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EXISTENCE OF QUASI-STATIONARY MEASURES FOR ASYMMETRIC ATTRACTIVE PARTICLE SYSTEMS ON $\mathbb{Z}^d$

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We show the existence of nontrivial quasi-stationary measures for conservative attractive particle systems on $\mathbb{Z}^d$ conditioned on avoiding an increasing local set $\mathcal{A}$. Moreover, we exhibit a sequence of measures $\{\nu_n\}$, whose ω-limit set consists of quasi-stationary measures. For zero-range processes, with stationary measure ν_ρ, we prove the existence of an $L^2(\nu_\rho)$ nonnegative eigenvector for the generator with Dirichlet boundary on $\mathcal{A}$, after establishing a priori bounds on the $\{\nu_n\}$.

1. Introduction. We consider the "processus des misanthropes," which includes the asymmetric exclusion process and zero-range processes. For concreteness, let us describe here the dynamics of a zero-range process. We denote the path of the process by $\{\eta_t, t \geq 0\}$ with $\eta_t(i) \in \mathbb{N}$ for $i \in \mathbb{Z}^d$. At site i and at time t, one of the $\eta_t(i)$ particles jumps to site j at rate $g(\eta_t(i))p(i,j)$, where

$$g : \mathbb{N} \to [0, \infty) \text{ is increasing, with } g(0) = 0, \qquad \sup_k \bigl(g(k+1) - g(k)\bigr) < \infty \tag{1.1}$$

and $p(\cdot,\cdot)$ is the transition kernel of a transient random walk. Under assumptions that we make precise later, the informal dynamics described above corresponds to a Markov process with stationary product measures $\{\nu_\rho, \rho > 0\}$ (see [1]).

Our motivation stems from statistical physics where such systems model a gas of charged particles in equilibrium under an electrical field. An interesting issue is the distribution of the occurrence time of density fluctuations in equilibrium. Thus, let Λ be a finite subset of $\mathbb{Z}^d$ and consider the event

$$\mathcal{A} = \left\{\eta : \frac{1}{|\Lambda|}\sum_{i\in\Lambda} \eta(i) > \rho'\right\}, \qquad \text{with } \rho' > \rho. \tag{1.2}$$

Let τ be the first time a trajectory $\{\eta_t : t \geq 0\}$ enters $\mathcal{A}$. As in [4, 5], we consider two complementary issues:

(i) estimating the tail of the distribution of τ;
(ii) characterizing the law of η_t at large time, conditioned on $\{\tau > t\}$, when the initial configurations are drawn from ν_ρ.

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We denote by $\mathcal{L}$ the generator of our process on the domain $\mathcal{D}(\mathcal{L})$, by $\{S_t, t \geq 0\}$ the associated semigroup and by P_μ the law of the process with initial probability μ. For any probability ν, we denote by $T_t(\nu)$ the law of η_t conditioned on $\{\tau > t\}$, with respect to P_ν. Thus, for φ continuous and bounded, $\int \varphi \, dT_t(\nu) := E_\nu[\varphi(\eta_t) | \tau > t]$.

Now, from a statistical physics point of view, a relevant issue is the existence of a limit for $T_t(\nu_\rho)$, the so-called Yaglom limit, say μ_ρ. A Yaglom limit is established by Kesten [13] for an irreducible positive recurrent random walk on $\mathbb{N}$ with bounded jump size and with $\mathcal{A} = \{0\}$. Also, a Yaglom limit is established in [5] for the symmetric simple exclusion process in dimension $d \geq 5$, relying strongly on the self-adjointness and attractiveness and establishing uniform $L^2(\nu_\rho)$ bounds for $\{dT_t(\nu_\rho)/d\nu_\rho, t \geq 0\}$. We refer to the Introduction of [12] for a review of countable Markov chains for which the Yaglom limit is established. This notion was introduced first by Yaglom [18] in 1947 for subcritical branching processes.

We note that the existence of μ_ρ implies trivially that there is $\lambda(\rho) \in [0, \infty]$ such that, for any $s > 0$,

$$P_{\mu_\rho}(\tau > s) = \lim_{t \to \infty} \frac{P_{\nu_\rho}(\tau > t + s)}{P_{\nu_\rho}(\tau > t)} = \exp(-\lambda(\rho) s), \tag{1.3}$$

which, in turn, implies readily that

$$\lambda(\rho) = -\lim_{t \to \infty} \frac{1}{t} \log(P_{\nu_\rho}(\tau > t)). \tag{1.4}$$

Thus, right at the outset, one faces three issues:

(i) When does the ratio in (1.3) have a limit? This is linked with a wide area of investigations (see, e.g., [9, 11, 13]).

(ii) Is there a formula for $\lambda(\rho)$? One recognizes in $\lambda(\rho)$ the logarithm of the spectral radius of $\mathcal{L} : L^\infty(\nu_\rho) \to L^1(\nu_\rho)$ with Dirichlet conditions on $\mathcal{A}$. When $\mathcal{L}$ is a second-order elliptic operator on a bounded domain, and when we work with the sup-norm topology, Donsker and Varadhan [10] give a variational formula for (1.4).

(iii) When is $\lambda(\rho)$ a positive real? In other words, what is the right scaling for large deviations for the occupation time of $\mathcal{A}$. For symmetric simple exclusion, it is shown in [2] and [4] that $\lambda(\rho) > 0$ if and only if $d \geq 3$.

Since $\{T_t, t \geq 0\}$ is a semigroup, the Yaglom limit, when it exists, is a fixed point of T_t for any t. Thus, a preliminary step is to characterize possible fixed points of $\{T_t\}$, which are called quasi-stationary measures. In other words, μ is quasi-stationary if there is $\lambda \geq 0$ such that, for any $\varphi \in \mathcal{D}(\mathcal{L})$ and any $t > 0$,

$$\int E_{\eta_0}[\varphi(\eta_t) \mathbb{1}_{\tau > t}] \, d\mu(\eta_0) = e^{-\lambda t} \int \varphi \, d\mu.$$

We note that, in our context, the Dirac measure on the empty configuration is trivially a quasi-stationary measure with $\lambda = 0$. Thus, by nontrivial quasi-stationary measure, we mean one corresponding to $\lambda > 0$. Finally, we note that, in dynamical systems, quasi-stationary measures are well studied and named after Pianigiani and Yorke [15], who prove their existence for expanding C^2-maps.

Assume that μ is a probability measure with support in $\mathcal{A}^c$ such that, for any $t \geq 0$, $T_t(\mu) = \mu$. By differentiating this equality at $t = 0$, we obtain, for φ in the domain of $\mathcal{L}$ with $\varphi|_{\mathcal{A}} = 0$,

$$\int \mathcal{L}(\varphi)\, d\mu = \int \mathcal{L}(\mathbb{1}_{\mathcal{A}^c})\, d\mu \int \varphi\, d\mu. \tag{1.5}$$

Moreover, assume that μ is absolutely continuous with respect to a measure ν and that $f := d\mu/d\nu \in L^2(\nu)$. If $\mathcal{L}^*$ denotes the adjoint operator in $L^2(\nu)$, then $f \in D(\mathcal{L}^*)$ and f is a nonnegative solution of

$$\mathbb{1}_{\mathcal{A}^c}\mathcal{L}^* f + \lambda f = 0 \quad \text{and} \quad \lambda = \int -\mathcal{L}(\mathbb{1}_{\mathcal{A}^c})\, d\mu.$$

Thus, the problem of quasi-stationary measure for attractive particle systems is a problem of finding principal eigenvectors in a context where we lack irreducibility conditions and where neither the space nor the operator is compact.

Equation (1.5) is the starting point of Ferrari, Kesten, Martínez and Picco [12], whose work we describe in some detail since ours builds upon it. These authors consider an irreducible, positive recurrent random walk, $\{X_t, t \geq 0\}$ on $\mathbb{N}$, with rates of jump $\{q(i,j),\ i,j \in \mathbb{N}\}$. They study the first time the origin is occupied, say τ, when there is $\lambda > 0$ and $i \in \mathbb{N} \setminus \{0\}$ such that $E_i[\exp(\lambda\tau)] < \infty$. Assuming that μ satisfies (1.5), one obtains, for any φ with $\varphi(0) = 0$,

$$\sum_{j \neq 0} \sum_{k \neq 0} \big(q(j,k) + q(j,0)\mu(k)\big)\big(\varphi(k) - \varphi(j)\big)\mu(j) = 0. \tag{1.6}$$

Thus, μ can be thought of as the invariant measure of a new random walk, say $\{X_t^\mu, t \geq 0\}$ on $\mathbb{N} \setminus \{0\}$ with rates $\{q(j,k) + q(j,0)\mu(k),\ j,k \in \mathbb{N} \setminus \{0\}\}$. When μ is such that $E_\mu[\tau] < \infty$, X_t^μ is positive recurrent and has a unique invariant measure ν, and this procedure defines a map $\mu \mapsto \Phi(\mu) = \nu$. Thus, the problem reduces to finding fixed points of Φ. They notice also that X_t^μ can be built from the walk X_t, by starting it afresh from a random site drawn from μ, each time X_t hits 0. Then, using this renewal representation, an expression of $\Phi(\mu)$ is obtained (see equation (2.4) of [12])

$$\Phi(\mu) = \frac{1}{E_\mu[\tau]} \int_0^\infty T_t(\mu) P_\mu(\tau > t)\, dt. \tag{1.7}$$

In our case, the Laplace-like transform (1.7) is a well-defined map, and as observed in [8], as soon as $E_\mu[\tau] < \infty$, μ is quasi-stationary if and only if $\Phi(\mu) = \mu$.

In [12], the authors study the sequence of iterates $\{\Phi^n(\delta_i)\}_{n \geq 1}$ for $i \in \mathbb{N} \setminus \{0\}$. They show that this sequence is tight and that any limit point belongs to $\mathcal{M}_\lambda$, the subspace of probability measures under which τ is an exponential time of parameter

$$\lambda = -\lim_{t\to\infty} \frac{1}{t} \log\big(P_{\delta_i}(\tau > t)\big) > 0.$$

Then the facts that $\Phi(\mathcal{M}_\lambda) \subset \mathcal{M}_\lambda$ and Φ is continuous on the compact set $\mathcal{M}_\lambda$ imply that Φ has a fixed point in $\mathcal{M}_\lambda$.

Though the irreducibility assumption no longer holds for attractive particle systems on $\mathbb{Z}^d$, we show that $\{\Phi^n(\nu_\rho)\}$ is tight through the a priori bounds $\Phi^n(\nu_\rho) \prec \nu_\rho$, where $\prec$ denotes stochastic domination. These bounds permit us to prove that, as soon as $\lambda(\rho) > 0$, τ is an exponential time of parameter $\lambda(\rho) > 0$ under any limit point of the iterates sequence. We establish that $\lambda(\rho) > 0$ in any dimensions for zero-range processes, whereas $\lambda(\rho) > 0$ is only proved to hold in dimensions larger or equal than 3 for exclusion processes.

Once $\lambda(\rho) > 0$ holds, we show that any limit point of the Cesaro mean $(\Phi(\nu_\rho) + \cdots + \Phi^n(\nu_\rho))/n$ is quasi-stationary. It is useful to have a sequence converging to a quasi-stationary measure. Indeed, through a priori bounds, one gets regularity of the limiting quasi-stationary measure. For instance, for zero-range processes, we can show that, in dimensions $d \geq 3$, quasi-stationary measures obtained as Cesaro limits have a density with respect to ν_ρ which is in any $L^p(\nu_\rho)$ for $p \geq 1$. In this way, we establish the existence of a Dirichlet eigenvector, say $f \in D(\mathcal{L}^*)$ with

$$\forall\, \eta \notin \mathcal{A}, \qquad \mathcal{L}^* f(\eta) + \lambda(\rho) f(\eta) = 0 \quad \text{and} \quad f|_{\mathcal{A}} = 0.$$

This, in turn, gives estimates for $P_{\nu_\rho}(\tau > t)$, improving on (1.4).

Finally, we note that a natural way to prove the existence of quasi-stationary measures for our particle systems on $\mathbb{Z}^d$ would have been to work first with finite-dimensional approximations, where we can rely on the Perron–Frobenius theory. This strategy, naively implemented, fails as is shown in a simple example in Section 5.

2. Notation and results.

We consider $\mathbb{N}^{\mathbb{Z}^d}$ with the product topology. The local events are the elements of the union of all σ-algebras $\sigma\{\eta(i), i \in \Lambda\}$ over a Λ finite subset of $\mathbb{Z}^d$. We start by recalling the definition of the "processus des misanthropes" [7]. The rates $\{p(i,j),\ i,j \in \mathbb{Z}^d\}$ satisfy:

(2.1)

- (i) $p(i,j) \geq 0,\ \sum_{i\in\mathbb{Z}^d} p(0,i) = 1$;
- (ii) $p(i,j) = p(0, j-i)$ (translation invariance);
- (iii) $p(i,j) = 0$ if $|i-j| > R$ for some fixed R (finite range);
- (iv) if $p_s(i,j) = p(i,j) + p(j,i)$, then, $\forall\, i \in \mathbb{Z}^d$, $\exists\, n$, $p_s^{(n)}(0,i) > 0$ (irreducibility);
- (v) $\sum_{i\in\mathbb{Z}^d} i p(0,i) \neq 0$ (drift).

Let $b : \mathbb{N} \times \mathbb{N} \to [0, \infty)$ be a function with:

$$(2.2)\quad \begin{aligned} &\text{(i)}\ b(0, \cdot) \equiv 0;\\ &\text{(ii)}\ n \mapsto b(n, m) \text{ is increasing for each } m;\\ &\text{(iii)}\ m \mapsto b(n, m) \text{ is decreasing for each } n;\\ &\text{(iv)}\ b(n, m) - b(m, n) = b(n, 0) - b(m, 0)\ \forall n, m \geq 1;\\ &\text{(v)}\ \Delta := \sup_n (b(n + 1, 0) - b(n, 0)) < \infty. \end{aligned}$$

Let $g : \mathbb{N} \to [0, \infty)$ satisfy (1.1) and let $g(1) = 1$. For any $\gamma \in [0, \sup_k g(k)[$, we define a probability θ_γ on $\mathbb{N}$ by

$$(2.3)\quad \theta_\gamma(0) = 1/Z(\gamma), \qquad \theta_\gamma(n) = \frac{1}{Z(\gamma)} \frac{\gamma^n}{g(1) \cdots g(n)} \qquad \text{when } n \neq 0,$$

where $Z(\gamma)$ is the normalizing factor. If we set $\Upsilon(\gamma) = \sum_{n=1}^\infty n\theta_\gamma(n)$, then $\Upsilon : [0, \sup_k g(k)[\to [0, \infty[$ is increasing. Let $\gamma : [0, \sup_\gamma \Upsilon(\gamma)) \to [0, \sup_k g(k))$ be the inverse of Υ and let ν_ρ be the product probability with marginal law $\theta_{\gamma(\rho)}$. Thus, we have

$$(2.4)\quad \forall i \in \mathbb{Z}^d, \qquad \int \eta(i)\, d\nu_\rho = \rho \quad \text{and} \quad \int g(\eta(i))\, d\nu_\rho = \gamma(\rho).$$

Following [1] (and [17], Section 2), let

$$\alpha(i) = \sum_{n=0}^{\infty} 2^{-n} p^n(i, 0)$$

and, for $\eta, \zeta \in \mathbb{N}^{\mathbb{Z}^d}$,

$$\|\eta - \zeta\| = \sum_{i \in \mathbb{Z}^d} |\eta(i) - \zeta(i)| \alpha(i).$$

Our state space is $\Omega = \{\eta : \|\eta\| < \infty\}$, and we call $\mathcal{C}_b$ the space of a bounded Lipshitz function from $(\Omega, \|\cdot\|)$ to $(\mathbb{R}, |\cdot|)$. In [1], it is shown that a semigroup can be constructed on $\mathcal{C}_b$ with generator

$$(2.5)\qquad \mathcal{L}_b \varphi(\eta) := \sum_{i, j \in \mathbb{Z}^d} p(i, j) b(\eta(i), \eta(j)) (\varphi(\eta^i_j) - \varphi(\eta)),$$

where $\eta^i_j(k) = \eta(k)$ if $k \notin \{i, j\}$, $\eta^i_j(i) = \eta(i) - 1$ and $\eta^i_j(j) = \eta(j) + 1$.

For a function b satisfying (2.2), we assume there is g as above, with $b(n, m - 1)g(m) = b(m, n - 1)g(n)$, which together with (2.2(iv)) and (2.1(i)), implies that $\{\nu_\rho, \rho \in [0, \sup_\gamma \Upsilon(\gamma))\}$ are invariant with respect to $\mathcal{L}_b$.

In [17], Section 2, $\mathcal{L}_b$ is extended to a generator, say $\mathcal{L}$, on $L^2(\nu_\rho)$ for any $\rho > 0$. It is also shown that $\mathcal{C}_b$ is a core for $\mathcal{L}$.

Now, if we choose $b(n, m) = g(n)$, we obtain the zero-range process. We describe a way of realizing this process, in a case like ours, where the labeling of particles is innocuous. We start with an initial configuration $\eta \in \Omega$. We label arbitrarily particles on each site i from 1 to $\eta(i)$. We associate to each particle a path $\{S_n,\ n \in \mathbb{N}\}$, paths being drawn independently from those of a random walk with rates $\{p(i,j)\}$. Then a particle labeled k at site i jumps with rate $g(k) - g(k-1)$. If it jumps on site j, it gets the last label. Also, the remaining particles at site i are relabeled from 1 to $\eta(i) - 1$. Now, as $\Delta := \sup_{k>1}(g(k) - g(k-1)) < \infty$, we can dominate the Poisson clocks with independent Poisson clocks of intensity Δ, so that each particle is coupled with a random walk wandering faster on the same path.

If we restrict the process to $\{0,1\}^{\mathbb{Z}^d}$ and choose $b(n,m) = 1$ if $n = 1, m = 0$ and $b(n,m) = 0$ otherwise, we obtain the exclusion process. The measure ν_ρ is then a product Bernoulli measure.

We consider also the adjoint (or time-reversed) of $\mathcal{L}$ in $L^2(\nu_\rho)$ as acting on bounded Lipshitz functions φ and ψ by

$$\int \mathcal{L}^*(\varphi)\psi \, d\nu_\rho := \int \varphi \mathcal{L}(\psi) \, d\nu_\rho. \tag{2.6}$$

With our hypothesis, $\mathcal{L}^*$ is again the generator of a "processus des misanthropes" on Ω, with the same functions b and g, but with $p^*(i,j) := p(j,i)$ (see, e.g., [6]). We denote by $\{S_t^*\}$ the associated semigroup, and by P_η^* the associated process with initial configuration $\eta \in \Omega$.

For convenience, we fix an integer k and Λ a finite subset of $\mathbb{Z}^d$, and set $\mathcal{A} := \{\eta : \sum_{i\in\Lambda} \eta(i) > k\}$. We consider a density $\rho > 0$ such that $\nu_\rho(\mathcal{A}^c) > 0$. We denote by $\bar{\mathcal{L}} := \mathbb{1}_{\mathcal{A}^c}\mathcal{L}$ and $\{\bar{S}_t, t \geq 0\}$, respectively, the generator and associated semigroup for the process killed on $\mathcal{A}$. A core of $\bar{\mathcal{L}}$ consists of bounded Lipshitz functions vanishing on $\mathcal{A}$.

For $\eta, \xi \in \Omega$, we say that $\eta \leq \xi$ if $\eta(i) \leq \xi(i)$ for all $i \in \mathbb{Z}^d$. Also, a function is increasing (resp. decreasing) if $\eta \leq \xi$ implies that $f(\eta) \leq f(\xi)$ [resp. $f(\eta) \geq f(\xi)$]; in particular, we say that $A \subset \Omega$ is increasing if $\mathbb{1}_A$ is increasing. Finally, for given probability measures ν, μ on Ω, we say that $\nu \prec \mu$ if $\int f\,d\nu \leq \int f\,d\mu$ for every increasing function f. We recall that the "processus des misanthropes" is an attractive process; that is, there is a coupling such that $P_{\eta,\zeta}(\eta_t \leq \zeta_t, \forall t) = 1$ whenever $\eta \leq \zeta$.

Since $\mathcal{A}$ is an increasing local event, attractiveness implies that, for any $t \geq 0$, both $P_\eta(\tau > t)$ and $P_\eta^*(\tau > t)$ are decreasing in η. As our product measure satisfies FKG's inequality, we have

$$\begin{aligned} P_{\nu_\rho}(\tau > t+s) &= \int \mathbb{1}_{\mathcal{A}^c}\bar{S}_{t+s}(\mathbb{1}_{\mathcal{A}^c})\,d\nu_\rho \\ &= \int \bar{S}_t(\mathbb{1}_{\mathcal{A}^c})\bar{S}_s^*(\mathbb{1}_{\mathcal{A}^c})\,d\nu_\rho \\ &\geq P_{\nu_\rho}(\tau > t)P_{\nu_\rho}(\tau > s). \end{aligned} \tag{2.7}$$

Also, it is easy to see that $\nu_\rho(\mathcal{A}^c) > 0$ implies that, for any $t \geq 0$, $P_{\nu_\rho}(\tau > t) > 0$ [this is true for short time by continuity, and one then uses (2.7) to extend it to any time]. Thus, the subadditivity of $t \mapsto -\log(P_{\nu_\rho}(\tau > t))$ [as seen in (2.7)] and $P_{\nu_\rho}(\tau > t) > 0$ imply the existence of the limit $\lambda(\rho) < \infty$ in (1.4).

A key, though elementary, observation of [8, 12] is as follows.

LEMMA 2.1. *Let μ be such that $E_\mu[\tau] < \infty$. Then, μ is quasi-stationary if and only if $\Phi(\mu) = \mu$.*

We recall that, for $\varphi \in \mathcal{C}_b$,

$$\int \varphi \, d\Phi(\mu) = \frac{\int_0^\infty \int \bar{S}_t(\varphi)\, d\mu\, dt}{\int_0^\infty \int \bar{S}_t(\mathbb{1}_{\mathcal{A}^c})\, d\mu\, dt}.$$

Thus, Lemma 2.1 follows readily: if μ is quasi-stationary, then it is obvious that $\Phi(\mu) = \mu$. Conversely, for any $\varphi \in \mathcal{C}_b$,

$$\int \bar{S}_s(\varphi)\, d\mu = \frac{1}{E_\mu[\tau]} \int_0^\infty \int \bar{S}_t(\bar{S}_s(\varphi))\, d\mu\, dt = \frac{1}{E_\mu[\tau]} \int_s^\infty \int \bar{S}_t(\varphi)\, d\mu\, dt,$$

which implies that

$$\int \bar{S}_s(\varphi)\, d\mu = \exp\left(-\frac{s}{E_\mu[\tau]}\right) \int \varphi \, d\mu.$$

Now, a key a priori bound relies on the notion of stochastic domination.

LEMMA 2.2. *Assume $\lambda(\rho) > 0$. If Φ^n denotes the nth iterate of Φ, then $\Phi^n(\nu_\rho) \prec \nu_\rho$. Also, $\{\Phi^n(\nu_\rho)\}$ is tight.*

This allows us to prove a result analogous to Lemma 3.2 of [12].

LEMMA 2.3. *Assume $\lambda(\rho) > 0$. Then, for any integer $k \geq 1$,*

$$\lim_{n\to\infty} \int \tau^k \, d\Phi^n(\nu_\rho) = \frac{k!}{\lambda(\rho)^k}.$$

Moreover, for any $s \geq 0$,

$$\lim_{n\to\infty} P_{\Phi^n(\nu_\rho)}(\tau > s) = \exp(-\lambda(\rho) s). \tag{2.8}$$

If we set $\bar{\nu}_n := (1/n)(\Phi(\nu_\rho) + \cdots + \Phi^n(\nu_\rho))$, then our existence result reads as follows.

THEOREM 2.4. *Assume $\lambda(\rho) > 0$. Then any limit point along a subsequence of $\{\bar{\nu}_n,\ n \in \mathbb{N}\}$, which we denote by μ_ρ, is a quasi-stationary measure corresponding to $\lambda(\rho)$.*

We prove Lemmas 2.2 and 2.3 and Theorem 2.4 in Section 3. We now give conditions under which $\lambda(\rho) > 0$. Note that in the symmetric case [4] established the following stronger result using spectral representation:

$$\begin{aligned} &\lim_{u\to\infty} \frac{P_{\nu_\rho}(\tau > u+s)}{P_{\nu_\rho}(\tau > u)} = e^{-\lambda_s(\rho)s}, \\ &\text{with } \lambda_s(\rho) = \inf\left\{ \frac{-\int f\mathcal{L}f\,d\nu_\rho}{\int f^2\,d\nu_\rho} : f \in D(\mathcal{L}),\ f|_{\mathcal{A}} = 0 \right\}. \end{aligned} \tag{2.9}$$

It was established in [4] that, for the symmetric exclusion process, $\lambda_s(\rho) > 0$ for $d \ge 3$ and that $\lambda_s(\rho) = 0$ for $d = 1$ and $d = 2$. Using the classical bound $\lambda(\rho) \ge \lambda_s(\rho)$ (see, e.g., [16], Lemma 4.1), we have the following result.

LEMMA 2.5. *For the exclusion process in $d \ge 3$, $\lambda(\rho)$ given by (1.4) is positive.*

For zero-range processes, we prove in Section 4 the following results.

LEMMA 2.6. *For zero-range processes in any dimensions, $\lambda(\rho) > 0$.*

Moreover, we have the following regularity result.

PROPOSITION 2.7. *For zero-range processes in $d \ge 3$, any limit points along a subsequence of $\{\bar{\nu}_n\}$, say μ_ρ, is absolutely continuous with respect to ν_ρ and $f := d\mu_\rho/d\nu_\rho \in L^p(\nu_\rho)$ for any $p \ge 1$. Thus, f is in the domain of $\bar{\mathcal{L}}^*$ and*

$$\bar{\mathcal{L}}^* f + \lambda(\rho) f = 0 \qquad a.s.\text{-}\nu_\rho. \tag{2.10}$$

As a consequence of the existence of an eigenvector of (2.10) in $L^p(\nu_\rho)$ for $p \ge 1$, we have estimates for the hitting time.

COROLLARY 2.8. *For zero-range processes in $d \ge 3$, let f be a solution of (2.10) and let g be a solution of the adjoint eigenvector equation. Then $\int fg\,d\nu_\rho$ is finite and positive, and, for any time t,*

$$\exp\big(-H(\tilde{\nu}_\rho, \nu_\rho)\big) \le \frac{P_{\nu_\rho}(\tau > t)}{\exp(-\lambda(\rho)t)} \le 1, \tag{2.11}$$

with

$$d\tilde{\nu}_\rho = \frac{fg\,d\nu_\rho}{\int fg\,d\nu_\rho} \quad \text{and} \quad H(\tilde{\nu}_\rho, \nu_\rho) = \int \log\left(\frac{d\tilde{\nu}_\rho}{d\nu_\rho}\right) d\tilde{\nu}_\rho < \infty.$$

In Section 5, we see, on the totally asymmetric simple exclusion process, why a naive finite-dimensional approximation of our problem yields "wrong" results.

Finally, let us mention some open problems. (i) A result similar to Proposition 2.7 should hold for the asymmetric exclusion process in $d \geq 3$. (ii) For asymmetric misanthrope processes, $\lambda(\rho)$ should be positive in any dimension, although the quasi-stationary measure μ_ρ should not be equivalent to ν_ρ but in $d \geq 3$. (iii) The Yaglom limit has not been established in the asymmetric case (or in $d = 3, 4$ for the symmetric simple exclusion [4]), and the existence of a limit for $\exp(\lambda(\rho)t) P_{\nu_\rho}(\tau > t)$ has also not been established. (iv) When the particle system is not attractive, the problem of hitting-time estimates and quasi-stationary measures is open (see some existence results in [4] in the self-adjoint case).

3. Existence. We begin with some useful expressions for the iterates $\nu_n := \Phi^n(\nu_\rho)$. If $\lambda(\rho) > 0$, then, $\forall n \in \mathbb{N}$, $\int_0^\infty u^n P_{\nu_\rho}(\tau > u)\, du$ is finite, and it follows easily by induction that

$$\int \varphi \, d\nu_n = \frac{\int_0^\infty \cdots \int_0^\infty \int \bar{S}_{t_1+\cdots+t_n}(\varphi)\, d\nu_\rho \prod_{i=1}^n dt_i}{\int_0^\infty \cdots \int_0^\infty \int \bar{S}_{t_1+\cdots+t_n}(\mathbb{1}_{\mathcal{A}^c}) \nu_\rho \prod_{i=1}^n dt_i} = \frac{\int_0^\infty u^{n-1} \int \bar{S}_u(\varphi)\, d\nu_\rho \, du}{\int_0^\infty u^{n-1} \int \bar{S}_u(\mathbb{1}_{\mathcal{A}^c})\, d\nu_\rho\, du}. \tag{3.1}$$

Taking $\varphi = \bar{S}_t(\mathbb{1}_{\mathcal{A}^c})$ in (3.1) yields

$$P_{\nu_n}(\tau > t) = \frac{\int_0^\infty u^{n-1} P_{\nu_\rho}(\tau > t + u)\, du}{\int_0^\infty u^{n-1} P_{\nu_\rho}(\tau > u)\, du}.$$

Integrating over t, we obtain

$$E_{\nu_n}[\tau] = \frac{1}{n} \frac{\int_0^\infty u^n P_{\nu_\rho}(\tau > u)\, du}{\int_0^\infty u^{n-1} P_{\nu_\rho}(\tau > u)\, du} = \frac{E_{\nu_\rho}[\tau^{n+1}]}{(n+1) E_{\nu_\rho}[\tau^n]}. \tag{3.2}$$

PROOF OF LEMMA 2.2. Let φ be an increasing function in $\mathcal{C}_b$. Then

$$\int \bar{S}_u \varphi \, d\nu_\rho = \int \mathbb{1}_{\mathcal{A}^c} E_\eta[\varphi(\eta_u) \mathbb{1}_{\{\tau > u\}}]\, d\nu_\rho = \int \varphi(\eta) \bar{S}_u^*(\mathbb{1}_{\mathcal{A}^c})(\eta)\, d\nu_\rho.$$

Now, we note that $\eta \mapsto \bar{S}_u^* \mathbb{1}_{\mathcal{A}^c}(\eta)$ is decreasing. Thus, by FKG's inequality, we have

$$\int \bar{S}_u \varphi \, d\nu_\rho \leq \int \varphi \, d\nu_\rho \int \bar{S}_u(\mathbb{1}_{\mathcal{A}^c})\, d\nu_\rho.$$

This implies that $\int \varphi\, d\nu_n \leq \int \varphi \, d\nu_\rho$ by (3.1) as we are assuming that $\lambda(\rho) > 0$. Consider now compact subsets of $\mathbb{N}^{\mathbb{Z}^d}$ of the type $K_{(k_i)} = \{\eta : \forall i \in \mathbb{Z}^d, \eta_i \leq k_i\}$. Since these compacts are decreasing, we have $\inf_n \nu_n(K_{(k_i)}) \geq \nu_\rho(K_{(k_i)})$. Moreover, for all $\varepsilon > 0$, a good choice of the sequence (k_i) ensures that $\nu_\rho(K_{(k_i)}) \geq 1 - \varepsilon$, and tightness follows. $\square$

PROOF OF LEMMA 2.3. The argument follows closely [12] (proofs of Lemma 3.2, Proposition 3.3 and Theorem 4.1), the main difference being that we replace irreducibility by stochastic domination. If $\nu_n = \Phi^n(\nu_\rho)$, then we show in three steps that $\lim E_{\nu_n}[\tau] = 1/\lambda(\rho)$.

STEP 1. We first prove that

$$\liminf E_{\nu_n}[\tau] = 1/\lambda(\rho) \quad \text{and} \quad P_{\nu_\rho}(\tau > t) \le \exp\bigl(-\lambda(\rho)t\bigr). \tag{3.3}$$

As in Proposition 3.3 of [12], if

$$\frac{1}{\lambda_\infty} = \liminf E_{\nu_n}[\tau] \qquad \text{then } \lambda_\infty \ge \lambda(\rho),$$

and there is a subsequence $\{n_k\}$ such that

$$\forall t > 0, \qquad \lim_{k\to\infty} P_{\nu_{n_k}}(\tau > t) = \exp(-\lambda_\infty t).$$

The inequality $\lambda_\infty \le \lambda(\rho)$ follows after observing that, as $\eta \mapsto P_\eta(\tau > t)$ is decreasing and as $\nu_n \prec \nu_\rho$, we have $P_{\nu_{n_k}}(\tau > t) \ge P_{\nu_\rho}(\tau > t)$. Thus,

$$\exp(-\lambda_\infty t) = \lim_{k\to\infty} P_{\nu_{n_k}}(\tau > t) \ge P_{\nu_\rho}(\tau > t). \tag{3.4}$$

This establishes that $\lambda_\infty = \lambda(\rho)$ and (3.3).

STEP 2. We show that

$$\lim_{n\to\infty}\left(\frac{E_{\nu_\rho}[\tau^n]}{n!}\right)^{1/n} = \frac{1}{\lambda(\rho)}. \tag{3.5}$$

First, by Step 1,

$$\begin{aligned} E_{\nu_\rho}[\tau^n] &= \int_0^\infty n u^{n-1} P_{\nu_\rho}(\tau > u)\,du \\ &\le \int_0^\infty n u^{n-1} \exp(-\lambda(\rho)u)\,du = \frac{n!}{\lambda(\rho)^n}. \end{aligned} \tag{3.6}$$

If we set $v_n = E_{\nu_\rho}[\tau^n]/n!$, we then have $\limsup v_n^{1/n} \le 1/\lambda(\rho)$. Now, by (3.2), $E_{\nu_n}[\tau] = v_{n+1}/v_n$. Since $\liminf E_{\nu_n}[\tau] = 1/\lambda(\rho)$, it follows that

$$\forall \varepsilon \in]0, 1/\lambda(\rho)[,\ \exists n_0,\ \forall n \ge n_0, \qquad v_n \ge v_{n_0}\left(\frac{1}{\lambda(\rho)} - \varepsilon\right)^{n-n_0}. \tag{3.7}$$

Thus, for any $\varepsilon > 0$, $\liminf v_n^{1/n} \ge 1/\lambda(\rho) - \varepsilon$, and this concludes Step 2.

STEP 3. We show that $\limsup E_{\nu_n}[\tau] \le 1/\lambda(\rho)$ by following the proof of Theorem 4.1 of [12]. We omit the argument here.

Finally, as in [12], it is now easy to conclude that for any integer $k \geq 1$ and $s > 0$,

$$E_{\nu_n}[\tau^k] = k! \prod_{j=1}^{k} E_{\nu_{n+j+1}}[\tau] \underset{n\to\infty}{\longrightarrow} \frac{k!}{\lambda(\rho)^k}$$

and

$$P_{\nu_n}(\tau > s) \underset{n\to\infty}{\longrightarrow} e^{-\lambda(\rho)s}.$$

□

PROOF OF THEOREM 2.4. For any integer n, set $\bar{\nu}_n = (\Phi(\nu_\rho) + \cdots + \Phi^n(\nu_\rho))/n$. Note that from Lemmas 2.2 and 2.3, we have

$$\bar{\nu}_n \prec \nu_\rho, \qquad E_{\bar{\nu}_n}[\tau^k] \underset{n\to\infty}{\longrightarrow} \frac{k!}{\lambda(\rho)^k}, \qquad P_{\bar{\nu}_n}(\tau > t) \underset{n\to\infty}{\longrightarrow} \exp(-\lambda(\rho)t). \tag{3.8}$$

As $\{\bar{\nu}_n\}$ is tight, let μ_ρ be a limit point along the subsequence $\{\bar{\nu}_{n_k}\}$. As $\mathcal{A}^c$ is local and $\bar{S}_t$ preserves $\mathcal{C}_b$, (3.8) implies that

$$P_{\mu_\rho}(\tau > t) = \lim_{k\to\infty} \bar{S}_t(\mathbb{1}_{\mathcal{A}^c})\,d\nu_{n_k} = \lim_{k\to\infty} P_{\bar{\nu}_{n_k}}(\tau > t) = e^{-\lambda(\rho)t}. \tag{3.9}$$

We now check that $\Phi(\mu_\rho) = \mu_\rho$, or, in other words, that, for $\varphi \in \mathcal{C}_b$,

$$\lambda(\rho) \int_0^\infty \int \bar{S}_t\varphi\,d\mu_\rho\,dt = \int \varphi\,d\mu_\rho. \tag{3.10}$$

Now, for all $t \geq 0$, the integrable bound

$$\left| \int \bar{S}_t\varphi\,d\bar{\nu}_{n_k} \right| \leq |\varphi|_\infty P_{\bar{\nu}_{n_k}}(\tau > t) \leq |\varphi|_\infty \left(1 \wedge \frac{\sup_n E_{\bar{\nu}_n}[\tau^2]}{t^2}\right) \leq \frac{C|\varphi|_\infty}{1+t^2}$$

by (3.8). Thus, $\lim_k \int \bar{S}_t\varphi\,d\bar{\nu}_{n_k} = \int \bar{S}_t\varphi\,d\mu_\rho$ implies, by dominated convergence, that

$$\lim_{k\to\infty} \int_0^\infty \left(\int \bar{S}_t\varphi\,d\bar{\nu}_{n_k} \right) dt = \int_0^\infty \left(\int \bar{S}_t\varphi\,d\mu_\rho \right) dt. \tag{3.11}$$

However, by definition of the iterates,

$$\int \varphi\,d\nu_{k+1} = \frac{\int \int_0^\infty \bar{S}_t(\varphi)\,dt\,d\nu_k}{E_{\nu_k}[\tau]}.$$

Thus,

$$\int\int_0^\infty (\bar{S}_t\varphi)\,dt\,d\bar{\nu}_{n_k} = \frac{1}{n_k} \sum_{i=1}^{n_k} E_{\nu_i}[\tau] \int \varphi\,d\nu_{i+1} \to \frac{1}{\lambda(\rho)} \int \varphi\,d\mu_\rho. \tag{3.12}$$

The result follows by (3.11) and (3.12). □

4. Positivity of $\lambda(\rho)$ and regularity. Let $\Re_i : \Omega \to \Omega$ with $\Re_i \eta(k) = \eta(k) + \delta_{i,k}$, where $\delta_{i,k} = 1$ if $i = k$ and $\delta_{i,k} = 0$ otherwise. For any $\varphi \in \mathcal{C}_b$, we have

$$\int g(\eta_i)\varphi \, d\nu_\rho = \gamma(\rho) \int \Re_i(\varphi) \, d\nu_\rho. \tag{4.1}$$

Note also that, as $k\Delta \geq g(k)$, we have

$$\int \eta_i \varphi \, d\nu_\rho \geq \frac{\gamma(\rho)}{\Delta} \int \Re_i(\varphi) \, d\nu_\rho. \tag{4.2}$$

PROOF OF LEMMA 2.6. We prove that $P_{\nu_\rho}(\tau > t) \leq \exp(-\lambda t)$ for $\lambda > 0$ by showing that

$$-\frac{dP_{\nu_\rho}(\tau > t)}{dt} = -\int \bar{S}_t(\bar{\mathcal{L}}\mathbb{1}_{\mathcal{A}^c}) \, d\nu_\rho \geq \lambda \int \bar{S}_t(\mathbb{1}_{\mathcal{A}^c}) \, d\nu_\rho. \tag{4.3}$$

Now,

$$-\bar{\mathcal{L}}\mathbb{1}_{\mathcal{A}^c}(\eta) = \sum_{i \notin \Lambda} \sum_{j \in \Lambda} p(i,j) g(\eta_i) \mathbb{1}_{\{\eta \notin \mathcal{A}, \eta^i_j \in \mathcal{A}\}}. \tag{4.4}$$

We set $\partial\mathcal{A} := \{\eta : \sum_\Lambda \eta(i) = k\}$ and note that since $g(0) = 0$, for any $i \notin \Lambda$ and any $j \in \Lambda$, $g(\eta_i)\mathbb{1}_{\{\eta \in \partial\mathcal{A}\}} = g(\eta_i)\mathbb{1}_{\{\eta \notin \mathcal{A}, \eta^i_j \in \mathcal{A}\}}$. Hence,

$$\begin{aligned} -\int \bar{S}_t(\bar{\mathcal{L}}\mathbb{1}_{\mathcal{A}^c}) \, d\nu_\rho &= -\int \bar{\mathcal{L}}\mathbb{1}_{\mathcal{A}^c} P^*_\eta(\tau > t) \, d\nu_\rho \\ &= \sum_{i \notin \Lambda, j \in \Lambda} p(i,j) \int_{\partial\mathcal{A}} g(\eta_i) P^*_\eta(\tau > t) \, d\nu_\rho \\ &= \gamma(\rho) \sum_{i \notin \Lambda, j \in \Lambda} p(i,j) \int_{\partial\mathcal{A}} P^*_{\Re_i \eta}(\tau > t) \, d\nu_\rho, \end{aligned}$$

where we have used (4.1) and the fact that $\partial\mathcal{A}$ is independent of η_i for $i \notin \Lambda$.

Since $\{(i,j) \in \Lambda^c \times \Lambda, \text{ s.t. } p(i,j) > 0\}$ is finite, we now have to prove that $\forall i \notin \Lambda, \exists \lambda_i > 0$ such that

$$\int_{\partial\mathcal{A}} P^*_{\Re_i \eta}(\tau > t) \, d\nu_\rho \geq \lambda_i \int P^*_\eta(\tau > t) \, d\nu_\rho.$$

This will be done in three steps.

STEP 1. We show that, for $i \notin \Lambda$, there is $\varepsilon_i > 0$ such that

$$P^*_{\Re_i \eta}(\tau > t) \geq \varepsilon_i P^*_\eta(\tau > t). \tag{4.5}$$

We need to couple two trajectories, say $\{\eta_t, \zeta_t\}$ differing by a particle at i at time 0, that is, $\zeta_0 = \Re_i \eta_0$. We describe a basic coupling. We tag the additional particle at i and call its trajectory $\{X(i,t), t > 0\}$. It follows the path $\{S_n, n \in \mathbb{N}\}$ of a random walk with rates $p(\cdot,\cdot)$ and jumps at the time marks of an η-dependent Poisson clock: at time t, its intensity is $g(\eta_t(X(i,t))+1)-g(\eta_t(X(i,t)))$. With this labeling, the motion of the additional particle does not perturb the η-particles. Thus, we call the additional particle a second-class particle. As $\Delta := \sup(g(k+1)-g(k)) < \infty$, we can couple $\{X(i,t), t>0\}$ with $\{\tilde{X}(i,t), t>0\}$, which follows the same path $\{S_n, n\in\mathbb{N}\}$, but with a Poisson clock of intensity Δ which dominates the clock of $\{X(i,t), t>0\}$. Thus,

$$S(\Lambda^c) = \inf\{t : X(i,t)\in\Lambda\} \geq \tilde{S}(\Lambda^c) = \inf\{t : \tilde{X}(i,t)\in\Lambda\}, \tag{4.6}$$

and under our coupling, we have that $\{S(\Lambda^c)<\infty\}\subset\{\tilde{S}(\Lambda^c)<\infty\}\subset\{S_n\in\Lambda, n\in\mathbb{N}\}$. Therefore,

$$\begin{aligned} 0 \leq P^*_\eta(\tau>t) - P^*_{\Re_i\eta}(\tau>t) &= P^*_\eta\big(\tau(\eta.)>t,\ \tau(\zeta.)\leq t\big) \\ &\leq P^*_\eta\big(\tau(\eta.)>t,\ S(\Lambda^c)<\infty\big) \\ &\leq P^*_\eta\big(\tau(\eta.)>t, \tilde{S}(\Lambda^c)<\infty\big) \\ &\leq \mathbb{P}_i(S_n\in\Lambda,\ n\in\mathbb{N})P^*_\eta(\tau>t). \end{aligned} \tag{4.7}$$

Now, as the walk is transient, $\varepsilon_i := \mathbb{P}_i(S_n\notin\Lambda, \forall n\in\mathbb{N})>0$, so that (4.5) holds.

STEP 2. It now remains to show that $\int_{\partial\mathcal{A}} P^*_\eta(\tau>t)\,d\nu_\rho \geq \lambda\int P^*_\eta(\tau>t)\,d\nu_\rho$ for some $\lambda>0$. This would be easily done by the FKG's inequality if $\partial\mathcal{A}$ were a decreasing event, which is not the case. However, $\mathcal{A}_0 := \{\eta : \sum_{i\in\Lambda}\eta(i)=0\}$ is a decreasing event, and the idea is to compare $\int_{\partial\mathcal{A}} P^*_\eta(\tau>t)\,d\nu_\rho$ with $\int_{\mathcal{A}_0} P^*_\eta(\tau>t)\,d\nu_\rho$. To this end, we are going to compare $P^*_\eta(\tau>t)$ for $\eta\in\partial\mathcal{A}$, with $P^*_{\Re_j^{-1}\eta}(\tau>t)$ for $j\in\Lambda$, so that we consider now the case where the second-class particle is initially in $j\in\Lambda$. We will ensure that, uniformly in $\eta\in\partial\mathcal{A}$, there is a positive probability that the second-class particle escapes Λ within a small time $\delta>0$. If the second-class particle finds itself on a site with k particles, it jumps with rate $\Delta_k := g(k+1)-g(k)$. We have $\Delta_1>0$, but could very well have $\Delta_k=0$ for $k>1$. Thus, the second-class particle can move for sure only when on an empty site. As in Step 1, we have a coupling $(\eta., \zeta.)$, where $\zeta_0=\Re_j\eta_0$. For convenience, we use the notation $P_{\eta,j}$ instead of P_ζ.

Thus, we impose on the η-particles starting on Λ the following constraints:

(i) They do not escape from Λ during $[0,\delta]$.

(ii) They empty one "path" joining j with $\partial\Lambda$ during $[0,\delta/3]$, while the second-class particle is frozen.

(iii) They remain still during $[\delta/3, 2\delta/3]$, while the second-class particle escapes Λ.

(iv) They go back to their initial configuration during $]2\delta/3,\delta]$.

More precisely, we let $\Gamma := \{j_1, \dots, j_n\}$ be the shortest path linking j to Λ^c, that is,

$$j_1 = j,\ j_2, \dots, j_{n-1} \in \Lambda,\ \ j_n \notin \Lambda, \qquad p(j_k, j_{k+1}) > 0 \qquad \text{for } k < n.$$

We note $i_j := j_n$, the end point of Γ, and for a subset A of $\mathbb{Z}^d$, we call $\sigma(A)$ the first time that an η-particle initially in A exits A. Also, let

$$D_\Lambda := \{\eta : \eta(j_k) = 0 \text{ for } k = 1, \dots, n-1\} \cap \partial\mathcal{A}.$$

Now, we say that $(\eta_., X(j, \cdot)) \in \mathcal{F}_{j,i_j}[0, \delta]$ if:

(i) $\sigma(\Lambda)(\eta_.) > \delta$;
(ii) on $[0, \delta/3]$, $X(j, \cdot) = j$ and $\eta_{\delta/3} \in D_\Lambda$;
(iii) on $[\delta/3, 2\delta/3]$, $\eta_.|_\Lambda = \eta_{\delta/3}|_\Lambda$ and $X(j, \cdot)$ reaches i_j before $2\delta/3$ along Γ and stays still;
(iv) on $[2\delta/3, \delta]$, $X(j, \cdot) = i_j$ and $\eta_.|_\Lambda = \eta_{\delta-t}|_\Lambda$.

We call $\tilde{\mathcal{F}}_{i_j,j}[0, \delta]$ the time-reversed event

$$\{(\eta_., X(i, \cdot)) \in \tilde{\mathcal{F}}_{i_j,j}[0, \delta]\} := \{(\eta_{\delta-\cdot}, X(j, \delta - \cdot)) \in \mathcal{F}_{j,i_j}[0, \delta]\}.$$

It is plain that

$$\lambda_1 := \inf_{\eta : \sum_{i\in\Lambda} \eta(i) \le k} \inf_{j\in\Lambda} P^*_{\eta,j}(\mathcal{F}_{j,i_j}[0, \delta]) > 0. \tag{4.8}$$

We prove in this step that there is $\lambda_2 > 0$ such that, for η such that $\sum_{i\in\Lambda} \eta(i) \le k - 1$,

$$\begin{aligned} P^*_{\mathfrak{R}_j\eta}(\tau > t) &= P^*_{\eta,j}(\tau(\zeta_.) > t) \\ &\ge \lambda_2 P^*_{\eta,j}(\tau(\eta_.) > t, \sigma(\Lambda^c) > \delta, \mathcal{F}_{j,i_j}[0, \delta]). \end{aligned} \tag{4.9}$$

From the time δ on, we couple through our basic coupling, the second-class particle with a random walk whose Poisson clock has intensity Δ, so that

$$\{\tilde{S}(\Lambda^c) \circ \theta_\delta = \infty\} \subset \{S(\Lambda^c) \circ \theta_\delta = \infty\}. \tag{4.10}$$

Note that if particles from outside Λ do not enter Λ during time $[0, \delta]$, if the second-class particle exits Λ before δ, not to ever enter again, and if $\{\tau(\eta_.) > t\}$, then $\{\tau(\zeta_.) > t\}$. In other words,

$$\{\tau(\eta_.) > t\} \cap \{\sigma(\Lambda^c) > \delta\} \cap \mathcal{F}_{j,i_j}[0, \delta] \cap \{S(\Lambda^c) \circ \theta_\delta = \infty\} \subset \{\tau(\zeta_.) > t\}. \tag{4.11}$$

Thus, by conditioning on $\sigma\{\zeta_s, s \le \delta\}$,

$$
\begin{aligned}
P^*_{\eta,j}\big(\tau(\zeta.) > t\big) &\ge P^*_{\eta,j}\big(\tau(\eta.) > t, \sigma(\Lambda^c) > \delta, \mathcal{F}_{j,i_j}[0,\delta], S(\Lambda^c)\circ\theta_\delta = \infty\big) \\
&\ge P^*_{\eta,j}\big(\tau\circ\theta_\delta(\eta.) > t, \sigma(\Lambda^c) > \delta, \mathcal{F}_{j,i_j}[0,\delta], \tilde{S}(\Lambda^c)\circ\theta_\delta = \infty\big) \\
&\ge E^*_{\eta,j}\big[\mathbb{1}_{\{\sigma(\Lambda^c)>\delta, \mathcal{F}_{j,i_j}[0,\delta]\}} P^*_{\eta_\delta,i_j}\big(\tau(\eta.) > t-\delta, \tilde{S}(\Lambda^c) = \infty\big)\big] \\
&\ge \mathbb{P}_{i_j}(S_n \notin \Lambda, \forall n \in \mathbb{N}) \\
&\qquad \times P^*_{\eta,j}\big(\tau(\eta.)\circ\theta_\delta > t-\delta, \sigma(\Lambda^c) > \delta, \mathcal{F}_{j,i_j}[0,\delta]\big).
\end{aligned}
$$

Under $\{\sigma(\Lambda^c) > \delta, \mathcal{F}_{j,i_j}[0,\delta]\}$, no η-particle enters or leaves Λ during time δ so that

$$
\begin{aligned}
&P^*_{\eta,j}\big(\tau(\eta.)\circ\theta_\delta > t-\delta, \sigma(\Lambda^c) > \delta, \mathcal{F}_{j,i_j}[0,\delta]\big) \\
&\qquad = P^*_{\eta,j}\big(\tau(\eta.) > t, \sigma(\Lambda^c) > \delta, \mathcal{F}_{j,i_j}[0,\delta]\big),
\end{aligned}
$$

and (4.9) follows once we recall that $\{S_n\}$ is transient and that $\{i_j;\ j \in \Lambda\}$ is finite.

STEP 3. We prove the result inductively. We fix one configuration in $\partial\mathcal{A}$: let $\{k_j, j \in \Lambda\}$ be integers such that

$$
\sum_{j\in\Lambda} k_j = k \quad \text{and} \quad \mathcal{B} := \{\eta : \eta_j = k_j, j \in \Lambda\}. \tag{4.12}
$$

Let j be such that $k_j > 0$. Then, using (4.2),

$$
\begin{aligned}
\int_{\partial\mathcal{A}} P^*_\eta(\tau > t)\, d\nu_\rho &\ge \int_{\mathcal{B}} P^*_\eta(\tau > t)\, d\nu_\rho \\
&= \int_{\mathcal{B}} \frac{\eta_j}{k_j} P^*_\eta(\tau > t)\, d\nu_\rho(\eta) \\
&\ge \frac{\gamma(\rho)}{\Delta k_j} \int_{\Re_j^{-1}\mathcal{B}} P^*_{\Re_j\eta}(\tau > t)\, d\nu_\rho(\eta) \\
&\ge \frac{\lambda_2\gamma(\rho)}{\Delta k_j} \int_{\Re_j^{-1}\mathcal{B}} P^*_{\eta,j}\big(\tau(\eta.) > t, \sigma(\Lambda^c) > \delta, \mathcal{F}_{j,i_j}[0,\delta]\big)\, d\nu_\rho.
\end{aligned}
$$

Using the stationarity of ν_ρ and reversing time on the interval $[0,\delta]$, the last integral becomes

$$
\int P_{\eta,i_j}\big(\tilde{\mathcal{F}}_{i_j,j}[0,\delta], \eta_\delta \in \Re_j^{-1}\mathcal{B}, \sigma(\Lambda^c) > \delta\big) P^*_\eta(\tau > t-\delta)\, d\nu_\rho(\eta).
$$

Note that in $\{\tilde{\mathcal{F}}_{i_j,j}[0,\delta], \eta_\delta \in \Re_j^{-1}\mathcal{B}, \sigma(\Lambda^c) > \delta\}$ the particles from inside and outside Λ do not interact and that $\tilde{\mathcal{F}}_{i,j}[0,\delta]$ imposes the same initial and final configuration for the η-particles in Λ, so that

$$
\begin{aligned}
&P_{\eta,i_j}\big(\tilde{\mathcal{F}}_{i_j,j}[0,\delta], \eta_\delta \in \Re_j^{-1}\mathcal{B}, \sigma(\Lambda^c) > \delta\big) \\
&\qquad = \mathbb{1}_{\mathcal{B}}(\Re_j(\eta)) P^*_{\eta,j}\big(\mathcal{F}_{j,i_j}[0,\delta]\big) P_\eta\big(\sigma(\Lambda^c) > \delta\big).
\end{aligned}
$$

Thus, from (4.8), there is $\tilde{\varepsilon} > 0$ such that

$$\int_{\mathcal{B}} P^*_\eta(\tau > t)\, d\nu_\rho \geq \tilde{\varepsilon} \int_{\mathfrak{R}_j^{-1}\mathcal{B}} P_\eta\big(\sigma(\Lambda^c) > \delta\big) P^*_\eta(\tau > t - \delta)\, d\nu_\rho(\eta). \tag{4.13}$$

We iterate the same procedure k times and end up with $\varepsilon > 0$ such that

$$\begin{aligned} &\int_{\mathcal{B}} P^*_\eta(\tau > t)\, d\nu_\rho \\ &\qquad \geq \varepsilon \int_{\prod_{j\in\Lambda} \mathfrak{R}_j^{-k_j}\mathcal{B}} P_\eta\big(\sigma(\Lambda^c) > k\delta\big) P^*_\eta(\tau > t - k\delta)\, d\nu_\rho(\eta). \end{aligned} \tag{4.14}$$

Finally, we note that

$$\eta \mapsto \mathbb{1}_{\prod_{j\in\Lambda} \mathfrak{R}_j^{-k_j}\mathcal{B}} = \mathbb{1}_{\{\eta:\,\eta(j)=0,\, j\in\Lambda\}},$$

$$\eta \mapsto P_\eta\big(\sigma(\Lambda^c) > k\delta\big), \qquad \eta \mapsto P^*_\eta(\tau > t - k\delta)$$

are decreasing functions. Thus, by the FKG's inequality,

$$\begin{aligned} &\int_{\mathcal{B}} P^*_\eta(\tau > t)\, d\nu_\rho \\ &\qquad \geq \varepsilon \nu_\rho\big(\{\eta : \eta(j) = 0,\, j \in \Lambda\}\big) P_{\nu_\rho}\big(\sigma(\Lambda^c) > k\delta\big) P_{\nu_\rho}(\tau > t). \end{aligned} \tag{4.15}$$

As $\mathcal{B} \subset \partial\mathcal{A}$, this step is concluded.

We establish in the next lemma that $P_{\nu_\rho}(\sigma(\Lambda^c) > k\delta) > 0$, which concludes the proof. □

LEMMA 4.1. *Let $\sigma(\Lambda^c)$ be the first time one particle starting outside Λ enters Λ. Then, for any $\kappa > 0$, $P_{\nu_\rho}(\sigma(\Lambda^c) > \kappa) > 0$.*

PROOF. We use the coupling described in Section 2. Thus, if $\tilde{\sigma}(\Lambda^c)$ is the stopping time corresponding to the coupled independent random walks, we have $\tilde{\sigma}(\Lambda^c) \leq \sigma(\Lambda^c)$. Thus,

$$\begin{aligned} &P_{\nu_\rho}\big(\sigma(\Lambda^c) > \kappa\big) \geq P_{\nu_\rho}\big(\tilde{\sigma}(\Lambda^c) > \kappa\big) \\ &\qquad = \int \prod_{i\notin\Lambda} \mathbb{P}\big(X(i,t) \notin \Lambda,\, \forall t \leq \kappa\big)^{\eta(i)}\, d\nu_\rho = \prod_{i\notin\Lambda} \frac{Z(\gamma(1-\delta_i))}{Z(\gamma)}, \end{aligned} \tag{4.16}$$

with $\delta_i = \mathbb{P}(X(i,t) \in \Lambda,\ t \leq \kappa)$. Now, by Jensen's inequality,

$$\frac{Z(\gamma(1-\delta))}{Z(\gamma)} \geq (1-\delta)^\rho.$$

Thus,

$$P_{\nu_\rho}\big(\sigma(\Lambda^c) > \kappa\big) \geq \left(\prod_{i\notin\Lambda}(1-\delta_i)\right)^\rho > 0 \quad \Longleftrightarrow \quad \sum_{i\in\mathbb{Z}^d} \delta_i < \infty. \tag{4.17}$$

Now, a particle starting on i reaches Λ within time κ if it makes at least $d(i,\Lambda)/R$ jumps within time κ (recall that R is the range of p). Thus, if $d(i)$ is the integer part of $d(i,\Lambda)/R$,

$$\mathbb{P}\big(X(i,t)\in\Lambda,\ t\le\kappa\big)\le\sum_{n\ge d(i)} e^{-\Delta\kappa}\frac{(\Delta\kappa)^n}{n!}\le\frac{(\Delta\kappa)^{d(i)}}{d(i)!}. \tag{4.18}$$

Hence, the series in (4.17) is converging. $\square$

PROOF OF PROPOSITION 2.7. The proof follows the same arguments as in the proof of Theorem 3(c) of [4] once inequality (4.5) is established with $\varepsilon_i=\mathbb{P}_i(S_n\notin\Lambda,\ \forall n\in\mathbb{N})$. It goes as follows. Let ν_ε be the product measure

$$d\nu_\varepsilon(\eta)=\prod_{i\in\Lambda} d\theta_{\gamma(\rho)}(\eta_i)\prod_{i\notin\Lambda} d\theta_{\varepsilon_i\gamma(\rho)}(\eta_i).$$

Let $\Lambda_n:=[-n;n]^d$ and let $\mathcal{G}_n$ be the σ-algebra $\sigma(\eta_i;i\in\Lambda_n)$. Then

$$\begin{aligned}&\nu_\rho\text{ p.s.}\qquad \frac{d\nu_\varepsilon}{d\nu_\rho}\bigg|_{\mathcal{G}_n}=\prod_{i\in\Lambda^c\cap\Lambda_n}\frac{\varepsilon_i^{\eta_i}Z(\gamma)}{Z(\varepsilon_i\gamma)},\\ &\nu_\varepsilon\text{ p.s.}\qquad \frac{d\nu_\rho}{d\nu_\varepsilon}\bigg|_{\mathcal{G}_n}=\prod_{i\in\Lambda^c\cap\Lambda_n}\frac{\varepsilon_i^{-\eta_i}Z(\varepsilon_i\gamma)}{Z(\gamma)}.\end{aligned} \tag{4.19}$$

Let $h(\alpha)$ denote the Laplace transform of θ_γ, that is, $h(\alpha)=Z(e^\alpha\gamma)/Z(\gamma)$. Note that h is defined for any α such that $e^\alpha\gamma<\sup g(k)$, and h is analytic in this domain. In particular, h is analytic in a neighborhood of 0. For all $i\notin\Lambda$, let α_i be defined by $e^{-\alpha_i}=\varepsilon_i$. A simple computation then yields, for all $p\ge1$,

$$\begin{aligned}\int\bigg(\frac{d\nu_\varepsilon}{d\nu_\rho}\bigg|_{\mathcal{G}_n}\bigg)^p d\nu_\rho&=\prod_{i\in\Lambda^c\cap\Lambda_n}\frac{Z(\varepsilon_i^p\gamma)}{Z(\gamma)}\frac{Z(\gamma)^p}{Z(\varepsilon_i\gamma)^p}=\prod_{i\in\Lambda^c\cap\Lambda_n}\frac{h(-p\alpha_i)}{h(-\alpha_i)^p},\\ \int\bigg(\frac{d\nu_\rho}{d\nu_\varepsilon}\bigg|_{\mathcal{G}_n}\bigg)^p d\nu_\varepsilon&=\prod_{i\in\Lambda^c\cap\Lambda_n}\frac{Z(\varepsilon_i^{-(p-1)}\gamma)}{Z(\gamma)}\frac{Z(\varepsilon_i\gamma)^{p-1}}{Z(\gamma)^{p-1}}\\ &=\prod_{i\in\Lambda^c\cap\Lambda_n}h\big(\alpha_i(p-1)\big)h(-\alpha_i)^{p-1}.\end{aligned} \tag{4.20}$$

The functions $m_p:\alpha\mapsto h(-p\alpha)/h(-\alpha)^p$ and $n_p:\alpha\mapsto h(\alpha(p-1))h(-\alpha)^{p-1}$ are analytic in a neighborhood of 0 and satisfy $m_p(0)=n_p(0)=1$, $m'_p(0)=n'_p(0)=0$, $m''_p(0)=n''_p(0)>0$ for $p>1$. Therefore, the products in (4.20) have finite limits when $n\to\infty$, as soon as $\sum_{i\in\Lambda^c}(1-\varepsilon_i)^2<+\infty$. In the asymmetric case, the Fourier transform of the Green function has a singularity at 0, which is square integrable as soon as $d\ge3$, so that the above series is convergent. Thus, for $d\ge3$, $d\nu_\varepsilon/d\nu_\rho|_{\mathcal{G}_n}$ is a $(P_{\nu_\rho},\{\mathcal{G}_n\})$ martingale, which is uniformly bounded in $L^p(\nu_\rho)$ for all $p \geq 1$. It follows from the martingale convergence theorem that ν_ε is absolutely continuous with respect to ν_ρ, with $d\nu_\varepsilon/d\nu_\rho \in L^p(\nu_\rho)$. In the same way, ν_ρ is absolutely continuous with respect to ν_ε, and $d\nu_\rho/d\nu_\varepsilon \in L^p(\nu_\varepsilon)$.

Following [4], we prove that this yields uniform $L^p(d\nu_\rho)$-estimates of $f_t := dT_t(\nu_\rho)/d\nu_\rho$ for $p \geq 1$. First of all, let us express the density of $\nu_t := T_t(\nu_\rho)$ with respect to ν_ρ. For φ continuous and bounded,

$$\int \varphi \, dT_t(\nu_\rho) = \frac{\int \bar{S}_t(\varphi)\mathbb{1}_{\mathcal{A}^c}\, d\nu_\rho}{\int \bar{S}_t(\mathbb{1}_{\mathcal{A}^c})\mathbb{1}_{\mathcal{A}^c}\, d\nu_\rho} = \int \varphi \frac{\bar{S}_t^*(\mathbb{1}_{\mathcal{A}^c})}{P^*_{\nu_\rho}(\tau > t)}\, d\nu_\rho,$$

so that ν_ρ-a.s. $f_t = P^*_\eta(\tau > t)/P^*_{\nu_\rho}(\tau > t)$.

Let $\mathcal{A}_0 = \{\eta;\, \forall i \in \Lambda,\, \eta_i = 0\}$. We prove now that, for any increasing function φ,

$$\int_{\mathcal{A}_0} \varphi \, d\nu_t \geq \frac{\nu_t(\mathcal{A}_0)}{\nu_\rho(\mathcal{A}_0)} \int_{\mathcal{A}_0} \varphi \, d\nu_\varepsilon. \tag{4.21}$$

To this end, let us write $\eta = (\eta_\Lambda, \eta_{\Lambda^c})$ for the decomposition of $\mathbb{N}^{\mathbb{Z}^d}$ in $\mathbb{N}^\Lambda \times \mathbb{N}^{\Lambda^c}$. Moreover, if μ is a probability measure on $\mathbb{N}^{\mathbb{Z}^d}$, let $\pi_{\Lambda^c}(\mu)$ denote its projection on $\sigma(\eta_i, i \in \Lambda^c)$. We have

$$\int_{\mathcal{A}_0} \varphi \, d\nu_t = \nu_\rho(\mathcal{A}_0) \int \varphi(0, \eta_{\Lambda^c}) f_t(0, \eta_{\Lambda^c}) \frac{d\nu_\rho}{d\nu_\varepsilon}(\eta_{\Lambda^c})\, d\pi_{\Lambda^c}(\nu_\varepsilon).$$

By (4.5), $\forall i \notin \Lambda$, $\Re_i f_t(0, \eta_{\Lambda^c}) \geq \varepsilon_i f_t(0, \eta_{\Lambda^c})$ and

$$\Re_i \frac{d\nu_\rho}{d\nu_\varepsilon} = \frac{1}{\varepsilon_i} \frac{d\nu_\rho}{d\nu_\varepsilon}.$$

Therefore, $f_t(0, \eta_{\Lambda^c})(d\nu_\rho/d\nu_\varepsilon)(\eta_{\Lambda^c})$ is an increasing function of η_{Λ^c}. Because $\pi_{\Lambda^c}(\nu_\varepsilon)$ is a product measure, it follows from FKG's inequality that

$$\int_{\mathcal{A}_0} \varphi \, d\nu_t \geq \nu_\rho(\mathcal{A}_0) \int \varphi(0, \eta_{\Lambda^c})\, d\pi_{\Lambda^c}(\nu_\varepsilon) \int f_t(0, \eta_{\Lambda^c}) \frac{d\nu_\rho}{d\nu_\varepsilon}(\eta_{\Lambda^c})\, d\pi_{\Lambda^c}(\nu_\varepsilon),$$

which is just (4.21).

We now apply (4.21) to the decreasing function $f_t^{p-1}(d\nu_\varepsilon/d\nu_\rho)^r$, $p \geq 1, r \geq 0$. We obtain

$$\begin{aligned}
\int_{\mathcal{A}_0} f_t^p \left(\frac{d\nu_\varepsilon}{d\nu_\rho}\right)^r d\nu_\rho &= \int_{\mathcal{A}_0} f_t^{p-1} \left(\frac{d\nu_\varepsilon}{d\nu_\rho}\right)^r d\nu_t \\
&\leq \frac{\nu_t(\mathcal{A}_0)}{\nu_\rho(\mathcal{A}_0)} \int_{\mathcal{A}_0} f_t^{p-1} \left(\frac{d\nu_\varepsilon}{d\nu_\rho}\right)^r d\nu_\varepsilon \\
&\leq \frac{\nu_t(\mathcal{A}_0)}{\nu_\rho(\mathcal{A}_0)} \int_{\mathcal{A}_0} f_t^{p-1} \left(\frac{d\nu_\varepsilon}{d\nu_\rho}\right)^{r+1} d\nu_\rho.
\end{aligned}$$

It follows by induction that, $\forall p, r \geq 0$,

$$\int_{\mathcal{A}_0} f_t^p \left(\frac{d\nu_\varepsilon}{d\nu_\rho}\right)^r d\nu_\rho \leq \left(\frac{\nu_t(\mathcal{A}_0)}{\nu_\rho(\mathcal{A}_0)}\right)^p \int_{\mathcal{A}_0} \left(\frac{d\nu_\varepsilon}{d\nu_\rho}\right)^{p+r} d\nu_\rho.$$

Taking $r = 0$ and applying once more FKG's inequality to the decreasing functions $\mathbb{1}_{\mathcal{A}_0}$ and f_t^p, we get, $\forall\, p \geq 1$,

$$\nu_\rho(\mathcal{A}_0) \int f_t^p \, d\nu_\rho \leq \int_{\mathcal{A}_0} f_t^p \, d\nu_\rho \leq \left(\frac{\nu_t(\mathcal{A}_0)}{\nu_\rho(\mathcal{A}_0)}\right)^p \int_{\mathcal{A}_0} \left(\frac{d\nu_\varepsilon}{d\nu_\rho}\right)^p d\nu_\rho,$$

so that, $\forall\, p \geq 1$,

$$\sup_t \int f_t^p \, d\nu_\rho \leq \frac{1}{\nu_\rho(\mathcal{A}_0)^{p+1}} \int_{\mathcal{A}_0} \left(\frac{d\nu_\varepsilon}{d\nu_\rho}\right)^p d\nu_\rho. \tag{4.22}$$

This, in turn, implies uniform $L^p(\nu_\rho)$-estimates for $d\Phi^n(\nu_\rho)/d\nu_\rho$. Indeed, using expression (3.1), if we define

$$\begin{aligned} dm_n(t) &= \frac{P_{\nu_\rho}(\tau > t) t^n \, dt}{\int_0^\infty P_{\nu_\rho}(\tau > t) t^n \, dt}, \quad \text{then} \\ \frac{d\Phi^n(\nu_\rho)}{d\nu_\rho} &= \int_0^\infty \frac{dT_t(\nu_\rho)}{d\nu_\rho} \, dm_{n-1}(t). \end{aligned} \tag{4.23}$$

Thus, using Hölder's inequality for $p \geq 1$,

$$\sup_{t>0} \int \left(\frac{dT_t(\nu_\rho)}{d\nu_\rho}\right)^p d\nu_\rho \leq C \quad \Longrightarrow \quad \sup_n \int \left(\frac{d\Phi^n(\nu_\rho)}{d\nu_\rho}\right)^p d\nu_\rho \leq C. \tag{4.24}$$

Moreover, we obtain the same uniform bounds for the Cesaro limit, and Proposition 2.7 follows. □

PROOF OF COROLLARY 2.8. We define the map Φ_* associated to the time-reversed dynamics. If ν is such that $E_\nu^*[\tau] < \infty$, then

$$\int \varphi \, d\Phi_*(\nu) = \frac{1}{E_\nu^*[\tau]} \int_0^\infty \int \bar{S}_t^*(\varphi) \, d\nu \, dt.$$

Our previous result (Proposition 2.7) holds equally for $\bar{\nu}_n^* := (1/n)(\Phi_*(\nu_\rho) + \cdots + \Phi_*^n(\nu_\rho))$, with the consequences that $\{\bar{\nu}_n^*, n \in \mathbb{N}\}$ is tight and $g_n := d\bar{\nu}_n^*/d\nu_\rho$ is uniformly in $L^p(\nu_\rho)$ for any $p \geq 1$ in dimensions $d \geq 3$. Let f_n be the density of $\bar{\nu}_n$ with respect to ν_ρ and assume that $\{f_n\}$ converge along a subsequence $\{n_k\}$ to the f solution of (2.10) and that $\{g_n\}$ converge along a subsequence $\{m_i\}$ to the g solution to the adjoint equation to (2.10). We can also assume that these convergences hold in weak $L^2(\nu_\rho)$. As f_n and g_n are decreasing functions, we have, by FKG's inequality,

$$\int f_{n_k} g_{m_i} \, d\nu_\rho \geq \int f_{n_k} \, d\nu_\rho \int g_{m_i} \, d\nu_\rho = 1.$$

After taking first the limit in k, and then in i, we obtain $\int fg\,d\nu_\rho \geq 1$. Also, this integral is finite by Cauchy–Schwarz. Thus, we can define $d\tilde{\nu}_\rho = fg\,d\nu_\rho/(\int fg\,d\nu_\rho)$. Let $dQ_t(\eta.)$ be the probability measure on paths, defined by

$$dQ_t(\eta.) := \frac{e^{\lambda(\rho)t} g(\eta_t) f(\eta_0)}{\int fg\,d\nu_\rho} \mathbb{1}_{\tau>t}\,dP_{\nu_\rho}(\eta.). \tag{4.25}$$

For φ such that $\varphi g \in L^2(\nu_\rho)$, we obtain, using (2.10),

$$\begin{aligned}\int \varphi(\eta_t)\,dQ_t(\eta.) &= \frac{\int E_\eta[\varphi(\eta_t)g(\eta_t)\mathbb{1}_{\tau>t}] f(\eta) e^{\lambda(\rho)t}\,d\nu_\rho(\eta)}{\int fg\,d\nu_\rho}\\ &= \frac{\int \bar{S}_t(\varphi g) f e^{\lambda(\rho)t}\,d\nu_\rho}{\int fg\,d\nu_\rho}\\ &= \frac{\int \varphi g \bar{S}_t^*(f) e^{\lambda(\rho)t}\,d\nu_\rho}{\int fg\,d\nu_\rho} = \int \varphi\,d\tilde{\nu}_\rho.\end{aligned}$$

Also, if φ is such that $\varphi f \in L^2(\nu_\rho)$,

$$\int \varphi(\eta_0)\,dQ_t(\eta.) = \frac{\int \bar{S}_t(g)\varphi f e^{\lambda(\rho)t}\,d\nu_\rho}{\int fg\,d\nu_\rho} = \int \varphi\,d\tilde{\nu}_\rho.$$

Now, by applying Jensen's inequality and recalling that $f, g \in L^p(\nu_\rho)$ for $p \geq 1$,

$$\begin{aligned}\log\bigl(P_{\nu_\rho}(\tau > t)\bigr) &= \log\biggl(\int fg\,d\nu_\rho\biggr) + \log\biggl(\int \frac{e^{-\lambda(\rho)t}}{g(\eta_t)f(\eta_0)}\,dQ_t(\eta.)\biggr)\\ &\geq \log\biggl(\int fg\,d\nu_\rho\biggr) - \int \log(g(\eta_t))\,dQ_t(\eta.)\\ &\quad - \int \log(f(\eta_0))\,dQ_t(\eta.) - \lambda(\rho)t\\ &\geq \log\biggl(\int fg\,d\nu_\rho\biggr) - \int \log(fg)\,d\tilde{\nu}_\rho - \lambda(\rho)t.\end{aligned}$$

This concludes the proof of the corollary. □

5. Example. Let us consider the totally asymmetric simple exclusion in one dimension. Thus,

$$\forall i \in \mathbb{Z}, \qquad p(i, i+1) = 1 \quad \text{and} \quad p(i,j) = 0 \qquad \text{if } j \neq i+1.$$

Let τ be the first time the origin is occupied. Let $\chi(\eta) := \inf\{k \geq 0 : \eta(-k) = 1\}$ and let N_t be a Poisson process of intensity 1. A simple computation yields

$$\begin{aligned}P_{\nu_\rho}(\tau > t) &= \int \mathbb{P}\bigl(N_t < \chi(\eta)\bigr)\,d\nu_\rho(\eta)\\ &= \sum_{k=1}^{\infty} \rho(1-\rho)^k \mathbb{P}(N_t < k) = (1-\rho)e^{-\rho t}.\end{aligned} \tag{5.1}$$

Thus,

$$\frac{P_{\nu_\rho}(\tau > t+s)}{P_{\nu_\rho}(\tau > t)} = e^{-\rho s} \quad \text{and} \quad \lambda(\rho) := \lim_t -\frac{1}{t}\log\bigl(P_{\nu_\rho}(\tau > t)\bigr) = \rho. \tag{5.2}$$

Following the approach of the proof of Theorem 3(c) of [4], it is easy to establish that the Yaglom limit exists and is

$$d\mu_\rho(\eta) = \prod_{i<0} d\mathcal{B}_\rho(\eta_i) \prod_{i\ge 0} d\mathcal{B}_0(\eta_i), \tag{5.3}$$

where $\mathcal{B}_\rho$ is the Bernoulli probability of parameter ρ. Can we approximate μ_ρ and $\lambda(\rho)$ by the corresponding quantities for the process on a large circle? The answer is no, as we shall see.

Let $\mathcal{C}_N = \{0, 1, \dots, N\}$, where sites N and 0 are identified, and consider the generator

$$\mathcal{L}_N\varphi = \sum_{i=0}^{N-1} \eta(i)\bigl(1-\eta(i+1)\bigr)\bigl(\varphi(\eta^i_{i+1}) - \varphi(\eta)\bigr), \tag{5.4}$$

with as invariant measure ν_N, which is the uniform measure on all configurations with $[\rho N]$ particles on $\mathcal{C}_N$.

Let $P_{\eta,N}$ be the law of the process generated by $\mathcal{L}_N$ and let η be in the support of ν_N. Then

$$P_{\eta,N}(\tau > t) = e^{-t} \sum_{k=1}^{\chi(\eta)-1} \frac{t^k}{k!}. \tag{5.5}$$

Thus, for a polynomial Q_N of degree at most N,

$$\begin{aligned} &P_{\nu_N,N}(\tau > t) = e^{-t} Q_N(t) \quad \Longrightarrow \\ &\lambda_N(\rho) := \lim_t -\frac{1}{t}\log\bigl(P_{\nu_N,N}(\tau > t)\bigr) = 1. \end{aligned} \tag{5.6}$$

Also, it is an easy computation that yields

$$\begin{aligned} &\lim_t \frac{P^*_{\eta,N}(\tau > t)}{P^*_{\nu_N,N}(\tau > t)} = \binom{N}{[\rho N]} \prod_{i=1}^{[\rho N]} \eta(-i) \quad \text{and} \\ &\lim_t \frac{P_{\nu_N,N}(\tau > t+s)}{P_{\nu_N,N}(\tau > t)} = e^{-s}. \end{aligned} \tag{5.7}$$

Thus, as in [4], one concludes the existence of a Yaglom limit μ_N concentrated on the configurations with particles occupying all $[\rho N]$ sites to the "left" of 0. Thus, μ_N and $\lambda_N(\rho)$ do converge, but to μ_1 and 1, respectively, and this approach misses all the μ_ρ with $\rho < 1$.

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