k,n}^{*}} |X_n(iq_{k,n})| \le u\Big\} + O\big(g_1(\gamma, S) + g_2(\delta)\big) + o(1) \tag{22}$$

$$= \prod_{k \in \bigcup_{h \in G} I_{h,n}} \mathbf{P}\Big\{\max_{iq_{k,n} \in J_{k,n}} |X_n(t)| \le u\Big\} + O\big(g_1(\gamma, S) + g_2(\delta)\big) + o(1) \tag{23}$$

$$= \exp\Big\{-\big(1 + o(1)\big) \sum_{k \in \bigcup_{h \in G} I_{h,n}} c_{k,n} u_{k,n}^{a} \exp(-u_{k,n}^2/2)\Big\} \tag{24}$$
$$+ O(g_1(\gamma, S) + g_2(\delta)) + o(1)$$

$$\to \exp\big(-\nu([0,1])\big) = \exp(-1) \qquad \text{as } n \to \infty. \tag{25}$$

Relations (20) and (21) [taking into account (19)] are implied by C1 and C2 as in Seleznjev (1993, 1996), Piterbarg and Seleznjev (1994) or Hüsler (1999). We are going to prove (22) by Berman's comparison lemma. (23) and (24) followed by similar arguments from C1 and C2. Relation (25) is implied by C4.

Now we show the approximation (22) by Berman's comparison lemma. We use the general form of this lemma given, for example, in Hüsler (1983) and Leadbetter, Lindgren and Rootzén (1983) with different boundary values $u_i = u/\sigma_n(iq_{k,n})$. The separation of the points $iq_{k,n}$ and $jq_{k',n}$, with $k < k'$, is at least $h_1^* + o(1)$ for sufficiently large n. We have by denoting $v_k = u_{k,n} = u/\sigma_{k,n}$ and

$r_n(i,j) = r_n(iq_{k,n}, jq_{k',n})$:

$$\begin{aligned}
&\left|\mathbf{P}\left\{\max_{iq_{k,n}\in J^*_{k,n}} |X_n(iq_{k,n})| \le u, k \in \bigcup_{h\in G} I_{h,n}\right\}\right.\\
&\qquad \left. - \prod_{k\in\bigcup_{h\in G} I_{h,n}} \mathbf{P}\left\{\max_{iq_{k,n}\in J^*_{k,n}} |X_n(iq_{k,n})| \le u\right\}\right| \qquad (26)\\
&\qquad \le C \sum_{k\le k'\in\bigcup_{h\in G} I_{h,n}} \sum_{iq_{k,n}\in J^*_{k,n}} \sum_{jq_{k',n}\in J^*_{k',n}} |r_n(i,j)| \exp\left(-\frac{u_i^2+u_j^2}{2(1+r_n(i,j))}\right)\\
&\qquad \le C \sum_{k\le k'\in\bigcup_{h\in G} I_{h,n}} \sum_{iq_{k,n}\in J^*_{k,n}} \sum_{jq_{k',n}\in J^*_{k',n}} |r_n(i,j)| \exp\left(-\frac{v_k^2+v_{k'}^2}{2(1+r_n(i,j))}\right).
\end{aligned}$$

This sum is split up by considering now the terms with $h < h' \in G$,

$$S_{h,h'} = \sum_{k\in I_{h,n}} \sum_{k'\in I_{h',n}} \sum_{iq_{k,n}\in J^*_{k,n}} \sum_{jq_{k',n}\in J^*_{k',n}} |r_n(i,j)| \exp\left(-\frac{v_k^2+v_{k'}^2}{2(1+r_n(i,j))}\right).$$

By the stated separation we have $|r_n(i,j)| \le \delta_0 < 1$ for some δ_0 depending on h_1^*. Then define $\gamma_0 > 0$ (large) and $\gamma(h,h') = \exp(\max(a_h, a_{h'})^2/4)$ to split the sum $S_{h,h'}$ into $S^{(1)}_{h,h'}$ with i,j such that $|jq_{k',n} - iq_{k,n}| < \gamma_0$, $S^{(2)}_{h,h'}$ with i,j such that $\gamma_0 \le |jq_{k',n} - iq_{k,n}| < \gamma(h,h')$, and $S^{(3)}_{h,h'}$ with i,j such that $|jq_{k',n} - iq_{k,n}| \ge \gamma(h,h')$.

We approximate each partial sum $S^{(j)}_{h,h'}$, $j = 1,2,3$, separately. For the first partial sums we derive for any h, using C2,

$$\begin{aligned}
\sum_{h':h<h'} S^{(1)}_{h,h'} &\le \sum_{k\in I_{h,n}} \sum_{iq_{k,n}\in J^*_{k,n}} \sum_{k'}{}^{*} S v_{k'}^{2/\alpha-2/\beta} \delta_0 \exp\left(-\frac{v_k^2+v_{k'}^2}{2(1+\delta_0)}\right)\\
&\le \sum_{k\in I_{h,n}} C(\gamma_0)(S v_k^{2/\alpha-2/\beta})^2 \delta_0 \exp\left(-\frac{v_k^2}{(1+\delta_0)}\right)
\end{aligned}$$

since there are at most a finite number $C(\gamma_0)$ of k' with $jq_{k',n} \in J^*_{k',n}$ such that $|jq_{k',n} - iq_{k,n}| < \gamma_0$ for fixed $iq_{k,n}$ (denoted by $\sum^*$), and $v_k < v_{k'}$. Thus this sum is bounded by

$$\sum_{k\in I_{h,n}} C(\gamma_0) S^2 v_k^a e^{-v_k^2/2} \exp\left(-\frac{v_k^2(1-\delta_0)}{2(1+\delta_0)}\right) v_k^{4/\alpha-4/\beta-a} = C(\gamma_0) S^2 f_h\, o(1)$$

uniformly in k and h, since $v_k \ge u_{n,\min} \to \infty$. Hence adding these terms results in

$$\sum_{h<h'} S^{(1)}_{h,h'} = o(1) S^2 \sum_h f_h = o(1)$$

as $n \to \infty$ for any fixed large S.

The second partial sums are handled in a similar way replacing δ_0 by $\delta_1 = \rho(\gamma_0)$ and $C(\gamma_0)$ by $O(\gamma(h,h'))$ (the sum is denoted again by $\sum^*$). δ_1 can be made small by taking γ_0 large. Take γ_0 such that $\delta_1 < 1/6$. For fixed $h < h'$ with $h, h' \in G$,

$$\begin{aligned}
S^{(2)}_{h,h'} &= \delta_1 \sum_{k\in I_{h,n}} \sum_{k'\in I_{h',n}}{}^{*} O\big(S^2 (v_k v_{k'})^{2/\alpha-2/\beta}\big) \exp\biggl(-\frac{v_k^2+v_{k'}^2}{2(1+\delta_1)}\biggr)\\
&= \delta_1 S^2 \sum_{k\in I_{h,n}} O(\gamma(h,h'))(v_k a_{h'})^{2/\alpha-2/\beta} \exp\biggl(-\frac{v_{k'}^2+a_{h'}^2}{2(1+\delta_1)}\biggr)\\
&= O(S^2) \sum_{k\in I_{h,n}} v_k^a e^{-v_k^2/2} a_{h'}^{4/\alpha-4/\beta-a} \exp\biggl(-\frac{a_{h'}^2(1-2\delta_1)}{2}\biggr)\gamma(h,h')\\
&= O(S^2) f_h f_{h'} a_{h'}^{4/\alpha-4/\beta-a} \exp\biggl(-\frac{a_{h'}^2(1-2\delta_1-1/3-1/2)}{2}\biggr)\\
&= O(S^2) f_h f_{h'} a_1^{4/\alpha-4/\beta-a} \exp\biggl(-\frac{a_1^2(1/6-2\delta_1)}{2}\biggr)\\
&= o(1) f_h f_{h'}
\end{aligned}$$

uniformly for any h, h' by the choice of $\gamma(h,h')$ and γ_0 such that $\delta_1 < 1/6$. Hence adding these sums,

$$\sum_{h<h'\in G} S^{(2)}_{h,h'} = o(1) \sum_h f_h \sum_{h'} f_{h'} = o(1)$$

as $n \to \infty$, for any S.

Finally we deal with the last partial sums $S^{(3)}_{h,h'}$ in a similar way. C4 implies

$$\sup_{|t-s|>u_{n,\min}} |r_n(t,s)|(\log u_{n,\min})^b \to 0 \qquad \text{as } n\to\infty. \tag{27}$$

Let $\delta_2 = \sup_{|t-s|\ge\gamma(h,h')} |r_n(t,s)|$, depending on h, h'. For fixed $h < h' \in G$,

$$\begin{aligned}
S^{(3)}_{h,h'} &\le \delta_2 S^2 \sum_{k\in I_{h,n}} \sum_{k'\in I_{h',n}} (v_k v_{k'})^{2/\alpha-2/\beta} \exp\biggl(-\frac{(v_k^2+v_{k'}^2)(1-\delta_2)}{2}\biggr)\\
&\le \delta_2 S^2 \sum_{k\in I_{h,n}} e^{-v_k^2/2} v_k^a\, 3a_h^{2/\alpha-2/\beta-a} \exp(9a_h^2\delta_2/2)\\
&\qquad\qquad \times \sum_{k'\in I_{h',n}} e^{-v_{k'}^2/2} v_{k'}^a\, 3a_{h'}^{2/\alpha-2/\beta-a} \exp(9a_{h'}^2\delta_2/2)\\
&\le C\delta_2 S^2 f_h f_{h'} a_{h'}^{4/\alpha-4/\beta-2a} \exp(9a_{h'}^2\delta_2)\\
&= o(1) f_h f_{h'}
\end{aligned}$$

uniformly in h, h' since $\delta_2 a_{h'}^2 = o(1/\log\gamma(h,h'))a_{h'}^2 = o(1)$ for $b \geq 1$, and similarly, $\delta_2 a_{h'}^{4/\alpha-4/\beta-2a} = o(1)$ for $b \geq 2/\alpha - 2/\beta - a$. Sum these terms to get

$$\sum_{h\neq h'\in G} S_{h,h'}^{(3)} = o(1)\sum_{h,h'} f_h f_{h'} = o(1)$$

as $n \to \infty$.

Adding the three sums of the partial sums $S_{h,h'}^{(j)}$, $j = 1, 2, 3$, together implies our claim

$$\mathbf{P}\Big\{\max_{t\in[0,T(n)]} |X_n(t)| \leq u\Big\} \to \exp\{-\nu([0,1])\} \qquad \text{as } n \to \infty.$$

The above proof is given for the whole interval $[0, T(n)]$ using the intervals $J_{k,n}$. It can be adapted easily for any Borel set $T(n)A \subset [0, T(n)]$ (with $A \subset [0,1]$) by restricting the derivation on intervals $J_{k,n}$ being subsets of $T(n)A$. It implies that

$$\begin{aligned}\mathbf{P}\Big\{\max_{t\in T(n)A} |X_n(t)| \leq u\Big\} &\sim \exp\Big\{-\sum_{k\leq n: J_{k,n}\subset T(n)A} c_{k,n}u_{k,n}^a \exp(-u_{k,n}^2/2)\Big\}\\ &\to \exp\{-\nu(A)\}\end{aligned}$$

as $n \to \infty$.

This implies finally that the point process N_n converges to a Poisson point process N with intensity defined by ν.

REMARK 4. We might also define the point process with respect to $\tau_{k,n} \in T(n)A$ instead of $J_{k,n}$; however, they do not differ asymptotically.

4.2. *Proof of Proposition* 1. Using C3 with $A = [0,1]$, and C5, for $x = 0$ and therefore $u_n = b_n$, we have

$$\begin{aligned}&\sum_{k=0}^{n-1} c_{k,n}(b_n/\sigma_{k,n})^a \exp\{-b_n^2/(2\sigma_{k,n}^2)\}\\ &\quad = nb_n^a e^{-b_n^2/2}\sum_{k=0}^{n-1}\frac{1}{n}\frac{c_{k,n}}{\sigma_{k,n}^a}\exp\Big\{-\frac{b_n^2(1-\sigma_{k,n}^2)}{2\sigma_{k,n}^2}\Big\}\\ &\quad = nb_n^a e^{-b_n^2/2}\int_0^1 \frac{c_{n,[nt]}}{\sigma_{[nt],n}^a}\exp\Big\{-\frac{b_n^2}{2}\frac{1-\sigma_{[nt],n}^2}{\sigma_{[nt],n}^2}\Big\}\,dt\\ &\quad = nb_n^a e^{-b_n^2/2}\int_0^1 \frac{c(t)}{\sigma^a(t)}\exp\Big\{-\frac{b_n^2(1-\sigma_{[nt],n}^2)}{2\sigma_{[nt],n}^2}\Big\}\,dt\big(1+o(1)\big)\end{aligned} \tag{28}$$

as $n \to \infty$. By assumption C3, the limit of the left-hand side of (28) equals 1; hence the same holds for the right-hand side. For the integral, using C5, we have

$$\int_0^1 \frac{c(t)}{\sigma^a(t)}\exp\Big\{-\frac{b_n^2(1-\sigma_{[nt],n}^2)}{2\sigma_{[nt],n}^2}\Big\}\,dt = I_{[0,1]}(b_n)\exp\{b_n^2\, o(1/\log n)\}$$

as $n \to \infty$. Taking the logarithm in (28) and dividing both parts by $\log n$, we get

$$\frac{b_n^2}{2\log n} = 1 + \frac{a\log b_n}{\log n} + b_n^2 o(1/\log^2 n) + \frac{\log I_{[0,1]}(b_n)}{\log n} + \frac{o(1)}{\log n}. \tag{29}$$

Consider $I_{[0,1]}(b_n)$. Note that $b_n \to \infty$ as $n \to \infty$. Take arbitrarily small $\varepsilon > 0$ and denote $A(\varepsilon) = \{t : 1 - \sigma^2(t) \le \varepsilon\}$. By continuity of $\sigma(t)$, $|A(\varepsilon)| > 0$, and for some positive constants C_0, C_1 and C_2,

$$C_0 \ge I_{[0,1]}(b_n) \ge \int_{A(\varepsilon)} \frac{c(t)}{\sigma^a(t)} \exp\left\{-\frac{b_n^2(1-\sigma^2(t))}{2\sigma^2(t)}\right\} dt \ge C_1|A(\varepsilon)|e^{-C_2\varepsilon b_n^2}.$$

Using this and (29), we get

$$2 \ge \limsup_{n\to\infty} b_n^2/\log n \ge \liminf_{n\to\infty} b_n^2/\log n \ge 2 - 2C_2\varepsilon,$$

which gives the first assertion of Proposition 1. Using now C6 we have ($x = 0$),

$$\int_0^1 \frac{c(t)}{\sigma^a(t)} \exp\left\{-\frac{b_n^2(1-\sigma^2_{[nt],n})}{2\sigma^2_{[nt],n}}\right\} dt$$

$$= I_{[0,1]}(b_n)e^{b_n^2 o(1/\log n)} = cb_n^{-\kappa}\big(1+o(1)\big)e^{b_n^2 o(1/\log n)}$$

as $n \to \infty$. Since already $b_n^2 \sim 2\log n$, the right-hand side equals $cb_n^{-\kappa}\,(1+o(1))$ as $n \to \infty$. Thus, from (28) we have that b_n is a root of the equation

$$1 + o(1) = cnb_n^{-\kappa} b_n^a e^{-b_n^2/2}\big(1+o(1)\big)$$

and it is easy to verify that b_n is as indicated in Proposition 1.

4.3. *Proof of Theorem* 3 (*Domination theorem*). The proof is organized as follows. First we derive the upper tail estimation, that is, the second inequality in (9). Then we prove the first inequality in (9), splitting up the interval $[0,\infty)$ into three intervals, $[0, Bb_n^2)$, $[Bb_n^2, 1.9b_n^2)$ and $[1.9b_n^2, \infty)$, with $1.9 > B > 0$, using separate arguments for each interval. Recall that for all n, $\max_{1\le k\le n} \sigma_{k,n} = 1$.

4.3.1. *Upper tail estimation.* For all large enough n, using C1, we have

$$\mathbf{P}\left\{\max_{t\in[0,T(n)]} |X_n(t)| > b_n + x/b_n\right\}$$

$$\le \sum_{k=0}^{n-1} \mathbf{P}\left\{\max_{t\in[t_k,t_{k+1}]} |X_n(t)| > b_n + x/b_n\right\}$$

$$\le 2\sum_{k=0}^{n-1} \frac{c_{k,n}}{\sigma^a_{k,n}}\,(b_n + x/b_n)^a \exp\left\{-\frac{(b_n + x/b_n)^2}{2\sigma^2_{k,n}}\right\}$$

$$\le 2n \max_{k<n} \frac{c_{k,n}}{\sigma^a_{k,n}} b_n^a \exp\{-b_n^2/2\}\, e^{-x} \le \text{const}\cdot e^{-x},$$

by Proposition 1, so the second part of (9) holds.

4.3.2. *Lower tail estimation.* Introduce a standardized form of the process $X_n(t)$, $\hat{X}_n(t) = X_n(t)/\sigma_n(t)$ and consider the set

$$A = A(n) = [0, T(n)] \cap \{t : \sigma_n(t) \ge 1/2\}.$$

For $x > 0$ we have

$$\mathbf{P}\Big\{\max_{t\in[0,T(n)]} |X_n(t)| < b_n - x/b_n\Big\}$$
$$\le \mathbf{P}\big\{\hat{X}_n(t) < \sigma_n(t)^{-1}(b_n - x/b_n) \ \text{ for any } t \in A(n)\big\}.$$

Let $k = [n^{d_1}]$, $0 < d < 1$, $l = [n^{d_2}]$, $0 < d_2 < d_1$; we specify the constants d_1 and d_2 later on.

Denote

$$\Delta_j = [t_{j(k+l),n}, t_{j(k+l)+k,n}] = \bigcup_{i=j(k+l)}^{j(k+l)+k-1} [t_{i,n}, t_{i+1,n}], \qquad j = 0, 1, 2, \ldots$$

and introduce independent copies $Y_{n,j}(t)$, $t \in \Delta_j$, of $\hat{X}_n(t)$, $t \in \Delta_j$, $j \ge 0$, respectively. Let Y be a Gaussian standard variable independent of $Y_{n,j}$, $j \ge 0$. Consider the Gaussian process

$$Z_n(t) = \sqrt{1 - \chi(l)}Y_{n,j}(t) + \sqrt{\chi(l)}Y, \qquad t \in \Delta_j, j \ge 0,$$

where $\chi(l) = L/\log(lh_1^*) \in (0, 1)$ for large enough n. Note that $\chi(l)$ also depends on n. We have

$$\rho_{n,Z}(t, s) = \mathbf{E}Z_n(t)Z_n(s) \ge r_n(t, s).$$

Indeed, for t and s belonging to different Δ_j, we have

$$\rho_{n,Z}(t, s) = \chi(l) \ge r_n(t, s).$$

For t and s from the same Δ_j,

$$\rho_{n,Z}(t, s) = \big(1 - \chi(l)\big)r_n(t, s) + \chi(l) = r_n(t, s) + \chi(l)\big(1 - r_n(t, s)\big) \ge r_n(t, s).$$

Using this and Slepian's inequality, we get for any $x > 0$,

$$\mathbf{P}\bigg\{\hat{X}_n(t) < \sigma_n(t)^{-1}\bigg(b_n - \frac{x}{b_n}\bigg) \text{ for any } t \in A(n)\bigg\}$$
$$\le \mathbf{P}\bigg\{\hat{X}_n(t) < \sigma_n(t)^{-1}\bigg(b_n - \frac{x}{b_n}\bigg) \text{ for any } t \in A(n) \cap \bigg(\bigcup_j \Delta_j\bigg)\bigg\}$$
$$\le \mathbf{P}\bigg\{Z_n(t) < \sigma_n(t)^{-1}\bigg(b_n - \frac{x}{b_n}\bigg) \text{ for any } t \in A(n) \cap \bigg(\bigcup_j \Delta_j\bigg)\bigg\}$$

$$= \int \prod_j \mathbf{P}\Big\{ Y_{n,j}(t) < \frac{\sigma_n(t)^{-1}(b_n - x/b_n) - v\sqrt{\chi(l)}}{\sqrt{1-\chi(l)}} \quad \text{for any } t \in A(n) \cap \Delta_j \Big\} \varphi(v)\, dv$$

$$\begin{aligned}(30) \quad &\leq \int_{-\infty}^{-x\sqrt{d_2/L}/2+4\sqrt{L/d_2}} \varphi(v)\, dv \\ &\quad + \int_{-x\sqrt{d_2/L}/2+4\sqrt{L/d_2}}^{+\infty} \\ &\qquad \times \prod_j \mathbf{P}\Big\{ X_n(t) < b_n - x/(2b_n) \\ &\qquad\qquad + \Big[-b_n + x/(2b_n) + \frac{b_n - x/b_n - v\sigma_n(t)\sqrt{\chi(l)}}{\sqrt{1-\chi(l)}} \Big] \\ &\qquad\qquad \text{for any } t \in A(n) \cap \Delta_j \Big\} \varphi(v)\, dv,\end{aligned}$$

where in the case $A(n) \cap \Delta_j = \varnothing$ the corresponding probability is assumed to be equal to 1, and n is sufficiently large.

Now we prove that as far as

$$v \geq -\frac{x}{2}\sqrt{\frac{d_2}{L}} + 4\sqrt{\frac{L}{d_2}},$$

the term in the last square brackets is nonpositive for any t and j. By algebraic manipulations, this holds if and only if

$$(31) \qquad v \geq -x \frac{1}{\sigma_n(t)\sqrt{\chi(l)}b_n}\left(1 - \frac{1}{2}\sqrt{1-\chi(l)}\right) + \frac{b_n(1-\sqrt{1-\chi(l)})}{\sigma_n(t)\sqrt{\chi(l)}}.$$

Recall that we consider positive x. For all sufficiently large n, the coefficient of $-x$ is bounded from below by some positive d_3 because $\chi(l) = L/\log(h_1^*[n^{d_2}])$ and by Proposition 1 we have $b_n^2 \leq 1.1 \cdot (2\log n)$ for all sufficiently large n. The second term in the right-hand side of (31) is bounded from above by

$$b_4\sqrt{\log n} \cdot \sqrt{L/\log(h_1^*[n^e])},$$

for some positive d_4, because $\sigma_n(t) \geq 1/2$ and

$$1 - \sqrt{1-\chi(l)} = \frac{\chi(l)}{1+\sqrt{1-\chi(l)}}.$$

Thus (30) is bounded by

$$(32) \qquad \Phi\left(-\frac{x}{2}\sqrt{\frac{d_2}{L}} + 4\sqrt{\frac{L}{d_2}}\right) + \prod_j \mathbf{P}\Big\{ \max_{t \in A(n)\cap\Delta_j} X_T(t) < b_n - x/2b_n \Big\}.$$

The first term here is integrable on $[0,\infty)$ and independent of n, so it may serve as a dominating function. For the second one in (32), we have

$$
\begin{aligned}
&\prod_j \mathbf{P}\Big\{\max_{t\in A(n)\cap\Delta_j} X_n(t) \le b_n - x/(2b_n)\Big\}\\
&\qquad = \exp\Big\{\sum_j \log\Big(1-\mathbf{P}\Big\{\max_{t\in A(n)\cap\Delta_j} X_n(t) > b_n - x/(2b_n)\Big\}\Big)\Big\}\\
&\qquad \le \exp\Big\{-\sum_j \mathbf{P}\Big\{\max_{t\in A(n)\cap\Delta_j} X_n(t) > b_n - x/(2b_n)\Big\}\Big\},
\end{aligned}\tag{33}
$$

because $\log(1-x) \le -x$ for all $x \in [0,1)$. Recall that the corresponding probability is assumed to be zero in the case $A(n)\cap\Delta_j = \varnothing$.

From now on we consider only nonempty sets $A(n)\cap\Delta_j$. We are in a position to derive a lower bound for the probability

$$
\mathbf{P}\Big\{\max_{t\in A(n)\cap\Delta_j} X_n(t) > b_n - x/(2b_n)\Big\}.\tag{34}
$$

We consider the probability for the case $j=0$. The derivations for $j>0$ follow the same steps.

4.3.3. *Estimation for the first interval.* Suppose now that $x \in (0, Bb_n^2]$, $0 < B < 1.9$, selecting the value B below. Note that

$$
b_n - x/(2b_n) > b_n - B\frac{b_n^2}{2b_n} = b_n(1-B/2) \to \infty
$$

as $n\to\infty$. Below we write simply t_i instead of $t_{i,n}$. By Bonferroni's inequality we have

$$
\begin{aligned}
&\mathbf{P}\Big\{\max_{t\in A(n)\cap\Delta_0} X_n(t) > b_n - x/(2b_n)\Big\}\\
&\qquad = \mathbf{P}\Big\{\max_{t\in A(n)\cap[0,t_k]} X_n(t) > b_n - x/(2b_n)\Big\}\\
&\qquad \ge \sum_{i=0}^{k-1} \mathbf{P}\Big\{\max_{t\in A(n)\cap[t_i,t_{i+1}]} X_n(t) > b_n - x/(2b_n)\Big\}\\
&\qquad\quad - \sum_{i=0}^{k-2} \mathbf{P}\Big\{\max_{t\in A(n)\cap[t_i,t_{i+1}]} X_n(t) > b_n - x/(2b_n),\\
&\qquad\qquad\qquad \max_{t\in A(n)\cap[t_{i+1},t_{i+2}]} X_n(t) > b_n - x/(2b_n)\Big\}
\end{aligned}\tag{35}
$$

$$- \sum_{l,j\,:\,l>j+2} \mathbf{P}\Big\{ \max_{t\in A(n)\cap[t_i,t_{i+1}]} X_n(t) > b_n - x/(2b_n),$$
$$\max_{t\in A(n)\cap[t_l,t_{l+1}]} X_n(t) > b_n - x/(2b_n) \Big\}.$$

Consider now the second sum in the right-hand part of (35). Similarly to (20), from C1, C2 [directly, without (19)], we get for the ith term

$$\begin{aligned}
&\mathbf{P}\Big\{ \max_{t\in A(n)\cap[t_i,t_{i+1}]} X_n(t) > b_n - x/(2b_n), \max_{t\in A(n)\cap[t_{i+1},t_{i+2}]} X_n(t) > b_n - x/(2b_n) \Big\} \\
&\quad = \mathbf{P}\Big\{ \max_{t\in A(n)\cap J_i^\delta} X_n(t) > b_n - x/(2b_n), \max_{t\in A(n)\cap J_{i+1}^\delta} X_n(t) > b_n - x/(2b_n) \Big\} \qquad (36)\\
&\qquad + \big(O\big(g_2(\delta) + g_1(\gamma, S)\big) + o(1)\big)\big(\mu_a(u_{i,n}) + \mu_a(u_{i+1,n})\big),
\end{aligned}$$

where now

$$u_{i,n} = \sigma_{i,n}^{-1}\big(b_n - x/(2b_n)\big).$$

For the first term in the right-hand part of (36) we have

$$\begin{aligned}
&\mathbf{P}\Big\{ \max_{t\in A(n)\cap J_i^\delta} X_n(t) > b_n - x/(2b_n), \max_{t\in A(n)\cap J_{i+1}^\delta} X_n(t) > b_n - x/(2b_n) \Big\} \\
&\quad \le \mathbf{P}\Big\{ \max_{(t,s)\in A(n)\cap J_i^\delta \times J_{i+1}^\delta} \big(\hat{X}_n(t) + \hat{X}_n(s)\big) > (\sigma_{i,n}^{-1} + \sigma_{i+1,n}^{-1})\big(b_n - x/(2b_n)\big) \Big\} \\
&\quad \le \mathbf{P}\Big\{ \max_{(t,s)\in A(n)\cap J_i^\delta \times J_{i+1}^\delta} \big(\hat{X}_n(t) + \hat{X}_n(s)\big) > 2b_n - \frac{x}{b_n} \Big\}.
\end{aligned}$$

The variance of the Gaussian field $\hat{X}_n(t) + \hat{X}_n(s)$ in the last probability does not exceed $2 + 2\rho(2\delta)$, thus this probability, by Fernique's inequality, is bounded for some C and ρ_1, $1 > \rho_1 > \rho(2\delta)$, from above by

$$C \exp\Big\{ -\frac{(b_n - x/(2b_n))^2}{1 + \rho_1} \Big\}.$$

Notice that from C1, symmetry of Gaussian distributions, and continuity of $X_n(t)$ it follows that

$$\mathbf{P}\{M_n(J_{k,n}) > u\} = 2\mathbf{P}\Big\{ \max_{t\in J_{k,n}} X_n(t) > u \Big\}(1 + \epsilon'_{k,n}),$$

with the same properties of $\epsilon'_{k,n}$. Therefore, summing, we have for the second sum in the right-hand part of (35),

$$\begin{aligned}
&\sum_{i=0}^{k-2} \mathbf{P}\biggl\{\max_{t\in A(n)\cap[t_i,t_{i+1}]} X_n(t) > b_n - x/(2b_n), \\
&\qquad\qquad \max_{t\in A(n)\cap[t_{i+1},t_{i+2}]} X_n(t) > b_n - x/(2b_n)\biggr\} \\
&\qquad \le C g_2(\delta) \sum_{i=0}^{k-1} \mathbf{P}\biggl\{\max_{t\in A(n)\cap[t_i,t_{i+1}]} X_n(t) > b_n - x/(2b_n)\biggr\} \\
&\qquad\quad + C(k-1)\exp\biggl\{-\frac{(b_n - x/(2b_n))^2}{1+\rho_1}\biggr\}.
\end{aligned} \tag{37}$$

The terms of the third sum in the right-hand side of (35) can be bounded in a similar way directly, without the passage to J_i^δ, for some C and the same ρ_1 by

$$\begin{aligned}
&\mathbf{P}\biggl\{\max_{t\in A(n)\cap[t_i,t_{i+1}]} X_n(t) > b_n - x/(2b_n), \max_{t\in A(n)\cap[t_l,t_{l+1}]} X_n(t) > b_n - x/(2b_n)\biggr\} \\
&\qquad \le C\exp\biggl\{-\frac{(b_n - x/(2b_n))^2}{1+\rho_1}\biggr\}.
\end{aligned}$$

Note that since $\delta < h_1^*$, we have $\rho(h_1^*) \le \rho(\delta)$. So, for sufficiently small δ,

$$\begin{aligned}
&\mathbf{P}\biggl\{\max_{t\in A(n)\cap\Delta_0} X_n(t) > b_n - x/(2b_n)\biggr\} \\
&\qquad \ge \frac{1}{2}\sum_{i=0}^{k-1} \mathbf{P}\biggl\{\max_{t\in A(n)\cap[t_i,t_{i+1}]} X_n(t) > b_n - x/(2b_n)\biggr\} \\
&\qquad\quad - C_2 k^2 \exp\biggl\{-\frac{(b_n - x/(2b_n))^2}{2(1+\rho_1)}\biggr\}.
\end{aligned} \tag{38}$$

Using C1, we have for sufficiently large n,

$$\begin{aligned}
&\mathbf{P}\biggl\{\max_{t\in[t_i,t_{i+1}]\cap A(n)} X_n(t) > b_n - x/(2b_n)\biggr\} \\
&\qquad \ge \frac{1}{2}\frac{c_{i,n}}{\sigma_{i,n}^a}\bigl(b_n - x/(2b_n)\bigr)^a \exp\biggl\{-\frac{1}{2}\frac{(b_n - x/(2b_n))^2}{\sigma_{i,n}^2}\biggr\} \\
&\qquad \ge \frac{1}{2}\frac{c_{i,n}}{\sigma_{i,n}^a}\bigl(b_n - x/(2b_n)\bigr)^a \exp\biggl\{-\frac{1}{2}\frac{(b_n - x/(2b_n))^2}{\sigma^2(i/n) - \epsilon_n/\log n}\biggr\},
\end{aligned}$$

where the sequence $\epsilon_n \to 0$ as $n \to \infty$ and does not depend on i. Continuing, the last term is larger than

$$\frac{1}{2}\frac{c_{i,n}}{\sigma_{i,n}^a}(b_n - x/(2b_n))^a \exp\left\{-\frac{1}{2}\frac{(b_n - x/(2b_n))^2}{\sigma^2(i/n)}\left(1 + \frac{\epsilon_n}{\sigma^2(i/n)\log n}\right)\right\}$$
$$\geq \frac{1}{2}\frac{c_{i,n}}{\sigma_{i,n}^a}(b_n - x/(2b_n))^a \exp\left\{-\frac{1}{2}\frac{(b_n - x/(2b_n))^2}{\sigma^2(i/n)} - \frac{b_n^2\epsilon_n(1 - x/(2b_n^2))^2}{2\sigma^4(i/n)\log n}\right\}$$
$$\geq \frac{1}{2}\frac{c_{i,n}}{\sigma_{i,n}^a}(b_n - x/(2b_n))^a \exp\left\{-\frac{(b_n - x/(2b_n))^2}{2\sigma^2(i/n)}\right\}\exp\{-\epsilon_n'(1 - x/(2b_n^2))^2\},$$

where $\epsilon_n' = 8b_n^2\epsilon_n/\log n \to 0$ as $n \to \infty$. Recall that $\sigma(t) \geq 1/2$ on $A(n)$.

Now we return to $[t_i, t_{i+1}]$ from $[t_i, t_{i+1}] \cap A(n)$. We have

$$\mathbf{P}\left\{\max_{t\in[t_i,t_{i+1}]\cap A(n)} X_n(t) > b_n - x/(2b_n)\right\}$$
$$= \mathbf{P}\left\{\max_{t\in[t_i,t_{i+1}]} X_n(t) > b_n - x/(2b_n)\right\}$$
$$- \mathbf{P}\left\{\max_{t\in[t_i,t_{i+1}]\cap A(n)} X_n(t) \leq b_n - x/(2b_n),\right.$$
$$\left.\max_{t\in[t_i,t_{i+1}]\cap A^C(n)} X_n(t) > b_n - x/(2b_n)\right\},$$

where $A^C(n) = \mathbb{R} \setminus A(n) = \{t : \sigma_n(t) \leq 1/2\}$. Using Fernique's inequality, we get for arbitrary $\varepsilon > 0$ and some constant C_3, that the last probability does not exceed

$$\mathbf{P}\left\{\max_{t\in[t_i,t_{i+1}]\cap A^C(n)} X_n(t) > b_n - x/(2b_n)\right\}$$
$$\leq C_3 \exp\left\{-\frac{(b_n - x/(2b_n))^2}{2(1/2 + \varepsilon)}\right\}.$$

This bound is uniform since $h_2^* \geq t_{i+1} - t_i \geq h_1^*$. Choosing $2\varepsilon = \rho_1/2$, hence from (38),

$$\mathbf{P}\left\{\max_{t\in A(n)\cap\Delta_0} X_n(t) > b_n - x/(2b_n)\right\}$$
$$\geq \frac{1}{2}\sum_{i=0}^{k-1}\mathbf{P}\left\{\max_{t\in[t_i,t_{i+1}]} X_n(t) > b_n - x/(2b_n)\right\}$$
$$- C_3k^2 \exp\left\{-\frac{(b_n - x/(2b_n))^2}{1 + \rho_1}\right\}.$$

Thus for all sufficiently large n,

$$\begin{aligned}
&\mathbf{P}\Big\{\max_{t\in\Delta_0} X_n(t) > b_n - x/(2b_n)\Big\}\\
&\quad\geq \frac{1}{2}\big(b_n - x/(2b_n)\big)^a \exp\{-\epsilon_n'\big(1 - x/(2b_n^2)\big)^2\}\\
&\qquad\times \sum_{i=0}^{k-1} \frac{c_{i,n}}{\sigma_{i,n}^a} \exp\bigg\{-\frac{(b_n - x/(2b_n))^2}{2\sigma^2(i/n)}\bigg\}\\
&\qquad - C_3 k^2 \exp\bigg\{-\frac{(b_n - x/2b_n^2)}{1+\rho_1}\bigg\}\\
&\quad= \frac{1}{2}\big(b_n - x/(2b_n)\big)^a \exp\{-\epsilon_n'\big(1 - x/(2b_n^2)\big)^2\}\\
&\qquad\times n\int_0^{(k-1)/n} \big(c([nt]/n) + o(1)\big) \exp\bigg\{-\frac{(b_n - x/(2b_n))^2}{2\sigma^2([nt]/n)}\bigg\}\,dt\\
&\qquad - C_3 k^2 \exp\bigg\{-\frac{(b_n - x/(2b_n))^2}{1+\rho_1}\bigg\}\\
&\quad\overset{\text{C5}}{\geq} \frac{1}{3}(1 - B/2)^a \exp\{-\epsilon_n'\} n \int_0^{(k-1)/n} c(t) \exp\bigg\{-\frac{(b_n - x/(2b_n))^2}{2\sigma^2(t)}\bigg\}\,dt\\
&\qquad - C_3 k^2 \exp\bigg\{-\frac{b_n^2(1 - B/2)^2}{1+\rho_1}\bigg\},
\end{aligned}$$

because from C5,

$$|\sigma(t) - \sigma([nt]/n)| = o(1/\log n), \tag{39}$$

and $b_n^2 o(1/\log n) \to 0$ as $n \to \infty$, by Proposition 1. Continuing with the lower approximation, we have

$$\begin{aligned}
&\geq \frac{1}{3} b_n^a (1 - B/2)^a e^{-\epsilon_n'} n \int_0^{(k+l)/n} c(t) \exp\bigg\{-\frac{(b_n - x/(2b_n))^2}{2\sigma^2(t)}\bigg\}\,dt\\
&\quad - \frac{1}{3} b_n^a (1 - B/2)^a e^{-\epsilon_n'} n \int_{(k-1)/n}^{(k+l)/n} c(t) \exp\bigg\{-\frac{(b_n - x/(2b_n))^2}{2\sigma^2(t)}\bigg\}\,dt\\
&\quad - C_3 k^2 \exp\bigg\{-\frac{b_n^2(1 - B/2)^2}{1+\rho_1}\bigg\}\\
&\geq c' b_n^a n \int_0^{(k+l)/n} c(t) \exp\bigg\{-\frac{(b_n - x/(2b_n))^2}{2\sigma^2(t)}\bigg\}\,dt\\
&\quad - c'' b_n^a n \frac{l+1}{n} \exp\bigg\{-\frac{1}{2} b_n^2 (1 - B/2)^2\bigg\} - C_3 k^2 \exp\bigg\{-\frac{b_n^2(1 - B/2)^2}{1+\rho_1}\bigg\}
\end{aligned}$$

$$= c' b_n^a n \int_0^{(k+l)/n} c(t) \exp\left\{-\frac{(b_n - x/(2b_n))^2}{2\sigma^2(t)}\right\} dt$$
$$- c'' b_n^a l \exp\left\{-\frac{1}{2} b_n^2 (1 - B/2)^2\right\} - C_3 k^2 \exp\left\{-\frac{b_n^2(1 - B/2)^2}{1 + \rho_1}\right\},$$

where c' and c'' are constants. Thus we have for general Δ_j,

$$\mathbf{P}\left\{\max_{t \in \Delta_j} X_n(t) > b_n - x/(2b_n)\right\}$$
$$\geq c' b_n^a n \int_{(k+l)j/n}^{((k+l)j+k+l)/n} c(t) \exp\left\{-\frac{(b_n - x/(2b_n))^2}{2\sigma^2(t)}\right\} dt$$
$$- c'' b_n^a l \exp\left\{-\frac{1}{2} b_n^2 (1 - B/2)^2\right\} - C_3 k^2 \exp\left\{-\frac{b_n^2(1 - B/2)^2}{1 + \rho_1}\right\}.$$

Using the above bounds, we approximate now (34) as

$$\sum_j \mathbf{P}\left(\max_{t \in A(n) \cap \Delta_j} X_n(t) > b_n - x/(2b_n)\right)$$
$$\geq c' b_n^a n \int_0^1 c(t) \exp\left\{-\frac{(b_n - x/(2b_n))^2}{2\sigma^2(t)}\right\} dt$$
$$- c'' b_n^a \frac{nl}{k+l} \exp\left\{-\frac{1}{2} b_n^2 (1 - B/2)^2\right\}$$
$$- C_3 k^2 \frac{n}{k+l} \exp\left\{-\frac{b_n^2(1 - B/2)^2}{1 + \rho_1}\right\}$$
$$= c' b_n^a n \int_0^1 c(t) \exp\left\{-\frac{b_n^2}{2\sigma^2(t)} + \frac{x}{2\sigma^2(t)} - \frac{x^2}{8 b_n^2 \sigma^2(t)}\right\} dt \tag{40}$$
$$- c'' b_n^a n^{1-d_1+d_2} \exp\left\{-\frac{1}{2} b_n^2 (1 - B/2)^2\right\}$$
$$- C_3 n^{1+d_1} \exp\left\{-\frac{b_n^2(1 - B/2)^2}{1 + \rho_1}\right\}$$
$$\geq c' e^{-B^2/8} e^{x/2} \left[b_n^a n \int_0^1 c(t) \exp\{-b_n^2/2\sigma^2(t)\} dt\right]$$
$$- c'' b_n^a n^{1-d_1+d_2} \exp\left\{-(1-\varepsilon)(1 - B/2)^2 \log n\right\}$$
$$- C_3 n^{1+d_1} \exp\left\{-2(1-\varepsilon)\frac{(1 - B/2)^2}{1 + \rho_1} \log n\right\},$$

where the last inequality holds by C1 and C3 with $A = [0, 1]$, for arbitrarily small positive ε and hence for sufficiently large n. Choose now B, $0 < B < 2$, d_1 and d_2

with respect to the following restrictions:

$$
\begin{aligned}
&(1-B/2)^2 > \frac{1}{2}(1+\rho_1), \qquad d_1 - d_2 > 1-(1-B/2)^2, \\
&d_1 < \frac{2(1-B/2)^2}{1+\rho_1} - 1 < 1
\end{aligned} \tag{41}
$$

and choose κ sufficiently small. Thus from the last chain of inequalities we have for some positive c_1 and c_2 and all sufficiently large n,

$$
\sum_j \mathbf{P}\Big\{\max_{t\in A(n)\cap\Delta_j} X_n(t) > b_n - x/(2b_n)\Big\} \ge c_1 e^{x/2} - c_2, \tag{42}
$$

and therefore, taking into account (32) and (33),

$$
\begin{aligned}
&\mathbf{P}\Big\{\max_{t\in[0,n]} X_n(t) \le b_n - x/b_n\Big\} \\
&\quad \le 1 - \Phi\left(\frac{x}{2}\sqrt{\frac{d_2}{L}} - 4\sqrt{\frac{L}{d_2}}\right) + \exp(c_2 - c_1 e^{x/2})
\end{aligned} \tag{43}
$$

which is the desirable dominating function.

4.3.4. *Estimation for the second interval.* Now we consider the case $x \in [Bb_n^2, 1.9b_n^2)$. Let $J \subseteq [0,1]$ be an interval such that $\sigma(t) \ge 1/2$ for all $t \in J$. Denote its length by $|J| > 0$. It is obvious that for all sufficiently large n, $\sigma_{i,n} > 1/3$ provided $[t_i, t_{i+1}] \subset T(n)J$ (which means $[t_i/T(n), t_{i+1}/T(n)] \subset J$). The intervals with the last property we will call "good" intervals. The number n_1 of "good" intervals is of order n, as $n \to \infty$, that is, $n_1 \asymp n$. We deal only with the "good" intervals, and renumber them from 1 to n_1. For $i = 1, 2, \dots, n_1$, let τ_i be a maximum point of the variance $\sigma_n(t)$ in the "good" interval $[t_i, t_{i+1}]$. Let $l \le n_1$ with $l \asymp n^c$ and $k = [n_1/l]$, where we choose c, $1 > c > 0$ later. We have

$$
\begin{aligned}
&\mathbf{P}\Big\{\max_{t\in[0,T(n)]} X_n(t) < b_n - x/b_n\Big\} \\
&\qquad \le \mathbf{P}\Big\{\max_{t\in T(n)J} X_n(t) < b_n - x/b_n\Big\} \\
&\qquad \le \mathbf{P}\big\{\max\{X_n(\tau_0), X_n(\tau_k), X_n(\tau_{2k}), \dots, X_n(\tau_{(l-1)k})\} < b_n - x/b_n\big\} \\
&\qquad = \mathbf{P}\big\{\hat{X}_n(\tau_{ik}) < \sigma_{ik,n}^{-1}(b_n - x/b_n) \text{ for any } i = 0, \dots, l-1\big\}
\end{aligned}
$$

Let $Z, Z_0, Z_1, \dots$ be independent, identically distributed Gaussian standard random variables, we have by Slepian's inequality,

$$\begin{aligned}
&\mathbf{P}\{\hat{X}_n(\tau_{ik}) \le \sigma_{ik,n}^{-1}(b_n - x/b_n) \text{ for any } i = 0, \dots, l-1\} \\
&\quad \le \mathbf{P}\{\sqrt{\chi(k)}Z + \sqrt{1-\chi(k)}Z_i \le \sigma_{ik,n}^{-1}(b_n - x/b_n) \text{ for any } i = 0, \dots, l-1\} \\
&\quad \le \mathbf{P}\left\{\max\{Z_0, \dots, Z_{l-1}\} \le \frac{\sigma_{ik,n}^{-1}(b_n - x/b_n) - \sqrt{\chi(k)}Z}{\sqrt{1-\chi(k)}}\right\} \\
&\quad = \int \prod_{i=0}^{l-1} \mathbf{P}\left\{Z_i \le \frac{\sigma_{ik,n}^{-1}(b_n - x/b_n) - \sqrt{\chi(k)}v}{\sqrt{1-\chi(k)}}\right\}\phi(v)\,dv \\
&\quad \le \int_{-\infty}^{-ax+b} \varphi(v)\,dv \\
&\qquad + \int_{-ax+b}^{\infty} \prod_{i=0}^{l-1} \mathbf{P}\left\{Z_i \le \frac{\sigma_{ik,n}^{-1}(b_n - x/b_n) - \sqrt{\chi(k)}v}{\sqrt{1-\chi(k)}}\right\}\phi(v)\,dv \\
&\quad \le \Psi(ax - b) + \prod_{i=0}^{l-1} \mathbf{P}\left\{Z_i \le \frac{\sigma_{ik,n}^{-1}(b_n - x/b_n) - \sqrt{\chi(k)}(-ax+b)}{\sqrt{1-\chi(k)}}\right\},
\end{aligned}$$

where we choose a small a and a large b, to be fixed later, with $\Psi(x) = 1 - \Phi(x)$. The last term equals

$$\begin{aligned}
&\prod_{i=0}^{l-1} \mathbf{P}\Bigg\{Z_1 \le b_n - x/(2b_n) \\
&\qquad - x\left[-1/(2b_n) + \frac{1}{b_n\sigma_{ik,n}\sqrt{1-\chi(k)}} - \frac{a\sqrt{\chi(k)}}{\sqrt{1-\chi(k)}}\right] \\
&\qquad - \left[b_n - \frac{b_n}{\sigma_{ik,n}\sqrt{1-\chi(k)}} + \frac{b\sqrt{\chi(k)}}{\sqrt{1-\chi(k)}}\right]\Bigg\} \\
&\quad \le \{\mathbf{P}\{Z_1 \le b_n - x/(2b_n)\}\}^l,
\end{aligned} \tag{44}$$

where the last inequality holds provided both expressions in the square brackets are nonnegative. The first one is nonnegative if

$$-\frac{1}{2b_n} + \frac{1}{b_n\sqrt{1-\chi(k)}} - \frac{a\sqrt{\chi(k)}}{\sqrt{1-\chi(k)}} \ge 0$$

or

$$-\frac{1}{2} + \frac{1}{\sqrt{1-\chi(k)}} - \frac{ab_n\sqrt{\chi(k)}}{\sqrt{1-\chi(k)}} \ge 0.$$

This is true for sufficiently large n provided

$$\frac{ab_n\sqrt{\chi(k)}}{\sqrt{1-\chi(k)}} < \frac{1}{2},$$

which in turn is valid for sufficiently small positive a, since $b_n^2 \asymp \log T$ and $\chi(k) = O(1/\log n) = O(1/\log T)$. The second expression in the square brackets is nonnegative if

$$b_n - \frac{3b_n}{\sqrt{1-\chi(k)}} + \frac{b\sqrt{\chi(k)}}{\sqrt{1-\chi(k)}} \ge 0 \quad \text{or} \quad \sqrt{1-\chi(k)} - 3 + \frac{b\sqrt{\chi(k)}}{b_n} \ge 0,$$

which is true for sufficiently large b. Recall that we consider "good" intervals.

We continue the approximation of (44). Since for large n, $b_n - x/(2b_n) \ge 0.05\, b_n$, we have,

$$\begin{aligned}
&\prod_{i=0}^{l-1} \mathbf{P}\{Z_1 \le b_n - x/(2b_n)\} \\
&\qquad \le \exp\{-l\mathbf{P}\{Z_1 > b_n - x/(2b_n)\}\} \\
&\qquad \le \exp\left\{-\frac{l}{\sqrt{2\pi}\, b_n} \exp\left\{-\frac{1}{2}(b_n - x/(2b_n))^2\right\}\right\} \\
&\qquad \le \exp\left\{-\frac{l}{\sqrt{2\pi}\, b_n} \exp\left\{\left(-\frac{1}{2}b_n^2 + \frac{1}{4}x - \frac{x^2}{8b_n^2}\right) + \frac{1}{4}x\right\}\right\} \\
&\qquad \le \exp\left\{-\frac{l}{\sqrt{2\pi}\, b_n} \exp\left\{\min_{x\in[Bb_n^2, 1.9b_n^2]}\left(-\frac{1}{2}b_n^2 + \frac{1}{4}x - \frac{x^2}{8b_n^2}\right) + \frac{1}{4}x\right\}\right\} \\
&\qquad \le \exp\left\{-\frac{l}{\sqrt{2\pi}\, b_n} \exp\left\{-\frac{1}{2}c_1 b_n^2\right\} e^{x/4}\right\},
\end{aligned}$$

where $c_1 = \min\{1 - B/4, 0.9525\} \in (0,1)$. Now choose l as the root of the equation

$$\frac{l}{\sqrt{2\pi}\, b_n} e^{-c_1 b_n^2/2} = 1,$$

that is, $l \sim n^{c_1} \asymp n_1^{c_1} \le n_1$ (it means that $k \asymp n^{1-c_1}$), and we have the desired domination on the interval $[Bb_n^2, 1.9b_n^2)$.

4.3.5. *Estimation for the third interval.* Finally, consider the case $x \ge 1.9b_n^2$. Since $\sigma(t) \le 1$, we have

$$\mathbf{P}\left\{\max_{t\in[0,T(n)]} X_n(t) \le b_n - x/b_n\right\} \le \mathbf{P}\{Y \le b_n - x/b_n\},$$

where Y is a standard Gaussian variable. Since $b_n - x/b_n \le -0.9b_n$,

$$\begin{aligned}
\mathbf{P}\{Y \le b_n - x/b_n\} &\le \frac{1}{\sqrt{2\pi}0.9b_n} \exp\left\{-\frac{1}{2}(b_n - x/b_n)^2\right\} \\
&\le \exp\left\{-\frac{1}{2}b_n^2 + x - x^2/2b_n^2\right\}
\end{aligned}$$

for sufficiently large n. In this case we use the inequality

$$-\frac{1}{2}b_n^2 + x - \frac{x^2}{2b_n^2} \le -0.1x. \tag{45}$$

Computing the roots of the parabola, we derive that (45) holds if $x \ge (1.1 + \sqrt{0.21})b_n^2$ and since $1.1 + \sqrt{0.21} \le 1.9$, we have for all $x \ge 1.9b_n^2$,

$$\mathbf{P}\{Y \le b_n - x/b_n\} \le e^{-0.1x}.$$

Thus we have all dominations and Theorem 3 holds.

4.4. *Proof of Theorem* 4. We have

$$\big(\mathbf{E}M_n(T)^p\big)^{1/p} = \left(\mathbf{E}\left(\frac{(M_n(T)-b_n)b_n}{b_n} + b_n\right)^p\right)^{1/p}.$$

Denote $\xi_n = (M_n(T) - b_n)b_n$. We write

$$\big(\mathbf{E}M_n(T)^p\big)^{1/p} = b_n\big(\mathbf{E}(1+b_n^{-2}\xi_n)^p\big)^{1/p}.$$

For the function $f_n(x) = (\mathbf{E}(1+x\xi_n)^p)^{1/p}$ we have

$$f_n'(x) = \big(\mathbf{E}(1+x\xi_n)^p\big)^{-1+1/p}\mathbf{E}(1+x\xi_n)^{p-1}\xi_n, \qquad f_n'(0) = \mathbf{E}\xi_n.$$

Further,

$$\begin{aligned} f_n'(x) - f_n'(0) = \mathbf{E}(1+x\xi_n)^{p-1}\xi_n\big[\big(\mathbf{E}(1+x\xi_n)^p\big)^{-1+1/p} - 1\big] \\ + \mathbf{E}\xi_n\big[(1+x\xi_n)^{p-1} - 1\big]. \end{aligned} \tag{46}$$

By Theorem 3, the first expectation in the right-hand part is bounded uniformly in n. Apply the following consequences of the triangle inequality:

$$(1+Cx)^p \ge \mathbf{E}(1+x\xi_n)^p \ge (1-Cx)^p,$$

where $x > 0$ and $C = (\mathbf{E}|\xi_n|^p)^{1/p}$, to get for all sufficiently small x,

$$\big|\big(\mathbf{E}(1+x\xi_n)^p\big)^{-1+1/p} - 1\big| \le C(p+1)x.$$

To bound the second term in the right-hand part of (46), use the inequality

$$|(1+y)^{p-1} - 1| \le 2^p(|y| + |y|^p), \qquad p \ge 1, -1 \le y < \infty.$$

Since by Theorem 3 moments of ξ_n are uniformly bounded, it follows that for some constant K and for all sufficiently small x, $|f_n'(x) - f_n'(0)|$ is bounded by Kx. From Theorems 2 and 3 it follows that $m_n = f_n'(0) = \mathbf{E}\xi_n \to c_e$ as $n \to \infty$. Hence, by Taylor, with $x = b_n^{-2}$,

$$\begin{aligned} f_n(x) &= f_n(0) + xf_n'(0) + x\big(f_n'(\theta x) - f_n'(0)\big) \\ &= 1 + b_n^{-2}\big(c_e + o(1)\big) + O(b_n^{-4}), \qquad 0 \le \theta \le 1, \end{aligned}$$

which implies the assertion of Theorem 4.

4.5. *Proof of Theorem* 5. Recall that the normalized deviation process $X_n(t) = \delta_n(\tau)/\sigma_n$, $\tau \in [0, 1]$, $t \in [0, n]$, $t = k-1+(\tau - \tau_{k-1})/h_k$, $h_k = \tau_k - \tau_{k-1}$, $k = 1, \dots, n$, and points τ_k belong to an $RS(h)$ design $T_n = \{\tau_0 = 0 < \tau_1 < \cdots < \tau_n = 1\}$. We denote the variance, covariance, correlation and structure function of X_n by $\sigma_n^2(t) := \mathbf{E}X_n(t)^2$, $R_n(t, s)$, $r_n(t, s) := R_n(t, s)/(\sigma_n(t)\sigma_n(s))$, and $d_n(t, s) := \mathbf{E}(X_n(t) - X_n(s))^2$, $t, s \in [0, n]$, respectively. Let $s_n(t) := \sigma_n^2 \sigma_n^2(t)$. We shall prove that assumptions B1–B3 for the Gaussian process $X(\tau)$, $\tau \in [0, 1]$, imply assumptions A1–A3, C4, C5 and C6 for the sequence of Gaussian processes $X_n(t) = \delta_n(\tau)/\sigma_n$, $t \in [0, n]$. Due to the time transformation (12), here we have $J_{k,n} = [k-1, k]$, $k = 1, \dots, n$. Derivations for continuous ($m = 0$) and differentiable ($m = 1$) cases are, to some extent, similar, therefore we will consider the cases in parallel, within every interval $J_{k,n}$.

We begin by verifying the assumptions A1–A3 of local behavior of X_n. First we investigate the asymptotic behavior of the variance $\sigma_n^2(t)$. For $\delta_n(\tau) = X(\tau) - L_n(\tau)$, $\tau \in [0, 1]$, in the continuous case ($m = 0$) we use the explicit formula

$$\begin{aligned} \sigma_n X_n(t) &= \delta_n(\tau) = X(\tau) - X(\tau_{k-1})(1 - \bar{t}\,) - X(\tau_k)\bar{t} \\ &= \big(X(\tau) - X(\tau_{k-1})\big)(1 - \bar{t}\,) + \big(X(\tau) - X(\tau_k)\big)\bar{t}\,, \end{aligned} \tag{47}$$

which follows from (4), where $\bar{t} := t - (k-1) = (\tau - \tau_{k-1})/h_k$, $\tau \in [\tau_{k-1}, \tau_k]$, $k = 1, \dots, n$. In the differentiable case ($m = 1$), it is more convenient to use the *Peano kernel* representation, $\tau \in [\tau_{k-1}, \tau_k]$, $k = 1, \dots, n$,

$$\sigma_n X_n(t) = \delta_n(\tau) = h_k \int_0^1 X^{(1)}(\tau_k + h_k v) K(v, \bar{t}\,)\, dv, \tag{48}$$

where

$$K(s, t) := \mathbf{I}_{[0,t]}(s) - t, \qquad t, s \in [0, 1], \tag{49}$$

is the corresponding Peano kernel for the two-point piecewise linear interpolation on $[0, 1]$, $\mathbf{I}_{[a,b]}(\cdot)$ denotes the indicator function of the interval $[a, b]$. Equation (48) follows directly from the definition of quadratic mean derivative $X^{(1)}(\cdot)$. Observe that

$$\int_0^1 K(s, t)\, ds = 0, \qquad t \in [0, 1]. \tag{50}$$

Throughout the proof, the next property of $C(t)$ will be used:

$$\begin{aligned} &C(t+s) = C(t)\big(1 + \theta(t, s)\big), \\ &\qquad \theta(t, s) \log |s| = o(1) \text{ as } s \to 0 \text{ uniformly in } t \in [0, 1] \end{aligned} \tag{51}$$

which follows directly from uniform continuity and positiveness of $C(t)$ and B1.

In order to ease the presentation, we introduce the following O-o notation which is similar to the standard one. Let $o_n(1)$ denote *any* sequence $\theta_{k,n}(t)$ such that

$$\max\big\{|\theta_{k,n}(t)|, t \in [k-1, k], k = 1, \dots, n\big\} \to 0 \qquad \text{as } n \to \infty \tag{52}$$

and $o_n(\delta_n) = \delta_n o_n(1)$ as usual. Let $I_{k,n}^{\delta} := \{v : |v - (t_{\max} + k - 1)| < \delta\}$, $\delta > 0$, where $t_{\max}$ denote the maximal point(s) of the function $S(t)$ (there are one or two maximum points) introduced in continuous and differentiable cases in (13) and (14), respectively. Denote by $o_{s,t}(1)$ *any* sequence $\theta_{k,n}(t,s)$ such that

$$\max\{|\theta_{k,n}(t,s)|, t, s \in I_{k,n}^{\delta}, k = 1, \dots, n, n \geq 1\} \to 0 \qquad \text{as } \delta \to 0 \tag{53}$$

(i.e., uniformly in all neighborhoods of variance maximal points). Uniformly bounded classes $O_n(1)$ and $O_{s,t}(1)$ are introduced in a similar way.

Consider first the continuous case, $m = 0$. Then q.m. deviation $s_n(t), t \in [k-1, k]$, can be represented as follows:

$$s_n(t) = d_X(\tau, \tau_{k-1})(1 - \bar{t}) + d_X(\tau, \tau_k)\bar{t} - d_X(\tau_{k-1}, \tau_k)\bar{t}(1 - \bar{t}), \tag{54}$$

where again $\bar{t} = t - (k-1) = (\tau - \tau_{k-1})/h_k$, $0 \leq \bar{t} \leq 1$, as above. From B1 we obtain

$$s_n(t) = C(\tau_k) h_k^{\alpha} s_0 S(\bar{t})\bigl(1 + o_n(1/\log n)\bigr), \tag{55}$$

where $S(v)$, $v \in [0,1]$, is defined by (13). Denote by $\beta = \beta(\alpha)$, where $\beta(\alpha_0) = 4$ and $\beta(\alpha) = 2$, otherwise. Then the function $S(v)$ can be represented by straightforward derivations near the maximum point(s) $t_{\max}$ in the following form:

$$S(v) = 1 - \bigl(\sigma_\alpha + \gamma(v)\bigr)|v - t_{\max}|^{\beta},$$

where $\gamma(v) \to 0$ as $v \to t_{\max}$, and

$$\sigma_\alpha = \begin{cases} \bigl(\alpha(3-\alpha)2^{1-\alpha} - 1\bigr)/s_0, & \text{if } \alpha > \alpha_0, \\ \alpha 2^{1-\alpha}(1-\alpha)(2-\alpha)(7-\alpha)/(3s_0), & \text{if } \alpha = \alpha_0, \\ -S''(t_{\max})/2, & \text{if } \alpha < \alpha_0. \end{cases}$$

In the differentiable case, $m = 1$, we introduce the following auxiliary random process, $Y_n(\bar{t}) = X^{(1)}(h_k \bar{t} + \tau_{k-1})$, $\bar{t} \in [0,1]$, $t \in [k-1,k]$, and write $d_{Y_n}(v,u) := \mathbf{E}(Y_n(v) - Y_n(u))^2$, $v, u \in [0,1]$. Then (48) and B1 imply [for details see Seleznjev (2000)] that q.m. deviation can be evaluated for $t \in [k-1,k]$ as

$$\begin{aligned} s_n(t) &= h_k^2/2 \int_0^1 \int_0^1 \bigl(d_{Y_n}(v,0) + d_{Y_n}(w,0) - d_{Y_n}(v,w)\bigr) K(v,\bar{t}) K(w,\bar{t})\, dv\, dw \\ &= C(\tau_k) h_k^{2+\alpha}/2 \biggl(\int_0^1 \int_0^1 (|v|^\alpha + |w|^\alpha - |v-w|^\alpha) \\ &\qquad\qquad \times K(v,\bar{t}) K(w,\bar{t})\, dv\, dw\biggr)\bigl(1 + o_n(1/\log n)\bigr) \\ &= s_0 C(\tau_k) h_k^{2+\alpha} S(\bar{t})\bigl(1 + o_n(1/\log n)\bigr), \end{aligned} \tag{56}$$

where $S(v)$ is defined by (14). It follows from (14) that $S(v) = S(1-v)$; $v \in [0,1]$, $\max_{[0,1]} S(v) = S(1/2) = 1$. The function $S(v)$ can be represented around the maximum point $t_{\max} = 1/2$ in the following form:

$$S(v) = 1 - \bigl(\sigma_\alpha + \gamma(v)\bigr)(v - t_{\max})^2,$$

where $\gamma(v) \to 0$ as $v \to t_{\max}$, and $\sigma_\alpha = (1 - (\alpha(\alpha+1) - 2)2^{-(\alpha+1)})/s_0$. The integral mean value theorem and the definition of the RS density $h(\cdot)$ imply that

$$h_k = 1/(h(w_k)n) \qquad \text{for some } w_k \in [\tau_{k-1}, \tau_k], k = 1, \dots, n. \tag{57}$$

Therefore, both for continuous and differentiable cases, we obtain the following representation:

$$s_n(t) = n^{-(2m+\alpha)} C(\tau) h(\tau)^{-(2m+\alpha)} s_0 S(\bar{t})\big(1 + o_n(1/\log n)\big) \qquad \text{as } n \to \infty$$

uniformly in $\tau \in [0, 1]$, $m = 0, 1$. Observe that the above functions of τ describe a global behavior of $s_n(t)$ whereas $S(\bar{t})$ describes the behavior of $s_n(t)$ within $J_{k,n}$. In particular,

$$\sigma_n = s_0^{1/2} \sigma_0 n^{-(m+\alpha/2)}\big(1 + o_n(1/\log n)\big) \sim s_0^{1/2} \sigma_0 n^{-(m+\alpha/2)} \qquad \text{as } n \to \infty.$$

Moreover,

$$\sigma_n^2(t) = \sigma_0^{-2} C(\tau_k) h(\tau_k)^{-(2m+\alpha)} S(\bar{t})\big(1 + o_n(1/\log n)\big), \qquad t \in [k-1, k],$$

and Theorem 5(i) follows. Hence in both continuous and differentiable cases, A1 holds for a one-periodic function $\sigma^\star(t)$, $t \geq 0$, which coincides with $\sqrt{S(t)}$ for $t \in [0, 1]$, where $S(t)$ and s_0 are defined above in Section 3 differently for the cases $m = 0, 1$. Furthermore, $\sigma_{k,n} = \sigma_0^{-1} C^{1/2}(\tau_k) h(\tau_k)^{-(m+\alpha/2)}$.

We turn now to the verification of A2. Represent the correlation function $r_n(t, s)$ in the following form:

$$r_n(t, s) = 1 - \frac{d_n(t, s)}{2\sigma_n(t)\sigma_n(s)} + \frac{(\sigma_n(t) - \sigma_n(s))^2}{2\sigma_n(t)\sigma_n(s)}, \qquad t, s \in [0, n], \tag{58}$$

Consider first the continuous case and let $\alpha_0 \leq \alpha < 2$, that is, $S(v)$, $v \in [0, 1]$, has the unique maximum point $t_{\max} = 1/2$. For the case $\alpha_0 > \alpha > 0$, arguments are similar. It follows directly from (54) and B2 that for all t, s in a δ-neighborhood of $t_{\max} + k - 1$, $t, s \in \{v : |v - k + 1/2| < \delta\}$; $\delta > 0$,

$$\begin{aligned} s_n(t) - s_n(s) &= h_k|t - s|\Big(\big|d_X^{(1,0)}\big(h_k(t + v_1) + \tau_{k-1}, \tau_{k-1}\big)\big| \\ &\qquad + \big|d_X^{(1,0)}\big(h_k(t + v_2) - \tau_k, \tau_k\big)\big|\Big) O_{s,t}(1) \\ &\quad + h_k^\alpha |t - s| O_{s,t}(1) \\ &= |t - s| h_k^\alpha O_{s,t}(1), \end{aligned} \tag{59}$$

where $0 \leq |v_1|, |v_2| \leq |t - s|$. Therefore, for points t, s near the maximum points $t_{\max} + k - 1$ with some positive C,

$$\big(\sigma_n(t) - \sigma_n(s)\big)^2 / \big(2\sigma_n(t)\sigma_n(s)\big) \leq C(t - s)^2.$$

Further, using (47) and B1 we obtain for $d_n(t,s)$, $\bar{t}-\bar{s}=t-s$,

$$\begin{aligned}\sigma_n^2 d_n(t,s) &= \mathbf{E}\big(X(h_k\bar{t}+\tau_{k-1})-X(h_k\bar{s}+\tau_{k-1})\\ &\qquad\qquad + (\bar{t}-\bar{s})\big(X(\tau_{k-1})-X(\tau_k)\big)\big)^2\\ &= d_X(h_k\bar{t}+\tau_{k-1},h_k\bar{s}+\tau_{k-1})+|t-s|^{(\alpha/2+1)}h_k^{(\alpha/2+1)}O_{s,t}(1)\\ &\quad + |t-s|^2h_k^\alpha O_{s,t}(1)\\ &= C(\tau_k)h_k^\alpha|t-s|^\alpha\big(1+o_{s,t}(1)\big).\end{aligned}\tag{60}$$

Finally, applying (59), (60) and the above estimates for $s_n(t)$, we obtain from (58) that

$$\begin{aligned}r_n(t,s) &= 1-\frac{C(\tau_k)h_k^\alpha|t-s|^\alpha(1+o_{s,t}(1))}{2\sigma_n^2\left(\sigma_n^2(t_{\max})+o_{s,t}(1)\right)}+(t-s)^2O_{s,t}(1)\\ &= 1-|t-s|^\alpha\big(b_{k,n}+o_{s,t}(1)\big),\end{aligned}\tag{61}$$

where $b_{k,n}={s_0}^{-1}/2+o(1)$ as $n\to\infty$. Thus A2 follows in the continuous case.

In the differentiable case, write first with $v=s-t$,

$$s_n(t+v)-s_n(t)=v\int_0^1 s_n'(t+vu)\,du.\tag{62}$$

Apply (56) and (49) for every k for the auxiliary process Y_n, to get for any t in a neighborhood of $t_{\max}+k-1=k-\frac{1}{2}$, that

$$\begin{aligned}s_n'(t) &= -h_k^2/2\bigg(\int_0^1\int_0^1 d_{Y_n}(v,w)K(u,\bar{t})K(w,\bar{t})\,du\,dw\bigg)'\\ &= h_k^2\bigg(\int_0^1\int_0^1\big(d_{Y_n}(u,w)-d_{Y_n}(\bar{t},w)\big)K(w,\bar{t})\,du\,dw\bigg)\\ &= s_0C(\tau_k)h_k^{2+\alpha}\big(S'(t)+o_n(1)\big).\end{aligned}\tag{63}$$

It follows from (62) and (63) with $S'(1/2)=0$ that

$$s_n(t)-s_n(s)=(t-s)h_k^{2+\alpha}o_{s,t}(1),\tag{64}$$

and therefore

$$\big(\sigma_n(t)-\sigma_n(s)\big)^2/\big(2\sigma_n(t)\sigma_n(s)\big)=(t-s)^2o_{s,t}(1).$$

For the second term in the right-hand side of (58), we have for any t, s, $v = s - t$ in a neighborhood of $t_{\max} + k - 1$,

$$
\begin{aligned}
d_n(t,s) &= h_k^2/2\Big(\int_0^1\int_0^1 d_{X_n}(u,w)\big(\mathbf{I}_{[\bar s,\bar t]}(u) - v\big)\big(\mathbf{I}_{[\bar s,\bar t]}(w) - v\big)\,du\,dw\Big)\\
&= v^2 h_k^2/2\Big(\int_0^1\int_0^1 \big(2d_{Y_n}(\bar t + vu, w) - d_{Y_n}(u,w)\\
&\qquad\qquad - d_{Y_n}(\bar t + vu, \bar t + vw)\big)\,du\,dw\Big)\\
&= (t-s)^2 h_k^2/2\Big(\int_0^1\int_0^1 \big(2d_{Y_n}(\bar t, w) - d_{Y_n}(u,w)\big)\,du\,dw + o_{s,t}(1)\Big)\\
&= (t-s)^2 h_k^2/2\Big(\int_0^1\int_0^1 \big(2d_{Y_n}(1/2, w) - d_{Y_n}(u,w)\big)\,du\,dw + o_{s,t}(1)\Big)\\
&= (t-s)^2 h_k^{2+\alpha}\big(b_n + o_{s,t}(1)\big),
\end{aligned}
\tag{65}
$$

where $b_n \to b_\alpha = (2^{-\alpha} - 1/(\alpha+2))/(2(\alpha+1)) > 0$ as $n \to \infty$, $0 < \alpha < 2$. Hence, A2 holds for $X_n(t)$ in both cases where $m = 0, 1$.

In both cases where $m = 0, 1$, condition A3 follows from the definition of the normalized deviation process $X_n(t)$. For example, consider the case $m = 0$ and $k = 1$. Then there exists $K > 0$ such that for any $t, s \in [0, 1]$,

$$
\begin{aligned}
&\big(\mathbf{E}\big(X_n(t) - X_n(s)\big)^2\big)^{1/2}\\
&\quad\le \sigma_n^{-1}\Big(\big(\mathbf{E}\big(X(h_1\bar t) - X(h_1\bar s)\big)^2\big)^{1/2} + \big(\mathbf{E}\big(L_n(h_1\bar t) - L_n(h_1\bar s)\big)^2\big)^{1/2}\Big)\\
&\quad\le \sigma_n^{-1}\big(d_X(h_1\bar s, h_1\bar t)^{1/2} + |\bar t - \bar s| d_X(h_1, 0)^{1/2}\big) \le K|t-s|^{\alpha/2}.
\end{aligned}
\tag{66}
$$

A similar argument works for the differentiable case $m = 1$.

Now we turn to conditions C4–C6 of a global behavior of X_n. Verification of condition C4 repeats similar steps in Seleznjev (1996) [cf. (A4) and (A5), respectively]. Henceforth, the following properties of regular designs will be used, which follow directly from the definition,

$$\tau_j - \tau_i \asymp (j-i)/n, \qquad j > i, \tag{67}$$

and for every finite k_0,

$$\tau_j - \tau_i = (j-i)h_i\big(1 + o(1)\big) \qquad \text{as } n \to \infty, \tag{68}$$

uniformly in $|j - i| \le k_0 < \infty$. Write $R_n(t,s)$ in the following form, $t, s \in [0, n]$:

$$
\begin{aligned}
R_n(t,s)\sigma_n^2 = 1/2\mathbf{E}_{\bar t,\bar s}\big[&d_X\big(h_{k_1}\bar t + \tau_{k_1-1}, h_{k_2}\eta + \tau_{k_2-1}\big)\\
&+ d_X\big(h_{k_1}\xi + \tau_{k_1-1}, h_{k_2}\bar s + \tau_{k_2-1}\big)\\
&- d_X\big(h_{k_1}\xi + \tau_{k_1-1}, h_{k_2}\eta + \tau_{k_2-1}\big)\\
&- d_X\big(h_{k_1}\bar t + \tau_{k_1-1}, h_{k_2}\bar s + \tau_{k_2-1}\big)\big],
\end{aligned}
\tag{69}
$$

where $t = \bar{t} + k_1 - 1$, $s = \bar{s} + k_2 - 1$, and ξ, η are independent standard Bernoulli random variables with "success" probabilities $\mathbf{P}\{\xi = 1\} = \bar{t}$ and $\mathbf{P}\{\eta = 1\} = \bar{s}$, respectively, and $\mathbf{E}_{\bar{t},\bar{s}}$ is the corresponding mathematical expectation. In the differentiable case, (48), (50), (68) and B1 imply

$$\begin{aligned} R_n(t,s)\sigma_n^2 &= h_{k_1}h_{k_2}/2 \int_0^1\int_0^1 \big[d_{X^{(1)}}(h_{k_1}v + \tau_{k_1-1}, \tau_{k_2-1}) \\ &\qquad + d_{X^{(1)}}(\tau_{k_1-1}, h_{k_2}w + \tau_{k_2-1}) \\ &\qquad - d_{X^{(1)}}(\tau_{k_1-1}, \tau_{k_2-1}) \\ &\qquad - d_{X^{(1)}}(h_{k_1}v + \tau_{k_1-1}, h_{k_2}w + \tau_{k_2-1})\big] \\ &\qquad \times K(v,\bar{t})K(w,\bar{s})\,dv\,dw \\ &= C(\tau_{k_1})h_{k_1}^{2+\alpha}/2\bigg(\int_0^1\int_0^1 \big(|v+k_1-k_2|^\alpha + |w+k_2-k_1|^\alpha \\ &\qquad - |v-w+k_1-k_2|^\alpha - |k_1-k_2|^\alpha\big) \\ &\qquad \times K(v,\bar{t})K(w,\bar{s})\,dv\,dw\bigg) \\ &\quad \times \big(1+o(1)\big) \end{aligned} \tag{70}$$

as $n \to \infty$, uniformly in $\bar{t}, \bar{s} \in [0,1]$, $|k_1 - k_2| \le k_0 < \infty$. Note that, for any $k_2 \ge k_1$,

$$\begin{aligned} &-\int_0^1\int_0^1 |v-w+k_2-k_1|^\alpha K(v,\bar{t})K(w,\bar{s})\,dv\,dw \\ &\quad = c_{1,\alpha}\mathbf{E}_{\bar{t},\bar{s}}\big[|t-\eta-k_2+1|^{2+\alpha} + |\xi+k_1-1-s|^{2+\alpha} \\ &\qquad - |\xi+k_1-\eta-k_2|^{2+\alpha} - |t-s|^{2+\alpha}\big], \end{aligned} \tag{71}$$

where $c_{1,\alpha} = 1/((\alpha+1)(\alpha+2))$. Denote $c_{0,\alpha} = 1/2$. B1, (69) and (70) imply

$$\begin{aligned} &C(\tau_{k_1})^{-1}h_{k_1}^{-(2m+\alpha)}R_n(t,s) \\ &\quad \to R(t,s) \\ &\quad = c_{m,\alpha}\mathbf{E}_{\bar{t},\bar{s}}\big[|t-\eta-k_2+1|^{2m+\alpha} + |\xi+k_1-1-s|^{2m+\alpha} \\ &\qquad - |\xi+k_1-\eta-k_2|^{2m+\alpha} - |t-s|^{2m+\alpha}\big] \end{aligned}$$

as $n \to \infty$, uniformly in $\bar{t}, \bar{s} \in [0,1]$, $|k_1 - k_2| \le k_0 < \infty$, $m = 0, 1$. Denote by $Z(t)$, $t \in [0, k_2]$, the Gaussian process with mean zero, covariance and correlation functions $R(t,s)$ and $r(t,s)$, respectively. For the continuous case, $Z(t)$ can be represented as $t = \bar{t} + k - 1$,

$$Z(t) = c_{0,\alpha}^{1/2}\big(B_\alpha(t) - \mathbf{E}_{\bar{t}}B_\alpha(\xi + k - 1)\big),$$

and for the differentiable case, (70) implies

$$Z(t) = c_{1,\alpha}^{1/2} \int_0^1 B_\alpha(v+k-1) K(v, \bar{t}\,)\, dv,$$

with the fractional Brownian motion $B_\alpha(t)$. The process $Z(t)$ is nondegenerate [see Seleznjev (1996), for the continuous case, and Piterbarg and Seleznjev (1994), for the differentiable case, respectively]. It means that for any $\varepsilon > 0$ there exists $\delta_1 > 0$ such that for the correlation function $r(t,s)$ we obtain

$$|r(t,s)| < \delta_1 < 1 \qquad \text{uniformly for } |t-s| > \varepsilon.$$

This implies for sufficiently large n, for any positive K and $\varepsilon > 0$ that there exists δ_2 such that $|r_n(t,s)| < \delta_2 < 1$ uniformly for $K > t-s| > \varepsilon$. Furthermore, in both cases, $m = 0, 1$, we have

$$r_n(t,s)|t-s|^{2-\alpha} = o(1) \qquad \text{as } t-s \to \infty$$

uniformly in n. In the case $m = 0$, (69), B2 and (67) imply that

$$\begin{aligned}
&|R_n(t,s)| \\
&\quad = \frac{1}{2\sigma_n^2} h_{k_1} h_{k_2} \big|\mathbf{E}_{\bar{t},\bar{s}}\big[(\bar{t}-\xi)(\bar{s}-\eta) \\
&\qquad\qquad \times d_X^{(1,1)}\big(\tau_{k_1} + \theta_1(\xi-\bar{t}\,)h_{k_1}, \tau_{k_2} + \theta_2(\eta-\bar{s})h_{k_2}\big)\big]\big| \\
&\quad \le C_1(h_{k_1}h_{k_2}/\sigma_n^2)|\tau_{k_2} - \tau_{k_1}|^{\alpha-2} \le C_2|t-s|^{\alpha-2},
\end{aligned}$$

for sufficiently large $|t-s|$, where $\bar{t} = t-(k_1-1)$, $\bar{s} = s-(k_2-1)$. In particular, we have that there exists $C > 0$ such that for $|t-s| > C$,

$$|r_n(t,s)| < \delta < 1 \qquad \text{uniformly in } n. \tag{72}$$

Thus, C4 follows. By using (70) and B2, similar arguments work for the differentiable case, $m = 1$.

Assumption C5 follows from (55) in the continuous case and from (56) in the differentiable case, with $\sigma(t) = \sigma_0^{-1} C(t)^{1/2} h(t)^{-(m+\alpha/2)}/2$, $t \in [0,1]$.

Assumption C6 is implied by B3.

Finally, assertions (ii)–(iv) of Theorem 5 follow from the corresponding assertions of Theorems 1, 2 and 4.

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