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ALGEBRAIC CONVERGENCE OF MARKOV CHAINS

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Algebraic convergence in the L^2-sense is studied for general time-continuous, reversible Markov chains with countable state space, and especially for birth–death chains. Some criteria for the convergence are presented. The results are effective since the convergence region can be completely covered, as illustrated by two examples.

1. Introduction. This paper is devoted to studying algebraic (or polynomial) L^2-convergence for reversible Markov chains. Roughly speaking, we are looking for slower than exponential convergence, for which there are a great many publications (see, e.g., [1, 5, 12] and the references within). However, the work on algebraic convergence is still limited; readers are urged to refer to [5, (II); 8, 13] for background and the present status of study on the topic. Additionally, a referee provided the recent preprints [10, 11] in which the same topic is studied using a different approach for time-discrete Markov processes with general state space.

Consider a reversible Markov process on a complete separable metric space $(E, \mathscr{E})$ with probability measure π. The process corresponds in a natural way to a strongly continuous semigroup (P_t) on $L^2(\pi)$ with generator L and domain $\mathscr{D}(L)$. It is said that the process has algebraic convergence in L^2-sense if there exists a functional $V : L^2(\pi) \to [0, \infty]$ and constants $C > 0$, $q > 1$ so that

$$\|P_t f - \pi(f)\|^2 \le C V(f)/t^{q-1}, \qquad t > 0,\ f \in L^2(\pi), \tag{1.1}$$

where $\|\cdot\|$ denotes the L^2-norm and $\pi(f) = \int f\,d\pi$.

The starting point of our study is the following result, taken from [13] which provides some necessary and sufficient conditions for algebraic L^2-convergence.

THEOREM A (Liggett–Stroock). *Let $1 < p, q < \infty$ such that $1/p + 1/q = 1$ and let $V : L^2(\pi) \to [0, \infty]$ satisfy $V(cf + d) = c^2 V(f)$ for all constants c and d. Consider the following two statements*:

(a) *There exists a constant $C' > 0$ such that*

$$\|f - \pi(f)\|^2 \le C' D(f)^{1/p} V(f)^{1/q} \qquad \text{for all } f \in \mathscr{D}(D), \tag{1.2}$$

where $D(f) := D(f, f)$ is the Dirichlet form of L with domain $\mathscr{D}(D)$.

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(b) *There exists a constant* $C > 0$ *so that* (1.1) *holds*.

We have the following conclusions:

(i) *If* (a) *holds and* V *satisfies the contraction*

$$V(P_t f) \le V(f), \qquad f \in L^2(\pi),\ t > 0, \tag{1.3}$$

then (b) *holds*.

(ii) *If* (b) *holds*, *then so does* (a) *if the process is reversible with respect to* π.

REMARK. (i) In condition (a), we use $D(f)$ instead of $-\int fLf$ appeared in [13]. The advantage of this was explained in [2], Sections 6.7 and 9.1.

(ii) If $p = 1$, then the process is in fact exponentially convergent. Hence we restrict ourselves to the case of $p > 1$ ($\Leftrightarrow q < \infty$).

(iii) If (a) is satisfied with

$$V(f) = \|f - \pi(f)\|^2 \quad \text{or} \quad V(f) = -\int fLf\, d\pi,$$

then the algebraic L^2-convergence is indeed exponential. Thus, none of these choices for V is useful in the present context. We will adopt several different types of V, given in (2.1), (2.4), (2.6) and Theorem C below.

(iv) Actually, what we are doing is to find some $q > 1$ such that $V(f) := \sup_{t>0} t^{q-1}\|P_t f - \pi(f)\|$ is bounded within a class of functions. Then this V should satisfy (1.3) and $V(cf + d) = c^2 V(f)$ for all constants c and d.

The main purpose of the paper is to work out some more explicit conditions for the Liggett–Stroock theorem in the context of Markov chains.

To have some feeling for what is going on in the paper, let us look at the following example, which will be treated in detail in the last section.

EXAMPLE 5.1. Consider an irreducible birth–death process with birth rates $b_i = i^r$ and death rates $a_i = i^r$ for some $r > 0$ and large i. Then the process is ergodic iff $r > 1$:

(i) Let $r > 1$. Then the process is L^2-exponential convergent iff $r \ge 2$ (cf. [4]). That is, with respect to $V(f) = \|f - \pi(f)\|$, the process has L^2-algebraic decay iff $r \ge 2$.

(ii) (Proposition 5.4) Let $r \in (1, 2)$. Then, with respect to $V_1 : V_1(f) = \sup_{k\ge 0}[(k+1)^s|f_{k+1} - f_k|]^2$ with $0 < s \le r - 1$, the process has L^2-algebraic decay iff $r \in (5/3, 2)$.

(iii) (Proposition 5.3) Let $r \in (1, 2)$. Then, with respect to $V_0 : V_0(f) = \sup_{i\ne j}(f_i - f_j)^2$, the process has L^2-algebraic decay for all $r \in (1, 2)$.

This example shows that algebraic convergence depends heavily on the functional V. Note that $V_0(f)$ is meaningful only for bounded f. This is rather restrictive but still enough to deduce the ergodicity of the process under (1.1). Clearly, for different V, the inequalities (1.1), (1.2) and the contraction (1.3) are all essentially different. Therefore the criteria developed in the paper are quite technical and depend on V case by case.

2. Main results. Let $Q = (q_{ij})$ be a regular and irreducible Q-matrix on a countable set E: $q_{ij} \ge 0$ $(i \ne j)$, $0 < q_i := -q_{ii} = \sum_{j \ne i} q_{ij} < \infty$. Assume that the corresponding Q-process $P(t) = (p_{ij}(t) : i, j \in E)$ is stationary having distribution (π_i), and $\pi_i q_{ij} = \pi_j q_{ji}$ for all $i, j \in E$. Then the corresponding operator $\Omega f(i) := \sum_j q_{ij}(f_j - f_i)$, $i \in E$, becomes symmetric on $L^2(\pi)$. Denote by $(D, \mathscr{D}(D))$ the Dirichlet form $D(f) = D(f, f) = \frac{1}{2}\sum_{i,j} \pi_i q_{ij}(f_j - f_i)^2$. Its domain is assumed to be $\mathscr{D}(D) = \{f \in L^2(\pi) : D(f) < \infty\}$.

We adopt the path technique developed in [9]. First, we define a graph structure associated with the matrix $Q = (q_{ij})$. We call $\langle i, j\rangle$ an edge if $q_{ij} > 0$ $(i \ne j)$. The adjacent edges $\langle i, i_1\rangle, \langle i_1, i_2\rangle, \dots, \langle i_n, j\rangle$ (i, j and i_k's are different) constitute a directional path from i to j. Due to the irreducibility of the Q-matrix, for each pair $i \ne j$, there exists a directional path from i to j. Choose and fix such a path γ_{ij}. Then fix all the selected paths, denoted by $\Gamma = \{\gamma_{ij}\}$. We have linear order for the vertices on each path. Then, for $e \in \gamma_{ij}$, we may write $e = \langle e_\ell, e_r\rangle$, where e_ℓ and e_r are the left and right vertices of e, respectively. Certainly, an edge may belong to several paths in Γ. Note that for birth–death processes, $\langle i, j\rangle$ is an edge if and only if $|i - j| = 1$. Then, for each pair $\{k, \ell\}$, $k < \ell$, there is exactly one path from k to ℓ: $\langle k, k+1\rangle, \langle k+1, k+2\rangle, \dots, \langle \ell - 1, \ell\rangle$. Define

$$\beta = \sup_i \#\{e : e \text{ is contained in a path in } \Gamma \text{ and } e_\ell = i\}.$$

For birth–death processes, we have $\beta = 1$.

Next, choose a symmetric function ϕ: $\phi_{ij} \equiv \phi(i, j) \ge 0$ and $\phi_{ij} = 0$ if and only if $i = j$. For instance, one may take ϕ_{ij} to be the geodesic distance between i and j on the graph. Then define

$$V_\delta(f) = \sup_{i \ne j} (f_j - f_i)^2 / \phi_{ij}^{2\delta}, \tag{2.1}$$

where $\delta = 0$ or 1. Note that $V_0(f)$ is independent of ϕ. As we will prove in Section 3, the contraction (1.3) is automatic for V_0. A sufficient condition for (1.3) with $V = V_1$ is the following: There exists a coupling operator $\widetilde{\Omega}$ so that

$$\widetilde{\Omega}\phi(i, j) \le 0 \qquad \text{for all } i \ne j \quad \text{and} \quad \widetilde{\Omega}\phi(i, i) = 0 \qquad \text{for all } i. \tag{2.2}$$

For the reader's convenience, we recall the definition of coupling operators. Because of the one-to-one correspondence of a Q-matrix and its operator just mentioned above, we need only define the coupling Q-matrices [of course, in the present discrete situation, one may simply use a coupling Q-matrix instead of the coupling operator used in (2.2)]. For a given Q-matrix $Q = (q_{ij})$, a coupling Q-matrix $(q_{(ij),(k\ell)} : (ij), (k\ell) \in E \times E)$ is described by the following marginality: $\sum_\ell q_{(ij),(k\ell)} = q_{ik}$ for all i, j, k and $\sum_k q_{(ij),(k\ell)} = q_{j\ell}$ for all i, j, ℓ. We refer to [2], Chapters 0 and 5, and [3] for various coupling operators.

Set

$$\sigma_1(e) = \sum_i \frac{\pi_i \phi_{i,e_\ell}^2}{\pi_{e_\ell}^2 q_{e_\ell e_r}} \left(\sum_{j : \gamma_{ij} \ni e} \frac{\pi_j}{\phi_{ij}^2} \right)^2, \qquad \sigma_2(e) = \frac{1}{\pi_{e_\ell} q_{e_\ell e_r}} \sum_{\{i,j\} : \gamma_{ij} \ni e} \frac{\pi_i \pi_j}{\phi_{ij}^2},$$

where $\{i, j\}$ denotes the disordered pair of i and j. We remark that the summation appearing in the first formula varies only over the pairs $\{i, j\} : \gamma_{ij} \ni e$. In [9], a geometric quantity κ, somehow like $\sup_e[\sigma_1(e) + \sigma_2(e)]$, was introduced in terms of the path length $|\gamma_{ij}|_P = \sum_{e \in \gamma_{ij}} (\pi_{e_\ell} p_{e_\ell, e_r})^{-1}$ to estimate the lower bound of the Poincaré constant [equivalently, the second largest eigenvalue of a transition probability matrix $P = (p_{ij})$]. As pointed out in [9], the quantity κ is a measure of bottlenecks. It will be small if it is possible to choose paths which do not traverse any edge too often. On the other hand, it was shown in [4, 6] that one has to use the weight function $w(e) = w(e_r) - w(e_\ell)$, where w is a mimic (varies case by case) of the eigenfunction corresponding to the eigenvalue, instead of $(\pi_{e_\ell} p_{e_\ell, e_r})^{-1}$ used above to obtain sharp estimates of the eigenvalue. Thus, even though the present situation is unrelated to the eigenvalues, the test function ϕ_{ij} (and its variants introduced below) plays a role similar to $|\gamma_{ij}|_w = \sum_{e \in \gamma_{ij}} w(e)$ used in [6], and then $\sup_e[\sigma_1(e) + \sigma_2(e)]$ plays a similar role as κ used in [9]. Certainly, the expressions of $\sigma_1(e)$ and $\sigma_2(e)$, and also the proof of Theorem 2.1 below, are much more complicated than those papers quoted above.

To state our result we need further notation. The technical conditions below are used only for the necessary part of our results. We say that the process has a finite range R if $q_{ij} = 0$ whenever $|j - i| > R$. We will use some function ρ on $E = \{0, 1, 2, \ldots\}$ having the following property:

(2.3) ρ is increasing, $\rho_0 = 0$ and there exists a constant c such that either (i) or (ii) holds:
(i) $\rho_N \le c\rho_{N/2}$ for all $N \ge 1$;
(ii) $\rho_{i+R} \le c\rho_i$ for all $i \ge 1$ but still $\sum_{N \ge 1} \rho_{NR}^{-\varepsilon} < \infty$ for all $\varepsilon > 0$.

A typical choice of ρ is as follows: There exist constants $\alpha > 0$, $c_1 > 0$ and $c_2 < \infty$ such that $c_1 \le \rho_i / i^\alpha \le c_2$ for all $i \ge 1$. Then the condition "$\rho_N \le c\rho_{N/2}$ for all $N \ge 1$" holds. Otherwise, let ρ satisfy the following: There exist constants $\alpha > 1$, $c_1 > 0$ and $c_2 < \infty$ such that $c_1 \le \rho_i / \alpha^i \le c_2$ for all $i \ge 1$. Then we do have "$\rho_{i+R} \le c\rho_i$ for all $i \ge 1$" and "$\sum_{N \ge 1} \rho_{NR}^{-\varepsilon} < \infty$ for all $\varepsilon > 0$."

Now, we can state our first criterion as follows:

THEOREM 2.1. (0) *If* (2.2) *is satisfied*, *then* (1.3) *holds with* $V = V_1$.

(i) *Let* (1.3) *hold. If* $\beta < \infty$, $\sup_e\{\sigma_1(e) + \sigma_2(e)\} < \infty$ *and* $\sum_{i,j} \pi_i\pi_j \times \phi_{ij}^{2(q+\delta-1)} < \infty$ *for some constant* $q > 1$, *then the Markov chain has algebraic decay with* $V = V_\delta$ ($\delta = 0$ *or* 1) *and the same* q.

(ii) *Conversely, let* $E = \{0, 1, 2, \dots\}$ *and suppose that the process has algebraic decay with respect to* V_δ ($\delta = 0, 1$) *and* $\phi_{ij} = |\rho_j - \rho_i|$ *for some function* ρ *satisfying* (2.3). *If moreover* $\sup_i \sup_{j\ge i+1} q_{ij}\phi_{ij}^2/\rho_i^\alpha \le c$ *for some constants* $c < \infty$ *and* $\alpha \in [0, 2)$, *then we have* $\sum_j \rho_j^k \pi_j < \infty$ *for all* $k < (2 - \alpha) \times (q - 1) + 2\delta$.

The next result is a straightforward consequence of, but more practical than, Theorem 2.1.

COROLLARY 2.2. *Theorem* 2.1(1) *holds if* $\sigma_1(e)$ *and* $\sigma_2(e)$ *are replaced by*

$$\sigma_1'(e) = \sup_i \frac{\phi_{i,e_\ell}}{\pi_{e_\ell}\sqrt{q_{e_\ell e_r}}} \sum_{j:\gamma_{ij}\ni e} \frac{\pi_j}{\phi_{ij}^2} \quad \textit{and} \quad \sigma_2'(e) = \frac{1}{\pi_{e_\ell}q_{e_\ell e_r}} \sup_i \sum_{j:\gamma_{ij}\ni e} \frac{\pi_j}{\phi_{ij}^2},$$

respectively.

We now introduce a different choice of V. Fix a reference point in E, say 0 for simplicity. For each $j \in E \setminus \{0\}$, choose a directional path, without loop, from 0 to j, denoted by γ_{0j}. Fix the family $\Gamma_0 = \{\gamma_{0j} : j \ne 0\}$ and define β as above. Next, choose $\phi : \phi_i > 0$ for $i \ne 0$ and $\phi_0 = 0$. Define

$$\widetilde{V}_\delta(f) = \sup_{i\ne 0} \big(f(i) - f(0)\big)^2/\phi_i^{2\delta}. \tag{2.4}$$

When $E = \{0, 1, 2, \dots\}$ and ϕ is increasing, for $\phi_{ij} := |\phi_j - \phi_i|$, it is easy to check that $\widetilde{V}_\delta(f) \le V_\delta(f)$ for each $\delta = 1$ or 0. Finally, set

$$\tilde{\sigma}_1(e) = \frac{\phi_{e_\ell}}{\pi_{e_\ell}\sqrt{q_{e_\ell e_r}}} \sum_{j:\gamma_{0j}\ni e} \frac{\pi_j}{\phi_j^2}, \qquad \tilde{\sigma}_2(e) = \frac{1}{\pi_{e_\ell}q_{e_\ell e_r}} \sum_{j:\gamma_{0j}\ni e} \frac{\pi_j}{\phi_j^2}.$$

THEOREM 2.3. (i) *Let*

$$\beta < \infty, \qquad \sup_e\{\tilde{\sigma}_1(e) + \tilde{\sigma}_2(e)\} < \infty \quad \textit{and} \quad \sum_j \pi_j\phi_j^{2(q+\delta-1)} < \infty \tag{2.5}$$

for some $q > 1$ *and* $\delta = 0$ *or* 1. *When* $\delta = 1$, *suppose additionally that* (1.3) *holds with* $V = \widetilde{V}_1$. *Then the Markov chain has algebraic decay with* $V = \widetilde{V}_1$ *when* $\delta = 1$ *and* $V = V_0$ *when* $\delta = 0$.

In particular, if $E = \{0, 1, 2, \dots\}$ *and* ϕ *is increasing, whenever* (2.2) *holds for* $\phi_{ij} := |\phi_i - \phi_j|$, *then condition* (2.5) *with* $\delta = 1$ *implies the algebraic decay of the Markov chain with respect to* V_1 *defined by* (2.1).

(ii) *Conversely, if* $E = \{0, 1, 2, \dots\}$, ϕ *is increasing and* $\rho = \phi$ *satisfies* (2.3), *the Markov chain has algebraic decay with respect to* $\widetilde{V}_\delta$ ($\delta = 0, 1$) *and*

$\sup_i \sup_{j\ge i+1} q_{ij}(\phi_i-\phi_j)^2/\phi_i^{\alpha} \le c$ *for some constants* $c<\infty$ *and* $\alpha\in[0,2)$, *then we have* $\sum_j \phi_j^k\pi_j<\infty$ *for all* $k<(2-\alpha)(q-1)+2\delta$.

Next, we consider positive recurrent birth–death processes. Then we have $E=\{0,1,2,\dots\}$, birth rate $b_i>0$ $(i\ge 0)$, death rate $a_i>0$ $(i\ge 1)$ and reversible measure π_i. Each edge has the form $e=\langle k,k+1\rangle,\ k\ge 0$. Obviously, $\beta=1$ and $R=1$. Let u_n be a positive sequence and set $\phi_{ij}=|\sum_{k<j}u_k-\sum_{k<i}u_k|$. Then we have

$$V_\delta(f)=\sup_{i\ne j}|f(i)-f(j)|^2/\phi_{ij}^{2\delta}=\sup_{k\ge 0}|f(k+1)-f(k)|^2/u_k^{2\delta} \tag{2.6}$$

and, for $e=\langle k,k+1\rangle$,

$$\sigma_1(e)=\frac{1}{\pi_k^2 b_k}\sum_{i\le k-1}\pi_i\phi_{ik}^2\Biggl(\sum_{j\ge k+1}\frac{\pi_j}{\phi_{ij}^2}\Biggr)^2,\qquad \sigma_2(e)=\frac{1}{\pi_k b_k}\sum_{i\le k}\pi_i\sum_{j\ge k+1}\frac{\pi_j}{\phi_{ij}^2},$$

$$\sigma_1'(e)=\frac{1}{\pi_k\sqrt{b_k}}\sup_{i\le k-1}\phi_{ik}\sum_{j\ge k+1}\frac{\pi_j}{\phi_{ij}^2},\qquad \sigma_2'(e)=\frac{1}{\pi_k b_k}\sup_{i\le k}\sum_{j\ge k+1}\frac{\pi_j}{\phi_{ij}^2}.$$

As a consequence of Theorem 2.1, we have the following result.

COROLLARY 2.4. (i) *Suppose that* $\sup_e\{\sigma_1(e)+\sigma_2(e)\}<\infty$ (*or sufficiently*, $\sup_e\{\sigma_1'(e)+\sigma_2'(e)\}<\infty$) *and* $\sum_{i,j}\pi_i\pi_j\phi_{ij}^{2(q+\delta-1)}<\infty$ *for some constant* $q>1$. *For* V_1, *suppose additionally that* $b_nu_n-a_nu_{n-1}$ *is nonincreasing* $(u_{-1}=0)$. *Then the birth–death process has algebraic decay with respect to* V_δ.

(ii) *Conversely, suppose that the process has algebraic decay with respect to* V_δ *and* $\phi_{ij}:=|\rho_i-\rho_j|$ *for some* ρ *satisfying* (2.3) *with* $R=1$. *If moreover* $\sup_i b_i(\rho_{i+1}-\rho_i)^2/\rho_i^{\alpha}\le c$ *for some constants* $c<\infty$ *and* $\alpha\in[0,2)$, *then we have* $\sum_j\rho_j^k\pi_j<\infty$ *for all* $k<(2-\alpha)(q-1)+2\delta$.

As a direct consequence of Theorem 2.3, we have the following result.

COROLLARY 2.5. *Let* $\phi_n=\sum_{i<n}u_i$ *for some positive sequence* (u_i) *and define* $\theta_1(n)=\phi_n\pi_n^{-1}b_n^{-1/2}\sum_{k=n+1}^{\infty}\pi_k/\phi_k^2$.

(i) *Suppose that the following conditions hold*:

(a) $\sup_n\theta_1(n)<\infty$;

(b) $\liminf_{k\to\infty}\phi_k\sqrt{b_k}>0$;

(c) $\sum_n\pi_n\phi_n^{2(q+\delta-1)}<\infty$.

Then (1.2) *holds with* $V=\widetilde{V}_\delta$. *For* V_1, *suppose additionally that* $b_nu_n-a_nu_{n-1}$ *is nonincreasing* $(u_{-1}=0)$. *Then the process has algebraic decay with respect to* V_δ.

(ii) *Conversely, suppose that the process has algebraic decay with respect to* $\widetilde{V}_\delta$ *and* $\rho = \phi$ *satisfies* (2.3) *with* $R = 1$. *If moreover* $\sup_i b_i u_i^2/\phi_i^\alpha \le c$ *for some constants* $c < \infty$ *and* $\alpha \in [0, 2)$, *then we have* $\sum_j \phi_j^k \pi_j < \infty$ *for all* $k < (2 - \alpha)(q - 1) + 2\delta$.

REMARK. (i) Let $\theta_2(n) = (\sum_{k=n+1}^\infty \pi_k/\phi_k^2)/(\pi_n b_n)$. Note that $\theta_1(n)$ and $\theta_2(n)$ play the some role as $\tilde{\sigma}_1(e)$ and $\tilde{\sigma}_2(e)$ used in Theorem 2.3. We mention that conditions (a) and (b) in Corollary 2.5 imply that $\sup_n \theta_2(n) < \infty$.

In fact,

$$\liminf_{k\to\infty} \phi_k \sqrt{b_k} > 0 \iff \sup_k \frac{1}{\phi_k\sqrt{b_k}} < \infty \iff \sup_n \frac{\theta_2(n)}{\theta_1(n)} < \infty.$$

This plus condition (a) implies that $\sup_n \theta_2(n) < \infty$.

(ii) Obviously, when $e = \langle n, n+1\rangle$, we have $\theta_1(n) \le \sigma_1'(e)$ and $\theta_2(n) \le \sigma_2'(e)$. Hence, $\sup_n \{\theta_1(n) + \theta_2(n)\} \le \sup_e \{\sigma_1'(e) + \sigma_2'(e)\}$.

The next result is a special case of Corollary 2.5.

COROLLARY 2.6. *The birth–death process has algebraic decay with respect to* V_0 *provided*

$$\liminf_{n\to\infty} n\left(\frac{a_{n+1}}{b_n} - 1\right) > 1,$$

$$\liminf_{n\to\infty} \frac{1}{\pi_n} \sum_{k\ge n+1} \pi_k > 0 \qquad \left(\text{or } \liminf_{n\to\infty} n^\alpha \sqrt{b_n} > 0\right)$$

and

$$\sup_n \frac{1}{\sqrt{b_n} n^\alpha \pi_n} \sum_{k\ge n+1} \pi_k < \infty$$

for some $\alpha > 0$.

For birth–death chains, the algebraic convergence was studied by Liggett [13], as a tool to deal with the critical case of attractive reversible nearest particle systems. To compare our results with known ones, we introduce two theorems taken from [13] as follows. The first result below was mentioned in the quoted paper without a proof. For completeness, we present a proof at the end of Section 4.

THEOREM B. *Let* $\phi_{ij} = |\sum_{k<j} u_k - \sum_{k<i} u_k|$ *for some positive sequence* (u_k) *and define* V_1 *as* (2.1). *Let* $\sigma_n = \sum_{k=n}^\infty \pi_k/\pi_n$. *If the following conditions hold*:

(i) $b_n u_n - a_n u_{n-1}$ *is nonincreasing* ($u_{-1} = 0$);

(ii) $\inf_{i \geq 0} b_i > 0$;
(iii) $\sup_n \sigma_n / n < \infty$;
(iv) $\sum_{n=0}^{\infty} u_n^2 n^{2q} \pi_n < \infty$;

then the process has algebraic decay with respect to V_1.

The next result is due to Liggett [13], Theorem 2.10 and Proposition 2.15:

THEOREM C. *Define* $\bar{V}(f) = \sup_{i \neq j} (|f_i - f_j|/|i - j|)^2 = \sup_k |f(k+1) - f(k)|^2$, *which is nothing new but* V_1 *with* $\phi_{ij} = |i - j|$. *If the following conditions hold*:

(i) $\inf_i b_i > 0,\ a_i \geq b_i$;
(ii) $\sup_i i(a_i - b_i) < \infty$;
(iii) $\sup_n \sigma(n)/n < \infty$;
(iv) $\sum_n (\log(n+2))^{3/2} n^{2q} \pi_n < \infty$;

then the process has algebraic decay with respect to $\bar{V}$.

Conversely, suppose that the process has algebraic decay with respect to $\bar{V}$ *and* $\sup_i b_i < \infty$, *then we have* $\sum_k k^{\alpha} \pi_k < \infty$ *for all* $\alpha < 2q$.

In general, the conditions of Theorem C is stronger than those of Corollary 2.5, as shown by Example 5.1, for which Theorem C is not available [since $\sum_n (\log(n+2))^{3/2} n^{2q} \pi_n = \infty$ for any $q > 1$ when $r \in (1, 2)$] but Corollary 2.5 is exact. Roughly speaking, the conditions of Corollary 2.5 (resp., Theorem 2.3) is stronger than those of Theorem B (resp., Theorem 2.1) since $\widetilde{V}_\delta \leq V_\delta$. However, the same example shows that, in some situation, Corollary 2.5 gives us the power $q \in (1, \infty)$, which can be much larger than $q \in (1, 3/2)$ provided by Theorem B. Among the corollaries, the conditions of Corollary 2.6 are the weakest but the corresponding conclusion (1.1) holds for a smaller class of functions. Besides, the two examples discussed in Section 5 are always (resp., partially) algebraically convergent with respect to V_0 (resp., V_1 or $\bar{V}$). We refer to Section 5 for details.

Finally, we examine a special birth–death process: $a_i = b_i = i^2$ for even number i and $a_i = b_i = i^{3/2}$ for odd i. It is easy to check that Corollary 2.6 fails for such an oscillation model. To handle it, we adopt the following comparison theorem: comparing the original process with the new one having $\tilde{a}_i = \tilde{b}_i = i^{3/2}$. The next result is parallel to Theorem 4.1.1 in [14].

THEOREM 2.7. *Let* $Q = (q_{ij})$ *and* $\widetilde{Q} = (\tilde{q}_{ij})$ *be two* Q-*matrices, reversible with respect to the distributions* π_i *and* $\tilde{\pi}_i$, *respectively. Suppose that* $\sup_{i \neq j} \tilde{\pi}_i \tilde{q}_{ij} / (\pi_i q_{ij}) < \infty$ *and* $\sup_i \pi_i / \tilde{\pi}_i < \infty$. *If moreover the* $\widetilde{Q}$-*process has algebraic decay with respect to* V_δ (*resp.*, $\widetilde{V}_\delta$), *then so does the* Q-*process provided it is* V_δ (*resp.*, $\widetilde{V}_\delta$)-*contractive*.

An immediate consequence of Theorem 2.7 is as follows. With respect to V_0 or $\widetilde{V}_0$, any local perturbation does not interfere with the algebraic convergence.

Theorem 2.1 is proved in the next section. The other results are proved in Section 4. In Section 5, two examples are discussed to illustrate the power of the results obtained in the paper.

3. Proof of Theorem 2.1. (A) First, we prove (1.3) under (2.2). Obviously, $V_\delta(cf+d)=c^2V_\delta(f)$ holds for all constants c and d.

Let $(x_t,\ y_t)$ be the Markov chain determined by the coupling operator $\widetilde{\Omega}$, starting from (i,j). Because $\widetilde{\Omega}\phi(i,j)\le 0$ for all $i\ne j$ and $\widetilde{\Omega}\phi(i,i)=0$ for all i, we have $E^{(i,j)}\phi_{x_t,y_t}\le\phi_{ij}$ (for more details of couplings, refer to [2, 3, 5, 7]). Then

$$\begin{aligned}\left|\frac{P_tf(i)-P_tf(j)}{\phi_{ij}}\right|^2&=\left|\frac{E^if(x_t)-E^jf(y_t)}{\phi_{ij}}\right|^2=\left|\frac{E^{(i,j)}(f(x_t)-f(y_t))}{\phi_{ij}}\right|^2\\&=\left|E^{(i,j)}\left[\frac{f(x_t)-f(y_t)}{\phi_{x_t,y_t}}\frac{\phi_{x_t,y_t}}{\phi_{ij}}\right]\right|^2\\&\le\sup_{k,\ell\in E}\left|\frac{f(k)-f(\ell)}{\phi_{k\ell}}\right|^2\left(\frac{E^{(i,j)}\phi_{x_t,y_t}}{\phi_{ij}}\right)^2\le V_1(f),\qquad i\ne j.\end{aligned}$$

Taking the supremum over all $i\ne j$ on the left-hand side yields $V_1(P_tf)\le V_1(f)$.

Next, we prove that (1.3) always holds for V_0. Actually, for any coupled process (x_t,y_t), we have

$$\begin{aligned}V_0(P_tf)&=\sup_{i\ne j}|P_tf(i)-P_tf(j)|^2=\sup_{i\ne j}|E^if(x_t)-E^jf(y_t)|^2\\&\le\sup_{i\ne j}E^{(i,j)}|f(x_t)-f(y_t)|^2\le\sup_{i\ne j}E^{(i,j)}V_0(f)=V_0(f),\qquad t\ge 0.\end{aligned}$$

However, the proof does not work when V_0 is replaced by $\widetilde{V}_0$ and so we do not consider the contraction for $\widetilde{V}_0$ in the study on the sufficient part of our results.

(B) Next, we prove part (i) of the theorem. Some ideas of the proof are taken from [6, 9, 13]. Let f satisfy $\pi(f)=0$ and $\|f\|^2=1$. Then we have

$$\begin{aligned}\mathrm{Var}_\pi(f)&=\frac12\sum_{i,j}\pi_i\pi_j(f_j-f_i)^2=\sum_{\{i,j\}}\pi_i\pi_j(f_j-f_i)^2\\&\le\left\{\sum_{\{i,j\}}\pi_i\pi_j\left(\frac{f_j-f_i}{\phi_{ij}}\right)^2\right\}^{1/p}\left\{\sum_{\{i,j\}}\pi_i\pi_j\left(\frac{f_j-f_i}{\phi_{ij}^\delta}\right)^2\phi_{ij}^{2(q+\delta-1)}\right\}^{1/q}\qquad(3.1)\\&=:\mathrm{I}^{1/p}\cdot\mathrm{II}^{1/q}.\end{aligned}$$

Put $f(e) = f_{e_r} - f_{e_\ell}$. Then

$$
\begin{aligned}
\mathrm{I} &= \sum_{\{i,j\}} \frac{\pi_i \pi_j}{\phi_{ij}^2} \biggl(\sum_{e \in \gamma_{ij}} f(e) \biggr)^2 \\
&= \sum_{\{i,j\}} \frac{\pi_i \pi_j}{\phi_{ij}^2} \sum_{e \in \gamma_{ij}} f(e) \biggl(\sum_{b \in \gamma_{i,e_\ell}} f(b) + \sum_{d \in \gamma_{e_\ell,j}} f(d) \biggr) \\
&= \sum_{\{i,j\}} \frac{\pi_i \pi_j}{\phi_{ij}^2} \biggl(\sum_{e \in \gamma_{ij}} f(e) \sum_{b \in \gamma_{i,e_\ell}} f(b) + \sum_{e \in \gamma_{ij}} f(e) \sum_{d \in \gamma_{e_\ell,j}} f(d) \biggr) \\
&= \sum_{\{i,j\}} \frac{\pi_i \pi_j}{\phi_{ij}^2} \biggl(\sum_{e \in \gamma_{ij}} f(e) \sum_{b \in \gamma_{i,e_\ell}} f(b) + \sum_{d \in \gamma_{ij}} f(d) \sum_{e \in \gamma_{i,d_r}} f(e) \biggr) \\
&= \sum_{\{i,j\}} \frac{\pi_i \pi_j}{\phi_{ij}^2} \biggl(\sum_{e \in \gamma_{ij}} f(e) \sum_{b \in \gamma_{i,e_\ell}} f(b) + \sum_{d \in \gamma_{ij}} f(d) \sum_{e \in \gamma_{i,d_\ell}} f(e) \\
&\qquad\qquad + \sum_{d \in \gamma_{ij}} f(d) \sum_{e = \langle d_\ell, d_r \rangle} f(e) \biggr) \\
&= \sum_{\{i,j\}} \frac{\pi_i \pi_j}{\phi_{ij}^2} \biggl(2 \sum_{e \in \gamma_{ij}} f(e) \sum_{b \in \gamma_{i,e_\ell}} f(b) + \sum_{e \in \gamma_{ij}} f(e)^2 \biggr) \\
&= 2 \sum_e f(e) \sqrt{\pi_{e_\ell} q_{e_\ell e_r}} \sum_{\{i,j\} : \gamma_{ij} \ni e} \frac{\pi_i \pi_j}{\phi_{ij}^2 \sqrt{\pi_{e_\ell} q_{e_\ell e_r}}} \sum_{b \in \gamma_{i,e_\ell}} f(b) \\
&\qquad + \sum_{\{i,j\}} \frac{\pi_i \pi_j}{\phi_{ij}^2} \sum_{e \in \gamma_{ij}} f(e)^2 \\
&\le 2 \biggl(\sum_e \pi_{e_\ell} q_{e_\ell e_r} f(e)^2 \biggr)^{1/2} \\
&\qquad \times \biggl(\sum_e \biggl[\sum_{\{i,j\} : \gamma_{ij} \ni e} \frac{\pi_i \pi_j}{\phi_{ij}^2 \sqrt{\pi_{e_\ell} q_{e_\ell e_r}}} \sum_{b \in \gamma_{i,e_\ell}} f(b) \biggr]^2 \biggr)^{1/2} \\
&\qquad + \sum_e \pi_{e_\ell} q_{e_\ell e_r} f(e)^2 \frac{1}{\pi_{e_\ell} q_{e_\ell e_r}} \sum_{\{i,j\} : \gamma_{ij} \ni e} \frac{\pi_i \pi_j}{\phi_{ij}^2}.
\end{aligned}
\tag{3.2}
$$

Here, we have used Schwarz's inequality in the last step. Note that $\sum_{\{i,j\} : \gamma_{ij} \ni e} = \sum_{i \in E} \sum_{j : \gamma_{ij} \ni e}$.

By using Schwarz's inequality again, we obtain

$$\sum_e \Bigg[\sum_{\{i,j\}:\gamma_{ij}\ni e} \frac{\pi_i\pi_j}{\phi_{ij}^2\sqrt{\pi_{e_\ell}q_{e_\ell e_r}}} \sum_{b\in\gamma_{i,e_\ell}} f(b)\Bigg]^2$$

$$= \sum_e \Bigg[\sum_i \Bigg(\frac{\sqrt{\pi_i\pi_{e_\ell}}}{\phi_{i,e_\ell}} \sum_{b\in\gamma_{i,e_\ell}} f(b)\Bigg)\Bigg(\frac{\sqrt{\pi_i}\phi_{i,e_\ell}}{\pi_{e_\ell}\sqrt{q_{e_\ell,e_r}}} \sum_{j:\gamma_{ij}\ni e} \frac{\pi_j}{\phi_{ij}^2}\Bigg)\Bigg]^2$$

$$\le \sum_e \Bigg[\sum_i \frac{\pi_i\pi_{e_\ell}}{\phi_{i,e_\ell}^2}\Bigg(\sum_{b\in\gamma_{i,e_\ell}} f(b)\Bigg)^2 \sum_i \frac{\pi_i\phi_{i,e_\ell}^2}{\pi_{e_\ell}^2 q_{e_\ell,e_r}}\Bigg(\sum_{j:\gamma_{ij}\ni e}\frac{\pi_j}{\phi_{ij}^2}\Bigg)^2\Bigg] \tag{3.3}$$

$$\le \Big\{\sup_e \sigma_1(e)\Big\} \sum_{e,i} \frac{\pi_i\pi_{e_\ell}}{\phi_{i,e_\ell}^2}[f_{e_\ell} - f_i]^2$$

$$\le \Big\{\sup_e \sigma_1(e)\Big\}\beta\cdot \mathrm{I}.$$

Here in the last step we have used the fact that a point e_ℓ occurs in $\sum_{e,i}$ at most β times. Combining (3.2), (3.3) with definition of $\sigma_2(e)$, we see that

$$\begin{aligned}\mathrm{I} &\le 2\sqrt{\sup_e \sigma_1(e)}\sqrt{\beta D(f)\mathrm{I}} + D(f)\sup_e \sigma_2(e)\\ &=: 2C_1\sqrt{\mathrm{I}\cdot D(f)} + D(f)C_2,\end{aligned}$$

where $C_1, C_2 < \infty$ by assumption. Solving the inequality, we get $\mathrm{I} \le D(f)[C_1 + \sqrt{C_1^2 + C_2}]^2$. Next,

$$\mathrm{II} = \sum_{\{i,j\}} \pi_i\pi_j\bigg(\frac{f_j - f_i}{\phi_{ij}^\delta}\bigg)^2 \phi_{ij}^{2(q+\delta-1)} \le V_\delta(f)\sum_{i,j}\pi_i\pi_j\phi_{ij}^{2(q+\delta-1)}.$$

Hence

$$\mathrm{Var}_\pi(f) \le C D(f)^{1/p} V_\delta(f)^{1/q},$$

where

$$C = \big(C_1 + \sqrt{C_1^2 + C_2}\big)^{2/p}\bigg(\sum_{i,j}\pi_i\pi_j\phi_{ij}^{2(q+\delta-1)}\bigg)^{1/q} < \infty$$

by assumption. By the Liggett–Stroock theorem, the process has algebraic decay.

(C) We now prove part (ii) of the theorem. We remark that condition "$\rho_{i+R} \le c'\rho_i$ for all $i \ge 1$" holds whenever $\rho_N \le c'\rho_{N/2}$ for all $N \ge 1$. To see this, let $i \ge R$ and $N = i + R$. Then $\rho_{i+R} = \rho_N \le c'\rho_{(i+R)/2} \le c'\rho_i$ since ρ_i is increasing in i.

On the other hand, since the set $\{i : i < R\}$ is finite, the inequality "$\rho_{i+R} \le c''\rho_i$ for all $i < R$" is automatic for some constant $c'' \le c'$. However, to simplify the notation, we will use the same c in these inequalities.

Assume that the process has algebraic decay. Let $m, N \in \mathbb{N}$ so that $\sum_i \rho_i^m \pi_i = \infty$ and let $f(k) = \rho_{k\wedge N}^m$. Then we have

$$\begin{aligned}
V_\delta(f) &= \max_{0\le i,j\le N,\ i\ne j} (\rho_i^m - \rho_j^m)^2/(\rho_i - \rho_j)^{2\delta} \\
&= \max_{0\le i<j\le N} \{\rho_j^{m-1}(1 + \rho_i/\rho_j + (\rho_i/\rho_j)^2 + \cdots + (\rho_i/\rho_j)^{m-1})\}^2 \\
&\qquad\qquad \times (\rho_j - \rho_i)^{2(1-\delta)} \\
&\le m^2 \rho_N^{2(m-\delta)}.
\end{aligned}$$

We now consider $D(f)$:

$$D(f) = \frac{1}{2} \sum_{0\le i,j\le N} \pi_i q_{ij} (\rho_j^m - \rho_i^m)^2 + \sum_{i<N,\ j>N} \pi_i q_{ij} (\rho_N^m - \rho_i^m)^2.$$

For the first term on the right-hand side, we have

$$\begin{aligned}
&\frac{1}{2} \sum_{0\le i,j\le N} \pi_i q_{ij} (\rho_j^m - \rho_i^m)^2 \\
&\quad = \sum_{0\le i<j\le N} \pi_i q_{ij} (\rho_j^m - \rho_i^m)^2 = \sum_{i=0}^{N} \pi_i \sum_{j=i+1}^{N} q_{ij} (\rho_j^m - \rho_i^m)^2 \\
&\quad = \sum_{i=0}^{N} \pi_i \sum_{j=i+1}^{N} q_{ij} (\rho_j - \rho_i)^2 [\rho_j^{m-1} + \rho_j^{m-2}\rho_i + \cdots + \rho_i^{m-1}]^2 \\
&\quad = \sum_{i=1}^{N} \pi_i \sum_{j=i+1}^{N\wedge(i+R)} q_{ij} (\rho_j - \rho_i)^2 [\rho_j^{m-1} + \rho_j^{m-2}\rho_i + \cdots + \rho_i^{m-1}]^2 \\
&\qquad + \pi_0 \sum_{j=1}^{N\wedge R} q_{0j} \rho_j^{2m} \\
&\quad \le c_1 m^2 \sum_{i=1}^{N} \pi_i \rho_i^{2m-2} \sum_{j=i+1}^{N\wedge(i+R)} q_{ij} (\rho_j - \rho_i)^2 + \pi_0 \sum_{j=1}^{N\wedge R} q_{0j} \rho_j^{2m} \\
&\quad \le c_1 c R m^2 \sum_{i=1}^{N} \pi_i \rho_i^{2m-2+\alpha} + c_2 m^2 \sum_{j=1}^{N\wedge R} \pi_j \rho_j^{2m-2+\alpha} \\
&\quad = m^2 (c_1 c R + c_2) \sum_{i=1}^{N} \pi_i \rho_i^{2m-2+\alpha},
\end{aligned}$$

where $c_1 = c_1(m) = \sum_{k=0}^{m-1} c^{2k} = (c^{2m} - 1)/(c - 1)$, $c_2 = c_2(m) = m^{-2}\pi_0 \times \max_{1\le j\le R} q_{0j}\rho_j^{2-\alpha}/\pi_j < \infty$ by assumption.

As for the second term, because of finite range, by condition (2.3) and the remark at the beginning of this part (C), we have $\sum_{j>N} q_{0j} = 0$ for $N > R$ and $\rho_N \le c\rho_i$ for all $1 \vee (N - R) \le i \le N - 1$. Then

$$\begin{aligned}
&\sum_{i,j\,:\,i<N<j} \pi_i q_{ij}(\rho_N^m - \rho_i^m)^2 \\
&\quad = \sum_{i=1}^{N-1} \pi_i(\rho_N^{m-1} + \rho_N^{m-2}\rho_i + \cdots + \rho_i^{m-1})^2(\rho_N - \rho_i)^2 \sum_{j>N} q_{ij} \\
&\qquad + \pi_0 \sum_{j>N} q_{0j}\rho_N^{2m} \\
&\quad \le \sum_{i=1\vee(N-R)}^{N-1} \pi_i \cdot c_1 \cdot m^2 \rho_i^{2(m-1)} \sum_{j=N+1}^{i+R} q_{ij}(\rho_j - \rho_i)^2 \\
&\quad \le c_1 cRm^2 \sum_{i=1}^{N} \pi_i \rho_i^{2m-2+\alpha}, \qquad N > R.
\end{aligned}$$

Finally, we get

$$D(f) \le C_1(m) \sum_{i=1}^{N} \pi_i \rho_i^{2(m-1)+\alpha}, \tag{3.4}$$

where $C_1(m) = m^2[2cRc_1(m) + c_2(m)] < \infty$ for all m by assumption.

Before moving further, we need an elementary result about the estimation of variation.

LEMMA 3.1. *Let f be an increasing function and define $h = f \circ g$ for some function g. Next, let $W > 0$ be a constant and set $h_W = h \wedge W$. Choose γ_M large enough so that $\pi(g > \gamma_M) \le 1/M$. Then we have*

$$\operatorname{Var}(h_W) \ge \left(\int_{[h\le W]} h^2\, d\pi\right)\left\{1 - \left(\frac{1}{\sqrt{M}} + \frac{f(\gamma_M)}{(\int_{[h\le M]} h^2\, d\pi)^{1/2}}\right)\right\}^2. \tag{3.5}$$

PROOF. Note that

$$\pi(h_W) - \pi\big(I_{[g\le\gamma_M]}h_W\big) = \pi\big(I_{[g>\gamma_M]}h_W\big) = \int_{[g>\gamma_M]} h_W\, d\pi \le \|h_W\|\sqrt{1/M}.$$

We have $\pi(h_W) \le \|h_W\|/\sqrt{M} + \pi(I_{[g\le\gamma_M]}h_W) \le \|h_W\|/\sqrt{M} + f(\gamma_M)$. Hence

$$\begin{aligned}
\operatorname{Var}(h_W) &= \|h_W\|^2 - \pi(h_W)^2 \ge \|h_W\|^2 - \left(\frac{\|h_W\|}{\sqrt{M}} + f(\gamma_M)\right)^2 \\
&= \|h_W\|^2\left\{1 - \left(\frac{1}{\sqrt{M}} + \frac{f(\gamma_M)}{\|h_W\|}\right)\right\}^2.
\end{aligned}$$

On the other hand, $\|h_W\|^2 = \int_{[h\le W]} h^2\,d\pi + W^2\pi[h > W] \ge \int_{[h\le W]} h^2\,d\pi$. From these two facts, we obtain (3.5). $\square$

Now, let $g_k = \rho_k,\ f(x) = x^m$ and $W = \rho_N^m$. Then we come back to $h_W(k) = \rho_{k\wedge N}^m$. The estimate (3.5) yields that

$$\left(\sum_{i=1}^N \pi_i\rho_i^{2m}\right)\left\{1 - \left[\frac{1}{\sqrt{M}} + \frac{\gamma_M^m}{\sqrt{\sum_{i=1}^\infty \pi_i\rho_{i\wedge N}^{2m}}}\right]\right\}^2 \le \mathrm{Var}(f).$$

Take $M = 16$. Since $\pi(\rho^{2m}) = \infty$, there exists $N_0 = N_0(m)$ such that

$$\tfrac{1}{2}\sum_{i=1}^N \pi_i\rho_i^{2m} \le \mathrm{Var}(f) \qquad \text{for all } N \ge N_0. \tag{3.6}$$

By Theorem A(ii), (1.2) holds with $V = V_\delta$. Combining (3.4), (3.6) with (1.2), we get

$$\begin{aligned}\sum_{j=1}^N \pi_j\rho_j^{2m} &\le C_2(m)\left(\sum_{j=1}^N \pi_j\rho_j^{2m-2+\alpha}\right)^{1/p}\rho_N^{2(m-\delta)/q}\\ &\le C_2(m)\left(\sum_{j=1}^N \pi_j\rho_j^{2m}\right)^{(m-1+\alpha/2)/(mp)}\rho_N^{2(m-\delta)/q},\end{aligned}$$

where in the last step, we have used the Schwarz's inequality. Therefore,

$$\sum_{j=1}^N \pi_j\rho_j^{2m} \le C_3(m)\rho_N^{2(m-\delta)mp/(q(mp-m+1-\alpha/2))}. \tag{3.7}$$

Now, we consider separately the two cases listed in (2.3). First, assume that there is $\varepsilon < 1$ such that $\inf_N \rho_{N/2}/\rho_N \ge \varepsilon$. Then, by (3.7), we have

$$\begin{aligned}\sum_{j=N/2}^N \pi_j\rho_j^k &= \sum_{j=N/2}^N \pi_j\rho_j^{2m+k-2m} \le \rho_{N/2}^{k-2m}\sum_{j=N/2}^N \pi_j\rho_j^{2m}\\ &\le \varepsilon^{k-2m}\rho_N^{k-2m}\sum_{j=N/2}^N \pi_j\rho_j^{2m}\\ &\le C_4(m)\rho_N^{k-2m+2(m-\delta)mp/(q(mp-m+1-\alpha/2))}.\end{aligned} \tag{3.8}$$

When $m \to \infty$, the power of ρ_N on the right-hand side converges to $k - (2-\alpha)\times(q-1) - 2\delta$. Thus, when $k < (2-\alpha)(q-1) + 2\delta$, we can choose and then fix m large enough so that the power just mentioned becomes negative, denoted by $-\gamma$. Since $(\rho_N/\rho_{N/2})^{-\gamma} \le \varepsilon^\gamma < 1$, by (3.8) and ratio test, we get

$$\sum_j \rho_j^k\pi_j = 1 + \sum_{\ell=0}^\infty \sum_{j\in\{2^\ell\le\rho_j\le 2^{\ell+1}-1\}} \rho_j^k\pi_j < \infty.$$

Second, by assumption, we have $\rho_{(N+1)R} \le c\rho_{NR}$ and $\sum_{N=1}^{\infty} \rho_{NR}^{-\varepsilon} < \infty$ for all $\varepsilon > 0$. Hence, by (3.7), we have

$$\begin{aligned}\sum_{j=NR}^{(N+1)R} \pi_j \rho_j^k &= \sum_{j=NR}^{(N+1)R} \pi_j \rho_j^{2m+k-2m} \le \rho_{NR}^{k-2m} \sum_{j=NR}^{(N+1)R} \pi_j \rho_j^{2m} \\ &\le c^{k-2m} \rho_{(N+1)R}^{k-2m} \sum_{j=NR}^{(N+1)R} \pi_j \rho_j^{2m} \\ &\le C_4(m) \rho_{(N+1)R}^{k-2m+2(m-\delta)mp/(q(mp-m+1-\alpha/2))}.\end{aligned} \tag{3.9}$$

So by (3.9), we get

$$\sum_j \rho_j^k \pi_j = \sum_{N=0}^{\infty} \sum_{j=NR}^{(N+1)R-1} \pi_j \rho_j^k \le \sum_{N=0}^{\infty} \sum_{j=NR}^{(N+1)R} \pi_j \rho_j^k < \infty.$$

Now, the proof of Theorem 2.1 is complete. □

4. Proofs of Theorem 2.3 and other results.

PROOF OF THEOREM 2.3. Let f satisfy $\pi(f) = 0$ and $\|f\|^2 = 1$. Then we have

$$\begin{aligned}\mathrm{Var}_\pi(f) &= \inf_c \sum_j \pi_j (f_j - c)^2 \le \sum_j \pi_j (f_j - f_0)^2 = \sum_j \pi_j \left(\frac{f_j - f_0}{\phi_j}\right)^2 \phi_j^2 \\ &\le \left\{\sum_j \pi_j \left(\frac{f_j - f_0}{\phi_j}\right)^2\right\}^{1/p} \left\{\sum_j \pi_j \left(\frac{f_j - f_0}{\phi_j^\delta}\right)^2 \phi_j^{2(q+\delta-1)}\right\}^{1/q} \\ &=: \mathrm{I}^{1/p} \cdot \mathrm{II}^{1/q}.\end{aligned}$$

The remainder of the proof is similar to that of Theorem 2.1. The key point is replacing $\sum_i$ used there by the single point $i = 0$. For instance, put $f(e) = f_{e_r} - f_{e_\ell}$. Then we have

$$\begin{aligned}\mathrm{I} &= \sum_j \frac{\pi_j}{\phi_j^2} \left(\sum_{e \in \gamma_{0j}} f(e)\right)^2 = \sum_j \frac{\pi_j}{\phi_j^2} \sum_{e \in \gamma_{0j}} f(e) \left(\sum_{b \in \gamma_{0,e_\ell}} f(b) + \sum_{d \in \gamma_{e_\ell, j}} f(d)\right) \\ &= \sum_j \frac{\pi_j}{\phi_j^2} \left\{2 \sum_{e \in \gamma_{0j}} f(e) \sum_{b \in \gamma_{0,e_\ell}} f(b) + \sum_{e \in \gamma_{0j}} f(e)^2\right\} \\ &= 2 \sum_e f(e) \sqrt{\pi_{e_\ell} q_{e_\ell e_r}} \sum_{j : \gamma_{0j} \ni e} \frac{\pi_j}{\phi_j^2 \sqrt{\pi_{e_\ell} q_{e_\ell e_r}}} \sum_{b \in \gamma_{0,e_\ell}} f(b) \\ &\quad + \sum_j \frac{\pi_j}{\phi_j^2} \sum_{e \in \gamma_{0j}} f(e)^2\end{aligned} \tag{4.1}$$

$$\le 2\biggl(\sum_e \pi_{e_\ell} q_{e_\ell e_r} f(e)^2\biggr)^{1/2} \biggl(\sum_e \biggl[\sum_{j\,:\,\gamma_{0j}\ni e} \frac{\pi_j}{\phi_j^2\sqrt{\pi_{e_\ell} q_{e_\ell e_r}}} \sum_{b\in\gamma_{0,e_\ell}} f(b)\biggr]^2\biggr)^{1/2}$$
$$+ \sum_e \pi_{e_\ell} q_{e_\ell e_r} f(e)^2 \frac{1}{\pi_{e_\ell} q_{e_\ell e_r}} \sum_{j\,:\,\gamma_{0j}\ni e} \frac{\pi_j}{\phi_j^2}.$$

Moreover,

$$\sum_e \biggl[\sum_{j,\gamma_{0j}\ni e} \frac{\pi_j}{\phi_j^2\sqrt{\pi_{e_\ell} q_{e_\ell e_r}}} \sum_{b\in\gamma_{0,e_\ell}} f(b)\biggr]^2$$
$$= \sum_e \biggl[\biggl(\frac{\sqrt{\pi_{e_\ell}}}{\phi_{e_\ell}} \sum_{b\in\gamma_{0,e_\ell}} f(b)\biggr)\biggl(\frac{\phi_{e_\ell}}{\pi_{e_\ell}\sqrt{q_{e_\ell,e_r}}} \sum_{j\,:\,\gamma_{0j}\ni e} \pi_j \frac{1}{\phi_j^2}\biggr)\biggr]^2$$
$$\le \Bigl\{\sup_e \tilde\sigma_1(e)\Bigr\}^2 \sum_e \frac{\pi_{e_\ell}}{\phi_{e_\ell}^2}\biggl(\sum_{b\in\gamma_{0,e_\ell}} f(b)\biggr)^2 \tag{4.2}$$
$$\le \Bigl\{\sup_e \tilde\sigma_1(e)\Bigr\}^2 \beta\cdot \mathrm{I}.$$

Combining (4.1) with (4.2), we see that

$$\mathrm{I} \le 2\Bigl\{\sup_e \tilde\sigma_1(e)\Bigr\}\sqrt{\beta D(f)\cdot \mathrm{I}} + D(f)\sup_e \tilde\sigma_2(e) =: 2C_1\sqrt{\mathrm{I}\cdot D(f)} + D(f)C_2.$$

Next,

$$\mathrm{II} = \sum_j \pi_j \biggl(\frac{f_j - f_0}{\phi_j^\delta}\biggr)^2 \phi_j^{2(q+\delta-1)} \le \widetilde V_\delta(f) \sum_j \pi_j \phi_j^{2(q+\delta-1)}.$$

In the particular case mentioned in Theorem 2.3(i), one may replace $\widetilde V_1$ by V_1 on the right-hand side since $\widetilde V_1(f) \le V_1(f)$. The remainder of the proof is almost the same as that of Theorem 2.1, the only place which needs a slight change is estimating $\widetilde V_\delta$ instead of V_δ at the beginning of the proof for Theorem 2.1(ii). □

To prove Corollaries 2.4 and 2.5, recall that for a positive recurrent birth–death process with birth rate $b_i > 0$ $(i \ge 0)$ and death rate $a_i > 0$ $(i \ge 1)$, the reversible measure (π_i) is $\pi_i = \mu_i/\mu,\ \mu_0 = 1,\ \mu_i = b_0 b_1 \cdots b_{i-1}/a_1 a_2 \cdots a_i,\ i \ge 1$, where $\mu = \sum_i \mu_i$.

LEMMA 4.1 ([3], Theorem 3.3). *Let (u_k) be a positive sequence on Z_+ and set $F(k) = \sum_{j<k} u_j$. Define $\rho(m,n) = |F(m) - F(n)|$. Then there exists a*

coupling operator $\widetilde{\Omega}$ (*the classical Doeblin's coupling, for instance*) *such that, for all* $j > i \geq 0$,

$$\widetilde{\Omega}\rho(i, j) = b_j u_j - a_j u_{j-1} - b_i u_i + a_i u_{i-1}, \qquad u_{-1} := 1. \tag{4.3}$$

PROOF OF COROLLARIES 2.4 AND 2.5. First, we prove $V_1(P_t f) \leq V_1(f)$. By Lemma 4.1, we know that there exists coupling operator $\widetilde{\Omega}$ satisfying (4.3). By the additional assumption in part (i) of the corollaries, we have $\widetilde{\Omega}\rho(i, j) \leq 0$ for all $i, j \in E$. Then applying part (A) of the proof of Theorem 2.1 gives the required assertion.

The other assertions of Corollary 2.4 follow from Theorem 2.1. The first assertion in Corollary 2.5(i) follows from the proof of Theorem 2.3. The other assertions of Corollary 2.5 are direct consequence of Theorem 2.3. We omit the details here. □

PROOF OF COROLLARY 2.6. To apply Corollary 2.5, take $\phi_n = n^\alpha$; then

$$\begin{aligned}\frac{\pi_n \phi_n^{2(q-1)}}{\pi_{n+1}\phi_{n+1}^{2(q-1)}} &= \frac{a_{n+1}}{b_n}\left(1 - \frac{1}{n+1}\right)^{2\alpha(q-1)} \\ &= \left(1 + \frac{1}{n} n\left(\frac{a_{n+1}}{b_n} - 1\right)\right)\left(1 - \frac{1}{n+1}\right)^{2\alpha(q-1)}.\end{aligned}$$

By the Gauss test, we have $\sum_n \pi_n \phi_n^{2(q-1)} < \infty$ once $\liminf_{n\to\infty} n(a_{n+1}/b_n - 1) - 2\alpha(q-1) > 1$, which is fulfilled for sufficiently small $q - 1 > 0$ by assumption. Condition (c) of Corollary 2.5 holds. Next,

$$\sum_{k\geq n+1} \pi_k k^{-2\alpha} \leq (n+1)^{-2\alpha} \sum_{k\geq n+1} \pi_k \leq n^{-2\alpha} \sum_{k\geq n+1} \pi_k.$$

Hence

$$\theta_1(n) = \frac{n^\alpha}{\sqrt{b_n}\pi_n} \sum_{k\geq n+1} \pi_k k^{-2\alpha} \leq \frac{1}{\sqrt{b_n} n^\alpha \pi_n} \sum_{k\geq n+1} \pi_k.$$

By assumption, we have $\sup_n \theta_1(n) < \infty$ and so condition (a) of Corollary 2.5 holds. Moreover,

$$\limsup_{n\to\infty} \frac{1}{\phi_n \sqrt{b_n}} \leq \sup_n \left\{\frac{\sum_{k\geq n+1} \pi_k}{\sqrt{b_n} n^\alpha \pi_n}\right\} \limsup_{m\to\infty} \frac{\pi_m}{\sum_{k\geq m+1} \pi_k} < \infty.$$

By assumption, condition (b) of Corollary 2.5 also holds. The required conclusion now follows from Corollary 2.5(i). □

PROOF OF THEOREM B. We have already proved that $V_1(P_t f) \leq V_1(f)$ in the proof of Theorem 2.1.

(a) Obviously,

$$D(f)=\sum_{k\ge 0}(f_{k+1}-f_k)^2 b_k\pi_k \ge \Big(\inf_{i\ge 0} b_i\Big)\sum_{k\ge 0}(f_{k+1}-f_k)^2\pi_k.$$

(b) Let $f\in L^2(\pi)$. Then

$$\begin{aligned}
&\sum_{n=0}^{\infty}\pi_n\left\{\sum_{k=0}^{n}|f_{k+1}-f_k|\right\}^2\\
&\quad\le 2\sum_{n=0}^{\infty}\pi_n\sum_{0\le j\le k\le n}|f_{j+1}-f_j|\,|f_{k+1}-f_k|\\
&\quad=2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}=2\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{n=k}^{\infty}=2\sum_{k=0}^{\infty}|f_{k+1}-f_k|\sigma_k\pi_k\sum_{j=0}^{k}|f_{j+1}-f_j|\\
&\quad\le 2\left\{\sum_{k=0}^{\infty}|f_{k+1}-f_k|^2\sigma_k^2\pi_k\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}|f_{j+1}-f_j|\right)^2\pi_k\right\}^{1/2}.
\end{aligned}$$

That is,

$$\sum_{n=0}^{\infty}\pi_n\left\{\sum_{k=0}^{n}|f_{k+1}-f_k|\right\}^2\le 4\sum_{k=0}^{\infty}|f_{k+1}-f_k|^2\sigma_k^2\pi_k.$$

On the other hand,

$$\begin{aligned}
\|f-\pi(f)\|^2&=\tfrac12\sum_{j,k=0}^{\infty}\pi_j\pi_k(f_k-f_j)^2\le\sum_{0\le j<k}\pi_j\pi_k\left\{\sum_{i=j}^{k-1}(f_{i+1}-f_i)\right\}^2\\
&\le\sum_{k=1}^{\infty}\pi_k\sum_{j=0}^{k-1}\pi_j\left\{\sum_{i=j}^{k-1}(f_{i+1}-f_i)\right\}^2\le\sum_{k=0}^{\infty}\pi_k\left\{\sum_{i=0}^{k}|f_{i+1}-f_i|\right\}^2.
\end{aligned}$$

Collecting the above two inequalities together, it follows that

$$\|f-\pi(f)\|^2\le 4\sum_{k=0}^{\infty}|f_{k+1}-f_k|^2\sigma_k^2\pi_k.$$

(c) By Schwarz's inequality, we get

$$\begin{aligned}
\|f-\pi(f)\|^2&\le 4\left\{\sum_{k=0}^{\infty}|f_{k+1}-f_k|^2\pi_k\right\}^{1/p}\left\{\sum_{k=0}^{\infty}|f_{k+1}-f_k|^2\sigma_k^{2q}\pi_k\right\}^{1/q}\\
&\le C D(f)^{1/p}V(f)^{1/q},
\end{aligned}$$

where $C=4(\inf_i b_i)^{-1/p}(\sup_k \sigma_k/k)^2\{\sum_{k=0}^{\infty}u_k^2k^{2q}\pi_k\}^{1/q}<\infty$ by assumption. $\square$

PROOF OF THEOREM 2.7. Let V denote either V_δ or $\widetilde{V}_\delta$ appearing in the theorem. Because the $\widetilde{Q}$-process has algebraic decay with respect to V, we have for some constants p, q and C that

$$\mathrm{Var}_{\tilde\pi}(f) \le C\widetilde{D}(f)^{1/p}V(f)^{1/q}, \qquad f \in \mathscr{D}(\widetilde{D}).$$

Next, by the assumptions of the theorem, we have $L^2(\tilde\pi) \subset L^2(\pi)$ and, moreover,

$$\begin{aligned}\frac{D(f)^{1/p}V(f)^{1/q}}{\mathrm{Var}_\pi(f)} &= \frac{[\frac{1}{2}\sum_{i,j}\pi_i q_{ij}(f_j - f_i)^2]^{1/p}V(f)^{1/q}}{\inf_{c\in\mathbb{R}}\sum_i \pi_i(f_i - c)^2}\\ &\ge \frac{\inf_{k\ne\ell}\{\pi_k q_{k\ell}/(\tilde\pi_k\tilde q_{k\ell})\}^{1/p}}{\sup_k\{\pi_k/\tilde\pi_k\}}\frac{\widetilde{D}(f)^{1/p}V(f)^{1/q}}{\mathrm{Var}_{\tilde\pi}(f)}\\ &\ge \frac{\inf_{k\ne\ell}\{\pi_k q_{k\ell}/(\tilde\pi_k\tilde q_{k\ell})\}^{1/p}}{\sup_k\{\pi_k/\tilde\pi_k\}}C, \qquad f \in L^2(\tilde\pi)\cap\mathscr{D}(D).\end{aligned} \tag{4.4}$$

The proof will be complete once we remove "$L^2(\tilde\pi)$" appearing at the end of (4.4). To do so, let $f \in \mathscr{D}(D)$ and set $f_M = (-M)\vee f \wedge M$ for constant $M > 0$. Then, by [2], Lemma 6.47, we have $f_M \in \mathscr{D}(D)$, $\|f_M - f\| \to 0$ and $D(f_M) \to D(f)$ as $M \to \infty$. Hence $\mathrm{Var}_\pi(f_M) \to \mathrm{Var}_\pi(f)$ as $M \to \infty$. The assertion now follows by replacing f with $f_M \in L^2(\tilde\pi)\cap\mathscr{D}(D)$ in (4.4) and then letting $M \to \infty$, since $V(f_M) \le V(f)$. $\square$

5. Two examples. In this section, we examine two examples of irreducible birth–death processes.

EXAMPLE 5.1. $a_i = b_i = i^r,\ i \gg 1.$

EXAMPLE 5.2. $a_i = 1,\ b_i = 1 - c/i,\ i \gg 1.$

It is easy to check that the process of Example 5.1 (resp., Example 5.2) is positive recurrent iff $r > 1$ (resp., $c > 1$). As was proved in [4], the first example has L^2-exponential convergence iff $r \ge 2$. However, the second example is never L^2-exponentially convergent for any c.

PROPOSITION 5.3. *With respect to* V_0, *Example* 5.1 (*resp.*, 5.2) *has algebraic decay for all* $r \in (1, 2)$ [*resp.*, $c \in (1, \infty)$].

PROOF. Simply take $\alpha = 1/2$ and $\alpha = 2$, respectively, for Examples 5.1 and 5.2 and then apply Corollary 2.6. $\square$

For the remainder of this section, we study the region of algebraic convergence with different V.

PROPOSITION 5.4. *With respect to* V_1 *defined by* (2.6) *in terms of the sequence* $u_n = (n+1)^{-s}$ *for some* $s \in (0, r-1]$, *Example* 5.1 *with* $r > 1$ *has algebraic decay iff* $r > 5/3$.

PROOF. Clearly, we need only prove the assertion for $r \in (1, 2)$ since the process has L^2-exponential convergence for all $r \geq 2$.

We should justify the power of the different results for this typical example.

The first proof of sufficiency. Use Corollary 2.5(i).

(i) Recall that $u_n = (n+1)^{-s}$, $s > 0$. We show that the upper bound $s \leq r-1$ listed in Proposition 5.4 comes from the additional condition in Corollary 2.5(i).

$$b_k u_k - a_k u_{k-1} = k^r \left(\frac{1}{(k+1)^s} - \frac{1}{k^s} \right) = k^{r-s} \left[\left(1 - \frac{1}{k+1} \right)^s - 1 \right].$$

Let $f(x) = x^{r-s}[1 - (1 - \frac{1}{x+1})^s],\ x \geq 1$. Then

$$f'(x) = (r-s)x^{r-s-1} \left[1 - \left(1 - \frac{1}{x+1} \right)^s \right] - s x^{r-s} \left(1 - \frac{1}{x+1} \right)^{s-1} \frac{1}{(1+x)^2}.$$

It is easy to prove that $f'(x) \geq 0$ if and only if $r - s \geq 1$. So, when $s \leq r-1$, the additional condition is satisfied with $u_n = (n+1)^{-s}$.

(ii) We prove that condition (a) of Corollary 2.5 is satisfied for all $s \leq r/2$. Since $s < 1$, we have $\phi_n = \sum_{k=0}^{n-1} u_k = \sum_{k=0}^{n-1} \frac{1}{(k+1)^s} \sim n^{1-s}$. Then

$$\theta_1(k) \sim \frac{k^{1-s}}{k^{-r} k^{r/2}} \sum_{n=k+1}^{\infty} \frac{n^{-r}}{n^{2-s}} \sim \frac{1}{k^{r/2-s}}.$$

So, when $s \leq r/2$, we get (a).

(iii) Because $\phi_k \sqrt{b_k} \sim k^{1-s} k^{r/2} = k^{1-s+r/2}$, condition (b) follows for all $s \leq 1 + r/2$.

(iv) Because $\sum_n \pi_n \phi_n^{2q} \sim \sum_n n^{-r} n^{2q(1-s)}$, if $2q < (r-1)/(1-s)$ $(s < 1)$, then we have $\sum_n \pi_n \phi_n^{2q} < \infty$. Combining this with condition $q > 1$, we get $(r-1)/(1-s) > 2$, that is, $s > (3-r)/2$.

Because of (i)–(iv), the process has algebraic decay whenever $(3-r)/2 < s \leq r-1$, namely $r > 5/3$. Choosing $s = r-1$, we obtain $q < (r-1)/[2(2-r)]$. It is clear that, when $r \to 2$, q is allowed to tend to ∞.

The second proof of sufficiency. Use Corollary 2.4(i).

(i) It is proved above that $b_n u_n - a_n u_{n-1}$ is nonincreasing whenever $s \leq r-1$. Set $\alpha = 1 - s > 0$. Then $1 > \alpha \geq 2 - r$.

(ii) Note that $\phi_{ij} = u_i + u_{i+1} + \cdots + u_{j-1} = (i+1)^{-s} + \cdots + j^{-s}$, and hence

$$\frac{1}{\alpha}\big[(j+1)^\alpha - (i+1)^\alpha\big] = \int_{i+1}^{j+1} \frac{dx}{x^{1-\alpha}} \geq \phi_{ij} \geq \int_i^j \frac{dx}{x^{1-\alpha}} = \frac{1}{\alpha}(j^\alpha - i^\alpha).$$

Consider condition $\sum_{i,j} \pi_i\pi_j\phi_{ij}^{2q} = 2\sum_{i<j} \pi_i\pi_j\phi_{ij}^{2q} < \infty$. Choose $i = 0$. We have

$$\begin{aligned}\sum_{j>0} \pi_j\phi_{0j}^{2q} < \infty \quad &\Longleftrightarrow \quad \sum_{j>0} j^{-r} j^{\alpha\cdot 2q} < \infty\\ &\Longleftrightarrow \quad r - 2q\alpha > 1\\ &\Longleftrightarrow \quad r > 1 + 2\alpha \qquad (\text{since } q > 1)\\ &\Longleftrightarrow \quad \alpha < \frac{r-1}{2}.\end{aligned}$$

Combining this with (i), we get $r > 5/3$. Then

$$\begin{aligned}\sum_{i<j} \pi_i\pi_j\phi_{ij}^{2q} &\le \sum_{i<j} \pi_i\pi_j[(j+1)^\alpha - (i+1)^\alpha]^{2q}\\ &\le \sum_i \pi_i \sum_{j\ge 1} \pi_j (j+1)^{2q\alpha} = \sum_{j\ge 1} \pi_j (j+1)^{2q\alpha}.\end{aligned}$$

The last sum is finite if and only if $q < (r-1)/(2\alpha)$.

(iii) Now, we consider condition $\sup_e \sigma_2'(e) < \infty$. Let $e = \langle k, k+1\rangle$. Then

$$\begin{aligned}\sigma_2'(e) &= \sup_{i\le k} \sum_{j\ge k+1} \frac{\pi_j}{\pi_k b_k \phi_{ij}^2} \sim \sup_{i\le k} \sum_{j\ge k+1} \frac{\pi_j}{\phi_{ij}^2} = \sum_{j\ge k+1} \frac{\pi_j}{\phi_j^2}\\ &\sim \int_{k+1}^\infty \frac{dx}{x^r[x^\alpha - k^\alpha]^2} \sim -\int_{k+1}^\infty \frac{1}{x^{r+\alpha-1}}\, d\bigg(\frac{1}{x^\alpha - k^\alpha}\bigg)\\ &= \frac{1}{(k+1)^{r+\alpha-1}[(k+1)^\alpha - k^\alpha]} - (r+\alpha-1)\int_{k+1}^\infty \frac{dx}{x^{r+\alpha}[x^\alpha - k^\alpha]}\\ &\sim \frac{1}{k^{r+\alpha-1+\alpha-1}} + \int_{k+1}^\infty \frac{dx}{x^{r+\alpha+\alpha-1}} \sim k^{-r-2\alpha+2}.\end{aligned}$$

Because $r + 2\alpha \ge r + 4 - 2r = 4 - r \ge 2$, the last term is bounded.

(iv) Finally, consider condition $\sup_e \sigma_1'(e) < \infty$.

$$\sigma_1'(e) = \sup_{i\le k-1} \frac{\phi_{ik}}{\pi_k\sqrt{b_k}} \sum_{j\ge k+1} \frac{\pi_j}{\phi_{ij}^2} \sim \sup_{i\le k-1} \frac{\phi_{ik}}{\sqrt{\pi_k}} \sum_{j\ge k+1} \frac{\pi_j}{\phi_{ij}^2}.$$

On the other hand,

$$\frac{\phi_{ik}}{\sqrt{\pi_k}} \sum_{j\ge k+1} \frac{\pi_j}{\phi_{ij}^2} \le \text{const}\cdot \frac{(k+1)^\alpha - (i+1)^\alpha}{k^{-r/2}} \sum_{j\ge k+1} \frac{1}{j^r[j^\alpha - i^\alpha]^2}.$$

We now adopt the continuous approximation. Note that $\sup_k f(k)/g(k) \le \sup_k f'(k)/g'(k)$. For $x \le k-1$, we get

$$\frac{(k+1)^\alpha - (x+1)^\alpha}{k^{-r/2}} \int_{k+1}^\infty \frac{dy}{y^r(y^\alpha - x^\alpha)^2}$$
$$\le \frac{1}{(-r/2)k^{-r/2+1}}$$
$$\times \left[\alpha(k+1)^{\alpha-1}\int_{k+1}^\infty \frac{dy}{y^r(y^\alpha - x^\alpha)^2} - \frac{(k+1)^\alpha - (x+1)^\alpha}{(k+1)^r((k+1)^\alpha - x^\alpha)^2}\right].$$

Note that

$$\frac{(k+1)^\alpha - (x+1)^\alpha}{(k+1)^r((k+1)^\alpha - x^\alpha)^2} \le \frac{1}{(k+1)^r[(k+1)^\alpha - x^\alpha]}$$
$$\le \frac{1}{(k+1)^r[(k+1)^\alpha - (k-1)^\alpha]} \sim k^{-r-\alpha+1}$$
$$= k^{-r+s} < \infty$$

and $\alpha(k+1)^{\alpha-1}\int_{k+1}^\infty \frac{dy}{y^r(y^\alpha - x^\alpha)^2} \le k^{-r-\alpha+2}$. When $r < 2$, we have $\frac{k^{-r-\alpha+2}}{k^{-r/2+1}} = k^{-r/2-\alpha+1} < \infty$ and so $\sup_e \sigma_1'(e) < \infty$. Because $\sup_e(\sigma_1'(e) + \sigma_2'(e)) < \infty \Longrightarrow \sup_e(\sigma_1(e) + \sigma_2(e)) < \infty$, by Corollary 2.4, the process has algebraic decay for $r > 5/3$.

The third proof of sufficiency. Use Theorem B.

As in the first proof, take $u_n = (n+1)^{1-r}$. In order for $\sum_{n=0}^\infty u_n^2 n^{2q}\pi_n < \infty$, we need $r > (2q+3)/3 > 5/3$. Namely, $q < (3r-3)/2$. When $r \in (5/3, 2)$, we get $q \in (1, 3/2)$, which is obviously not good.

Proof of necessity. Finally, we prove that, when $r \le 5/3$, the process is not algebraic-convergent with respect to the functional V_1 given in the proposition.

Suppose that the process has algebraic decay when $1 < r \le 5/3$ and the convergence power is $q - 1 > 0$. Let $\rho_n = \sum_{k=0}^{n-1} u_k$. We now apply Corollary 2.4(ii). Because $b_n = n^r$, $\rho_{n+1} - \rho_n = u_n = n^{-s}$ and $\rho_n \sim n^{1-s}$, we have $b_n(\rho_{n+1} - \rho_n)^2 = n^{r-2s} \sim \rho_n^{(r-2s)/(1-s)}$. Hence there exists a constant c such that $b_n(\rho_{n+1} - \rho_n)^2 \le c\rho_n^{(r-2s)/(1-s)}$ for all n. Moreover, $(r-2s)/(1-s) < 2$ since $r < 2$. Then, by Corollary 2.4(ii), we have $\sum_n \pi_n \rho_n^k < \infty$ for all $k < 2q - \alpha(q-1) = (2-\alpha)q + \alpha$, where $\alpha = (r-2s)/(1-s)$. Because $q > 1$, we have $\sum_n \pi_n\rho_n^k < \infty$ for all $k \le 2$. Since $\sum_n \pi_n \rho_n^2 \sim \sum_n n^{-r+2(1-s)} < \infty$, this implies that $-r + 2(1-s) < -1$. That is, $s > (3-r)/2$. Combining this with $s < r-1$ gives us $r > 5/3$. □

PROPOSITION 5.5. *With respect to* $\bar{V}$ *defined in Theorem* C, *Example* 5.2 *with* $c > 1$ *has algebraic decay iff* $c > 3$.

PROOF. First we use Theorem B to prove sufficiency.

Choose $u_n \equiv 1$. Then $\phi_{ij} = \left|\sum_{k<j} u_k - \sum_{k<i} u_i\right| = |j-i|$ and $\bar{V} = \sup_{j\neq i}(|f_j - f_i|/|j-i|)^2$, which is the same as V_1 given in Theorem B.

Now we check the conditions in Theorem B. First, it is quite obvious that conditions (i) and (ii) are satisfied when $c > 0$. Second, consider condition (iii): $\sup_n \sigma_n/n < \infty$. Fix n. Let $c > 1$ and $x_k = \pi_k/\pi_n$, $k \geq n$. Then $x_{k+1}/x_k = b_k = 1 - c/k < (1-1/k)^s = (1/k)^s/[1/(k-1)]^s$ for all $s \in (1, c)$. Thus, $x_k = x_k/x_n \leq (n-1)^s/(k-1)^s$, $k \geq n$. Therefore, we get

$$\sigma_n = \sum_{k=n}^{\infty} x_k \leq \sum_{k=n}^{\infty}\left(\frac{n-1}{k-1}\right)^s \leq (n-1)^s \int_{n-1}^{\infty} \frac{dx}{x^s} = \frac{n-1}{s-1}.$$

Hence $\sup_n \sigma_n/n \leq 1/(s-1) < \infty$.

Finally, we check condition (iv). Apply the Kummer test to $\sum_n n^{2q}\pi_n$. Set $x_n = n^{2q}\pi_n$ and $y_n = n$. Then

$$\begin{aligned} y_n \frac{x_n}{x_{n+1}} - y_{n+1} &= \frac{n \cdot n^{2q}}{(n+1)^{2q}(1-c/n)} - (n+1) \sim \frac{(c-2q-1)n^{2q+1}}{n^{2q+1}} \\ &= c - 2q + 1, \end{aligned}$$

where "$\sim$" comes from $(n+1)^{2q+1} \sim n^{2q+1} + (2q+1)n^{2q} + \cdots$. So $\sum_n n^{2q}\pi_n$ is finite whenever $c > 2q+1$. That is, the process is algebraic-convergent whenever $c > 3$.

Now we prove the process is not algebraic-convergent when $c \leq 3$. Suppose that the process has algebraic decay. Since $\sup_k b_k \leq 1 < \infty$, by Theorem C [or Corollary 2.4(ii)], we must have $\sum_k k^\alpha \pi_k < \infty$ for all $\alpha < 2q$. However, $q > 1$; the conclusion should hold for $\alpha = 2$, that is, $\sum_k k^2\pi_k < \infty$. We prove that this is impossible when $c \leq 3$. Let $x_n = n^2\pi_n$ and apply the Gauss test. We have

$$\begin{aligned} \frac{x_n}{x_{n+1}} &= \frac{n^2}{(n+1)^2(1-c/n)} = 1 + \frac{c-2}{n} + \frac{1}{n^2}\frac{3(c-1) + (3c-1)/n + c/n^2}{(1+1/n)^2(1-c/n)} \\ &\sim 1 + \frac{c-2}{n} + \frac{M}{n^2}. \end{aligned}$$

So, $\sum_n x_n$ is finite if and only if $c - 2 > 1 \Longleftrightarrow c > 3$. $\square$

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