

## PARTIAL IMMUNIZATION PROCESSES

BY ALAN STACEY

*University of Cambridge*

Partial immunization processes are generalizations of the contact process in which the susceptibility of a site to infection depends on whether or not it has been previously infected. Such processes can exhibit a phase of weak survival, in which the process survives but drifts off to infinity, even on graphs such as  $\mathbb{Z}^d$ , where no such phase exists for the contact process. We establish that whether or not strong survival occurs depends only on the rate at which sites are reinfected and not on the rate at which sites are infected for the first time. This confirms a prediction by Grassberger, Chaté and Rousseau. We then study the processes on homogeneous trees, where the behaviour is also related to that of the contact process whose infection rate is equal to the reinfection rate of the partial immunization process. However, the phase diagram turns out to be substantially richer than that of either the contact process on a tree or partial immunization processes on  $\mathbb{Z}^d$ .

**1. Introduction.** In this paper we shall be concerned with a family of processes, which we shall call partial immunization processes, and which generalize the widely studied contact process. We begin by defining the *partial immunization process* with parameters  $(\lambda_n, \lambda_o)$  (short for  $\lambda_{\text{new}}$  and  $\lambda_{\text{old}}$ ) on a graph  $G = (V, E)$ ; all graphs are assumed to be connected and of bounded degree and throughout most of this paper we shall be interested in the case when  $G$  is either  $\mathbb{Z}^d$  or the infinite homogeneous tree in which each vertex has  $d$  neighbours,  $\mathbb{T}_d$ .

A *partial immunization process* is a continuous-time Markov process with state space  $\{-1, 0, 1\}^V$ , so each vertex can be in state  $-1$ ,  $0$  or  $1$ . We shall denote the state of the process at time  $t$  by  $\eta_t$ , so the state of a vertex  $v$  is  $\eta_t(v)$ . The state of each vertex  $v$  flips according to the following rules:

$$\begin{aligned} -1 &\rightarrow 1 && \text{at rate } \lambda_n \cdot \#\{w : w \sim v \text{ and } \eta_t(w) = 1\}, \\ 0 &\rightarrow 1 && \text{at rate } \lambda_o \cdot \#\{w : w \sim v \text{ and } \eta_t(w) = 1\}, \\ 1 &\rightarrow 0 && \text{at rate } 1, \end{aligned}$$

where  $w \sim v$  means that  $w$  is joined to  $v$  by an edge (which we will denote  $wv$ ). For general information about interacting particle systems, such as the fact that the above rates do indeed define a unique process (subject to some basic constraints, such as right-continuity of sample paths), see Liggett [14]. For basic graph-theoretic terminology see Bollobás [4].

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Received June 2001; revised April 2002.

AMS 2000 subject classification. 60K35.

Key words and phrases. Contact process, immunization, tree, weak survival, critical value, phase diagram.

We shall think of the process as modelling the spread of a disease. If  $\eta(v) = 1$ , we call  $v$  *infected*; if  $\eta_t(v) = 0$ , we call  $v$  *previously infected*; and if  $\eta_t(v) = -1$ , we call  $v$  *never infected*. We shall usually start the process at time 0 with a certain subset,  $A$ , of vertices (often just a single distinguished vertex  $O$ , the *root* or *origin*) in state 1 and all the other vertices in state  $-1$ . Note then that any vertex ever in state 0 (“previously infected”) will indeed, at some previous time, have been in state 1. We shall use  $\eta_t^A$  to denote the process with this starting set and we will abbreviate  $\eta_t^{\{O\}}$  by  $\eta_t^O$ .

In the case  $\lambda_o = \lambda_n$ , where the states  $-1$  and  $0$  are essentially equivalent, this process reduces to the very widely studied contact process introduced by Harris [9] in 1974; see [16] for a recent detailed account. States of the contact process will be identified with subsets of  $V$ , corresponding to sets of infected vertices. In the contact process an infected vertex recovers at rate 1; an uninfected vertex becomes infected at a rate proportional to the number of infected neighbours. Partial immunization processes differ in that the constant of proportionality may be different according to whether or not the uninfected vertex in question has been previously infected. If  $\lambda_o > \lambda_n$ , then a previously infected vertex is more susceptible to future infections. If  $\lambda_o < \lambda_n$ , then previously infected vertices have been partially immunized.

We shall say that a partial immunization process *survives* if

$$\mathbb{P}(\forall t, \eta_t^O(v) = 1 \text{ for some } v) > 0$$

and that it *survives strongly* if

$$\mathbb{P}(\forall T, \exists t > T \text{ with } \eta_t^O(O) = 1) > 0.$$

If the process survives, but does not survive strongly, we say that it *survives weakly*. These definitions agree with standard definitions for the contact process and we now summarize the most fundamental facts which are known in that case. For the contact process on an infinite connected graph of bounded degree,  $G$ , it is well known that there exist critical values  $0 < \lambda_1(G) \leq \lambda_2(G) < \infty$  with the following property. Denoting by  $\lambda$  the common value of  $\lambda_o$  and  $\lambda_n$ , then the process does not survive if  $\lambda < \lambda_1$ , it survives weakly if  $\lambda_1 < \lambda < \lambda_2$  and it survives strongly if  $\lambda > \lambda_2$ . On  $\mathbb{Z}^d$  (for any  $d \geq 1$ ) it is known—but far from straightforward to prove—that  $\lambda_1 = \lambda_2$ , so there is no phase of weak survival [2]. In contrast, on  $\mathbb{T}_d$ , one has  $\lambda_1 < \lambda_2$  (for  $d \geq 3$ , note that  $\mathbb{T}_2 = \mathbb{Z}$ ) [18, 15, 22]. See also [16] for an overview (and for proofs).

Partial immunization processes on  $\mathbb{Z}^d$  have been studied by Durrett and Schinazi [7], and independently by Grassberger, Chaté and Rousseau [8]. The special case when  $\lambda_o = 0$  is known, for obvious reasons, as the *forest fire model* and had been studied earlier by Kulasmaa [11]. In this special case, it is not hard to see that there exists a critical  $\lambda_f = \lambda_f(G) \in (0, \infty]$  such that if  $\lambda_n > \lambda_f$ , then one has weak survival and if  $\lambda_n < \lambda_f$ , one has no survival. It is clear that strong survival cannot occur in this case. Furthermore,  $\lambda_f(\mathbb{Z}^d)$  is finite if and only if  $d \geq 2$ .

In [8], simulations were reported which suggested that whether or not the partial immunization process on  $\mathbb{Z}^d$  survives strongly depends only on the value of  $\lambda_o$  (provided  $\lambda_n > 0$ ). In other words, letting  $\lambda_c$  denote (as usual) the common value of  $\lambda_1$  and  $\lambda_2$  on  $\mathbb{Z}^d$ , the process survives strongly if and only if  $\lambda_o > \lambda_c$ . In independent work, Durrett and Schinazi [7] proved that  $\lambda_o > \lambda_c$  is equivalent to a different notion, which they called persistence. The process  $\eta_t$  is said to be *persistent* if

$$\liminf_{t \rightarrow \infty} \mathbb{P}(\eta_t^O(O) = 1) > 0.$$

The question of strong survival was not explicitly addressed in [7]. It is trivial that persistence implies strong survival. The reverse implication is known to hold for the contact process on  $\mathbb{Z}^d$  and trees  $\mathbb{T}_d$ ; it is a consequence of the complete convergence theorem (see [16] for an explanation and references) which is far from easy. There are, however, graphs for which the reverse implication does not hold; see [20].

The first principal result of this paper is that strong survival cannot occur on  $\mathbb{Z}^d$  for  $\lambda_o < \lambda_c$ , whatever the value of  $\lambda_n$ . Combined with the reverse implication (up to behaviour at the critical value) in [7] this will establish the following theorem, suggested by [8].

**THEOREM 1.1.** *Let  $\eta_t$  be the partial immunization process on  $\mathbb{Z}^d$  with parameters  $(\lambda_n, \lambda_o)$ . Let  $\lambda_c$  be the contact process critical value on the same graph. Then if  $\lambda_o > \lambda_c$  and  $\lambda_n > 0$ ,  $\eta_t$  survives strongly. If  $\lambda_o < \lambda_c$ , then  $\eta_t$  does not survive strongly.*

In recent years, questions traditionally asked about processes on  $\mathbb{Z}^d$  have been considered on more general graphs, particularly homogeneous trees. The main interest in studying the contact process on trees is that different behaviour from that on  $\mathbb{Z}^d$  can occur:  $\lambda_1 < \lambda_2$  so there is a phase of weak survival. We shall consider partial immunization processes on trees. It turns out that the phase diagram is significantly richer than is the case either for the process on  $\mathbb{Z}^d$  or for contact processes on trees. We prove the following theorem.

**THEOREM 1.2.** *Let  $\eta_t$  be the partial immunization process on  $\mathbb{T}_d$  ( $d \geq 3$ ) with parameters  $(\lambda_n, \lambda_o)$ , where  $\lambda_n$  is assumed to be nonzero. Let  $\lambda_1 < \lambda_2$  be the critical values for the contact process on the same graph.*

(i) *If  $\lambda_o < \lambda_1$ , then for sufficiently large  $\lambda_n$  one has weak survival and for sufficiently small  $\lambda_n$  one has no survival. Strong survival cannot occur.*

(ii) *If  $\lambda_1 < \lambda_o < \lambda_2$ , then for sufficiently large  $\lambda_n$  (dependent on  $\lambda_o$ ) one has strong survival. For sufficiently small  $\lambda_n$  one has no survival. There is an open interval of values of  $\lambda_n$  for which weak survival occurs; indeed, this conclusion holds if  $\lambda_1 < \lambda_n < \lambda_2$ .*

(iii) *If  $\lambda_o > \lambda_2$ , then strong survival occurs.*

In the next section of this paper, we explain the graphical representation of processes and state some background results which are needed. In Section 3 we prove Theorem 1.1 and in Section 4 we prove the results for trees. In the final section we discuss further questions, particularly the issue of monotonicity of the process as the parameters vary.

**2. Preliminaries.** The graphical representation is an important tool in the study of certain interacting particle systems and was first introduced by Harris [10] for the contact process. Here we describe a graphical representation for partial immunization processes. Some familiarity with the representation for the contact process is advisable (see [5] or [16]).

Given a graph  $G = (V, E)$ , parameters  $\lambda_n$  and  $\lambda_o$  and  $\eta_0 : V \rightarrow \{-1, 0, 1\}$ , we give a construction of the partial immunization process on  $G$  with parameters  $(\lambda_n, \lambda_o)$  and starting state  $\eta_0$ . Let us suppose that  $\lambda_n > \lambda_o$  (the other case is similar). We begin with the set  $V \times [0, \infty)$ , thinking of each  $\{v\} \times [0, \infty)$  as a timeline. To each vertex  $v \in V$  we associate an independent Poisson process of rate 1, and for the arrival times  $t$  of this process we place a *recovery mark* at  $(v, t)$ . To each edge  $e = \{v, w\}$  of the graph we associate four independent Poisson processes. At the arrival times,  $t$ , of a Poisson process of rate  $\lambda_o$  we place a *stable infection arrow*,  $(\overrightarrow{vw}, t)$ , which points from  $(v, t)$  to  $(w, t)$ . At the arrival times of a Poisson process of rate  $\lambda_n - \lambda_o$  we place similar *virgin infection arrows*. The other two Poisson processes associated with  $e$  are much the same, but the infection arrows go from  $w$  to  $v$ .

One constructs the process  $\eta_t$  by setting  $\eta_t(v) = 1$  (so  $v$  is infected at time  $t$ ) if there is an infection trail (which we are about to define) from  $(w, 0)$  to  $(v, t)$  for some  $w$  with  $\eta_0(w) = 1$ . We will set  $\eta_t(v) = 0$  if there is no such infection trail and either there is some time  $s \in [0, t)$  with  $\eta_s(v) = 1$ , or  $\eta_0(v) = 0$ . Otherwise we set  $\eta_t(v) = -1$ . An *infection trail* from  $(w, s)$  to  $(v, t)$ , (for  $s \leq t$ ) is, as for the contact process, a path that goes along the timelines of  $V \times [0, \infty)$  in the direction of increasing time and along infection arrows in the direction of the arrows, without passing through any recovery marks; additionally, we require that it only uses virgin infection arrows— $(\overrightarrow{vw}, t)$ , say—if  $w$  is in the state  $-1$  immediately prior to time  $t$ . In other words, virgin infection arrows may not be used to infect previously infected vertices.

There is some danger of circularity in the above definition. The process  $\eta_t$  is defined in terms of infection trails, but to define an infection trail one needs to know which vertices have been previously infected. It is clear that on a finite graph there are no problems: there are almost surely, in any finite time period, only finitely many arrows and recovery marks, and the effect of each can be considered in chronological order. For an infinite graph one can, however, reduce to the finite case, in the following way. To any given time period, one associates the spanning subgraph of  $G$  whose edge set consists precisely of those edges along which there is some infection arrow (in either direction) during that time. It is not

too hard to show that there is some (deterministic)  $\delta > 0$  with the property that in any time period of length  $\delta$ , the associated subgraph almost surely contains only finite connected components. One can therefore increment time in steps of length  $\delta$  and thereby consider only finite graphs at each step. The details of this general argument can be found in [6].

In the case when  $\lambda_n < \lambda_o$  the graphical representation will contain stable infection arrows and *reinfection arrows*. Reinfection arrows can only be used to infect vertices which are in state 0. In either case, the term *stable infection trail* will refer to an infection trail all arrows of which are stable.

We shall make considerable use of the graphical representation in our proofs. One very useful property, which we explore a little here, is that it gives an easy way to couple slightly different processes. Processes with the same parameters and different starting states on the same graph are immediately coupled by the representation. With a little more work, other couplings are possible. In particular, if we consider only the stable infection arrows, we obtain the graphical representation for the contact process with parameter  $\min(\lambda_n, \lambda_o)$ . Any infection trail in this contact process is automatically a valid infection trail in the partial immunization process. On the other hand, if we treat all the arrows in the partial immunization process as stable, we obtain the graphical representation of the contact process with parameter  $\max(\lambda_n, \lambda_o)$ . In this case we have the converse implication, that a valid trail in the partial immunization process is necessarily a trail in this contact process. We thereby obtain the following result.

PROPOSITION 2.1. *Let  $\eta_0 : V \rightarrow \{-1, 0, 1\}$  be a starting state for a partial immunization on a graph  $G = (V, E)$  of bounded degree and let  $A = \eta_0^{-1}(\{1\})$ . Let  $\lambda_n, \lambda_o \geq 0$ . Then one can construct, on a common probability space, the following processes:*

- $\eta_t$ , the partial immunization process with parameters  $(\lambda_n, \lambda_o)$  and starting state  $\eta_0$ ;
- $\xi_t^A$ , the contact process with parameter  $\min(\lambda_n, \lambda_o)$  and initial set of infected sites  $A$ ;
- $\Xi_t^A$ , the contact process with parameter  $\max(\lambda_n, \lambda_o)$  and initial set of infected sites  $A$ ;

in such a way that, for all  $t$  and for all  $v \in V$ ,

$$\xi_t^A(v) = 1 \quad \Rightarrow \quad \eta_t(v) = 1$$

and

$$\eta_t(v) = 1 \quad \Rightarrow \quad \Xi_t^A(v) = 1.$$

Proposition 2.1 says that  $\Xi_t$  stochastically dominates  $\eta_t$  and  $\eta_t$  dominates  $\xi_t$  (if we consider states 0 and  $-1$  to be the same for this purpose). For more information

about this concept see, for example, [14]. Certain questions about stochastic domination are not as straightforward for partial immunization processes as they are for the contact process. For example, if  $\lambda_o < \lambda_n^1 < \lambda_n^2$ , then it is not clear that the process with parameters  $(\lambda_o, \lambda_n^2)$  dominates that with parameters  $(\lambda_o, \lambda_n^1)$ . See Section 5 for a more detailed discussion.

We now collect together a miscellany of results about the contact process which we shall need to use. For reasons of brevity, we state the results tersely and give references for further information.

It is not too hard to show that, starting from a finite number of infected sites, the contact process with any parameter on any graph of bounded degree does not explode: at any time there are only finitely many infected sites and the process can be regarded as a Markov chain on the set of finite subsets of the vertex set; see [16], I(1.19).

The contact process is self-dual. If  $(\xi_t^A)$  and  $(\xi_t^B)$  are contact processes with the same parameter on the same graph with starting sets  $A$  and  $B$ , respectively, then for all  $t$ ,

$$(2.1) \quad \mathbb{P}(\xi_t^A \cap B \neq \emptyset) = \mathbb{P}(\xi_t^B \cap A \neq \emptyset).$$

This duality relation is easily seen by reversing the arrows and the direction of time in the graphical representation. We will use it most often in the case when  $B$  is a singleton. For more information about duality, see [14].

On a graph with a distinguished vertex,  $O$ , the distance of another vertex  $v$  from  $O$  (i.e., the number of edges in the shortest path from  $O$  to  $v$ ) will be denoted  $|v|$ . The ball of radius  $n$  about  $O$  is

$$B_n = \{v \in V : |v| \leq n\}.$$

The *boundary*,  $\partial W$ , of a set of vertices  $W \subset V$ , shall mean the internal boundary,  $\{w \in W : \exists v \in V \setminus W \text{ with } vw \in E\}$ .

In the subcritical phase of the contact process on a reasonable graph, it is expected that a number of quantities decay exponentially as a function of time or distance. For the process on  $\mathbb{Z}^d$  these results were established by Bezuidenhout and Grimmett [3] (see also [16]). The one result of this kind we shall need is the following.

**THEOREM 2.2.** *Let  $\xi_t^O$  denote the contact process on  $\mathbb{Z}^d$ , with parameter  $\lambda < \lambda_c$ , starting from a single infection at the origin. Then there exists  $\gamma > 0$  such that for all  $v \in \mathbb{Z}^d$ ,*

$$\mathbb{P}(v \in \xi_t^O \text{ for some } t) \leq e^{-\gamma|v|}.$$

The remainder of our preliminary results concern the contact process on trees. Letting  $\xi_t^O$  denote the process on  $\mathbb{T}_d$  starting from a single infection at the root, as usual, let  $v_n$  be some vertex at distance  $n$  from  $O$  and define

$$(2.2) \quad u_n = \mathbb{P}(v_n \in \xi_t^O \text{ for some } t).$$

It is not hard to show [16] that

$$(2.3) \quad \beta(\lambda) = \lim_{n \rightarrow \infty} (u_n)^{1/n}$$

exists and satisfies  $u_n \leq \beta^n$ . We shall need the following properties of  $\beta$ , which were established in several stages [13, 21, 12] (see also [16]).

**THEOREM 2.3.** *For the contact process on  $\mathbb{T}_d$  ( $d \geq 3$ ) the  $\beta$  function, defined by (2.3), is continuous and strictly increasing on  $[0, \lambda_2]$  with:*

- (i)  $\beta(\lambda_1) = 1/(d - 1)$ ;
- (ii)  $\beta(\lambda_2) = 1/\sqrt{d - 1}$ ;
- (iii)  $\beta(\lambda) = 1$  for  $\lambda > \lambda_2$ .

Note that our  $\mathbb{T}_d$  is denoted  $T_{d-1}$  by some authors.

We often regard  $\mathbb{T}_d$  as being arranged in *levels*, indexed by  $\mathbb{Z}$ , in such a way that every vertex in level  $n$  has one neighbour in level  $n - 1$  (its parent) and  $d - 1$  neighbours in level  $n + 1$  (its children). The root is in level 0. As usual for rooted trees, the transitive closure of the children relation is the descendant relation. The induced subtree whose vertex set consists precisely of  $O$  and its descendants [so that level  $n$  contains  $(d - 1)^n$  vertices for  $n \geq 0$ ] will be denoted  $\mathbb{T}'_d$ . It is not too surprising that the behaviour of the contact process does not change very much if we restrict our attention to  $\mathbb{T}'_d$ . The following lemma is a particular case of this fact combined with the fact (more important for us) that we lose essentially nothing by replacing  $\mathbb{P}(v_n \in \xi_t^O$  for some  $t$ ) in the definition of  $u_n$  by  $\sup_t \mathbb{P}(v_n \in \xi_t^O)$ , effectively enabling us to fix a specific choice of  $t$ . We assume that the  $v_n$  of (2.2) was chosen so as to lie in  $\mathbb{T}'_d$ .

**LEMMA 2.4.** *Let  $\beta$  be as defined in (2.3) for the contact process on  $\mathbb{T}_d$  and let  $\xi'_t$  be the contact process with the same parameter on the subtree  $\mathbb{T}'_d$ . Then*

$$\beta = \lim_{n \rightarrow \infty} \left[ \sup_t P(v_n \in \xi'_t) \right]^{1/n}.$$

Lemma 2.4 is (a trivial variant of) Lemma I.4.53 of [16], owing to Salzano and Schonmann [19].

Finally, we quote some results about the contact process on finite trees. We shall use  $\mathbb{T}_d^h$  to denote the finite subtree whose vertex set consists of the root,  $O$ , and its descendants down to level  $h$ . The contact process on  $\mathbb{T}_d^h$  is a finite state-space Markov chain with an absorbing state to which all other states lead, namely the state with no vertex infected. Therefore the contact process will almost surely reach this state eventually. The time it takes to reach this state when  $h$  is large is closely related to the behaviour of the process on the infinite tree  $\mathbb{T}_d$ . We will use the following two instances of this principle which say, roughly, that in the

intermediate phase the process on  $\mathbb{T}_d^h$  survives for time linear in  $h$  and in the strong survival phase it survives for a time which is almost exponential in the number of vertices of  $\mathbb{T}_d^h$ . For further information, see [23].

**THEOREM 2.5.** *Suppose  $d \geq 3$  and let  $\lambda_1(\mathbb{T}_d) < \lambda < \lambda_2(\mathbb{T}_d)$ . Then there exist constants  $c, C > 0$  with the following property. Let  $\xi_t^h$  be the contact process with parameter  $\lambda$  on  $\mathbb{T}_d^h$  starting with all sites infected and let  $\tau_h^h = \inf\{t > 0 : \xi_t^h = \emptyset\}$  be the time until the absorbing state is hit. Then*

$$\mathbb{P}(ch < \tau_h^h < Ch) \rightarrow 1 \quad \text{as } h \rightarrow \infty.$$

**THEOREM 2.6.** *Let  $d \geq 3, a < 1, \lambda > \lambda_2(\mathbb{T}_d)$  and let  $\tau_h^O$  be the extinction time for the contact process with parameter  $\lambda$  on  $\mathbb{T}_d^h$  starting from a single infection at the root. Then there exist  $p > 0, c > 0$  and  $\alpha > 1$  such that for all  $h$ ,*

$$\mathbb{P}(\tau_h^O \geq c\alpha^{(a(d-1))^h}) \geq p.$$

Note that this last theorem is vacuous if  $a \leq 1/(d - 1)$  and strongest for  $a$  close to 1. It is not too hard to see that it cannot be improved to  $a > 1$ , but it remains open whether it holds with  $a = 1$ .

**3. Processes on  $\mathbb{Z}^d$ .** The purpose of this section is to prove that, for a partial immunization process with parameters  $(\lambda_n, \lambda_o)$  on  $\mathbb{Z}^d$ , if  $\lambda_o$  is less than the contact process critical value  $\lambda_c$ , then strong survival does not occur. This will establish Theorem 1.1.

Throughout this section we consider such a process on  $\mathbb{Z}^d$  starting from a single infection at  $O, \eta_t^O$ , with all other vertices initially “never infected.” We assume that  $\lambda_o < \lambda_c(\mathbb{Z}^d)$ . If  $\lambda_n < \lambda_c$ , then, by Proposition 2.1, the process is stochastically dominated by the contact process with parameter  $\max(\lambda_n, \lambda_o)$ , so does not survive. Therefore we may assume that  $\lambda_n \geq \lambda_c > \lambda_o$  and we wish to show that strong survival does not occur.

We will make use of the graphical representation described in detail in Section 2. As  $\lambda_n > \lambda_o$  we have *stable arrows* from each vertex to each neighbouring vertex at the arrival times of a Poisson process of rate  $\lambda_o$  and *virgin arrows* between such vertices at rate  $\lambda_n - \lambda_o$ . A *stable infection trail* from  $(v, t)$  to  $(w, s)$  (for vertices  $v, w$  and times  $t, s$ ) is an infection trail which does not contain any virgin arrows.

For our process  $\eta_t^O$  and for each vertex  $v \in \mathbb{Z}^d$ , we let  $A_v$  denote the event that there is a **stable** infection trail from  $(v, \tau(v))$  to  $(O, t)$  for some time  $t$ , where

$$\tau(v) = \inf\{s : \eta_s(v) = 1\};$$

if there is no time at which  $v$  is infected, then the event  $A_v$  does not occur. Now, conditional on the finiteness of  $\tau(v)$ , the probability of  $A_v$  is exactly equal to the probability that  $v$  is ever infected, starting from a single infection at the origin, in a



contact process with parameter  $\lambda_o$ . By Theorem 2.2, there exists a constant  $\gamma > 0$  so that

$$(3.1) \quad \mathbb{P}(A_v) \leq e^{-\gamma|v|}.$$

Therefore, letting

$$(3.2) \quad N = \inf \left\{ n : \text{the event } \bigcap_{v:|v|>n} A_v^c \text{ occurs} \right\},$$

by the first Borel–Cantelli lemma,  $N$  is almost surely finite. Let  $S$  be the event that strong survival occurs, that is,  $S = \{\forall T, \exists t > T \text{ with } \eta_t^O(O) = 1\}$ . Let us suppose that  $\mathbb{P}(S) > 0$ , with view to a contradiction. Then it is easily seen via a conditional form of the second Borel–Cantelli lemma, that conditional on  $S$ , the quantity

$$(3.3) \quad T_0(n) = \inf\{t > 0 : \eta_t^O(v) \neq -1, \forall v \in B_n\}$$

is almost surely finite for any  $n$ ; in particular,  $T_0 = T_0(N)$ , for the random integer  $N$  defined by (3.2), is finite. Then let  $I = (\eta_{T_0}^O)^{-1}(\{1\})$  be the set of infected sites at time  $T_0$ . For each  $v \in I$ , let

$$T_1(v) = \sup\{t \geq T_0 : \exists \text{ a stable infection trail from } (v, T_0) \text{ to } (O, t)\},$$

where the supremum of the empty set is taken to be  $T_0$  in this case.

Now, letting  $\xi_t$  be the set of vertices  $w$  such that there is a stable infection trail from  $(v, T_0)$  to  $(w, t)$ , then  $(\xi_t)_{t \geq T_0}$  is a realization of the contact process with parameter  $\lambda_o$  starting from a single infection at  $v$  at time  $T_0$ . Since  $\lambda_o < \lambda_c(\mathbb{Z}^d)$ , this contact process does not survive. Therefore, for each  $v \in I$ , one has that  $T_1(v)$  is almost surely finite. Since  $I$  is itself almost surely finite, then

$$T_1 = \max_{v \in I} T_1(v)$$

is finite a.s. Note that if  $t \geq T_1$ , then there is no stable infection trail from  $(v, T_0)$  to  $(O, t)$  for any  $v \in I$ . We claim that, in fact,  $\eta_t^O(O) = 0$  for all  $t \geq T_1$ , contradicting strong survival. To see this, suppose not. Then, since  $I$  is the set of infected sites at time  $T_0$  ( $\leq T_1$ ) there must be, for some  $v \in I$ , an infection trail from  $(v_0, T_0)$  to  $(O, t_0)$  for some  $t_0 \geq T_1$ . By the definition of  $T_1$  (and the right-continuity of the process) this cannot be a stable infection trail, so it contains some virgin arrows. Let  $(e, s)$  be the last virgin arrow on the infection trail, where  $e$  is directed from vertex  $v_1$  to vertex  $v_2$ . Then  $s = \tau(v_2)$  is the time of first infection of  $v_2$ , so, by the definition of  $T_0$ ,  $v_2 \notin B_N$ . However, then, by (3.2),  $A_{v_2}$  does not occur, that is, there is no stable infection trail from  $(v_2, s)$  to the origin. However, the remainder of the infection trail beyond  $(e, s)$  leading to  $(O, t_0)$  constitutes precisely such a trail, since  $(e, s)$  was the last virgin arrow on the trail from  $(v_0, T_0)$  to  $(O, t_0)$ . This gives the desired contradiction, completing the proof of Theorem 1.1.

**4. Processes on trees.** In this section we establish Theorem 1.2 as a series of separate propositions. Some parts follow very easily from known results, whereas others require considerable work. We start with part (i) which deals with the case  $\lambda_0 < \lambda_1$ .

**PROPOSITION 4.1.** *Let  $d \geq 3$  and  $\lambda_0 < \lambda_1(\mathbb{T}_d)$ . Then for  $\lambda_n < \lambda_1(\mathbb{T}_d)$  the partial immunization process on  $\mathbb{T}_d$  with parameters  $(\lambda_n, \lambda_0)$  does not survive. There exists  $\lambda_k = \lambda_k(\lambda_0)$  such that if  $\lambda_n > \lambda_k$ , then the partial immunization process survives weakly. For no values of  $\lambda_n$  does it survive strongly.*

**PROOF.** The first conclusion follows immediately from Proposition 2.1, since if  $\lambda_0, \lambda_n < \lambda_1$ , then the partial immunization process is dominated by the contact process with parameter  $\max(\lambda_n, \lambda_0)$  which does not survive.

The fact that strong survival cannot occur, however large  $\lambda_n$  is, follows from the same argument used to prove Theorem 1.1. To see this, suppose  $\lambda_n \geq \lambda_1 > \lambda_0$  and, for each vertex  $v$ , let the event  $A_v$  be as defined in the proof of Theorem 1.1. Then if  $v$  is at distance  $n$  from  $O$ ,

$$\mathbb{P}(A_v) = u_n(\lambda_0) \leq (\beta(\lambda_0))^n,$$

where  $u_n$  is defined by (2.2). Since  $\lambda_0 < \lambda_1$ , Theorem 2.3 implies that  $\beta(\lambda_0) < 1/(d - 1)$ . Therefore,

$$(4.1) \quad \sum_{v \in \mathbb{T}_d} \mathbb{P}(A_v) < \infty.$$

The fact that (4.1) holds is precisely what enables the remainder of the proof of Theorem 1.1 to apply unchanged to this case, showing that strong survival cannot occur.

We now show how a branching process comparison establishes that if  $\lambda_n(d - 1)/(\lambda_n + 1) > 1$ , then weak survival occurs and that, furthermore, this bound is tight if  $\lambda_0 = 0$ . Let  $Z_n$ , a random subset of  $\partial B_n$ , be defined recursively as follows. Let  $Z_0 = \{O\}$ . A vertex  $v$  in  $\partial B_n$  ( $n \geq 1$ ) lies in  $Z_n$  if its parent,  $w$  (i.e., unique neighbour in  $\partial B_{n-1}$ ), lies in  $Z_{n-1}$  and there is an infection arrow from  $w$  to  $v$  between the time of first infection of  $w$  and the time of the first recovery at  $w$  after that time. It is then easily seen that all the vertices in  $Z_n$  are infected at some time (and that when  $\lambda_0 = 0$ ,  $Z_n$  consists of all the vertices of  $\partial B_n$  that are ever infected). Furthermore,  $(|Z_n|)_{n \geq 0}$  is a simple branching process, the offspring distribution of which has mean  $(d - 1)\lambda_n/(\lambda_n + 1)$  (other than for the first generation). Therefore if  $d\lambda_n/(\lambda_n + 1) > 1$ , this branching process has a positive probability of non-extinction, which implies survival of the underlying partial immunization process. So we may take  $\lambda_k = 1/(d - 2)$ .  $\square$

We now turn to the intermediate case when  $\lambda_1(\mathbb{T}_d) < \lambda_0 < \lambda_2(\mathbb{T}_d)$ . To demonstrate what happens when  $\lambda_n$  is large, we shall need a lemma, and to state

this lemma, we shall need some further notation. Recall that  $\mathbb{T}'_d$  is the subtree of  $\mathbb{T}_d$  obtained by deleting the edge connecting the root to its parent and then taking the connected component containing the root in the resulting graph. Therefore  $\mathbb{T}'_d$  contains  $(d - 1)^n$  vertices at level  $n$  for each  $n \geq 0$ . Let  $v_n$  be a vertex at level  $n$  in this tree and let  $v_0e_1v_1e_2 \cdots e_nv_n$  be the path from  $O (= v_0)$  to  $v_n$ . In the graphical representation of a partial immunization process on  $\mathbb{T}_d$  or on  $\mathbb{T}'_d$ , an *immediate infection trail* (or i.i.t.) from  $(O, s)$  to  $(v_n, S)$  is a sequence of infection arrows  $(e_1, t_1), \dots, (e_n, t_n)$  with the following properties, where, for convenience, we set  $t_0 = s$  and  $t_{n+1} = S$ :

1. There is no recovery mark at any  $v_i$  ( $0 \leq i \leq n$ ) between time  $t_i$  and time  $t_{i+1}$ .
2. For  $1 \leq i \leq n$ , there is no infection arrow  $(e_i, t)$  for any  $t$  with  $t_{i-1} \leq t < t_i$ .
3. There is no infection arrow  $(e, t)$  for any edge  $e$  leading from  $v_n$  to any vertex at level  $n + 1$  for any time  $t$  with  $t_n \leq t \leq S$ .

Immediate infection trails are the fastest possible infection trails in a sense we now make precise. We consider a partial immunization process on  $\mathbb{T}_d$  in the case when  $\lambda_n \geq \lambda_0$ , starting from an initial configuration in which  $O$  is infected and all the other vertices of  $\mathbb{T}'_d$  are in the “never infected” state. Vertices of  $\mathbb{T}_d$  outside  $\mathbb{T}'_d$  may begin in any state. Then, if there is an immediate infection trail from  $(O, 0)$  to  $(v_n, T)$ , then  $v_n$  is infected at time  $T$  and, using the above notation,  $t_n$  is the time of first infection of  $v_n$ ; this is true regardless of activity on other parts of the tree.

Similarly, given a descendant  $v_p$  of  $v_n$ , at level  $p$  ( $p > n$ ) we may define an immediate infection trail from  $(v_n, S)$  to  $(v_p, T)$ . Property 3 of immediate infection trails enables us to piece together such a trail with a trail from  $(O, s)$  to  $(v_n, S)$  (where  $s < S < T$ ) to obtain an immediate infection trail from  $(O, s)$  to  $(v_p, T)$ .

We shall need to know that when  $\lambda_n$  is large, many of the vertices at a given level can be reached by immediate infection trails lasting a particular time. Let  $V_t^n$  be those vertices  $v$  of  $\mathbb{T}'_d$  at level  $n$  for which there is an immediate infection trail from  $(O, 0)$  to  $(v, t)$ . Then we need the following.

LEMMA 4.2. *Let  $\lambda_0, d \geq 3$  and  $a < 1$  be fixed. Then for  $\lambda_n$  sufficiently large we can find  $n_0 > 0$  and  $t_0 > 0$  such that*

$$(4.2) \quad \mathbb{P}(|V_{j t_0}^{j n_0}| \geq ((d - 1)a)^{j n_0}) \geq \varepsilon$$

for all sufficiently large  $j$ .

PROOF. We may restrict our attention to the case  $\lambda_n > \lambda_0$ . Given any two neighbouring vertices  $v$  and  $w$ , and starting from any time, the probability that there is an infection arrow from  $v$  to  $w$  before there is a recovery mark at  $v$  is exactly  $\lambda_n/(\lambda_n + 1)$ . Let us choose  $\lambda_n$  so that  $\lambda_n/(\lambda_n + 1) = b$  for some  $b$

chosen strictly between  $a$  and 1. Let  $v_n$  be some fixed vertex at level  $n$  in  $\mathbb{T}'_d$ . Then the probability that there is some immediate infection trail from  $(O, 0)$  to  $v_n$  is exactly  $b^n$ . Conditional on the existence of such a (necessarily unique) i.i.t.,  $(e_1, t_1), \dots, (e_n, t_n)$ , the times  $t_{i+1} - t_i$  are independent, each having the distribution of the random variable  $T_\lambda$  conditional on  $T_\lambda < T_1$ , where  $T_\lambda$  and  $T_1$  are independent exponentially distributed random variables with parameters  $\lambda$  and 1, respectively. Since this (conditional) distribution has some finite mean,  $\mu$  and variance  $\sigma^2$ , we may apply the central limit theorem (more elementary methods will also suffice) to approximate the distribution of  $t_n$ . In particular, the probability that  $t_n$  satisfies  $\mu n/2 < t_n < 3\mu n/2$  (conditional on the existence of the i.i.t.) tends to 1 as  $n \rightarrow \infty$ . Therefore for  $n$  sufficiently large we may find some fixed time  $t = t(n) \in (\mu n/2 + 1, 3\mu n/2)$  (dependent on  $n$ , but nonrandom) such that with probability at least  $b^n/2\mu n$  there is an i.i.t. as above with  $t_n \in (t - 1, t)$ . In an interval of length 1 (or less) the probability of seeing no recovery mark at  $v_n$  or infection arrow leading from  $v_n$  to one of its children is (at least)  $e^{-(1+(d-1)\lambda)}$ . Therefore with probability at least  $b^n e^{-(1+(d-1)\lambda)}/2\mu n$  there is an immediate infection trail from  $(O, 0)$  to  $(v_n, t(n))$ . Now fix  $n_0$  to be a value of  $n$  so that this probability is at least  $2a^n$ . Let  $t_0 = t(n_0)$ .

Since there are  $(d - 1)^{n_0}$  vertices in  $\mathbb{T}'_d$  at level  $n_0$  and each has the same probability to lie in  $V_{t_0}^{n_0}$ , the above calculation shows that  $\mathbb{E}(|V_{t_0}^{n_0}|) \geq 2((d - 1)a)^{n_0}$ . Now let  $A_j$  consist of those vertices,  $v$ , at level  $jn_0$ , with the property that there is an i.i.t. from  $(O, 0)$  to  $(v, jt_0)$  that can be obtained by piecing together  $j$  distinct i.i.t.s, from  $(O, 0)$  to  $(v_{n_0}, t_0)$ , from  $(v_{n_0}, t_0)$  to  $(v_{2n_0}, 2t_0)$  and so forth, where each  $v_{in_0}$  is at level  $in_0$ . Note that  $A_1$  is exactly equal to  $V_{t_0}^{n_0}$  and, more generally,  $A_j$  is a subset of  $V_{jt_0}^{jn_0}$ . We can use  $A_j$  to obtain information about  $V_{jt_0}^{jn_0}$ : letting  $Z_j = |A_j|$ , it is clear that  $(Z_j)$  is precisely a branching process whose offspring distribution is equal to the distribution of  $|V_{t_0}^{n_0}|$ ; since  $A_j \subseteq V_{jt_0}^{jn_0}$  we see that  $(|V_{jt_0}^{jn_0}|)_{j=0}^\infty$  stochastically dominates  $(Z_j)$ . Let  $m = \mathbb{E}Z_1$ , so  $m \geq 2((d - 1)a)^{n_0}$ , and let  $W_j = Z_j/m^j$ . It is well known (see, e.g., [1]) that  $(W_j)$  is an  $L^2$ -bounded martingale and converges almost surely to some random variable  $W$  with  $\mathbb{E}W = 1$ . For some  $\varepsilon > 0$  we have that  $\mathbb{P}(W \geq 1) = 2\varepsilon$ . It follows that for  $j$  sufficiently large,  $\mathbb{P}(W_j \geq 1/2) \geq \varepsilon$  and hence  $\mathbb{P}(|V_{jt_0}^{jn_0}| \geq m^j/2) \geq \varepsilon$ . Since  $m^j/2 \geq ((d - 1)a)^{jn_0}$  (with some room to spare), the desired conclusion (4.2) follows.  $\square$

The preceding lemma is required in the proof of the final part of the following proposition, which deals with the intermediate case.

**PROPOSITION 4.3.** *Suppose that  $\lambda_1(\mathbb{T}_d) < \lambda_0 < \lambda_2(\mathbb{T}_d)$ . Then there exist positive  $\lambda_\alpha$  and  $\lambda_\beta$  such that if  $\lambda_n < \lambda_\alpha$ , then the partial immunization process with parameters  $(\lambda_n, \lambda_0)$  does not survive and if  $\lambda_n > \lambda_\beta$ , then it survives strongly. If  $\lambda_1 < \lambda_n < \lambda_2$ , then the partial immunization process survives weakly.*

PROOF. The final part follows immediately from Proposition 2.1: if  $\lambda_n, \lambda_o \in (\lambda_1, \lambda_2)$ , then the partial immunization process in question dominates the contact process with parameter  $\min(\lambda_n, \lambda_o)$  and is dominated by the contact process with parameter  $\max(\lambda_n, \lambda_o)$ ; both of these contact processes survive weakly, so the partial immunization process does also.

We now show that for  $\lambda_n$  sufficiently small, the partial immunization process does not survive, and we begin by outlining the idea of the proof. We assume throughout that  $\lambda_n < \lambda_o$ ; indeed we know from the previous paragraph that it is futile to consider any  $\lambda_n > \lambda_1$ . Consider  $B_h$ , a ball of radius  $h$  centered on the root  $O$  of  $\mathbb{T}_d$ . The contact process with parameter  $\lambda_o$ , restricted to  $B_h$ , loosely speaking survives only for a time  $Ch$ , where  $C$  is a constant depending on  $\lambda_o$ ; the same holds for the partial immunization process in question since it is stochastically dominated by this contact process. The partial immunization process  $\eta_t^O$  on the whole of  $\mathbb{T}_d$  is, trivially, equivalent to the process restricted to the ball  $B_h$  until it reaches the boundary of  $B_h$ . Therefore if this unrestricted process is to survive (forever) it must reach the boundary of  $B_h$  by time  $Ch$ . However, the chance that it does so tends to zero as  $h \rightarrow \infty$  provided we choose a sufficiently small  $\lambda_n$ .

Consider  $\eta_t^O$ , the partial immunization process with parameters  $(\lambda_n, \lambda_o)$ , constructed using a graphical representation. Let  $\zeta_t^h$  be the same process with infections outside  $B_h$  forbidden, constructed from the same graphical representation after deleting infection arrows which go outside  $B_h$ . Let  $C$  be as in Theorem 2.5 with  $\lambda = \lambda_o$ , so for the contact process with parameter  $\lambda_o$  one has  $\mathbb{P}(\tau_{2h}^{2h} > 2Ch) \rightarrow 0$  as  $h \rightarrow \infty$ . Since  $B_h$  is isomorphic to a subgraph of  $\mathbb{T}_d^{2h}$  and the partial immunization process is dominated by this contact process,

$$(4.3) \quad \mathbb{P}(\zeta_{2Ch}^h(v) = 1 \text{ for some } v) \rightarrow 0 \quad \text{as } h \rightarrow \infty.$$

Note that  $C$  depends on  $\lambda_o$ , but not  $\lambda_n$  (provided  $\lambda_n < \lambda_o$ ). In what follows we shall consider a fixed value of  $\lambda_o$ .

We now construct a pure birth process  $\xi_t$  which dominates  $\eta_t^O$  in the sense that  $\eta_t^O(v) = 1 \Rightarrow \xi_t(v) = 1$ . At each time  $t$  and for each vertex  $v$  in  $\mathbb{T}_d$ , the state  $\xi_t(v)$  is either 0 or 1. For  $v \in \mathbb{T}_d$  let  $e_1, \dots, e_n$  be the sequence of distinct edges on the unique path from  $O$  to  $v$ . Set  $\xi_t(v)$  to be 1 if there is a sequence of stable infection arrows  $(e_1, t_1), \dots, (e_n, t_n)$  with  $0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq t$ . Note that by the assumptions  $\lambda_n \leq \lambda_1 < \lambda_o$ , the graphical representation contains stable arrows and reinfection arrows. Note also that recovery marks in the graphical representation have no significance for this pure birth process. The required domination property needs only to be checked at the time a vertex first becomes infected in the partial immunization process; this is easily done by induction on the total number of vertices that have been infected (in the partial immunization process) at any time prior to that point.

Let us fix a vertex  $v$  at distance  $n$  from  $O$  and estimate the probability that  $v \in \xi_t$ . The process  $\xi_t$  has infection rate  $\lambda_n$ , so one has

$$(4.4) \quad \mathbb{P}(v \in \xi_t) = \mathbb{P}(X_1 + X_2 + \dots + X_n \leq t),$$

where  $X_1, \dots, X_n$  are independent random variables that are distributed exponentially with parameter  $\lambda_n$ . It is easily seen (and well known) that the sum of these random variables has the gamma distribution with parameters  $\lambda_n$  and  $n$ , that is,  $X_1 + \dots + X_n$  has probability density function

$$(4.5) \quad \frac{e^{-\lambda_n s} \lambda_n^n s^{n-1}}{(n-1)!}, \quad s \geq 0.$$

Discarding the exponential factor in (4.5), integrating from 0 to  $t$  and using (4.4), one obtains

$$(4.6) \quad \mathbb{P}(v \in \xi_t) \leq \frac{(\lambda_n t)^n}{n!}.$$

Now in the case when  $v$  lies on the boundary of  $B_h$ , so  $n = h$ , and taking  $t = 2Ch$  in (4.6), we obtain  $\mathbb{P}(v \in \xi_{2Ch}) \leq (2\lambda_n Ch)^h / h!$ . Since there are  $d(d-1)^{h-1}$  vertices at distance  $h$  from  $O$ , summing over these vertices gives

$$(4.7) \quad \mathbb{P}(\forall v \in \partial B_h, v \notin \xi_{2Ch}) \geq 1 - \frac{c_d (2\lambda_n (d-1)Ch)^h}{h!},$$

where  $c_d = d/(d-1)$ . By Stirling's formula,  $h! \sim \sqrt{2\pi h} h^h e^{-h}$ , so if  $\lambda_n < 1/2C(d-1)e$  the right-hand side of (4.7) tends to 1 as  $h \rightarrow \infty$ .

Now, if  $\forall v \in \partial B_h, v \notin \xi_{2Ch}$ , then no site outside  $B_h$  has been infected up to time  $2Ch$ , so  $\eta_{2Ch}^O = \xi_{2Ch}^h$ . Combining (4.7) and (4.3) gives, for  $\lambda_n < 1/2C(d-1)e$ ,

$$\mathbb{P}(\eta_{2Ch}^O(v) = 1 \text{ for some } v) = o(h) \quad \text{as } h \rightarrow \infty.$$

Setting  $\lambda_\alpha = 1/2C(d-1)e$ , this implies for  $\lambda_n < \lambda_\alpha$  that the partial immunization process does not survive, as required.

Finally, we must show that strong survival occurs for  $\lambda_n$  sufficiently large. Since  $\lambda_o > \lambda_1$ , by Theorem 2.3 we have that  $\beta = \beta(\lambda_o) > 1/(d-1)$ . Choose  $\gamma$  with  $\beta > \gamma > 1/(d-1)$ . Then, letting  $\xi_t^O$  be the contact process with parameter  $\lambda_o$  on  $\mathbb{T}'_d$  and letting  $v_n$  be a vertex of  $\mathbb{T}'_d$  at distance  $n$  from  $O$ , we have by Lemma 2.4 that for  $n$  sufficiently large there exists  $s_0$  with

$$(4.8) \quad \mathbb{P}(v_n \in \xi_{s_0}^O) \geq \gamma^n.$$

Now choose  $a < 1$  with  $(d-1)a\gamma > 1$ . We shall show that if  $\lambda_n$  is large enough for the conclusion of Lemma 4.2 to hold (with this value of  $a$ ), then the partial immunization process on  $\mathbb{T}_d$  with parameters  $(\lambda_n, \lambda_o)$  survives strongly. Pick such a  $\lambda_n$  and let  $\varepsilon, n_0$  and  $t_0$  be as Lemma 4.2. Now let  $j$  and  $s_0$  be chosen such that (4.8) holds with  $n = jn_0$ , such that (4.2) holds, and finally (for reasons which will become clear) such that

$$(4.9) \quad \left(1 - \frac{\varepsilon \gamma^{jn_0}}{2}\right)^{((d-1)a)^{jn_0}} \leq \frac{1}{2}.$$

Note that (4.9) holds for  $j$  sufficiently large because

$$\begin{aligned} \left(1 - \frac{\varepsilon\gamma^{jn_0}}{2}\right)^{((d-1)a)^{jn_0}} &\leq \left[\exp\left(-\frac{\varepsilon\gamma^{jn_0}}{2}\right)\right]^{((d-1)a)^{jn_0}} \\ &= \exp\left(-\frac{\varepsilon((d-1)a\gamma)^{jn_0}}{2}\right) \\ &\rightarrow 0 \quad \text{as } j \rightarrow \infty \text{ since } (d-1)a\gamma > 1. \end{aligned}$$

Now consider the partial immunization process with parameters  $(\lambda_n, \lambda_o)$  on  $\mathbb{T}_d$  constructed via a graphical representation. We say that an initial configuration on  $\mathbb{T}_d$  is *virgin* if  $O$  is infected and the other vertices of the subtree  $\mathbb{T}'_d$  are in the “never infected” state. We claim that for each  $k \in \mathbb{N}$ , we can find an event  $A_k$ , which depends only on the graphical representation within the tree  $\mathbb{T}'_d$  (i.e., on infection arrows corresponding to edges of  $\mathbb{T}'_d$  and recovery marks at vertices of  $\mathbb{T}'_d$ ) between time 0 and  $k(s_0 + jt_0)$  with  $\mathbb{P}(A_k) \geq \varepsilon\gamma^{jn_0}/2$  and such that if  $A_k$  occurs and the starting configuration is virgin, then  $O$  is infected at time  $k(s_0 + jt_0)$ . Since this probability is bounded away from 0, independent of  $k$ , and since, in particular, it holds for the standard starting configuration with  $O$  infected and all other vertices never infected, the claim implies strong survival.

We establish the claim by induction. It is trivial for  $k = 0$ , so we now show it for a particular value of  $k$ , assuming it holds for  $k - 1$ . Let  $E$  be the event that  $|V_{jt_0}^{jn_0}| \geq ((d-1)a)^{jn_0}$ . By (4.2),  $\mathbb{P}(E) \geq \varepsilon$ . Now for each  $v \in V_{jt_0}^{jn_0}$  we consider the subtree whose vertices are precisely  $v$  and all its descendants, which we denote by  $\mathbb{T}_d^v$ . We note that if the initial configuration is virgin and  $v \in V_{jt_0}^{jn_0}$ , then the configuration at time  $jt_0$  must be *virgin for  $v$*  in the sense that  $v$  is infected and all the other vertices of  $\mathbb{T}_d^v$  are in the “never infected” state. In this case, for each  $v \in V_{jt_0}^{jn_0}$ , there is, by the induction hypothesis—translated by time  $jt_0$  and so that  $v$  is the root—some event  $A_{k-1}^v$ , which depends only on the graphical representation on  $\mathbb{T}_d^v$  between times  $jt_0$  and  $jt_0 + (k-1)(s_0 + jt_0)$ , with  $\mathbb{P}(A_{k-1}^v) \geq \varepsilon\gamma^{jn_0}/2$  and such that the occurrence of  $A_{k-1}^v$  implies that  $v$  is infected at time  $jt_0 + (k-1)(s_0 + jt_0)$ . By independence of the  $A_{k-1}^v$  for different  $v$ , we have that, conditional on  $E$ ,

$$\begin{aligned} (4.10) \quad \mathbb{P}\left(\bigcup_{v \in V_{jt_0}^{jn_0}} A_{k-1}^v\right) &\geq 1 - \left(1 - \frac{\varepsilon\gamma^{jn_0}}{2}\right)^{((d-1)a)^{jn_0}} \\ &\geq \frac{1}{2} \quad \text{by (4.9).} \end{aligned}$$

Now if  $\bigcup_{v \in V_{jt_0}^{jn_0}} A_{k-1}^v$  occurs, let  $v$  be the first vertex (in some arbitrary ordering) in  $V_{jt_0}^{jn_0}$  for which  $A_{k-1}^v$  occurs. Then let  $C$  be the event that there is a stable

infection trail lying inside  $\mathbb{T}'_d$  from  $(v, jt_0 + (k - 1)(s_0 + jt_0))$  to  $(O, k(s_0 + jt_0))$ . The probability of  $C$  (conditional on the existence of such a  $v$ ) is—by the duality of the contact process (2.1)—exactly the probability on the left-hand side of (4.8), so is at least  $\gamma^{jn_0}$  by choice of  $j$ . Now, setting

$$A_k = E \cap \left( \bigcup_{v \in V^{jn_0}_{jt_0}} A_{k-1}^v \right) \cap C,$$

$\mathbb{P}(A_k) \geq \varepsilon \gamma^{jn_0}/2$  by the fact that  $\mathbb{P}(E) \geq \varepsilon$ ,  $\mathbb{P}(C) \geq \gamma^{jn_0}$  and (4.10). The event  $A_k$  has the other required properties by construction and the induction step is complete.  $\square$

We now complete the proof of Theorem 1.2 by dealing with the case when  $\lambda_o > \lambda_2$ .

**PROPOSITION 4.4.** *Let  $\lambda_o > \lambda_2(\mathbb{T}_d)$  and  $\lambda_n > 0$ . Then the partial immunization process on  $\mathbb{T}_d$  with parameters  $(\lambda_n, \lambda_o)$  survives strongly.*

**PROOF.** We may assume that  $\lambda_n \leq \lambda_2$  since the partial immunization process otherwise dominates the contact process with parameter  $\min(\lambda_n, \lambda_o)$  which survives strongly. In particular, we have  $\lambda_o > \lambda_n$  so the graphical representation of the process contains stable arrows and reinfection arrows. In this situation, the graphical representation enables us to couple two processes and preserve a great deal of monotonicity. To be precise, suppose that  $H$  is a subgraph of  $G$  and suppose that  $\eta_0 : V(G) \rightarrow \{-1, 0, 1\}$  and  $\xi_0 : V(H) \rightarrow \{-1, 0, 1\}$  are states for the partial immunization process on  $G$  and  $H$ , respectively. Then the graphical representation gives us a natural coupling of the partial immunization processes,  $\eta_t$  and  $\xi_t$ , with these starting states [with the same parameters,  $(\lambda_n, \lambda_o)$ ]. In the case when  $\lambda_o \geq \lambda_n$ , this coupling will preserve monotonicity: if  $\eta_t(v) \geq \xi_t(v)$  for all  $v \in V(H)$  when  $t = 0$ , then this relation will hold for all  $t$ . Loosely speaking, the process cannot do better if we forget about certain infections or certain parts of the graph. We shall make extensive use of this fact, most heavily in the induction step toward the end of this proof.

We shall show that local survival occurs for the process on  $\mathbb{T}'_d$ , the rooted tree in which all vertices have  $d - 1$  children and the root has no parent. Recall that the subgraph of  $\mathbb{T}'_d$  spanned by the vertices within distance  $n$  of the root is denoted  $\mathbb{T}^n_d$ . Throughout what follows, we shall consider the process on  $\mathbb{T}'_d$  starting from a state in which the root,  $O$ , is infected, the other vertices of  $\mathbb{T}^n_d$  (for some  $n$  to be fixed) are in the previously infected state, 0, and the other vertices of  $\mathbb{T}'_d$  are in the never infected state. It is clear that it is sufficient to prove local survival starting from this state.

Our proof will proceed very roughly as follows. Starting from the above state, the process restricted to  $\mathbb{T}^n_d$  would be identical to the contact process with



parameter  $\lambda_0$  so, by Theorem 2.6, has a reasonable probability of surviving for an extremely long time. Therefore the same is true for the unrestricted partial immunization process. If it does survive for a long time, then it is very likely that other nearby copies of  $\mathbb{T}_d^n$  will reach a state equivalent to (or better than) the starting state of the first copy. These, in turn, may infect nearby copies and so forth. A branching process argument then enables us to show that this implies that there is a positive probability that the root is infected at arbitrarily large times.

We shall need to make careful use of some conditional probabilities and independence. In what follows,  $\mathcal{F}_t$  denotes the  $\sigma$ -field generated by the process up to time  $t$ .

Fix  $a < 1$  chosen so that  $a(d - 1) > 1$  and take constants  $p > 0$ ,  $c > 0$  and  $\alpha > 1$  so that the conclusion of Theorem 2.6 holds (with  $\lambda = \lambda_0$  and  $h = n$ ). Now choose further constants as follows. Choose  $N$  such that  $(1 - p^2/8)^N < 1/2$ . Let  $\delta = e^{-1}(1 - e^{-\lambda n})$ ; note that if a vertex  $v$  is infected at time  $t$  and  $w$  is a neighbouring vertex, then the probability (conditional on  $\mathcal{F}_t$ ) that  $w$  is infected at time  $t + 1$  is at least  $\delta$ . We will also need the (more trivial) fact that  $\delta$  is a lower bound on the conditional probability that  $v$  is still infected at time  $t + 1$ . We now choose  $n$  satisfying three conditions, all of which hold for  $n$  sufficiently large. The first such condition is that

$$(4.11) \quad (d - 1)^{2n} (1 - \delta^{3n})^{\lfloor c\alpha^{(a(d-1))^n} / (6n) \rfloor} \leq \frac{P}{4}.$$

Note that (4.11) holds for  $n$  sufficiently large since the left-hand side is at most  $\exp(2n \log(d - 1) - \delta^{3n} \lfloor c\alpha^{(a(d-1))^n} / (6n) \rfloor)$ , which tends to 0 as  $n \rightarrow \infty$ . We shall similarly require that

$$(4.12) \quad (1 - \delta^{3Nn})^{\lfloor c\alpha^{(a(d-1))^n} / (6n) \rfloor} \leq \frac{P}{4}.$$

The final condition which  $n$  must satisfy is rather simpler:

$$(4.13) \quad (d - 1)^n \geq N.$$

We shall show the following for the partial immunization process on  $\mathbb{T}'_d$  with the starting state defined above (in which the root is infected and the other vertices of  $\mathbb{T}'_d$  are “previously infected”): for any  $j \geq 0$ , starting from a single infection at the root, with probability  $\rho = p^2/8$  the origin is infected at some time after  $8jn$ . This will establish that on  $\mathbb{T}'_d$ , and hence on  $\mathbb{T}_d$ , the process survives strongly.

We proceed by induction on  $j$ . The result trivially holds for  $j = 0$ . Now let us fix some  $j > 0$  and suppose the result holds for smaller values.

Let  $E_l$  be the event that at time  $3nl$  some vertex between generation 0 and  $n$  (inclusive) is infected, and let  $E = \bigcap_{l=0}^{2L} E_l$ , where  $L = \lfloor c\alpha^{(a(d-1))^n} / (6n) \rfloor$ . By Theorem 2.6,  $\mathbb{P}(E) \geq p$ , since if the process restricted to a certain subgraph survives for a certain time, then throughout that time period there is always some vertex of that graph which is infected. Note that here we are making use of

the natural coupling, referred to in the first paragraph of this proof, between the process on  $(G =) \mathbb{T}_d^n$  and the process on the subgraph  $(H =) \mathbb{T}_d^n$ .

Now let  $v_1, \dots, v_N$  be  $N$  distinct vertices at generation  $n$ , chosen arbitrarily in advance; note that  $N$  such exist by (4.13). For each such  $v_i$ , let  $\mathbb{T}_{v_i}$  denote the tree spanned by  $v_i$  and its descendants within distance  $n$ , so  $\mathbb{T}_{v_i}$  is isomorphic to  $\mathbb{T}_d^n$ .

Let  $V = V(\mathbb{T}_{v_1}) \cup \dots \cup V(\mathbb{T}_{v_N})$ . Our first step will be to show that if  $E$  happens, then it is likely that all the vertices of  $V$  have been infected by time  $3nL$ . Suppose that  $v \in V$  and  $w$  is a vertex of  $\mathbb{T}_d^n$ . Then the distance from  $w$  to  $v$  is at most  $3n$ . Then, for any  $t$ , the probability that there is a stable infection trail from  $(w, t)$  to  $(v, t + 3n)$  is at least  $\delta^{3n}$ . Now for each such  $v$  and  $t$ , let  $I_v^t$  be the event that  $v$  is infected at time  $t$  and let

$$I_v = \bigcup_{l=0}^{L-1} I_v^{3nl+3n}.$$

Note that  $I_v^l \in \mathcal{F}_t$  and that on the event  $E$  defined above, one has for all  $l < L$  (in fact, for all  $l \leq 2L$ ),  $\mathbb{P}(I_v^{3nl+3n} \mid \mathcal{F}_{3nl}) \geq \delta^{3n}$ . A (careful) manipulation of conditional expectations yields

$$\begin{aligned} \mathbb{P}\left(\bigcap_{l=0}^{L-1} (I_v^{3nl+3n})^c \cap E\right) &\leq (1 - \delta^{3n})^L \\ &\leq \frac{P}{4(d-1)^{2n}} \quad \text{by (4.11)} \end{aligned}$$

(where  $A^c$  denotes the complement of an event  $A$ ). Since  $V$  contains fewer than  $(d-1)^{2n}$  vertices, it follows that

$$(4.14) \quad \mathbb{P}\left(\bigcup_{v \in V} \bigcap_{l=0}^{L-1} (I_v^{3nl+3n})^c \cap E\right) \leq \frac{P}{4}.$$

Let  $I$  be the event  $\bigcap_{v \in V} I_v$ . It follows from (4.14) that

$$(4.15) \quad \mathbb{P}(E \cap I) \geq \mathbb{P}(E) - p/4.$$

Note that if  $I$  happens, then at time  $3nL$  (and all subsequent times) all the vertices of  $V$  are in state 0 or 1.

Our next step is to show that, on  $E$ , there is a high probability of finding some time between  $3nL$  and  $6nL$  at which all the vertices  $v_1, \dots, v_N$  are simultaneously infected.

Let  $v$ , as before, be any vertex of  $\mathbb{T}_d^n$ . Given some time  $t$ , let  $G_i$  be the event that there is an infection trail from  $(v, t)$  to  $(v_i, t + 3n)$ ; we have already seen that  $\mathbb{P}(G_i) \geq \delta^{3n}$ . The events  $\{G_i\}_{i=1}^N$  are monotone increasing and hence positively correlated (see [14]). Therefore if  $v$  is infected at time  $t$ , then, with probability (conditional on  $\mathcal{F}_t$ ) at least  $\delta^{3Nn}$ ,  $v_1, \dots, v_N$  are all infected [via infection trails from  $(v, t)$ ] at time  $t + 3n$ .

Let  $A_l$  be the event that  $v_1, \dots, v_N$  are all infected at time  $3ln$ . Then  $A_l \in \mathcal{F}_{3ln}$  and  $\mathbb{P}(A_{l+1} \mid \mathcal{F}_{3ln}) \geq \delta^{3Nn}$  on  $E$  for  $L \leq l < 2L$ . This implies (much as before) that

$$(4.16) \quad \mathbb{P}\left(\bigcup_{l=L}^{2L-1} A_{l+1} \cap E\right) \geq \mathbb{P}(E) - (1 - \delta^{3Nn})^L$$

$$\geq \mathbb{P}(E) - p/4 \quad \text{by (4.12).}$$

Let  $A = \bigcup_{l=L}^{2L-1} A_{l+1}$ . Combining (4.15) and (4.16) together with the fact that  $\mathbb{P}(E) \geq p$  yields

$$(4.17) \quad \mathbb{P}(A \cap I) \geq p/2.$$

If  $A \cap I$  occurs, then there is some time, at least  $6n$  (since  $L \geq 1$ ), at which  $v_1, \dots, v_N$  are all infected and at which none of the vertices in  $V$  is in the state  $-1$ . Let  $T$  be the least such time.

Now let  $\mathbb{T}'_{v_i}$  be the subtree spanned by  $v_i$  and *all* its descendants; so the first  $n$  generations of  $\mathbb{T}'_{v_i}$  form  $\mathbb{T}_{v_i}$ . We apply the induction hypothesis to each of the trees  $\mathbb{T}'_{v_i}$  (using the fact that the state of each  $\mathbb{T}_{v_i}$  at time  $T$  dominates the chosen starting state of  $\mathbb{T}'_d$  and using the observation made in the first paragraph of this proof with  $G = \mathbb{T}'_d$  and  $H = \mathbb{T}'_{v_i}$ ): there is some event— $B_i$ , say—with probability at least  $\rho$ , which depends only on the graphical representation within the tree  $\mathbb{T}'_{v_i}$  beyond time  $T$ , such that if  $B_i$  holds, then  $v_i$  is infected at a time beyond  $T + 8(j - 1)n$ . So, conditional on the existence of  $T$ , the probability that some  $v_i$  is infected at time beyond  $T + 8(j - 1)n$  is at least  $1 - (1 - \rho)^N$ , which exceeds  $1/2$  by choice of  $N$ . Let the least such time be  $S$ . Then we have that  $S \geq 6n + 8(j - 1)n$  and, using (4.17), the nonconditional probability that  $S$  exists satisfies

$$(4.18) \quad \mathbb{P}(S \text{ exists}) \geq p/4.$$

If  $S$  exists, let  $i_0$  be the least  $i$  such that  $v_i$  is infected at time  $S$ . Theorem 2.6 again gives that (conditional on the existence of  $S$ ), with probability at least  $p$ ,  $\mathbb{T}_{v_{i_0}}$  remains infected until time at least  $S + c\alpha^{(a(d-1))^n}$ . Letting this event play much the same role as  $E$  in the first stages of this proof, one can see that if this happens one has essentially  $\lfloor c\alpha^{(a(d-1))^n} / 2n \rfloor$  independent opportunities to reinfect the origin, each with probability  $\delta^{2n}$  of success (since any vertex of  $\mathbb{T}_{v_{i_0}}$  is within distance  $2n$  of  $O$ ). The probability (again conditional on the existence of  $S$ ) that  $O$  is reinfected at some time exceeding  $S + 2n$  is therefore at least

$$(4.19) \quad p - (1 - \delta^{2n})^{\lfloor c\alpha^{(a(d-1))^n} / (2n) \rfloor} \geq \frac{p}{2},$$

where the inequality follows from a greatly weakened (4.11). Combining (4.18) and (4.19), we see that the (nonconditional) probability that  $O$  is infected at some time exceeding  $8jn$  is at least  $p^2/8 (= \rho)$ , as required to complete the induction step.  $\square$

**5. Other questions.** The graphical representation for the contact process enables one to easily answer the most basic questions of stochastic monotonicity. If  $\xi_t^A$  is the contact process with a parameter  $\lambda$  on a graph  $G$  with starting set  $A$ , and if  $\Xi_t^B$  is the contact process with parameter  $\mu$  on a graph  $H$  with starting set  $B$ , and if  $\lambda \leq \mu$ ,  $G \subseteq H$  and  $A \subseteq B$ , then  $\Xi_t^B$  stochastically dominates  $\xi_t^A$ : the two processes can be coupled so that  $\xi_t^A \subseteq \Xi_t^B$  for all  $t$ . The coupling is immediate from the graphical representation, noting that a process with parameter  $\mu$  is obtained from a process with parameter  $\lambda$  by the addition of arrows according to Poisson processes of rate  $\mu - \lambda$ .

The picture is not so straightforward for partial immunization processes, since if a vertex is infected at some time, it may be less likely to be infected at later times due, indeed, to partial immunization. Indeed, if one considers the forest fire model (the case  $\lambda_o = 0$ ) on the connected graph with two vertices starting with one infected vertex, a straightforward calculation shows that stochastic monotonicity, as  $\lambda_n$  varies, fails.

Here we make a few simple observations about when monotonicity (of various kinds) can be deduced. In the region  $\lambda_o \geq \lambda_n$  one has all the monotonicity one might hope for: increasing  $\lambda_n$  or  $\lambda_o$  corresponds to the addition of reinfection arrows or the conversion of reinfection arrows into stable infection arrows; neither of these processes can cause an infected site (at some particular time) to be uninfected. Likewise, an increase in the set of initially infected sites (or the replacement of initial  $-1$  states by initial  $0$  states) can only result in extra infections at later times. Every previously valid infection trail remains a valid infection trail.

Proposition 2.1 leads to essentially the only other situation in which we are able to deduce that a partial immunization process with parameters  $(\lambda_n, \lambda_o)$  dominates one with parameters  $(\lambda'_n, \lambda'_o)$  (on the same graph and with the same initial configuration). If  $\min(\lambda_n, \lambda_o) \geq \max(\lambda'_n, \lambda'_o)$ , then we can find a contact process which lies between the two partial immunization processes.

There are instances where weaker forms of monotonicity can be seen to hold. Suppose that  $\eta_t$  is a forest fire process ( $\lambda_o = 0$ ) with parameter  $\lambda_n$  and some fixed starting state, and let  $E_t$  be the forest fire process with parameter  $\Lambda_n > \lambda_n$  and the same starting state. Let

$$(5.1) \quad A = \bigcup_{t \geq 0} \{v : \eta_t(v) = 1\}; \quad B = \bigcup_{t \geq 0} \{v : E_t(v) = 1\}.$$

An easy argument, which replaces the forest fire model with a locally dependent percolation model [11], shows that  $B$  stochastically dominates  $A$ . The coupling implied by this argument is quite different from the graphical representation coupling (which shows that the same result holds for the contact process). This suggests the following question.

QUESTION 5.1. Let  $\eta_t$  be the partial immunization process with parameters  $(\lambda_n, \lambda_o)$  on a given graph with a fixed starting state. Let  $E_t$  be the process with parameters  $(\Lambda_n, \Lambda_o)$  on the same graph and with the same starting state. Let  $A$  and  $B$  be as defined by (5.1). Suppose that  $\Lambda_n \geq \lambda_n$  and  $\Lambda_o \geq \lambda_o$ . Is it necessarily the case that  $B$  stochastically dominates  $A$ ?

Perhaps the simplest question to state, of this general kind, is whether or not increasing the parameters  $\lambda_o$  and  $\lambda_n$  can actually decrease the survival probability of the process? We have not been able to answer this question, but we tentatively conjecture the following.

CONJECTURE 5.2. Let  $\theta(\lambda, \mu)$  [respectively,  $\Theta(\lambda, \mu)$ ] be the probability of survival (respectively, strong survival) of the partial immunization process with parameter  $(\lambda, \mu)$  and the standard starting state on some (connected, locally finite) vertex transitive graph  $G$ . Suppose that  $\Lambda_n \geq \lambda_n$  and  $\Lambda_o \geq \lambda_o$ . Then:

- (i)  $\theta(\Lambda_n, \Lambda_o) \geq \theta(\lambda_n, \lambda_o)$ ;
- (ii)  $\Theta(\Lambda_n, \Lambda_o) \geq \Theta(\lambda_n, \lambda_o)$ .

Finally, we remark that our results in this paper apply only to the specific graphs  $\mathbb{Z}^d$  and  $\mathbb{T}_d$ . It is natural to try to extend the result for  $\mathbb{Z}^d$  to a more general class of amenable graphs and the results for  $\mathbb{T}_d$  to a general class of nonamenable graphs. (See, e.g., [17].) However, this is not essentially a problem about partial immunization processes, since (except for a rather limited class of treelike graphs [22]) it remains open even for the contact process. As we have seen in this paper, if we know a great deal about the contact process, then we are able to use this information to prove fundamental results for more complicated processes.

## REFERENCES

- [1] ATHREYA, K. B. and NEY, P. E. (1972). *Branching Processes*. Springer, Berlin.
- [2] BEZUIDENHOUT, C. and GRIMMETT, G. (1990). The critical contact process dies out. *Ann. Probab.* **18** 1462–1482.
- [3] BEZUIDENHOUT, C. and GRIMMETT, G. (1991). Exponential decay for subcritical and percolation processes. *Ann. Probab.* **19** 984–1009.
- [4] BOLLOBÁS, B. (1998). *Modern Graph Theory*. Springer, New York.
- [5] DURRETT, R. (1988). *Lectures Notes on Particle Systems and Percolation*. Wadsworth, Belmont, CA.
- [6] DURRETT, R. (1995). *Ten Lectures on Particle Systems. Lecture Notes in Math.* **1608**. Springer, New York.
- [7] DURRETT, R. and SCHINAZI, R. (2000). Boundary modified contact processes. *J. Theoret. Probab.* **13** 575–594.
- [8] GRASSBERGER, P., CHATÉ, H. and ROUSSEAU, G. (1997). Spreading in media with long-time memory. *Phys. Rev. E* **55** 2488–2495.
- [9] HARRIS, T. E. (1974). Contact interactions on a lattice. *Ann. Probab.* **2** 969–988.
- [10] HARRIS, T. E. (1978). Additive set-valued Markov processes and graphical methods. *Ann. Probab.* **6** 355–378.

- [11] KULASMAA, K. (1982). The spatial general epidemic and locally dependent random graphs. *J. Appl. Probab.* **19** 745–758.
- [12] LALLEY, S. (1999). Growth profile and invariant measures for the weakly supercritical contact process on a homogeneous tree. *Ann. Probab.* **27** 206–225.
- [13] LALLEY, S. and SELLKE, T. (1998). Limit set of a weakly supercritical contact process on a homogeneous tree. *Ann. Probab.* **26** 644–657.
- [14] LIGGETT, T. M. (1985). *Interacting Particle Systems*. Springer, New York.
- [15] LIGGETT, T. M. (1996). Multiple transition points for the contact process on the binary tree. *Ann. Probab.* **24** 1675–1710.
- [16] LIGGETT, T. M. (1999). *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*. Springer, New York.
- [17] LYONS, R. (2000). Phase transitions on nonamenable graphs. *J. Math. Phys.* **41** 1099–1126.
- [18] PEMANTLE, R. (1992). The contact process on trees. *Ann. Probab.* **20** 2089–2116.
- [19] SALZANO, M. and SCHONMANN, R. H. (1998). A new proof that for the contact process on homogeneous trees local survival implies complete convergence. *Ann. Probab.* **26** 1251–1258.
- [20] SALZANO, M. and SCHONMANN, R. H. (1999). The second lowest extremal invariant measure of the contact process II. *Ann. Probab.* **27** 845–875.
- [21] SCHONMANN, R. H. (1998). The triangle condition for contact processes on homogeneous trees. *J. Statist. Phys.* **90** 1429–1440.
- [22] STACEY, A. M. (1996). The existence of an intermediate phase for the contact process on trees. *Ann. Probab.* **24** 1711–1726.
- [23] STACEY, A. M. (2001). The contact process on finite trees. *Probab. Theory Related Fields* **121** 551–576.

DEPARTMENT OF PURE MATHEMATICS  
AND MATHEMATICAL STATISTICS  
CENTRE FOR MATHEMATICAL SCIENCES  
WILBERFORCE ROAD  
CAMBRIDGE CB3 0WB  
ENGLAND  
E-MAIL: ams14@dpmms.cam.ac.uk