The Annals of Applied Probability
2003, Vol. 13, No. 2, 490–500

MULTITYPE BRANCHING LIMIT BEHAVIOR

BY HARRY COHN AND QIAO WANG

University of Melbourne

For a multitype branching process in varying environment convergent in probability, a certain sequence of linear combinations of the type sizes is shown to possess some convergence properties. This sequence turns out to be instrumental in deriving a condition for continuity of the limiting distribution function. An application to an L_1 convergent process whose offspring mean matrices are weakly ergodic is also given.

1. Introduction and summary. Define a multitype (p-type) branching process in a varying environment, which will be the object of our study. We restrict our attention to a process initiated at times in the index set $\{0, 1, 2, \dots\}$ and as labels for the types we choose $1, \dots, p$.

We shall assume that our population is initiated by a single ancestor of type 1 at time 0. The individuals will be labeled such that (n, i, l) denote the lth individual of type i in generation n. The p-type branching process in varying environment considered here is a vector-valued, nonnegative random process $\{\mathbf{Z}_n\} = \{(Z_n^{(1)}, \dots, Z_n^{(p)})\}$, where $Z_n^{(j)}$ stands for the nth generation size of type j particles with $j = 1, \dots, p$.

Denote

$$\mathbf{Z}_n(r, i, l) = \bigl(Z_n^{(1)}(r, i, l), \dots, Z_n^{(p)}(r, i, l)\bigr) \tag{1}$$

the sizes of the nth generation populations of various types stemming from (r, i, l).

The basic decomposition of $\mathbf{Z}_n$ with respect to the ancestry of the nth generation of individuals, that is, the previous rth generation, is given by

$$\mathbf{Z}_n = \sum_{i=1}^{p} \sum_{l=1}^{Z_r^{(i)}} \mathbf{Z}_n(r, i, l), \qquad r \le n, \tag{2}$$

where, conditionally on $\mathbf{Z}_0, \dots, \mathbf{Z}_r$, the random vectors $\{\mathbf{Z}_n(r, i, l),\ l = 1, \dots\}$ are independent and identically distributed but with r, i and n-dependent distributions.

Write $\mathcal{F}_n$ for the σ-field generated by $\mathbf{Z}_0, \dots, \mathbf{Z}_n$. The branching process $\{\mathbf{Z}_n\}$ is a time-inhomogeneous Markov process, that is, $P(\mathbf{Z}_{n+1} = \mathbf{i} | \mathbf{Z}_n) = P(\mathbf{Z}_{n+1} = \mathbf{i} | \mathcal{F}_n)$ holds for any n and $\mathbf{i} = (i_1, \dots, i_p)$.

In the classical Galton–Watson setting, convergence properties are proven for suitably normed variables derived from $\{Z_n^{(i)}\}$ for an arbitrary type i. In the varying

Received May 2001; revised January 2002.
AMS 2000 subject classification. 60J80.
Key words and phrases. Martingale, varying environment, multitype, concentration function, continuity, space–time harmonic function.

environment setting, the natural object of study appears to be linear combinations of the type sizes of successive generations, the so-called c-counted process to be defined next.

Let $c = (c(n, j), j = 1, \ldots, p, n = 0, 1, \ldots)$ stand for an arbitrary array of nonnegative constants and define a c-counted process to be

$$Z_n^c = \sum_{j=1}^{p} c(n, j) Z_n^{(j)}, \qquad n = 0, 1, \ldots. \tag{3}$$

Similarly, a c-counted daughter process is defined by

$$Z_n^c(r, i, l) = \sum_{j=1}^{p} c(n, j) Z_n^{(j)}(r, i, l), \tag{4}$$

where $\{Z_n^{(j)}(r, i, l), l = 1, \ldots, i = 1, \ldots, p\}$ are independent given $\mathbf{Z}_r$. Furthermore, we get, for $t > n$,

$$Z_t^c = \sum_{i=1}^{p} \sum_{l=1}^{Z_n^{(i)}} Z_t^c(n, i, l), \tag{5}$$

where $\{Z_t^c(n, i, l), l = 1, \ldots, i = 1, \ldots\}$ are independent given $\mathbf{Z}_n$.

In this paper we shall assume that $\{Z_n^c\}$ converges in probability to a limit W. Convergence to a limit W includes the case when the limit is defective. That is, $P(W = \infty) > 0$ is allowed. In general, W may depend on c.

Using (5) and then taking the limits as $t \to \infty$ yields

$$W = \sum_{i=1}^{p} \sum_{l=1}^{Z_n^{(i)}} W(n, i, l) \qquad \text{a.s.}, \tag{6}$$

where $W(n, i, l)$ is the contribution to W of the line of descent of the lth individual of type i of the nth generation.

Let $\{X_n(i, j)\}$ be the offspring variables of $\{\mathbf{Z}_n\}$, where $X_n(i, j)$ is the number of offspring of type j of an individual of type i of the nth generation. By $X_n(i, j, l)$ we denote the offspring variables of the lth individual of type i of the nth generation. It is assumed that $\{X_n(i, j, l);\ l = 1, 2, \ldots\}$ are i.i.d copies of $X_n(i, j)$. By (5) we can write Z_{n+1}^c in terms of the offspring variables of the nth generation as

$$Z_{n+1}^c = \sum_{i=1}^{p} \sum_{l=1}^{Z_n^{(i)}} \sum_{j=1}^{p} c(n+1, j) X_n(i, j, l) = \sum_{i=1}^{p} \sum_{l=1}^{Z_n^{(i)}} Z_{n+1}^c(n, i, l), \tag{7}$$

where $\{Z_{n+1}^c(n, i, l), l = 1, 2, \ldots, i = 1, \ldots, p\}$ are independent given $\mathbf{Z}_n$.

In what follows 1_A will denote the indicator function of a set A. We shall say that $\lim_{n\to\infty} A_n = A$ a.s. if $\lim_{n\to\infty} 1_{A_n} = 1_A$ a.s.

We shall need to consider the notion of concentration function due to Paul Levy. For an updated account of its use see Petrov [12]. Define the concentration function of a random variable X by $Q(X;\lambda) = \sup_x P(x \le X \le x+\lambda)$. In particular when $\lambda = 0$ we write $Q(X) = \sup_x P(X = x)$. Denote

$$Y_n(i) = \sum_{j=1}^{p} \gamma(n+1, j) X_n(i, j)$$

and $\mathbf{Y}_n = (Y_n(1), \ldots, Y_n(p))$, where $\{\gamma(n,i)\}$ are some constants.

In the one type case, a.s. convergence is equivalent to weak convergence, which always holds for some suitably chosen norming constants. It was shown in Cohn [4] that the continuity of the limiting distribution for suitably normed processes holds barring a very restrictive case when the offspring variables converge fastly to constants. Recently, a number of papers (Biggins, Cohn and Nerman [2], Cohn [3, 5], Hattori [8] and Jones [10]) have dealt with convergence of multitype processes in varying environment. An extension of some continuity results of Cohn [4] to the multitype case was given by Jones [11].

Unlike the one type case, a.s. convergence is no longer equivalent to weak convergence for multitype processes as shown in the example of Nerman (see [2]) for a two type process convergent in distribution but not in probability.

A certain linear combination of the type sizes will turn out to play a key role in our study. It will help to describe the case when $\{Z_n^c\}$ has a discrete limit distribution and will thereby provide a sufficient condition for such a limit to be continuous. The result is then applied to multitype processes with weakly ergodic offspring mean matrices.

Interestingly, it appears that the case when only convergence in distribution holds may allow for a discrete limit under considerably less restrictive assumptions than for processes convergent in probability.

2. Convergence in probability and almost sure

THEOREM 1. *Suppose that $\{Z_n^c\}$ converges in probability to a limit W, and that $P(W = \alpha) > 0$ for some $\alpha > 0$. Then there exists an array of nonnegative constants $\gamma = (\gamma(n, j), n = 0, 1, \ldots, j = 1, \ldots, p)$ such that*

$$\lim_{n\to\infty} \{Z_n^\gamma = \alpha\} = \{W = \alpha\} \qquad a.s.$$

PROOF. Notice that in view of the Markov property of $\{\mathbf{Z}_n\}$, the sequence of random variables $\{P(W = \alpha|\mathbf{Z}_n)\}$ is a martingale that converges a.s. to $1_{\{W=\alpha\}}$ as $n \to \infty$. Thus, by (6) we conclude that

$$P\left(\sum_{i=1}^{p}\sum_{l=1}^{Z_n^{(i)}} W(n,i,l) = \alpha \,\middle|\, \mathbf{Z}_n\right) \to 1_{\{W=\alpha\}} \qquad \text{a.s.} \tag{8}$$

as $n \to \infty$. It follows from (8) that there must exist a sequence of vectors $\{\mathbf{z}_n = (z_1^{(n)}, \ldots, z_n^{(p)})\}$ such that

$$P\left(\sum_{j=1}^{p} \sum_{l=1}^{z_n^{(j)}} W(n, j, l) = \alpha\right) \to 1 \tag{9}$$

as $n \to \infty$, where $\{W(n, j, l)\}$ are independent random variables.

By a property that goes back to Paul Levy (see, e.g., Petrov [12]) the concentration function of a sum of independent random variables is exceeded by the sum of the concentration functions of its summands. Therefore, for any type i, there are some constants $\{\alpha_n^{(i)}\}$ such that

$$P\left(\sum_{l=1}^{z_n^{(j)}} W(n, j, l) = \alpha_n^{(j)}\right) \to 1, \tag{10}$$

where $\sum_{i=1}^{p} \alpha_n^{(i)} = \alpha$. Now by (10) and by Lemma 7 of Cohn [4] there exists an array of nonnegative constants $\gamma = (\gamma(n, j), n = 0, 1, \ldots j = 1, \ldots, p)$ such that

$$\lim_{n\to\infty} P\big(W(n, j, l) = \gamma(n, j)\big) = 1. \tag{11}$$

By (10) and (11) we get that

$$\lim_{n\to\infty} P\left(\sum_{j=1}^{p} \sum_{l=1}^{Z_n^{(j)}} W(n, j, l) = \alpha \middle| \mathbf{Z}_n\right) = \lim_{n\to\infty} P\left(\sum_{j=1}^{p} \sum_{l=1}^{Z_n^{(j)}} \gamma(n, j) = \alpha \middle| \mathbf{Z}_n\right) \qquad \text{a.s.}$$

Thus

$$\lim_{n\to\infty} P(W = \alpha | \mathbf{Z}_n) = \lim_{n\to\infty} P\left(\sum_{j=1}^{p} \gamma(n, j) Z_n^{(j)} = \alpha \middle| \mathbf{Z}_n\right) \qquad \text{a.s.} \tag{12}$$

Since $\{\sum_{j=1}^{p} \gamma(n, j) Z_n^{(j)} = \alpha\}$ is measurable with respect to $\mathbf{Z}_n$, we get

$$P\left(\sum_{j=1}^{p} \gamma(n, j) Z_n^{(j)} = \alpha \middle| \mathbf{Z}_n\right) = 1_{\{\sum_{j=1}^{p} \gamma(n,j) Z_n^{(j)} = \alpha\}} \qquad \text{a.s.} \tag{13}$$

which together with (12) and (8) yields

$$1_{\{\sum_{j=1}^{p} \gamma(n,j) Z_n^{(j)} = \alpha\}} \to 1_{\{W = \alpha\}} \qquad \text{a.s.} \tag{14}$$

as $n \to \infty$. Thus

$$\lim_{n\to\infty} \{Z_n^{\gamma} = \alpha\} = \{W = \alpha\} \qquad \text{a.s.},$$

where $Z_n^{\gamma} = \sum_{j=1}^{p} \gamma(n, j) Z_n^{(j)}$, which completes the proof. $\square$

THEOREM 2. *Suppose that* $\{Z_n^c\}$ *converges in probability to a limit* W *with* $P(0 < W < \infty) > 0$. *Then there exists a* γ-*counted process*, $\{Z_n^\gamma\}$, *that converges a.s. to* W.

PROOF. Consider first the case when $P(W = \alpha) > 0$ for some $\alpha > 0$. Using that $\{P(W \le x|\mathbf{Z}_n)\}$ is a martingale, and taking (11) into account, we can argue as in the proof of Theorem 1 by replacing "$= \alpha$" by "$\le x$" in (12)–(14) to conclude that for any real x,

$$\lim_{n\to\infty}\left\{\sum_{j=1}^{p}\gamma(n,j)Z_n^{(j)} \le x\right\} = \{W \le x\} \qquad \text{a.s.}$$

as $n \to \infty$. We are now in a position to apply Proposition 2 of Cohn [5] and conclude the proof in this case.

It remains to consider the case when $P(W \le x)$ is continuous at all $x > 0$. Notice that by (6) we get

$$P(W \le x|\mathbf{Z}_n) = P\left(\sum_{i=1}^{p}\sum_{l=1}^{Z_n^{(i)}} W(n,i,l) \le x \,\middle|\, \mathbf{Z}_n\right) \qquad \text{a.s.} \tag{15}$$

Some of $\{W(n,i,l)\}$ may be 0. Applying the martingale convergence theorem to (15) yields

$$\lim_{n\to\infty} P\left(\sum_{i=1}^{p}\sum_{l=1}^{Z_n^{(i)}} W(n,i,l) \le x \,\middle|\, \mathbf{Z}_n\right) = 1_{\{W\le x\}} \qquad \text{a.s.}$$

Thus there exists a sequence of p-dimensional vectors $\{\mathbf{x}_n\}$ such that

$$\lim_{n\to\infty} P\left(\sum_{i=1}^{p}\sum_{l=1}^{x_n^{(i)}} W(n,i,l) \le x\right) = 1.$$

Notice that we may choose $x_n^{(i)} \to \infty$ for any type i as otherwise by an argument already employed above some of $W(n,i,l)$ would be discrete, contradicting that $P(W \le x)$ is continuous at $x > 0$. Since $\{W(n,i,l), l = 1,2,\ldots\}$ are independent and identically distributed, we get that the limiting distribution of any weakly convergent subsequence of $\{\sum_{l=1}^{x_n^{(i)}} W(n,i,l)\}$ as $n \to \infty$ must be infinitely divisible. However, infinitely divisible distributions with bounded support are degenerate. Thus, we deduce that any weakly convergent subsequence of $\{\sum_{l=1}^{x_n^{(i)}} W(n,i,l)\}$ converges in probability to a constant, say c_i, that may depend on type and subsequence. Assume first that all c_i are positive. Then we may choose some numbers $\{\gamma(n,i)\}$ such that 1 is the q-quantile of $\sum_{l=1}^{(\gamma(n,i))^{-1}} W(n,i,l)$ for

$i \in \{1, \dots, p\}$, where q is some number in $(0, 1)$. In this way we get

$$\sum_{l=1}^{(\gamma(n,i))^{-1}} W(n, i, l) \xrightarrow{p} 1. \tag{16}$$

Now we can use the reasoning of Theorem 15 of [5] to deduce that $\sum_{l=1}^{Z_n^{(i)}} W(n, i, l)$ and $\gamma(n, i)Z_n^{(i)}$ are a.s. conditional on $\{Z_n\}$ asymptotically equivalent (that is, $\lim_{n\to\infty} P(|\sum_{l=1}^{Z_n^{(i)}} W(n, i, l) - \gamma(n, i)Z_n^{(i)}| > \epsilon|Z_n) = 0$ a.s.). This entails that $\{Z_n^{\gamma}\}$ converges a.s. to W.

It remains to consider the case when $\sum_{l=1}^{Z_n^{(i)}} W(n, i, l)$, or a subsequence thereof, converges in probability to 0 for some type i. In such a case it is easy to see that i has no contribution to W, and therefore we may take $\gamma(n, i) = 0$ and complete the proof. □

3. Continuity of the limiting distribution function

LEMMA 3. *There exists a sequence of p-dimensional nonnegative vectors, $\{\mathbf{z}_k\}$ with $\sum_{i=1}^{p} \gamma(k, i)z_k^{(i)} = \alpha$ such that*

$$P\left(\bigcap_{k=r}^{n}\{Z_{k+1}^{\gamma} = \alpha\}\right) \le \prod_{k=r}^{n} P\left(\sum_{i=1}^{p}\sum_{l=1}^{z_k^{(i)}} Z_{k+1}^{\gamma}(k, i, l) = \alpha\right) P(Z_r^{\gamma} = \alpha). \tag{17}$$

PROOF. Assume that for some r and $n > r$,

$$P(Z_{n+1}^{\gamma} = \alpha, Z_n^{\gamma} = \alpha, \dots, Z_r^{\gamma} = \alpha) > 0. \tag{18}$$

The Markov property of $\{\mathbf{Z}_n\}$ yields

$$\begin{aligned} &P(Z_{n+1}^{\gamma} = \alpha | Z_n^{\gamma} = \alpha, \dots, Z_r^{\gamma} = \alpha) \\ &\quad = \sum_{\mathbf{x}\in\mathbf{A}_n} \frac{P(Z_{n+1}^{\gamma} = \alpha | \mathbf{Z}_n = \mathbf{x}) P(\mathbf{Z}_n = \mathbf{x}, Z_{n-1}^{\gamma} = \alpha, \dots, Z_r^{\gamma} = \alpha)}{P(Z_n^{\gamma} = \alpha, \dots, Z_r^{\gamma} = \alpha)}, \end{aligned} \tag{19}$$

where $\mathbf{A}_n = \{\mathbf{x} = (x_1, \dots, x_p) : \gamma(n, 1)x_1 + \dots + \gamma(n, p)x_p = \alpha\}$. Notice now that

$$\sum_{\mathbf{x}\in\mathbf{A}_n} \frac{P(\mathbf{Z}_n = \mathbf{x}, Z_{n-1}^{\gamma} = \alpha, \dots, Z_r^{\gamma} = \alpha)}{P(Z_n^{\gamma} = \alpha, \dots, Z_r^{\gamma} = \alpha)} = 1. \tag{20}$$

Thus in view of (19) and (20) there must be some δ_{n+1} with

$$\delta_{n+1} \le \max_{\mathbf{x}\in\mathbf{A}_n} P(Z_{n+1}^{\gamma} = \alpha | \mathbf{Z}_n = \mathbf{x}) \tag{21}$$

such that

$$P(Z_{n+1}^{\gamma} = \alpha | Z_n^{\gamma} = \alpha, \dots, Z_r^{\gamma} = \alpha) = \delta_{n+1}. \tag{22}$$

From (21) and (22) we deduce by induction that there exist some vectors $\{\mathbf{z}_m = (z_1^{(m)}, \dots, z_p^{(m)}) \in \mathbf{A}_m, m = r, \dots, n\}$ such that

$$\begin{aligned} &P(Z_{n+1}^\gamma = \alpha, Z_n^\gamma = \alpha, \dots, Z_r^\gamma = \alpha) \\ &\quad \le P(Z_{n+1}^\gamma = \alpha | \mathbf{Z}_n = \mathbf{z}_n) \cdots P(Z_{r+1}^\gamma = \alpha | \mathbf{Z}_r = \mathbf{z}_r) P(\mathbf{Z}_r = \mathbf{z}_r) \end{aligned} \tag{23}$$

and an appeal to (17) completes the proof. □

For the next result we shall need a result on independent random variables that was derived in [4], Lemma 9.

LEMMA 4. *Suppose that* $\{\xi_i^{(n)}, i = 1, \dots, m_n\}$ *is, for each* n, *a sequence of nonnegative, independent and identically distributed random variables. The following conditions are equivalent*:

(i) $$\sum_{n=1}^\infty \bigl(1 - P(\xi_1^{(n)} + \cdots + \xi_{m_n}^{(n)} = m_n k_n)\bigr) < \infty;$$

(ii) $$\sum_{n=1}^\infty m_n \bigl(1 - P(\xi_1^{(n)} = k_n)\bigr) < \infty.$$

THEOREM 5. *Suppose that* $\{Z_n^c\}$ *converges in probability to* W *with* $P(0 < W < \infty) > 0$. *If*

$$\sum_{n=1}^\infty \min_{i \in \{1, \dots, p\}} \bigl(\gamma(n, i)\bigr)^{-1} \bigl(1 - Q\bigl(Y_n(i)\bigr)\bigr) = \infty, \tag{24}$$

then W *assumes a distribution function which is continuous at* $x > 0$.

PROOF. By Theorem 1 and Lemma 3, it will be sufficient to prove that

$$\prod_{n=r}^\infty P\left(\sum_{i=1}^p \sum_{l=1}^{z_n^{(i)}} Z_{n+1}^\gamma(n, i, l) = \alpha\right) = 0, \tag{25}$$

where $Z_{k+1}^\gamma(k, i, l) = \sum_{j=1}^p \gamma(n+1, j) X_n(i, j, l)$.

By an elementary property, (25) is equivalent to

$$\sum_{n=1}^\infty \left(1 - P\left(\sum_{i=1}^p \sum_{l=1}^{z_n^{(i)}} Z_{n+1}^\gamma(n, i, l) = \alpha\right)\right) = \infty. \tag{26}$$

Recall now that for fixed n, the variables $\{Z_{n+1}^\gamma(n, i, l)\}$ are independent, and for fixed n and i $\{Z_{n+1}^\gamma(n, i, l), l = 1, 2, \dots\}$ are identically distributed. Since the concentration function of a sum of independent random variables does not exceed the sum of the concentration functions of its summands, we get

$$
\begin{aligned}
P\Biggl(\sum_{i=1}^{p}\sum_{l=1}^{z_n^{(i)}} Z_{n+1}^{\gamma}(n,i,l)=\alpha\Biggr) &\le Q\Biggl(\sum_{i=1}^{p}\sum_{l=1}^{z_n^{(i)}} Z_{n+1}^{\gamma}(n,i,l)\Biggr)\\
&\le \sum_{i=1}^{p} Q\Biggl(\sum_{l=1}^{z_n^{(i)}} Z_{n+1}^{\gamma}(n,i,l)\Biggr).
\end{aligned}
\tag{27}
$$

In view of (26), (27) and Lemma 4, we need to show that

$$
\sum_{i=1}^{p}\sum_{n=1}^{\infty} z_n^{(i)}\bigl(1-Q(Y_n(i))\bigr)=\infty. \tag{28}
$$

Recall now that $\sum_{i=1}^{p}\gamma(n,i)z_n^{(i)}=\alpha$, which entails $\gamma(n,i_n)z_n^{(i_n)}\ge \alpha/p$ for at least one i_n and all $n=1,2,\dots$. This boils down to

$$
\sum_{i=1}^{p}\sum_{n=1}^{\infty} z_n^{(i)}\bigl(1-Q(Y_n(i))\bigr)\ge \frac{\alpha}{p}\sum_{n=1}^{\infty}\min_{i\in\{1,\dots,p\}}\bigl(\gamma(n,i)\bigr)^{-1}\bigl(1-Q(Y_n(i))\bigr),
$$

and now (28) follows from (24). $\square$

REMARK. It is possible to have more than one rate of growth for $\{\mathbf{Z}_n\}$ as in the one type case (see MacPhee and Schuh [13]). This happens if $P(W^c=\infty)>0$, and there is another c-sequence, say c', such that $\{Z_n^{c'}\}$ converges a.s. to a limit $W^{c'}$ as $n\to\infty$ with $P(W^{c'}>0)=P(\{W^{c'}>0\}\cap\{W^c=\infty\})$.

4. Martingales and space–time harmonic functions. Consider the case when the multitype process $\{\mathbf{Z}_n\}$ has finite expectations. Let $\{M_n\}=\{(M_n(i,j))\}$ be the mean matrices, where $M_n(i,j)$ is the expected number of offspring of type j produced by one particle of type i of the nth generation. Define ${}^kM^{k-1}=I$, where I is the identity matrix. For $n\ge 1$ it will be seen that if ${}^1M^n=({}^1M^n(i,j))=M_1\cdots M_{n-1}$, then ${}^1M^n(i,j)=E(Z_n^{(j)}|\mathbf{Z}_0=\mathbf{e}_i)$, where $\mathbf{e}_i$ is the p-dimensional vector with 1 in the ith place and 0 elsewhere. We shall assume that ${}^mM^n(i,j)>0$ for each m, i and j, for n large enough (which may depend on m).

The Markov property of $\mathbf{Z}_n$ makes $\{X_n\}$ with

$$
X_n=\lim_{m\to\infty}E(Z_m^c|\mathbf{Z}_n)\qquad \text{a.s.} \tag{29}
$$

a martingale provided that such limits exist. Since $\{X_n\}$ is nonnegative, the a.s. convergent limit, say X, always exists but it may be null. If $\{Z_n^c\}$ converges in L_1, the limit X is not identically null, and

$$
E(W|\mathbf{Z}_n)=\lim_{m\to\infty}E(Z_m^c|\mathbf{Z}_n)=X_n\qquad \text{a.s.} \tag{30}
$$

By (29) and (30) we get $X=W$ a.s.

On the other hand, the martingales $\{X_n\}$ turn out to be associated with the class of space–time harmonic functions $h = \{h_n\}$ for $\{M_n\}$ defined by some column vectors $\{h_n\}$ such that $M_n h_{n+1} = h_n$. Indeed, write

$$h(n, i) = E(W|\mathbf{Z}_n = \mathbf{e}_i). \tag{31}$$

After an easy calculation we deduce that $h = \{h_n\}$ defined by (31) is indeed a space–time harmonic function for $\{M_n\}$.

In general there are t extremal space–time harmonic functions $h_1, \dots, h_t$ with $t \le p$ where h is said to be extremal if for any other space–time harmonic function h', $h' \le h$ implies $h = Kh'$, where K is a constant. It was proven in [6] that to each extremal space–time harmonic function h_k there corresponds a sequence of sets $\{E_n^{(k)}\}$ such that for $j_n \in E_n^{(k)}$ and fixed k in $1, \dots, t$,

$$\lim_{n\to\infty} \frac{{}^mM^n(i, j_n)}{{}^mM^n(l, j_n)} = \frac{h_k(m, i)}{h_k(m, l)}.$$

The harmonic functions attached to a sequence $\{M_n\}$ belong to the convex hull of the extremal harmonic functions (see the discussion in Cohn and Nerman [6]).

An important instance of space–time harmonic function is the so called weakly ergodic case, defined for a sequence of matrices $\{M_n\}$, when for arbitrary m, i, l there exist $\gamma_m(i, l)$ such that $0 < \gamma_m(i, l) < \infty$, and

$$\lim_{n\to\infty} \frac{{}^mM^n(i, j)}{{}^mM^n(l, j)} =: \gamma_m(i, l) \tag{32}$$

for any j.

There are a number of criteria for weak ergodicity (see Cohn and Nerman [6] and the references therein). A simple rule due to Hajnal [7] requires

$$\sum_{n=1}^{\infty} \sqrt{\min_{i,j,k,l} \frac{M_n(i, k)M_n(j, l)}{M_n(i, l)M_n(j, k)}} = \infty. \tag{33}$$

If weak ergodicity holds, there exists only one harmonic function—up to a multiplicative constant (see [6]):

$$h(m, i) = \lim_{n\to\infty} \frac{{}^mM^n(i, k)}{{}^1M^n(1, k)} \tag{34}$$

for any type k.

THEOREM 6. *Suppose that $\{\mathbf{Z}_n\}$ is a p-type weakly ergodic multitype branching process in varying environment and that $\{Z_n^{(i)}/{}^1M^n(1, i)\}$ converges in L_1 as $n \to \infty$. Then the $\{Z_n^{(i)}/{}^1M^n(1, i)\}$ have the same limit variable W for any $i = 1, \dots, p$. In addition, $\{X_n\}$, defined by*

$$X_n = h(n, 1)Z_n^{(1)} + \cdots + h(n, p)Z_n^{(p)}, \tag{35}$$

where

$$h(n,i)=\left[\sum_{j=1}^{p}\gamma_n(i,j)^1M^n(1,j)\right]^{-1} \tag{36}$$

is a martingale that converges a.s. to W. If

$$\sum_{n=1}^{\infty}\bigl({}^1M^n(1,1)\bigr)^{-1}(1-Q_n)=\infty,$$

where $Q_n = \max_{i\in\{1,\dots,p\}} Q(Y_n(i))$, then the distribution function of W is continuous outside 0.

For necessary and sufficient conditions ensuring the convergence assumption of this result see the criteria of [2] and [5].

PROOF. Using (29)–(31) and simple manipulations yield (35).

On the other hand, (32) and (34) yield

$$\begin{aligned}h(m,i)&=\lim_{n\to\infty}\frac{{}^mM^n(i,k)}{{}^1M^n(1,k)}=\lim_{n\to\infty}\frac{{}^mM^n(i,k)}{\sum_{j=1}^{p}{}^1M^m(1,j)^mM^n(j,k)}\\&=\left[\sum_{j=1}^{p}\gamma_n(i,j)^1M^n(1,j)\right]^{-1}\end{aligned}$$

which proves (36). Since there exists—up to an equivalence—only one space–time harmonic function, the limit of $\{X_n\}$ is uniquely determined. Applying the martingale convergence theorem to (30) we get that $\lim_{n\to\infty} X_n = W$. Thus, all convergent $\{Z_n^c\}$ are asymptotically equivalent. In particular, $\{Z_n^{(i)}/E(Z_n^{(i)})\}$ have the same limit, W, for all i, a case that parallels the classical convergence result for the supercritical multitype Galton–Watson process (see Athreya and Ney [1]).

It is easy to see that in view of (35), the conditions of Theorem 5 are satisfied. This completes the proof. □

REFERENCES

[1] ATHREYA, K. B. and NEY, P. (1972). *Branching Processes.* Springer, New York.

[2] BIGGINS, J. D., COHN, H. and NERMAN, O. (1999). Multitype branching processes. *Stochastic Process. Appl.* **83** 357–400.

[3] COHN, H. (1989). On the growth of the multitype supercritical branching process in a random environment. *Ann. Probab.* **17** 1118–1123.

[4] COHN, H. (1996). On the asymptotic patterns of supercritical branching processes in varying environments. *Ann. Appl. Probab.* **6** 896–902.

[5] COHN, H. (2001). Convergence in probability and almost sure with applications. *Stochastic Process. Appl.* **94** 135–154.

[6] COHN, H. and NERMAN, O. (1990). On products of nonnegative matrices. *Ann. Probab.* **18** 1806–1815.
[7] HAJNAL, J. (1976). On products of nonnegative matrices. *Math. Proc. Cambridge Philos. Soc.* **79** 521–530.
[8] HATTORI, T. (1997). Asymptotically one-dimensional diffusions and scale-irregular gaskets. *J. Math. Sci. Univ. Tokyo* **4** 229–278.
[9] JAGERS, P. (1975). *Branching Processes with Biological Applications.* Wiley, New York.
[10] JONES, O. D. (1997). On the convergence of the multitype branching processes with varying environment. *Ann. Appl. Probab.* **7** 772–801.
[11] JONES, O. D. (1999). Continuity for multi-type branching processes with varying environments. *J. Appl. Probab.* **36** 139–145.
[12] PETROV, V. V. (1995). *Limit Theorems of Probability Theory*: *Sequences of Independent Random Variables.* Clarendon Press, Oxford.
[13] MACPHEE, I. M. and SCHUH, H.-J. (1983). A Galton–Watson process in varying environment with essentially constant means and two rates of growth. *Austral. J. Statist.* **25** 329–338.

DEPARTMENT OF MATHEMATICS
AND STATISTICS
UNIVERSITY OF MELBOURNE
PARKVILLE, VICTORIA 3052
AUSTRALIA
E-MAIL: harry@ms.unimelb.edu.au