The Annals of Applied Probability
2003, Vol. 13, No. 2, 475–489

HARMONIC MOMENTS AND LARGE DEVIATION RATES FOR SUPERCRITICAL BRANCHING PROCESSES

BY P. E. NEY AND A. N. VIDYASHANKAR

University of Wisconsin–Madison and University of Georgia

Let $\{Z_n, n \geq 1\}$ be a single type supercritical Galton–Watson process with mean $EZ_1 \equiv m$, initiated by a single ancestor. This paper studies the large deviation behavior of the sequence $\{R_n \equiv \frac{Z_{n+1}}{Z_n} : n \geq 1\}$ and establishes a "phase transition" in rates depending on whether r, the maximal number of moments possessed by the offspring distribution, is less than, equal to or greater than the Schröder constant α. This is done via a careful analysis of the harmonic moments of Z_n.

1. Introduction. Let $\{Z_n : n \geq 1\}$ be a single type Galton–Watson process with $Z_0 \equiv 1$. Let $\{p_j : j \geq 0\}$ denote the offspring distribution function, m the offspring mean (assumed to be > 1) and $f(s)$, $0 \leq s \leq 1$, the probability generating function, that is,

$$P(Z_1 = j) = p_j, \qquad j \geq 0, \qquad f(s) = \sum_{j \geq 0} p_j s^j \quad \text{and} \quad m = \sum_{j \geq 1} j p_j.$$

Let $\gamma = f'(q)$, where $q = P(Z_n = 0 \text{ for some } n \geq 1)$ is the extinction probability. Let $\{\xi_{n,i} : i \geq 1, n \geq 1\}$ be i.i.d. random variables with $P(\xi_{n,i} = k) = p_k$, interpreted as the number of offspring of the ith parent in the nth generation. Let $\alpha \equiv \frac{-\log \gamma}{\log m}$. Drawing on the functional iterations literature, α is frequently called the Schröder constant. In the same spirit, offspring distributions with $\alpha = \infty$ (i.e., $p_0 + p_1 = 0$) are said to be of Böttcher type.

In this paper we obtain sharp rate estimates for the large deviation behavior of the statistic $R_n \equiv \frac{Z_{n+1}}{Z_n}$. This statistic has been used in the estimation of the amplification rate in a quantitative polymerase chain reaction (PCR) experiment (see [9, 10]) where only Z_n and Z_{n+1} are observed. In fact under such an observation scheme R_n is the nonparametric maximum likelihood estimator of m (see [6]). A natural question concerning the Bahadur efficiency (see [4]) of this estimator leads to considering the large deviation behavior of the statistic R_n. A more interesting statistical question in the context of quantitative PCR experiments concerns the estimation of Z_0, the initial number of molecules used for the amplification process. Even though Z_0 is not in general consitently estimable with this data, one can obtain quantitative information about Z_0 (see [12]) in these specific binary cases.

Received January 2002; revised May 2002.
AMS 2000 subject classifications. 60J80, 60F10.
Key words and phrases. Branching processes, harmonic moments, large deviations.

The key technical tool needed in this paper involves determination of the exact asymptotic behavior of the harmonic moments of Z_n, namely,

$$\tau_n(r) \equiv E(Z_n^{-r} | Z_n > 0), \qquad r > 0,$$

under no moment restrictions on Z_n other than the finiteness of the mean. Incidentally, the quantity $\tau_n(r)$ arises in various other settings as well. First, in the study of the kin number problem, the quantity $\tau_n(r)$ arises in the expression for the generating function of the generation sizes when the ego is sampled from the nth generation (see [11]). Second, $\tau_n(r)$ plays an essential role when one investigates the rate of convergence in the central limit theorem for the quantity $(R_n - m)$ (see [8]).

Athreya and Vidyashankar [3] showed that, under an exponential moment hypothesis, $\gamma^{-n} P(R_n \geq a | Z_n > 0)$ $(a > m)$ converges to a limit which is finite and positive. Athreya [1] improved the result by reducing the moment assumption to $E(Z_1^r) < \infty$, where $r \geq 2$ and satisfies the conditions that $\gamma m^r > 1$. A natural question is to clarify the role played by the quantity γm^r; that is, is it just a technical condition that is needed for the proof to work or does it play an intrisic role in determining the rates of convergence? Furthermore, one would like to reduce this assumption further to a familiar condition such as $E(Z_1 \log Z_1) < \infty$ or to just $m < \infty$. We will see (Theorems 2 and 3 below) that there is a "phase transition" in the rate of convergence of R_n to m depending on whether the number of moments r (assumed to be greater than 1) of Z_1, namely, $E(Z_1^r)$, is greater than, equal to or less than α.

Partial results on $\tau_n(r)$ are known in the literature under restrictive assumptions and some parts were left as a conjecture (see [15]). Indeed, Heyde and Brown [8], when studying the rate of convergence in the central limit theorem, encountered the quantity $\tau_n(\frac{1}{2})$. They conjectured that under some conditions $\tau_n(1) \sim m^{-n}$. Further, they provide an example to show, under some other conditions, that $\tau_n(1) \sim nm^{-n}$ as $n \to \infty$. The example indeed is consistent with our general result. Nagaev [13] showed that $\tau_n(1) = O(\rho^n)$, where $1 > \rho^2 > \max(0, m^{-1})$, under the assumption that $E(Z_1 \log Z_1) < \infty$. Pakes [15], still under the assumption that $E(Z_1 \log Z_1) < \infty$, established the correct asymptotic behavior of $\tau_n(1)$ when $p_1 m \neq 1$ thereby (i) proving the conjecture of Heyde and Brown for the case $\gamma m < 1$ and $p_0 + p_1 > 0$ and (ii) giving the "correct" rate of convergence to 0 when $\gamma m > 1$. He furthermore conjectured that if $\gamma m = 1$, $\tau_n(1) \sim nm^{-n}$ as $n \to \infty$.

We provide a unified treatment to the asymptotic behavior of $\tau_n(r)$ under no assumption other than the finiteness of the mean. In the process, we settle the conjecture of Pakes when $p_1 m = 1$ and $E Z_1 \log Z_1 < \infty$, but also obtain the correct rate of convergence of $\tau_n(r)$ to 0 when $E(Z_1 \log Z_1) = \infty$ and $m < \infty$. Furthermore, our results settle the conjecture of Heyde and Brown even when

$p_1 = 0$, the case left open by Pakes. From a more technical perspective, our Lemma 2 below provides results relating the rate of "slow down" p_1^n and rate of growth c_n.

Our treatment is based on the study of certain integrals when the integrand has an isolated singularity and is related to the approach we developed in the study of local limit theory for branching processes (see [14]). Indeed, in that paper we established that the behavior $P(Z_n = v_n)$ is dictated by the range of values of $p_1 m$. It turns out that a similar phenomenon occurs in the analysis of $\tau_n(r)$ as well.

The rest of the paper is structured as follows: Section 2 states and proves the main result concerning the harmonic moments while Section 3 deals with large deviations issues. Section 4 provides some concluding remarks.

2. Harmonic moments. In this section we state and prove the main result concerning the rate of convergence of $\tau_n(r)$ to 0 as $n \to \infty$ under the assumption $1 < m < \infty$. We begin with notation and some preliminary work. Recall that $\{W_n \equiv \frac{Z_n}{m^n} : n \geq\}$ is a nonnegative martingale sequence with respect to the sequence of σ-fields $\{\mathcal{H}_n(\equiv \sigma\langle Z_1, \dots, Z_n\rangle) : n \geq 1\}$ and hence $W_n \to W$ almost surely (a.s.) as $n \to \infty$. It is also well known (see [2]) that the necessary and sufficient condition for W to be a nondegenerate random variable is that $E(Z_1 \log Z_1) < \infty$. Furthermore, it was shown by Seneta and later strengthened by Heyde (see [2]) that, when $E(Z_1 \log Z_1) = \infty$, there exists a sequence $\{c_n : n \geq 1\}$ such that $W_n^{\mathrm{SH}} \equiv \frac{Z_n}{c_n}$ converges to a limit W^{SH} a.s. as $n \to \infty$, and that W^{SH} is a nondegenerate random variable (see [2]). The sequence $\{c_n : n \geq 1\}$ is usually called the Seneta-constants and has the property that $\frac{c_{n+1}}{c_n} \uparrow m$ as $n \uparrow \infty$. This sequence will play a crucial role in our analysis.

Frequently, in the supercritical case, there is no loss of generality in assuming that the extinction probability $q = 0$. We will follow with this custom. This implies that $\gamma = p_1$. The analysis of $\tau_n(r)$ is facilitated by the following expression for the reciprocal of a positive random variable X^r, namely,

$$\frac{1}{X^r} = \frac{1}{\Gamma(r)} \int_0^\infty e^{-uX} u^{r-1}\, du. \tag{1}$$

Now with $X = Z_n$ and taking expectations we get, using Tonelli's theorem, that

$$\begin{aligned} \tau_n(r) \equiv E\left(\frac{1}{Z_n^r}\right) &= \frac{1}{\Gamma(r)} \int_0^\infty f_n(e^{-u}) u^{r-1}\, du \\ &\equiv \frac{1}{\Gamma(r)} I_n(r), \end{aligned} \tag{2}$$

where $f_n(\cdot)$ is the nth iterate of f.

We are now ready to state the main result of this paper.

THEOREM 1. *Assume* $1 < m < \infty$. *Let*

$$A_n(r) = \begin{cases} p_1^{-n}, & \text{if } p_1 m^r > 1, \\ p_1^{-n}\left(\sum_{k=0}^{(n-1)} p_1^{-k} c_k^{-r}\right)^{-1}, & \text{if } p_1 m^r = 1, \\ c_n^r, & \text{if } p_1 m^r < 1. \end{cases} \tag{3}$$

Then

$$\lim_{n\to\infty} A_n(r) E\left(\frac{1}{Z_n^r}\right) = \begin{cases} \dfrac{1}{\Gamma(r)} \displaystyle\int_0^\infty Q(e^{-v}) v^{r-1}\, dv, & \text{if } p_1 m^r > 1, \\ \dfrac{1}{\Gamma(r)} \displaystyle\int_1^m Q(\phi^{\mathrm{SH}}(v)) v^{r-1}\, dv, & \text{if } p_1 m^r = 1, \\ \dfrac{1}{\Gamma(r)} \displaystyle\int_0^\infty \phi^{\mathrm{SH}}(v) v^{r-1}\, dv, & \text{if } p_1 m^r < 1, \end{cases}$$

where $\phi^{\mathrm{SH}}(v) = \lim_{n\to\infty} \phi_n^{\mathrm{SH}}(v) \equiv E(e^{-vW_n^{\mathrm{SH}}})$ *and the limits are positive and finite.*

REMARK 1. Stated differently, the above theorem says that $\tau_n(r)$ decays at the rate $A_n(r)$ whose values depend on whether r is greater than, equal to or less than α.

COROLLARY 1. *If* $E(Z_1 \log Z_1) < \infty$, *then* $\lim_{n\to\infty} \frac{1}{n} \log E(\frac{1}{Z_n^r}) = \max(\log p_1, -r \log m)$.

EXAMPLE 1. The example of Heyde and Brown [8] considers the offspring distribution whose generating function is given by $f(s) = s(m - (m-1)s^k)^{-1/k}$, where k is a positive integer. Here $p_1 m = m^{1-1/k}$, so $p_1 m > 1$ $(= 1)$ corresponds to $k > 1$ $(= 1)$.

NOTATION 1. From now on we abreviate

$$p_1 m^r = \rho, \qquad (p_1^n c_n^r)^{-1} = b_n, \qquad \sum_{k=0}^{n-1} b_k = B_n. \tag{4}$$

Using Seneta's argument [17] one can show that

$$c_n = m^n L^n(m^n) \qquad \text{where } L(x) \text{ is slowly varying at } \infty \tag{5}$$

and decreasing, with $L(x) \searrow (EW)^{-1}$, $EW < \infty$ if and only if $EZ_1 \log Z_1 < \infty$. Hence,

$$p_1^n c_n^r = \rho^n L^r(m^n). \tag{6}$$

The proof of the theorem involves a detailed analysis of the integral occurring on the RHS of (2) and is broken into several lemmas. We begin with the following decomposition of $I_n(r)$:

$$
\begin{aligned}
I_n(r) &= \int_0^{c_n^{-1}} f_n(e^{-u})u^{r-1}\,du + \int_{c_n^{-1}}^{c_0^{-1}} f_n(e^{-u})u^{r-1}\,du \\
&\quad + \int_{c_0^{-1}}^{\infty} f_n(e^{-u})u^{r-1}\,du \\
&= J_n(1) + J_n(2) + J_n(3).
\end{aligned}
\tag{7}
$$

Recall that $f_n(s) \to 0$ as $n \to \infty$ for $0 \le s < 1$. Lemma 0 collects some other well-known properties of $f_n(s)$.

LEMMA 0. *If* $p_1 \neq 0$,

$$
\frac{f_n(s)}{p_1^n} \equiv Q_n(s) \nearrow Q(s) \qquad \textit{uniformly for } s \in [0,\ b],\ b < 1, \tag{8}
$$

as $n \to \infty$ *and* $Q(\cdot)$ *satisfies the functional equation*

$$
Q(f(s)) = p_1 Q(s), \qquad Q(1) = \infty, \qquad Q(0) = 0. \tag{9}
$$

Furthermore, $Q(\cdot)$ *has a power series expansion given by*

$$
Q(s) = \sum_{k \ge 1} q_k s^k \qquad \textit{for } 0 \le s < 1. \tag{10}
$$

Our next lemma shows that $Q(\cdot)$ is integrable as long as one stays away from 1.

LEMMA 1. $\int_x^\infty Q(e^{-u})u^{r-1}\,du < \infty$ *for any* $x > 0$.

PROOF. By a change of variable the integral equals $\int_0^{e^{-x}} \frac{Q(u)}{u}(-\log u)^{r-1}\,du$. Since $\frac{Q(u)}{u} \le C < \infty$ for $0 < u < 1$, the result follows. $\square$

The following lemma relates the behavior of p_1^n and c_n^r, using the notation in (4).

LEMMA 2.

(a)
$$
\lim_{n\to\infty} B_n \begin{cases} < \infty, & \textit{if } \rho > 1, \\ = \infty, & \textit{if } \rho \le 1, \end{cases}
$$

(b)
$$
\lim_{n\to\infty} \frac{B_n}{b_n} = \begin{cases} \dfrac{\rho}{1-\rho}, & \textit{if } \rho < 1, \\ \infty, & \textit{if } \rho \ge 1. \end{cases}
$$

PROOF. (a) If $\rho \neq 1$, the result follows from the ratio test. If $\rho = 1$, then $B_n = \sum_{k=0}^{n} L^{-r}(m^k)$, which diverges since $L^{-r}(m^n)$ is bounded away from 0.

(b) Note that

$$\frac{B_n}{b_n} = \sum_{j=0}^{\infty} \rho^j \left(\frac{L(m^n)}{L(m^{n-j})} \right)^r I_{\{j \leq n-1\}}.$$

If $\rho < 1$, the result follows by dominated convergence; if $\rho = 1$, the result follows by Fatou's lemma, by taking the lim inf inside the sum. The case $\rho > 1$ is trivial. $\square$

For future reference we note that

$$p_1^n A_n(r) \begin{cases} \equiv 1, & \text{if } \rho > 1, \\ \to 0, & \text{if } \rho \leq 1, \end{cases} \tag{11}$$

and

$$c_n^r A_n(r) \begin{cases} \to 0, & \text{if } \rho \geq 1, \\ \equiv 1, & \text{if } \rho < 1. \end{cases} \tag{12}$$

LEMMA 3. (a) *If* $\rho \geq 1$, *then* $\lim_{n \to \infty} A_n(r) J_n(1) = 0$.

(b) *If* $\rho < 1$, *then* $\lim_{n \to \infty} A_n(r) J_n(1) = \int_0^1 \phi^{\mathrm{SH}}(v) v^{r-1}\, dv < \infty$.

PROOF. (a) Recall that, for $v \geq 0$,

$$\phi_n^{\mathrm{SH}}(v) \equiv E(e^{-vW_n}) \quad \longrightarrow \quad E(e^{-vW}) \equiv \phi^{\mathrm{SH}}(v).$$

Hence by a change of variable

$$c_n^r J_n(1) = \int_0^1 \phi_n^{\mathrm{SH}}(v) v^{r-1}\, dv \to \int_0^1 \phi^{\mathrm{SH}}(v) v^{r-1}\, dv \qquad \text{as } n \to \infty. \tag{13}$$

If $\rho > 1$, then

$$A_n(r) J_n(1) = p_1^{-n} J_n(1) = p_1^{-n} c_n^{-r} \int_0^1 \phi_n^{\mathrm{SH}}(v) v^{r-1}\, dv \to 0 \qquad \text{as } n \to \infty$$

by Lemma 2(a) and (13).

If $\rho = 1$, then

$$A_n(r) J_n(1) = \frac{b_n}{B_n} \int_0^1 \phi_n^{\mathrm{SH}}(v) v^{r-1}\, dv \to 0 \qquad \text{as } n \to \infty$$

by Lemma 2(b) and (13).

If $\rho < 1$, then $A_n(r) = c_n^r$ and the conclusion follows from (13). $\square$

Our next lemma is similar in spirit to a lemma by Dubuc and Seneta [5] for characteristic functions, but the proof in our case is much simpler.

LEMMA 4. *The following estimate holds*:

$$\sup_{k\ge 1}\sup_{x\ge 1}\phi_k^{\mathrm{SH}}(x)\equiv b<1. \tag{14}$$

PROOF. Note

$$\sup_{x\ge 1}\phi_k^{\mathrm{SH}}(x)=\phi_k^{\mathrm{SH}}(1)\to\phi^{\mathrm{SH}}(1)\qquad\text{as } k\to\infty.$$

Thus, given $\varepsilon>0$, there exists a k_0 such that, for all $k\ge k_0$, $\phi_k^{\mathrm{SH}}(1)\le\phi^{\mathrm{SH}}(1)+\varepsilon$. Furthermore since $\sup_{k\le k_0}\sup_{x\ge 1}\phi_k^{\mathrm{SH}}(x)<1$ and ε is arbitrary, the lemma follows. $\square$

LEMMA 5.

$$\lim_{n\to\infty}A_n(r)J_n(2)=\begin{cases}\displaystyle\int_0^{1/c_0}Q(e^{-v})v^{r-1}\,du, & \textit{when } \rho>1,\\ \displaystyle\int_1^{m}Q(\phi^{\mathrm{SH}}(v))v^{r-1}\,dv, & \textit{when } \rho=1,\\ \displaystyle\int_1^{\infty}\phi^{\mathrm{SH}}(v)v^{r-1}\,dv, & \textit{when } \rho<1,\end{cases}$$

and the limits are positive and finite.

PROOF. Decomposing the integral $J_n(2)$, we get

$$\int_{c_n^{-1}}^{c_0^{-1}}f_n(e^{-u})u^{r-1}\,du=\sum_{k=1}^{n}\int_{c_k^{-1}}^{c_{k-1}^{-1}}f_n(e^{-u})u^{r-1}\,du;$$

changing the variable $v=c_k u$ yields

$$J_n(2)=\sum_{k=1}^{n}\int_1^{c_kc_{k-1}^{-1}}f_{n-k}\big(\phi_k^{\mathrm{SH}}(u)\big)\,du. \tag{15}$$

Now letting

$$\int_1^{c_kc_{k-1}^{-1}}Q_n\big(\phi_k^{\mathrm{SH}}(u)\big)\,du=x_{n,k}$$

some algebra shows that

$$p_1^{-n}J_n(2)=\sum_{k=1}^{n}b_kx_{n-k,k}. \tag{16}$$

However,

$$x_{n,k}\nearrow x_k\equiv\int_1^{c_kc_{k-1}^{-1}}Q\big(\phi_k^{\mathrm{SH}}(v)\big)v^{r-1}\,du \tag{17}$$

as $n \nearrow \infty$, so by the monotone convergence theorem

$$\sum_{k=1}^{n} b_k x_{n-k,k} \nearrow \sum_{k\geq 1} b_k x_k \tag{18}$$

as $n \nearrow \infty$. When $\rho > 1$, the last sum is finite since $\sup_{k\geq 1} x_k < \infty$ by Lemma 4 and $\sum_{k\geq 1} b_k < \infty$ by Lemma 2(a). To identify the sum, change the variable back, $u = \frac{v}{c_k}$ to yield

$$\begin{aligned}\sum_{k\geq 1} b_k x_k &= \sum_{k\geq 1} \int_{c_k^{-1}}^{c_{k-1}^{-1}} \frac{Q(f_k(e^{-u}))}{p_1^k} u^{r-1}\,du \\ &= \int_0^{c_0^{-1}} Q(e^{-u}) u^{r-1}\,du \qquad \text{[using the functional equation (9)]},\end{aligned}$$

proving the lemma when $\rho > 1$. When $\rho = 1$, then, by (16) and the definition of $A_n(r)$,

$$A_n(r) J_n(2) = \frac{1}{B_n} \sum_{k=1}^{n} b_k (x_{n-k,k} - x_k) + \frac{1}{B_n} \sum_{k=1}^{n} b_k x_k.$$

To treat the first term, note that by Lemmas 4 and 0, for some $0 < b < 1$,

$$\lim_{n\to\infty} |x_{n,k} - x_k| \leq \lim_{n\to\infty} C \sup_{y\in[0,b]} |Q_n(y) - Q(y)| = 0$$

uniformly in k. Then using the fact that $\frac{b_{n-k}}{B_n} \to 0$ for all fixed k [by Lemma 2(b)] the first term on the right-hand side of (19) $\to 0$ as $n \to \infty$. It is easily shown that $\lim_{n\to\infty} \frac{1}{B_n} \sum_{k=1}^{n} b_k x_k = x$, proving the lemma when $\rho = 1$.

When $\rho < 1$,

$$A_n(r) J_n(2) = c_n^r \sum_{k=1}^{n} \int_{c_k^{-1}}^{c_{k-1}^{-1}} f_n(e^{-u}) u^{r-1}\,du.$$

Changing variable $u = \frac{v}{c_k}$ yields after some manipulation

$$A_n(r) J_n(2) = \sum_{j=0}^{\infty} a(n, j),$$

where we have set

$$a(n, j) = \frac{c_n^r}{c_{n-j}^r} I_{\{j\leq n\}} \left| \int_1^{c_{n-j} c_{n-j-1}^{-1}} f_j\bigl(\phi_{n-j}^{\mathrm{SH}}(v)\bigr) v^{r-1}\,dv \right|. \tag{19}$$

Now, by Lemma 4,

$$a(n, j) \leq \text{const} \cdot \frac{c_n^r}{c_{n-j}^r} I_{\{j\leq n\}} f_j(\beta)$$

for some $\beta < 1$, while

$$f_j(\beta) \le \text{const} \cdot \begin{cases} p_1^j, & \text{if } p_1 > 0, \\ \beta^{k_0^n}, & \text{if } p_1 = 0 \text{ and } k_0 = \inf\{k : p_k > 0\}. \end{cases} \tag{20}$$

Setting

$$b(n, j) = \begin{cases} \dfrac{c_n^r}{c_{n-j}^r} I_{\{j \le n\}} p_1^j, & \text{if } p_1 > 0, \\ \dfrac{c_n^r}{c_{n-j}^r} I_{\{j \le n\}} \beta^{k_0^j}, & \text{if } p_1 = 0, \end{cases}$$

we see that

(i) $$a(n, j) \le b(n, j),$$

(ii) $$b(n, j) \nearrow b(j) = \begin{cases} m^{jr} p_1^j = \rho^j, & \text{if } p_1 > 0, \\ m^{jr} \beta^{k_0^j}, & \text{if } p_1 = 0, \end{cases}$$

(iii) $$\sum_{j=0}^{\infty} b(n, j) \to \sum_{j=0}^{\infty} b(j) \qquad \text{as } n \to \infty$$

by the monotone convergence theorem.

Hence by the generalized dominated convergence theorem (see [16], Theorem 16, page 89),

$$\lim_{n\to\infty} \sum_{j=0}^{\infty} a(n, j) = \sum_{j=0}^{\infty} \lim_{n\to\infty} a(n, j) = \sum_{j=0}^{\infty} m^{jr} \int_1^m f_j(\phi^{\mathrm{SH}}(v)) v^{r-1}\, dv.$$

Using the functional equation $f(\phi^{\mathrm{SH}}(u)) = \phi^{\mathrm{SH}}(mu)$, the above reduces to

$$\begin{aligned} \sum_{j=0}^{\infty} m^{jr} \int_1^m \phi^{\mathrm{SH}}(m^j v) v^{r-1}\, dv &= \sum_{j\ge 0} \int_{m^j}^{m^{j+1}} \phi^{\mathrm{SH}}(u) u^{r-1}\, du \\ &= \int_1^{\infty} \phi^{\mathrm{SH}}(u) u^{r-1}\, du. \end{aligned}$$

□

LEMMA 6. *If* $\rho \le 1$, *then* $A_n(r) J_n(3) \to 0$.
If $\rho > 1$, *then* $A_n(r) J_n(3) \to \int_{1/c_0}^{\infty} Q(e^{-v}) v^{r-1}\, dv$.

PROOF. Suppose $p_1 \ne 0$. Then

$$\begin{aligned} p_1^{-n} J_n(3) &= \int_{c_0^{-1}}^{\infty} \frac{f_n(e^{-v})}{p_1^{-n}} v^{r-1}\, dv = \int_{c_0^{-1}}^{\infty} Q_n(e^{-v}) v^{r-1}\, dv \\ &\to \int_{1/c_0}^{\infty} Q(e^{-v}) v^{r-1}\, dv \qquad \text{by the monotone convergence theorem.} \end{aligned} \tag{21}$$

The result when $\rho > 1$ follows [since $A_n(r) = p_1^{-n}$ in this case].

If $\rho < 1$,

$$A_n(r)J_n(3) = (p_1^n c_n^r)\frac{J_n(3)}{p_1^n} = \rho^n L^r(m^n)\frac{J_n(3)}{p_1^n} \to 0$$

by (21).

If $p_1 = 0$, then, using the estimate in (20),

$$c_n^r J_n(3) = c_n^r \int_{c_0^{-1}}^{\infty} f_n(e^{-u})u^{r-1}\,du \le c_n^r \int_{c_0^{-1}}^{n} e^{-uk_0^n} u^{r-1}\,du,$$

which converges to 0 as $n \to \infty$.

If $\rho = 1$, then

$$A_n(r)J_n(3) = \left(\sum_{k=0}^{n-1} \frac{1}{p_1^k c_k^r}\right)^{-1} \frac{J_n(3)}{p_1^n} \to 0$$

by (21) and Lemma 2(a). $\square$

PROOF OF THEOREM 1. The proof follows from (2), (7) and Lemmas 3, 5 and 6. $\square$

3. Large deviations. In this section we deal with large deviation rates for the convergence of R_n to m. The large deviations of R_n when Z_1 satisfies an exponential moment hypothesis have been treated in [3]. When Z_1 has finite rth moment with $r > \alpha$ (where α is the Schröder constant) the result has been treated in [1]. In this section we show (see Theorems 2 and 3 below) that there is a "phase transition" in the rates of convergence depending on whether $r > \alpha$ or $r = \alpha$ or $r < \alpha$. We begin with a general result illustrating this phenomenon. Let $\bar{\xi}_n = n^{-1}\sum_{i=1}^n \xi_{n,i}$.

THEOREM 2. *If, for some set $D \in R$, constant $C_1(D) < \infty$ and $r > 0$,*

$$P(\overline{\xi}_n \in D) \le \frac{C_1(D)}{n^r}, \tag{22}$$

then

$$\limsup_{n\to\infty} A_n(r)P(R_n \in D) \le C_1(D)B, \tag{23}$$

where B is a finite positive constant. Furthermore, if

$$P(\overline{\xi}_n \in D) \ge \frac{C_2(D)}{n^r}, \tag{24}$$

then

$$\liminf_{n\to\infty} A_n(r)P(R_n \in D) \ge C_2(D)B. \tag{25}$$

PROOF. Conditioning on Z_n yields

$$
\begin{aligned}
P(R_n \in D) &= \sum_{k\geq 1} P(\overline{\xi}_k \in D) P(Z_n = k) \\
&\leq \sum_{k\geq 1} \frac{C_1(D)}{k^r} P(Z_n = k) \\
&= E(Z_n^{-r}).
\end{aligned}
$$

The result (23) now follows from Theorem 1, and (25) follows similarly. □

REMARK 2. It should be noted that if $E(Z_1^r) < \infty$ for $r > \alpha$, then under (22) one can establish the existence of the limit in (23). This was carried out in [1]. If $r \leq \alpha$, then under (22) it is clear that $\limsup_{n\to\infty} p_1^{-n} P(|R_n - m| > a) = \infty$; that is, p_1^n is *not* the correct rate of convergence. Theorem 2 shows that the behavior of $P(R_n > a)$ is different for $r \leq \alpha$ and $r = \alpha$ and the rate involves r explicitly.

A natural next question is whether it is possible to bring out the phase transition in rates in a more precise form than is given in Theorem 2. The answer to this question is in the affirmative if one makes a more detailed assumption about the tails of the offspring distribution. We illustrate this phenomenon with the following class of distributions having Pareto-type tails.

THEOREM 3. *Assume that the offspring distribution satisfies*

$$
P(Z_1 \geq j) \sim c_1 j^{1-\omega} \qquad \text{as } j \to \infty, \tag{26}
$$

where $\omega > 2$. *Then* $\lim_{n\to\infty} A_n(\omega - 2) P(|R_n - m| > a)$ *exists and is finite and positive*.

REMARK 3. The limiting constant in the theorem depends on whether $\rho_1 m^{w-2} > 1$ or is less than or equal to 1, and can be determined using Theorem 1.

PROOF. Let $X_1, X_2, \ldots$ be i.i.d. with $X_1 \stackrel{d}{=} Z_1 - m$ and let $\overline{X_k} = \frac{1}{k}\sum_{i=1}^k X_i$. Conditioning on Z_n one gets

$$
P(|R_n - m| > a) = \sum_{k\geq 1} P(|\overline{X_k}| > a) P(Z_n = k). \tag{27}
$$

Using Heyde's theorem (see [7]), given $\varepsilon > 0$, there exists a $k_0(\varepsilon)$ such that, for all $k \geq k_0$,

$$
(1-\varepsilon) k P(|X_1| > ka) \leq P(|S_k| > ka) \leq (1+\varepsilon) k P(|X_1| > ka). \tag{28}
$$

Now using (26) it follows that there exists a constant c_0 such that, for $k \geq k_0$,

$$
c_0 a^{1-\omega}(1-\varepsilon) k^{-(\omega-2)} \leq P(|S_k| > ka) \leq c_0 a^{1-\omega}(1+\varepsilon) k^{-(\omega-2)}. \tag{29}
$$

Thus using (29) on (27) one gets

$$P(|R_n - m| > a) \geq c_0 a^{1-\omega}(1-\varepsilon)\left(\sum_{k>k_0} k^{-(\omega-2)} P(Z_n = k) + D_1(n, k_0)\right) \tag{30}$$

$$= c_0 a^{1-\omega}(1-\varepsilon)\big(D_0(n) - D_2(n, k_0) + D_1(n, k_0)\big), \tag{31}$$

where

$$D_0(n) = \sum_{k\geq 1} k^{-(\omega-2)} P(Z_n = k) = E(Z_n^{-(w-2)}), \tag{32}$$

$$D_1(n, k_0) = \sum_{k=1}^{k_0} P(|S_k| > ka) P(Z_n = k) \tag{33}$$

and

$$D_2(n, k_0) = \sum_{k=1}^{k_0} k^{-(\omega-2)} P(Z_n = k). \tag{34}$$

If $p_1 m^{\omega-2} \leq 1$, then, by (11),

$$\lim_{n\to\infty} A_n(\omega-2) P(Z_n = k) = 0. \tag{35}$$

Thus, by Theorem 1,

$$\liminf_{n\to\infty} A_n(\omega-2) P(|R_n - m| > a) \geq c_0 a^{1-\omega}(1-\varepsilon) L, \tag{36}$$

where

$$L = \lim_{n\to\infty} A_n(\omega-2) E(Z_n^{-(\omega-2)}) \tag{37}$$

is explicitly identified in Theorem 1. Next using the upper bound from (29) on (27) we get

$$P(|R_n - m| > a) \leq c_0 a^{1-\omega}(1+\varepsilon)\big(D_0(n) + D_1(n, k_0) - D_2(n, k_0)\big). \tag{38}$$

Then, from (11) and Theorem 1,

$$\limsup_{n\to\infty} A_n(\omega-2) P(|R_n - m| > a) \leq c_0 a^{1-\omega}(1+\varepsilon) L. \tag{39}$$

The arbitrariness of ε concludes the proof of the theorem when $p_1 m^{(\omega-2)} \leq 1$. When $p_1 m^{(\omega-2)} > 1$, then by the upper bound on (29) there exists a constant C such that

$$P(|\overline{X_k}| > a)\frac{P(Z_n = k)}{p_1^n} \leq C k^{-(\omega-2)} \frac{P(Z_n = k)}{p_1^n} \qquad \text{for all } k \geq 1. \tag{40}$$

The RHS of the above expression converges to $Ck^{-(\omega-2)}q_k$, where q_k is defined in (10). Furthermore, by Theorem 1,

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{k\geq 1} k^{-(\omega-2)}\frac{P(Z_n=k)}{p_1^n} &= \lim_{n\to\infty} p_1^{-n}E(Z_n^{-(\omega-2)})\\
&= \big(\Gamma(2-\omega)\big)^{-1}\int_0^\infty Q(e^{-u})u^{\omega-3}\,du \qquad (41)\\
&= \sum_{k\geq 1} k^{2-\omega}q_k < \infty.
\end{aligned}
$$

Now,

$$
\frac{P(|R_n-m|>a)}{p_1^n} = \sum_{k\geq 0} P(|\overline{X_k}|>a)\frac{P(Z_n=k)}{p_1^n} \tag{42}
$$

and by (40) and (41) one can apply the generalized version of the dominated convergence theorem (see [16], page 89) to take the limit as $n\to\infty$ inside the sum in (42) to conclude that

$$
\lim_{n\to\infty}\frac{P(|R_n-m|>a)}{p_1^n} = \sum_{k\geq 0} P(|\overline{X_k}|>a)q_k. \tag{43}
$$

This completes the proof of the theorem. □

REMARK 4. As noted in Remark 1, if $\omega > 2+\alpha$ (where α is the Schröder constant), then $P(|R_n-m|>a)$ decays at the rate p_1^n, regardless of ω. However, the above results show that if $\omega \leq \alpha+2$, then the rate explicitly involves ω. When $p_1=0$ then $p_1 m^{\omega-2}<1$ and the rate of convergence is given by $m^{(\omega-2)n}$. Thus as ω increases, the rate of convergence also increases. This suggests, in general, that if the offspring distributions possess exponential moments and $p_1=0$, then the rate of convergence should be investigated by considering the rate of convergence of $\log P(|R_n-m|>a)$. This was considered in [14] and the results are quite different from the case $p_1>0$.

4. Conclusions. In this paper we developed the rate of convergence of the harmonic moments under various regimes, namely, $p_1 m^r > 1, = 1$ or < 1, under the weak moment condition, $m<\infty$. This weakening of the assumption not only brought to the fore the relationship between c_n and p_1^n but also brought out the "phase transition" in large deviation rates of convergence.

Extension of these results to multitype is challenging and the authors are considering these extensions. In conclusion we mention another application of the results developed in this paper, namely, to the rate of convergence in the central limit theorem, a problem first considered by Heyde and Brown [8] which initiated the work on harmonic moments:

THEOREM 4. *Assume that* $E(Z_1^3) < \infty$. *Let* $\sigma_2 = \mathrm{Var}(Z_1)$. *Then the following hold*:

(i) $$A_n\big(\tfrac{1}{2}\big) \sup_x \big|P\big((m^2 - m)^{1/2}\sigma^{-1} Z_n^{-1/2} m^n (W - W_n) \le x\big) - \Phi(x)\big| \le KC\sigma^{-3}(m^2 - m)^{1/2} E|W - 1|^3;$$

(ii) $$A_n\big(\tfrac{1}{2}\big) \sup_x \big|P\big(\sigma_r^{-1} Z_n^{-1/2}(Z_{n+r} - m^r Z_n) \le x\big) - \Phi(x)\big| \le KC\sigma_r^{-3} E|Z_r - m^r|^3;$$

where K *is the universal constant in the Berry–Esseen bound*, $A_n(\frac{1}{2})E(\frac{1}{Z_n})^{1/2} \le C$, $\sigma_r^2 = \sigma^2 m^r(m^r - 1)(m^2 - m)^{-1}$ *and*

$$A_n\big(\tfrac{1}{2}\big) = \begin{cases} p_1^{-n}, & \text{if } p_1 m^{1/2} > 1, \\ p_1^{-n} n, & \text{if } p_1 m^{1/2} = 1, \\ m_n^r, & \text{if } p_1 m^{1/2} < 1. \end{cases}$$

PROOF. Since $E(Z_1^3) < \infty$, one can replace c_n in Theorem 1 by m^n. The result follows from Theorem 2 of Heyde and Brown [8] and Theorem 1. □

Acknowledgments. We thank the referee for a careful reading of the manuscript and several useful suggestions that led to substantial improvements of the paper. Part of this work was carried out while the second author was visiting the Center for Mathematical Sciences at the University of Wisconsin–Madison, during the years 2000 and 2001 and he would like to thank the center for research facilities and travel support. The first author would like to thank the University of Georgia's Department of Statistics for support during this work.

REFERENCES

[1] ATHREYA, K. B. (1994). Large deviation rates for branching processes. I. Single type case. *Ann. Appl. Probab.* **4** 779–790.

[2] ATHREYA, K. B. and NEY, P. (1972). *Branching Processes.* Springer, Berlin.

[3] ATHREYA, K. B. and VIDYASHANKAR, A. N. (1993). Large deviation results for branching processes. In *Stochastic Processes. A Festschrift in Honour of Gopinath Kallianpur* (S. Cambanis, J. K. Ghosh, R. L. Karandikar and P. K. Sen, eds.) 7–12. Springer, New York.

[4] BAHADUR, R. R. (1960). On the asymptotic efficiency of tests and estimates. *Sankhyā* **22** 229–252.

[5] DUBUC, S. and SENETA, E. (1976). The local limit theorem for Galton–Watson processes. *Ann. Probab.* **4** 490–496.

[6] GUTTORP, P. (1991). *Statistical Inference for Branching Processes.* Wiley, New York.

[7] HEYDE, C. C. (1968). On large deviation probabilities in the case of attraction to a non-normal stable law. *Sankhyā Ser. A* **30** 253–258.

[8] HEYDE, C. C. and BROWN, B. M. (1971). An invariance principle and some convergence rate results for branching processes. *Z. Wahrsch. Verw. Gebiete* **20** 271–278.

[9] JACOB, C. and PECCOUD, J. (1996). Inference on the initial size of a supercritical branching process from migrating binomial observations. *C. R. Acad. Sci. Paris Sèr. I* **322** 875–888.

[10] JACOB, C. and PECCOUD, J. (1998). Estimation of the parameters of a branching process from migrating binomial observations. *Adv. in Appl. Probab.* **30** 948–967.

[11] JOFFE, A. and WAUGH, W. A. O'N. (1982). Exact distributions of kin numbers in a Galton–Watson process. *J. Appl. Probab.* **19** 767–775.

[12] KARRER, E. E., LINCOLN, S., BENNETT, A. B., BOSTOCK, R. M., LUCAS, W. J., GILCHRIST, D. G. and ALEXANDER, D. (1995). In situ isolation of mRNA from individual plant of cell-specific cDNA libraries. *Proc. Natl. Acad. Sci. USA* **92** 3814–3818.

[13] NAGAEV, A. V. (1967). On estimating the expected number of direct descendants of a particle in a branching process. *Theory Probab. Appl.* **12** 314–320.

[14] NEY, P. E. and VIDYASHANKAR, A. N. (2000). Local limit theory and large deviations for supercritical branching processes. Technical Report Stat 2000-25, Univ. Georgia, Athens.

[15] PAKES, A. G. (1975). Non-parametric estimation in the Galton–Watson process. *Math. Biosci.* **26** 1–18.

[16] ROYDEN, H. L. (1968). *Real Analysis,* 2nd ed. Macmillan, New York.

[17] SENETA, E. (1974). Regularly varying functions in the theory of simple branching processes. *Adv. in Appl. Probab.* **6** 408–420.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN
MADISON, WISCONSIN 53706

DEPARTMENT OF STATISTICS
UNIVERSITY OF GEORGIA
ATHENS, GEORGIA 30602
E-MAIL: anand@stat.uga.edu