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DYNAMIC AND EQUILIBRIUM BEHAVIOR OF CONTROLLED LOSS NETWORKS

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We study the behavior of large loss networks in which the offered traffic is subject to acceptance controls. Hunt and Kurtz proved a functional law of large numbers for the dynamics of such networks, as capacity and offered traffic are allowed to increase in proportion. However, limiting dynamics were not in general uniquely identified. We establish further results identifying these dynamics under given conditions. We also investigate the existence of fixed points for these dynamics and relate them to limiting equilibrium behavior, permitting the investigation of common modelling assumptions. We study in detail single- and two-resource networks and we give an example of bistability for the former.

1. Introduction. We study the dynamic and equilibrium behavior of large loss networks in which the offered traffic is subject to acceptance controls. Such networks were considered by Hunt and Kurtz (1994) (referred to henceforth as HK), who established rigorous results for their asymptotic behavior as capacity and offered traffic are allowed to increase in proportion. We develop these results further, so that they may be applied to deduce detailed behavior in networks.

The results are particularly appropriate to the effective control of modern high-capacity communications networks in which traffic of widely differing characteristics is integrated. In general they remain qualitatively correct for smaller capacity networks, and, further, are readily modified to model accurately their quantitative behavior. [See, e.g., Bean, Gibbens, and Zachary (1994, 1995) and Moretta (1995). For a review of earlier work and summary, without proofs, of present results, see Zachary (1996).]

The mathematical framework is the same as that of HK. Consider a sequence of loss networks, indexed by a scale parameter N. All members of the sequence are identical except in respect of capacities and call arrival rates, and are identically controlled. (As defined more precisely below, capacities and call arrival rates are essentially proportional to N.) Resources (or links) are indexed in a finite set $\mathscr{J}$ and call types in a finite set $\mathscr{R}$. For the Nth member of the sequence, each resource $j \in \mathscr{J}$ has integer capacity $C_j(N)$, and calls of each type $r \in \mathscr{R}$ arrive as a Poisson process of rate $\kappa_r(N)$. Each such call, if accepted, simultaneously requires an integer A_{jr} units of the capacity of

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each resource j for the duration of its holding time, which is exponentially distributed with mean $1/\mu_r$ (where $\mu_r \geq 0$). All arrival streams and holding times are independent.

Let $n^N(t) = (n_r^N(t),\ r \in \mathscr{R})$, where $n_r^N(t)$ is the number of calls of type r in progress at time t, and let $m^N(t) = (m_j^N(t),\ j \in \mathscr{J})$ where $m_j^N(t) = C_j(N) - \sum_{r\in\mathscr{R}} A_{jr} n_r^N(t)$ is the free capacity of resource j at time t. Let $\mathbb{Z}_+ \cup \{\infty\}$ be topologized according to the one-point compactification of $\mathbb{Z}_+$, and give $E = (\mathbb{Z}_+ \cup \{\infty\})^J$, where $J = |\mathscr{J}|$, the corresponding product topology. A call of type r arriving at time t is accepted if and only if $m^N(t-)$ belongs to some acceptance region $\mathscr{A}_r$ (independent of N) in E, which is well behaved in that its indicator function $I_{\mathscr{A}_r}$ is continuous with respect to the above topology. [Of course, the process $m^N(\cdot)$ only takes values in $\mathbb{Z}_+^J$.] HK show that this framework permits the modelling of a wide variety of control mechanisms, including most of those employed in practical applications to communications networks.

Now suppose that, as $N \to \infty$, for all $j \in \mathscr{J}$, $r \in \mathscr{R}$,

$$\frac{1}{N} C_j(N) \to C_j, \qquad \frac{1}{N} \kappa_r(N) \to \kappa_r. \tag{1.1}$$

For each N define the normalized process $x^N(\cdot) = n^N(\cdot)/N$. We are interested in the existence and characterization of any possible "fluid limit" process $x(\cdot)$ of these normalized processes [see, e.g., Kelly (1991)].

Any such limit necessarily takes values in the space $X = \{x \in \mathbb{R}_+^R\colon \sum_r A_{jr} x_r \leq C_j$ for all $j \in \mathscr{J}\}$, where $R = |\mathscr{R}|$. For each $x \in X$, let $m_x(\cdot)$ be the Markov process on E with transition rates given by

$$m \to \begin{cases} m - A_r, & \text{at rate } \kappa_r I_{\{m \in \mathscr{A}_r\}}, \\ m + A_r, & \text{at rate } \mu_r x_r, \end{cases}$$

where A_r denotes the vector $(A_{jr},\ j \in \mathscr{J})$ and $\infty \pm a = \infty$ for any $a \in \mathbb{Z}_+$. Note that the process $m_x(\cdot)$ is reducible, and so does not always have a unique invariant distribution. HK, Theorem 3, show that, provided the distribution of $x^N(0)$ converges weakly to that of $x(0)$, the sequence of processes $x^N(\cdot)$ is relatively compact in $D_{\mathbb{R}^R}[0,\infty)$ and any weakly convergent subsequence has a limit $x(\cdot)$ which obeys the relation

$$x_r(t) = x_r(0) + \int_0^t (\kappa_r \pi_u(\mathscr{A}_r) - \mu_r x_r(u))\,du, \tag{1.2}$$

where, for each t, π_t is some invariant distribution of the Markov process $m_{x(t)}(\cdot)$ and additionally satisfies, for all j,

$$\pi_t\{m\colon m_j = \infty\} = 1 \quad \text{if } \sum_{r\in\mathscr{R}} A_{jr} x_r(t) < C_j. \tag{1.3}$$

This result involves a separation, in the limit, of the time scales of the processes $x^N(\cdot)$ and $m^N(\cdot)$; see HK and Bean, Gibbens, and Zachary (1995). The condition (1.3) ensures that, for times t such that $\sum_{r\in\mathscr{R}} A_{jr}x_r(t) < C_j$, the corresponding dynamics of the process $x(\cdot)$ are as they would be in the absence of the constraint j.

Under appropriate conditions (see, e.g., Sections 3 to 5) there exists a function π' on X (each value of which is a probability distribution on E) with the property that, for *all* convergent subsequences, we may take $\pi_t = \pi'_{x(t)}$ in (1.2). We may then define a *velocity field* $v = (v_r,\ r \in \mathscr{R})$ on X by $v_r(x) = \kappa_r \pi'_x(\mathscr{A}_r) - \mu_r x_r$, so that (1.2) becomes

$$x_r(t) = x_r(0) + \int_0^t v_r(x(u))\,du. \tag{1.4}$$

In Section 2 we assume that such a velocity field v exists. We consider the existence of fixed points for the dynamics of the process $x(\cdot)$ and relate these to the limiting equilibrium behavior of the processes $x^N(\cdot)$ and $m^N(\cdot)$. In particular we give weak convergence results for the case where there is a single fixed point.

Section 3 considers further the general problem of identifying the distributions π_t, $t \geq 0$, and derives some conditions under which this is possible. In Section 4 we study the single resource case $J = 1$. Here it is relatively straightforward to identify the velocity field v (which always exists). However, interesting bistable behavior may occur, and we give an example of this. Similarly, in Section 5 we study the two-resource case. We give an essentially complete analysis, deriving conditions for the existence of a velocity field and identifying it. We show how these results may be applied to deduce both dynamic and equilibrium behavior in a simple example, and show also that here the commonly assumed product form for the limiting distribution of $m^N(\cdot)$ is incorrect.

2. Fixed points and equilibrium behavior.

Assume throughout the present section that there does exist a distribution-valued function π', and hence a velocity field v, on X as defined above. Define $x \in X$ to be a *fixed point* of the limiting dynamics $x(\cdot)$ if $v(x) = 0$. The conditions of the following theorem are usually easy to verify in applications.

THEOREM 2.1. *Suppose that, for all $t > 0$, $x(t)$ is a uniquely defined and continuous function of $x(0)$. Then there exists at least one fixed point for the process $x(\cdot)$.*

PROOF. Define the mapping θ: $\mathbb{R}_+ \times X \to X$ by $\theta(t, x(0)) = x(t)$. Define the sequence of sets $A_n = \{x \in X: x = \theta(2^{-n}, x)\}$, for $n = 0, 1, \dots$. Since X is compact and convex, Brouwer's fixed point theorem implies that, for all $n \geq 0$, the set A_n is nonempty; further, the uniqueness of $x(\cdot)$, given $x(0)$, implies that if $x \in A_n$ then $\theta(k2^{-n}, x) \in A_n$ for all integer $k > 0$. In particular $A_n \subseteq A_{n-1}$ for all $n \geq 1$. Since X is compact and the sets A_n are closed (by the hypothesized continuity), standard results show that $\bigcap_{n=0}^{\infty} A_n \neq \varnothing$. It now follows from the continuity of $x(\cdot)$ that there exists a point x such that $\theta(t, x) = x$ for all t so that, from (1.4), $v(x) = 0$. □

We expect that, for large N, once the process $x^N(\cdot)$ is close to any fixed point x, especially one which is asymptotically stable in the usual terminology of dynamical systems, it will remain close to it for an extended period of time, and that the distribution of $m^N(\cdot)$ over that period will similarly remain close to π'_x [see HK, Kelly (1991) and Bean, Gibbens and Zachary (1995)]. Further, the following result is not surprising.

THEOREM 2.2. *Suppose that v is such that there exists a single fixed point $\bar{x}$ and that, for all $x(0)$, $x(t) \to \bar{x}$ as $t \to \infty$. Then* (i) *the invariant distribution of the process $x^N(\cdot)$ converges weakly to the distribution concentrated on $\bar{x}$, and* (ii) *the invariant distribution of the process $m^N(\cdot)$ converges weakly to $\pi'_{\bar{x}}$.*

PROOF. The proof is an adaptation of the theory of Section 2 of HK. Start each process $x^N(\cdot)$ at time 0 with its invariant distribution ϕ^N, so that both $x^N(\cdot)$ and $m^N(\cdot)$ are stationary. Define the random measure ν^N on $[0, \infty) \times E$ by

$$\nu^N((0, t) \times \Gamma) = \int_0^t I_{\{m^N(u) \in \Gamma\}}\, du,$$

for all $t \in [0, \infty)$ and Γ in the σ-algebra on E generated by the open sets. The first part of condition (1.1) ensures that the sequence ϕ^N is relatively compact, and so there exists a subsequence in which ϕ^N converges weakly to ϕ, say. From Lemma 1 of HK there exists a further subsequence in which $(x^N(\cdot), \nu^N)$ converges weakly to a limit $(x(\cdot), \nu)$, with respect to the topology of that paper. By Theorem 3 of HK the process $x(\cdot)$ is continuous and satisfies (1.4). For all $t \geq 0$, $x^N(t)$ has distribution ϕ^N, and so $x(t)$ has distribution ϕ. Hence, in all convergent subsequences, ϕ is the distribution concentrated on the globally asymptotically stable fixed point $\bar{x}$, so that the result (i) follows.

It follows from Lemma 2 of HK, the proof of Theorem 3 of that paper and the hypothesis of the present theorem that the limiting (random) measure ν satisfies

$$\nu((0, 1) \times \Gamma) = \int_0^1 \pi'_{x(u)}(\Gamma)\, du, \tag{2.1}$$

for all Γ in the above σ-algebra on E. Since the continuous process $x(\cdot)$ may be almost surely identified with the fixed-point $\bar{x}$, the right-hand side of (2.1) is almost surely equal to $\pi'_{\bar{x}}(\Gamma)$. It follows by bounded convergence that the expectation of $\nu^N((0, 1) \times \Gamma)$ converges to $\pi'_{\bar{x}}(\Gamma)$. By stationarity, this expectation is equal to the probability of the set Γ under the invariant distribution of the process $m^N(\cdot)$. Hence this distribution converges weakly to $\pi'_{\bar{x}}$, in all convergent subsequences, and so result (ii) follows. □

Observe that the invariant distribution of the process $m^N(\cdot)$ determines the equilibrium acceptance probabilities for the various call types; see Section 5 for an important application.

3. Drifts. Consider now the general problems of identifying the distributions π_t of (1.2) and determining any velocity field which exists. For each subset $\mathscr{S}$ of $\mathscr{J}$, let $E_{\mathscr{S}} = \{m \in E: m_j < \infty$ if and only if $j \in \mathscr{S}\}$. We assume (without loss of generality—see HK) that the matrix of capacity requirements (A_{jr}) and the acceptance regions $\mathscr{A}_r$ are such that, for each $x \in X$ and $\mathscr{S} \subseteq \mathscr{J}$, there is at most a single invariant distribution $\pi_x^{\mathscr{S}}$ of the Markov process $m_x(\cdot)$ on E which assigns probability one to the set $E_{\mathscr{S}}$. [The distribution $\pi_x^{\mathscr{S}}$ may also be thought of as the invariant distribution of the obvious projection of the process $m_x(\cdot)$ onto $\mathbb{Z}_+^{\mathscr{S}}$.] Note that $E_{\varnothing}$ contains the single point $(\infty, \dots, \infty)$ and so the distribution $\pi_x^{\varnothing}$ always exists.

For each $x \in X$ define $\mathscr{B}(x) = \{\mathscr{S} \subseteq \mathscr{J}: \sum_r A_{jr} x_r = C_j$ for all $j \in \mathscr{S}$ and $\pi_x^{\mathscr{S}}$ exists$\}$. Then, from the results of HK described in Section 1, it follows that there exist nonnegative functions $\lambda^{\mathscr{S}}(\cdot)$, $\mathscr{S} \subseteq \mathscr{J}$, such that, for all t,

$$\pi_t = \sum_{\mathscr{S} \in \mathscr{B}(x(t))} \lambda^{\mathscr{S}}(t) \pi_{x(t)}^{\mathscr{S}}, \tag{3.1}$$

where, necessarily,

$$\sum_{\mathscr{S} \in \mathscr{B}(x(t))} \lambda^{\mathscr{S}}(t) = 1, \tag{3.2}$$

and where additionally we make the convention that $\lambda^{\mathscr{S}}(t) = 0$ if $\mathscr{S} \notin \mathscr{B}(x(t))$. Identification of π_t, $t \geq 0$, thus reduces to identification of the functions $\lambda^{\mathscr{S}}(\cdot)$.

Now define, for each x, each $\mathscr{S} \subseteq \mathscr{J}$ such that $\pi_x^{\mathscr{S}}$ exists, and each $j \in \mathscr{J}$,

$$\alpha_j^{\mathscr{S}}(x) = \sum_{r \in \mathscr{R}} A_{jr} \{\kappa_r \pi_x^{\mathscr{S}}(\mathscr{A}_r) - \mu_r x_r\}. \tag{3.3}$$

For each x, the $\alpha_j^{\mathscr{S}}(x)$ have natural interpretations as drifts for both the processes $x(\cdot)$ and $m_x(\cdot)$. In particular

$$\alpha_j^{\mathscr{S}}(x) = 0 \quad \text{if } j \in \mathscr{S}. \tag{3.4}$$

This follows from the observation that, in equilibrium, the jth component of the restriction of the process $m_x(\cdot)$ to $E_{\mathscr{S}}$ has zero drift for each $j \in \mathscr{S}$. A formal proof may be given analogously to that of Lemma 4 of HK. Also, from (1.2), (3.1)–(3.3), for each $j \in \mathscr{J}$, $\sum_{\mathscr{S} \in \mathscr{B}(x(t))} \lambda^{\mathscr{S}}(t) \alpha_j^{\mathscr{S}}(x(t))$ is the drift rate of the process $\sum_{r \in \mathscr{R}} A_{jr} x_r(\cdot)$ at time t. Thus, using also (3.4), we have immediately the following lemma, which encapsulates the idea of *x-feasibility* of Hunt (1990, 1995) and is frequently useful in deducing that $\lambda^{\mathscr{S}}(t) = 0$ for appropriate $\mathscr{S}$ and t.

LEMMA 3.1. *Let $j \in \mathscr{J}$ and suppose that, for all t in some interval T, $\sum_{r\in\mathscr{R}} A_{jr}x_r(t) = C_j$ and $\lambda^{\mathscr{S}}(t)\alpha_j^{\mathscr{S}}(x(t)) \geq 0$ for all $\mathscr{S} \in \mathscr{B}(x(t))$ with $j \notin \mathscr{S}$. Then, for almost all $t \in T$, $\lambda^{\mathscr{S}}(t)\alpha_j^{\mathscr{S}}(x(t)) = 0$ for all $\mathscr{S} \in \mathscr{B}(x(t))$.*

For general networks, Lemma 3.1 yields the following partial result, whose conditions are satisfied in many applications.

THEOREM 3.2. *Suppose that $\mathscr{S} \subseteq \mathscr{J}$ is such that, for all t in some interval T, $\mathscr{S} \in \mathscr{B}(x(t))$ and if $\mathscr{S}' \in \mathscr{B}(x(t))$ then $\mathscr{S}' \subseteq \mathscr{S}$. Suppose further that, for all $t \in T$,*

$$\text{if } \mathscr{S}' \in \mathscr{B}(x(t)) \text{ and } j \in \mathscr{S} \setminus \mathscr{S}', \text{ then } \alpha_j^{\mathscr{S}'}(x(t)) > 0. \tag{3.5}$$

Then $\pi_t = \pi_{x(t)}^{\mathscr{S}}$ for almost all $t \in T$.

PROOF. For each $j \in \mathscr{S}$, apply Lemma 3.1 to deduce that, under the conditions of the theorem, for almost all $t \in T$, $\lambda^{\mathscr{S}'}(t) = 0$ for all $\mathscr{S}' \in \mathscr{B}(x(t))$ with $j \notin \mathscr{S}'$. □

More generally, it seems natural to conjecture that the condition (3.5) is unnecessary for Theorem 3.2 to hold. This is so for single- and two-resource networks, for which we give a full analysis in the following two sections.

We first require some additional notation. Partition the set X by defining, for each $\mathscr{S} \subseteq \mathscr{J}$, $X_{\mathscr{S}} = \{x \in X\colon \sum_r A_{jr}x_r(t) = C_j$ if and only if $j \in \mathscr{S}\}$. We shall find it convenient to write X_j for $X_{\{j\}}$, and shall make similar notational simplifications elsewhere. Note in particular that, from (3.1), $\pi_t = \pi_{x(t)}^{\varnothing}$ whenever $x(t) \in X_{\varnothing}$.

4. Single-resource systems.

Consider further the single-resource case $\mathscr{J} = \{1\}$. It is convenient to write C for C_1, A_r for A_{1r}, and $\alpha^{\mathscr{S}}(x)$ for $\alpha_1^{\mathscr{S}}(x)$.

Here $E = \mathbb{Z}_+ \cup \{\infty\}$. Define $\mathscr{R}^* = \{r \in \mathscr{R}\colon \infty \in \mathscr{A}_r\}$. It follows from the assumed continuity of each $I_{\mathscr{A}_r}$ at ∞ that $\mathscr{R}^*$, $\mathscr{R}\setminus\mathscr{R}^*$, are the sets of call types which are accepted, respectively, rejected, for all sufficiently large values of the free capacity in the network.

Note that, from (3.3), $\alpha^{\varnothing}(x) = \sum_{r\in\mathscr{R}} A_r\{\kappa_r I_{\{r\in\mathscr{R}^*\}} - \mu_r x_r\}$. This quantity is also the drift rate *towards* the origin of the restriction of the process $m_x(\cdot)$ to $\mathbb{Z}_+$, except in some finite neighborhood of the origin. Elementary Lyapounov techniques for irreducible processes with such partial spatial homogeneity [see, e.g., Fayolle, Malyshev and Menshikov (1995)] now show that the distribution π_x^1 $(= \pi_x^{\{1\}})$ exists if and only if $\alpha^{\varnothing}(x) > 0$. It follows that $\{1\} \in \mathscr{B}(x)$ if and only if $x \in X_1^+ = \{x \in X_1\colon \alpha^{\varnothing}(x) > 0\}$.

Further, the processes $x(\cdot)$ and $\alpha^{\varnothing}(x(\cdot))$ are continuous and, on the set $X_{\varnothing}$, the latter is the drift rate of the process $\sum_r A_r x_r(\cdot)$. Hence, if t is such that $x(t) \in X_1^+$, then there exists $t' > t$ such that $x(u) \in X_1^+$ for all $u \in [t, t')$. It follows that the set $\{t\colon x(t) \in X_1^+\}$ is a countable union of intervals, to each of which Lemma 3.1 may be applied to deduce that $\lambda^{\varnothing}(t) = 0$ for almost all t in this set.

It now follows, using also (3.1) and (3.2), that a velocity field for the limit process $x(\cdot)$ may be defined everywhere on X, the function π' being given by

$$\pi'_x = \begin{cases} \pi_x^{\varnothing}, & \text{if } x \in X \setminus X_1^+, \\ \pi_x^1, & \text{if } x \in X_1^+. \end{cases} \tag{4.1}$$

(This result is also given, for the case $\mathscr{R}^* = \mathscr{R}$, by Lemma 4 of HK.)

It is readily verified that, for each r, $\pi'_x(\mathscr{A}_r)$ is Lipschitz continuous on each of the sets $X \setminus X_1^+$ and X_1 [see Bean, Gibbens and Zachary (1995)]. Hence trajectories of the process $x(\cdot)$ are uniquely defined functions of their positions at time 0 and discontinuities in the velocity of any trajectory occur only at times of passage from $X_{\varnothing}$ to X_1^+. (Passage from X_1^+ to $X_{\varnothing}$ is clearly impossible.) It follows from standard arguments for dynamical systems that, for each t, $x(t)$ is a continuous function of $x(0)$ and so, by Theorem 2.1, the process $x(\cdot)$ always possesses at least one fixed point.

Bean, Gibbens and Zachary (1995) consider in detail the (apparently more natural) case $\mathscr{R}^* = \mathscr{R}$. They show how the distribution π_x^1, where it exists, may be determined precisely, and give sufficient conditions for the existence of a single fixed point. There are also practical circumstances where we might have $\mathscr{R}^* \neq \mathscr{R}$, and much of the analysis of that paper extends to this case [for further details see Zachary (1996)]. However here, and provided also $\sum_{r \in \mathscr{R}} A_r \kappa_r I_{\{r \in \mathscr{R}^*\}} / \mu_r \le C$ so that $x^{(1)} \in X \setminus X_1^+$ given by $x_r^{(1)} = \kappa_r I_{\{r \in \mathscr{R}^*\}} / \mu_r$ is a fixed point, there may also be further fixed points in the set X_1^+. This is illustrated by the following example of (at least) bistability in which, for appropriate parameter values, there are at least three fixed points, at least two of which are asymptotically stable. This example illustrates in detail behavior which corresponds closely to what happens in a more complex network if its control strategy is such that congested states of the network are self-perpetuating. [See, e.g., Gibbens, Hunt and Kelly (1990).]

Suppose that there are two call types, with $A_r = 1$ for $r = 1, 2$. Let $\mu_1 = 1$, $\mu_2 C < \kappa_1 < C$ and assume $\kappa_2 > 0$. Let $\mathscr{A}_1 = \{m\colon m > 0\}$ and let $\mathscr{A}_2 = \{m\colon 0 < m \le s\}$ for some $s > 0$. Then, for all $x = (x_1, x_2) \in X$, $\alpha^{\varnothing}(x) = \kappa_1 - x_1 - \mu_2 x_2$, and it is readily verified that there exists a unique point $\hat{x} = (\hat{x}_1, \hat{x}_2) \in X_1$ such that $\alpha^{\varnothing}(\hat{x}) = 0$ and that $X_1^+ = \{x \in X_1\colon x_1 < \hat{x}_1\}$. It follows easily that, provided the process $x(\cdot)$ remains within $X \setminus X_1^+$—which, for example, will certainly be the case if $x_2(0) \le \hat{x}_2$—its trajectories tend to the (asymptotically stable) fixed point $x^{(1)} = (\kappa_1, 0)$.

However, once within the set X_1^+, the process $x(\cdot)$ can only leave it at the point $\hat{x}$. Note also that, for all $x \in X$, $v_1(x) = \kappa_1 \pi'_x(\mathscr{A}_1) - x_1$. It follows from the continuity of $\pi'_x(\mathscr{A}_1)$ on the set X_1 that, as $x \to \hat{x}$ in X_1^+, $v_1(x) \to v_1(\hat{x}) = \kappa_1 - \hat{x}_1 > 0$. We also have $v_1(0) > 0$. By considering the birth and death process on $\mathbb{Z}_+$ of which π_x^1, $x \in X_1^+$, is the invariant distribution, it is not difficult to show that, if μ_2 is close to (but less than) κ_1/C, then $v_1(x) > 0$ for all $x \in X_1^+$. Thus, in this case, all those trajectories of the process $x(\cdot)$ which do enter or start at time zero in the set X_1^+ leave it again at the point $\hat{x}$ and tend to the unique fixed point $x^{(1)}$.

Similarly we may show that, for sufficiently small μ_2, $v_1(x) < 0$ for some $x \in X_1^+$ and hence there exist at least two fixed points of the process $x(\cdot)$ in the set X_1^+. In the case where there are exactly two such points $x^{(2)}$, $x^{(3)}$, with $x_1^{(2)} < x_1^{(3)}$, the point $x^{(2)}$ attracts those trajectories of $x(\cdot)$ which enter the set X_1^+ at points x satisfying $x_1 < x_1^{(3)}$, while those trajectories which enter X_1^+ at points x satisfying $x_1 > x_1^{(3)}$ tend, as in the earlier case, to the fixed point $x^{(1)}$. The points $x^{(1)}$ and $x^{(2)}$ are thus asymptotically stable, while $x^{(3)}$ has a domain of attraction of Lebesgue measure zero.

Figure 1 illustrates a numerical example in which $C = 1000$, $\kappa_1 = 500$, $\mu_1 = 1.0$, $\kappa_2 = 700$, $\mu_2 = 0.1$ and $s = 5$. There are three fixed points as above. The upper panel shows trajectories of the limit process $x(\cdot)$ for several initial positions. The locations of the two asymptotically stable fixed points $x^{(1)}$, $x^{(2)}$ are as indicated. The curved solid line separates the domains of

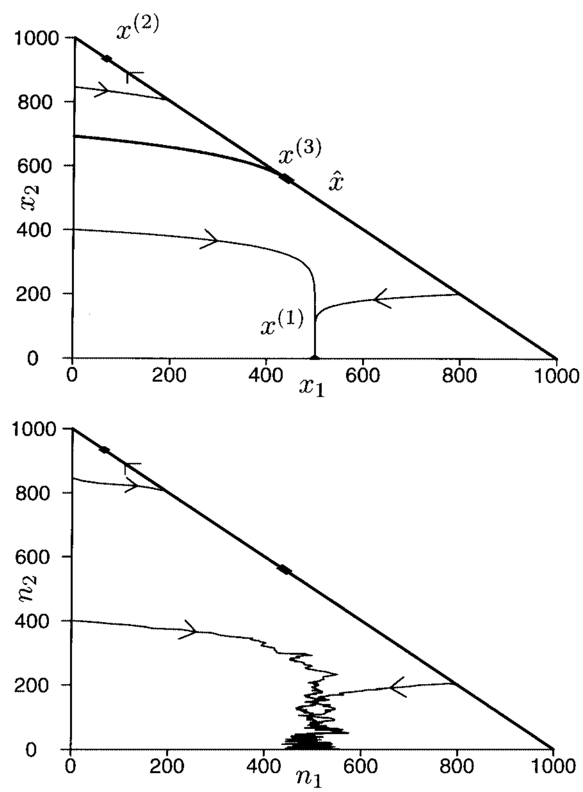

FIG. 1. *Two-type system: analytical and simulation results.*

attraction of these two points. It is of course also a trajectory of $x(\cdot)$ and intersects X_1 at the unstable fixed point $x^{(3)}$ (which in this example lies very close to $\hat{x}$). The lower panel shows simulated trajectories of the process $x^1(\cdot)$ $(= n^1(\cdot))$. Here C is sufficiently large that the process $x^1(\cdot)$ should be reasonably well approximated by $x(\cdot)$ and indeed the bistable behavior of $x^1(\cdot)$ is clearly evident. However, this process is of course ergodic, so that, over sufficiently long time periods, it alternates between typically lengthy residences in the neighborhoods of $x^{(1)}$ and $x^{(2)}$.

5. Two-resource networks. Consider now the two-resource case $\mathscr{J} = \{1, 2\}$. Our interest is again in describing both the dynamic and equilibrium behavior of the process $x(\cdot)$. By the remark following (1.3), for times t such that $x(t) \notin X_{12}$, the behavior of $x(\cdot)$ may be described as in the previous section, so that it is sufficient to consider further the case $x(t) \in X_{12}$.

For either $j \in \mathscr{J}$, let j' denote its complement in $\mathscr{J}$. Note that, as in the previous section, for any x and j, the distribution π_x^j exists, and so also $\alpha_{j'}^j(x)$ is defined, if and only if $\alpha_j^{\varnothing}(x) > 0$. (Recall that the restriction of the process $m_x(\cdot)$ to E_j is essentially one dimensional.) For each j, define the function β_j on X by

$$\beta_j(x) = \begin{cases} \alpha_j^{j'}(x), & \text{if } \alpha_{j'}^{\varnothing}(x) > 0, \\ \alpha_j^{\varnothing}(x), & \text{if } \alpha_{j'}^{\varnothing}(x) \le 0. \end{cases} \tag{5.1}$$

As in the previous section, the continuity of the indicator functions $I_{\mathscr{A}_r}$ ensures that, for each x, the restriction of the process $m_x(\cdot)$ to $E_{12} = \mathbb{Z}_+^2$ possesses a property of partial spatial homogeneity. Each $\beta_j(x)$ also has an interpretation as an averaged drift for this restricted process, and standard results for such processes [see, e.g., Fayolle, Malyshev and Menshikov (1995), or Zachary (1995), Theorem 2.4] show that the distribution π_x^{12} does not exist whenever $\beta_1(x) \wedge \beta_2(x) < 0$. Standard arguments for one-dimensional processes, as in the previous section, also show that each of functions β_j is continuous on X.

Define X_{12}^+, X_{12}^0, X_{12}^- to be the sets of x in X_{12} such that $\beta_1(x) \wedge \beta_2(x)$ is, respectively, greater than, equal to, and less than 0. Further partition X_{12}^- as $V_1 \cup V_2 \cup W^- \cup W^+$ where

$$V_j = \{x \in X_{12}^-: \beta_j(x) > 0\}, \qquad j = 1, 2,$$

$$W^- = \{x \in X_{12}^-: \beta_1(x) \vee \beta_2(x) \le 0,\ \alpha_1^{\varnothing}(x) \vee \alpha_2^{\varnothing}(x) \le 0\},$$

$$W^+ = \{x \in X_{12}^-: \beta_1(x) \vee \beta_2(x) \le 0,\ \alpha_1^{\varnothing}(x) \wedge \alpha_2^{\varnothing}(x) > 0\}.$$

[That V_1, V_2, W^- and W^+ do form a partition of X_{12}^- follows from (5.1).] The following theorem identifies π_t uniquely for t such that $x(t) \in X_{12}^+ \cup V_1 \cup V_2 \cup W^-$ (so that a unique velocity field exists within this region). For t such that $x(t) \in W^+$, π_t is not in general uniquely determined (see below). We discuss subsequently behavior when $x(t)$ lies in the remaining "boundary set" X_{12}^0, which is typically of Lebesgue measure zero in X.

THEOREM 5.1. *For almost all t:*

(i) *if $x(t) \in X_{12}^{+}$, then $\pi_t = \pi_{x(t)}^{12}$;*

(ii) *if $x(t) \in V_j$, then $\pi_t = \pi_{x(t)}^{j}$;*

(iii) *if $x(t) \in W^{-}$, then $\pi_t = \pi_{x(t)}^{\varnothing}$;*

(iv) *if $x(t) \in W^{+}$, then $\pi_t = \lambda^1(t)\pi_{x(t)}^1 + \lambda^2(t)\pi_{x(t)}^2$, for some nonnegative $\lambda^1(t)$, $\lambda^2(t)$, necessarily summing to one.*

PROOF. For each j, let $T_j = \{t\colon \sum_{r\in\mathscr{R}} A_{jr}x_r(t) = C_j,\ \alpha_j^{\varnothing}(x(t)) > 0,\ \beta_j(x(t)) > 0\}$. It follows as in the previous section, by using the continuity of $\alpha_j^{\varnothing}$ and β_j and considering drifts, that T_j is a countable union of intervals. From the definition of β_j and the convention following equation (3.2), we may apply Lemma 3.1 to each of these to deduce that

$$\lambda^{\varnothing}(t) = \lambda^{j'}(t) = 0 \quad \text{for almost all } t \in T_j. \tag{5.2}$$

To prove (ii) note first that, if $x \in V_j$, then $\alpha_j^{\varnothing}(x) > 0$ [for otherwise we would have, from (5.1), $\alpha_{j'}^{\varnothing}(x) = \beta_{j'}(x) < 0$ and so $\alpha_j^{\varnothing}(x) = \beta_j(x) > 0$—a contradiction]. Hence $\{t\colon x(t) \in V_j\} \subseteq T_j$ and the result now follows from (3.1) and (3.2) on recalling that π_x^{12} does not exist for $x \in V_j$.

To prove (i) suppose that t is such that $x(t) \in X_{12}^{+}$. Then, easily from (5.1), $\alpha_1^{\varnothing}(x(t)) \vee \alpha_2^{\varnothing}(x(t)) > 0$. Suppose $\alpha_1^{\varnothing}(x(t)) > 0$. Then, as above, there exists $t' > t$ such that $[t, t') \subset T_1$ and $\alpha_2^1(u) = \beta_2(u) > 0$ for all $u \in [t, t')$. It follows from (5.2) and further consideration of drifts that also $x(u) \in X_{12}^{+}$ for all $u \in [t, t')$, and hence, again from (5.2) followed by a further use of Lemma 3.1—this time for the case $j = 2$—we have that $\lambda^{\varnothing}(u) = \lambda^1(u) = \lambda^2(u) = 0$ for almost all $u \in [t, t')$. The result (i) now follows easily, again using (3.1) and (3.2).

To prove (iii) observe that, for all $x \in W^{-}$, none of the distributions π_x^1, π_x^2 and π_x^{12} exists, so that this result is immediate.

Finally, let $T' = \{t\colon x(t) \notin X_{\varnothing},\ \alpha_1^{\varnothing}(x(t)) \wedge \alpha_2^{\varnothing}(x(t)) > 0\}$. As before we may show that T' is a countable union of intervals, to each of which a coupling argument, identical to that of the proof of Lemma 2 of Hunt (1995), may be applied to deduce that $\pi_t(E_{\varnothing}) = 0$ and so $\lambda^{\varnothing}(t) = 0$ for almost all $t \in T'$. Since $\{t\colon x(t) \in W^{+}\} \subseteq T'$ and the distribution π_x^{12} does not exist for $x \in W^{+}$, the result (iv) now follows. $\square$

We now consider the identification of π_t for $x(t) \in X_{12}^{0}$. Under the condition

$$(\infty, \infty) \in \mathscr{A}_r \quad \text{for all } r \in \mathscr{R}, \tag{5.3}$$

it is not difficult to show that the distribution π_x^{12} also fails to exist for x such that $\beta_1(x) \wedge \beta_2(x) = 0$: the condition implies, from (3.3), that $\alpha_j^{\varnothing}(x) \geq \beta_j(x)$, $j = 1, 2$, and then the above assertion follows from, for example, Zachary (1995), Theorem 2.4, except in the "zero-drift" case $\alpha_j^{\varnothing}(x) = \beta_j(x) = 0$, $j = 1, 2$. This latter case is here easily handled by, for example, the simple Lyapounov function $f(m) = m_1$. Hence, under condition (5.3), we may replace X_{12}^- by $X_{12}^- \cup X_{12}^0$ in the definitions of V_1, V_2, W^- and W^+. Theorem 5.1 will continue to hold as stated and will now identify π_t everywhere.

Condition (5.3) corresponds to the requirement that calls of all types are accepted provided the free capacity of each resource in the network exceeds some given value. This condition will be satisfied in most applications. Elsewhere it seems very unlikely that the boundary region X_{12}^0 (which may well fail to exist at all) will cause problems.

At least under condition (5.3), when W^+ is empty, the dynamics of the process $x(\cdot)$ are readily inferred from Theorem 5.1 and the slightly more detailed description of behavior given in its proof. In particular, the process will usually only remain within the set X_{12} for a nonzero length of time while in the set X_{12}^+, and, if $\beta_1(x(t)) \wedge \beta_2(x(t))$ should fall below zero, will then depart to X_1^+ or X_2^+. Further, only mild regularity conditions are required to show, as in Section 2, that there exists at least one fixed point for the process $x(\cdot)$.

Hunt (1995) gives an example in which the set W^+ is nonempty [and for which (5.3) also fails to hold]. Here the process $x(\cdot)$ generally departs the set W^+ immediately, indeterminately to either X_1^+ or X_2^+—corresponding to the fact that the sequence of processes $x^N(\cdot)$ may have different limits in different subsequences. The example confirms that the result (iv) of Theorem 5.1 is the best possible statement of behavior in the set W^+. It is therefore important for the control of networks to have conditions which ensure that this set is empty. Some such conditions are given by Moretta (1995), and by Zachary (1996), and suggest that in particular W^+ is always empty under the condition (5.3).

We conclude with an example which also provides a counterexample to a commonly assumed result. Let $C_1 = C_2 = C$. Assume three call types and let $A = (A_{jr})$ be given by

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Let $\kappa_1 = \kappa_2 = \kappa$ and $\kappa_3 = \kappa'$ where $\kappa + \kappa' > C$. Further let $\mu_r = 1$ for all r. Finally take $\mathscr{A}_1 = \{m\colon m_1 > 0\}$, $\mathscr{A}_2 = \{m\colon m_2 > 0\}$ and $\mathscr{A}_3 = \{m\colon m_1 \wedge m_2 > s\}$ where $s > 0$ is a *trunk reservation* parameter which restricts acceptance of calls of type 3 when either resource in the network is close to capacity.

For $j = 1, 2$ and all $x \in X$, $\alpha_j^{\varnothing}(x) = \kappa + \kappa' - (x_j + x_3) \geq \kappa + \kappa' - C > 0$, so that in particular the distribution π_x^j exists. Note also that $\pi_x^j(\mathscr{A}_{j'}) = 1$. It follows, using also (3.3) and (3.4), that, for each j and for each $x \in X_j \cup X_{12}$,

$$\kappa \pi_x^j(\mathscr{A}_j) + \kappa' \pi_x^j(\mathscr{A}_3) - C = \alpha_j^j(x) = 0, \tag{5.4}$$

and hence $\beta_{j'}(x) = \alpha_{j'}^j(x) = \kappa + \kappa' \pi_x^j(\mathscr{A}_3) - (x_{j'} + x_3) \geq \kappa(1 - \pi_x^j(\mathscr{A}_j))$. Since the distribution π_x^j is here independent of x for $x \in X_j \cup X_{12}$, it follows that $\beta_{j'}(x)$ is positive and bounded away from zero on this set. Thus, from (4.1) and Theorem 5.1, for each $\mathscr{S} \subseteq \{1, 2\}$, $\pi_t = \pi_{x(t)}^{\mathscr{S}}$ whenever $x(t) \in X_{\mathscr{S}}$. In particular, a velocity field v may be defined everywhere on X.

It now follows easily that the process $x(\cdot)$ enters the set X_{12} (both resources fill to capacity) within a finite time and remains within it thereafter. In particular, any fixed point necessarily lies in this set. It follows, using also (5.4), that such fixed points x are the solutions of

$$\kappa' \pi_x^{12}(\mathscr{A}_3) = x_3. \tag{5.5}$$

In general the distribution π_x^{12} must be determined numerically. However, there are good reasons for believing it to be relatively insensitive to variation of x within X_{12} [see Moretta (1995) for some numerical investigations], so that we may reasonably expect that there exists a single fixed point $\bar{x} \in X_{12}$ such that, for all $x(0)$, $x(t) \to \bar{x}$ as $t \to \infty$. It then follows in particular, from Theorem 2.2, that the invariant distribution of $m^N(\cdot)$ converges to $\pi_{\bar{x}}^{12}$. We now show that this limit distribution does *not* have a product form, contrary to the assumption of many routines used to calculate call acceptance probabilities in applications. (HK, in a similar but more complex argument, show that this limit distribution does not have the *particular* product form which corresponds to the well-known generalized Erlang fixed point approximation.)

Suppose instead that we may write $\pi_{\bar{x}}^{12}(m) = \bar{\pi}_1(m_1)\bar{\pi}_2(m_2)$ for some distributions $\bar{\pi}_1$, $\bar{\pi}_2$ on $\mathbb{Z}_+$. Recall that $\pi_{\bar{x}}^{12}$ is the invariant distribution of the Markov process $m_{\bar{x}}$ restricted to $\mathbb{Z}_+^2$. Since $s > 0$, the balance equation associated with the point $m = (0, 0)$ is

$$\bar{\pi}_1(0)\bar{\pi}_2(0)(x_1 + x_2 + x_3) = [\bar{\pi}_1(0)\bar{\pi}_2(1) + \bar{\pi}_1(1)\bar{\pi}_2(0)]\kappa. \tag{5.6}$$

Further, for $j = 1, 2$, consideration of the balance of probability flux between the set $\{m \in \mathbb{Z}_+^2 \colon m_j = 0\}$ and its complement shows that $\bar{\pi}_j(0)(x_j + x_3) = \bar{\pi}_j(1)\kappa$. Substitution of this result, for each j, into (5.6) implies that $x_3 = 0$, in contradiction of (5.5).

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