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ASYMPTOTIC BEHAVIOR OF SOME INTERACTIVE POPULATION FLOW MODELS

By Wolfgang Stadje

University of Osnabrück

This paper is concerned with Markov chain models for flows of a finite population among a set of groups, where the individuals base their decisions on which group to go next partially on the current frequency distribution (profile). For a certain class of these models, the transition matrix of the profile process is analyzed algebraically, leading to surprisingly simple asymptotic results. Furthermore, in a model with after effects, the absorption probabilities are derived.

1. Introduction. In social science, Markov chains are often used to model population flows among various "states." As typical examples of states or categories which have been considered, we mention occupational or other social classes, geographical regions, brands, investment allocations of firms, and political or religious affiliations (see [1] for general background and references). In the early models, the transitions of the individuals form independent Markov chains so that social interaction is ruled out. However, in many applications, the effects on the decisions of individuals of imitation, fashion, popularity, contagion, and so on cannot be ignored. In order to take into account interactions among individuals, Conlisk [5] introduced the concept of an interactive Markov chain (IMC) as a framework for stochastic flows of this kind. In an IMC, the next state of an individual depends on his current state and on the current frequency distribution of the population among the states. Basically, the idea can already be found in the analysis of social mobility by Matras [12] and in the brand choice model of Smallwood [13] (see also [14]). Regarding other models of consumer choice behavior, we mention [9] and [15] and the references given there. Conlisk's original model (which he also studies in [6, 7, 8]) was a deterministic recursion which was intended to serve as an approximation for the implied randomly fluctuating process. This underlying stochastic structure was later called "finite population model" and investigated by Brumelle and Gerchak [4], Lehoczky [11], Gerchak [10] and Bartholomew [2, 3]. IMC's were further discussed in Bartholomew [1] Sections 2.5 and 2.3.

IMC models have the undesirable property that the fluctuating individuals possess no individual traits, since their movements are all governed by the same Markovian mechanism; at any time the transition probabilities are assumed to be identical for all individuals residing in the same state. To remedy this deficiency, one obviously has to allow an individual's transition rule to

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depend not only on his current state and the allocation of the entire population, but also on the individual himself. An example involving distinguishable individuals can be found in [5], Section 5, but apparently this idea has not been investigated any further.

In this paper we study two models incorporating this additional feature. As a warm-up, let us start with the following Markov chain. The notation which we now introduce is retained throughout the entire paper. Consider n individuals moving among m exclusive groups. The variable $t = 0, 1, 2, \ldots$ denotes discrete time. Then let

$$\xi_t(i, \nu) = \begin{cases} 1, & \text{if the } i\text{th individual belongs} \\ & \text{to the } \nu\text{th group at time } t, \\ 0, & \text{otherwise,} \end{cases}$$

for $t \in \mathbb{Z}_+$, $i = 1, \ldots, n$ and $\nu = 1, \ldots, m$. The $(n \times m)$-random matrix

$$\xi_t = (\xi_t(i, \nu) \mid i = 1, \ldots, n,\ \nu = 1, \ldots, m)$$

of 0's and 1's describes the allocation of the individuals to groups at time t. The initial allocation $\xi_0 = (\xi_0(i, \nu))$ is fixed. Clearly, every ξ_t has exactly one 1 in every row. Now let $\mathcal{F}_t = \sigma(\xi_0, \ldots, \xi_t)$ be the σ-field generated by $\xi_0, \ldots, \xi_t$ and suppose that the rows $\xi_{t+1}^{(i)} = (\xi_{t+1}(i, \nu))_{\nu=1,\ldots,m}$ of ξ_{t+1} are conditionally independent, given $\mathcal{F}_t$. For $\nu = 1, \ldots, m$ let e_ν be the unit row vector with the 1 in νth position; that is, $e_\nu = (\delta_{\nu,\nu'})_{\nu'=1,\ldots,m}$, where $\delta_{\nu,\nu'} = 1$ (0) if $\nu = \nu'$ $(\nu \neq \nu')$. We assume that the conditional distribution of $\xi_{t+1}^{(i)}$ is given by

$$P\big(\xi_{t+1}^{(i)} = e_\nu \mid \mathcal{F}_t\big) = \sum_{j=1}^{n} \beta_{i,j}\xi_t(j, \nu), \tag{1.1}$$

where the coefficients $\beta_{i,j}$ satisfy $\beta_{i,j} \geq 0$ and $\sum_{j=1}^{n} \beta_{i,j} = 1$.

According to (1.1), the individual i goes to group ν with a probability that is a weighted average of the current allocation; the weights attributed by individual i to the different members of the population may vary with i. Clearly, $(\xi_t)_{t\in\mathbb{Z}_+}$ is a Markov chain with stationary transition probabilities having as state space the set $\mathcal{X}$ of all 0–1-matrices of size $n \times m$ with all row sums equal to 1.

If the matrix $(\beta_{i,j})$ is irreducible, eventually all individuals will end up in one of the groups so that there are m absorbing states. For arbitrary $\xi \in \mathcal{X}$, let $\pi_\nu(\xi)$ be the probability that $\xi_t^{(1)} = \cdots = \xi_t^{(n)} = e_\nu$ for some $t \in \mathbb{R}_+$ (i.e., the probability of absorption in group ν), given that $\xi_0 = \xi$. The following proposition shows how to compute $\pi_\nu(\xi)$.

PROPOSITION 1. *Assume that $(\beta_{i,j})$ is irreducible and let $(\sigma_i)_{i=1,\ldots,n}$ be the corresponding stationary distribution. Then*

$$\pi_\nu(\xi) = \sum_{i:\,\xi(i,\nu)=1} \sigma_i.$$

PROOF. For the Markov chain on the finite state space $\{1, \dots, n\}$ with transition probabilities $\beta_{i,j}$ there is exactly one stationary distribution $(\sigma_i)_{i=1,\dots,n}$, which is characterized by the relations

$$\sigma_i \geq 0, \qquad \sum_{i=1}^{n} \sigma_i = 1, \qquad \sigma_j = \sum_{i=1}^{n} \sigma_i \beta_{i,j}.$$

Let $Z_t^{(\nu)} = \sum_{i=1}^n \sigma_i \xi_t(i,\nu)$. Multiplying both sides of (1.1) by σ_i and summing over i we find that $Z_{t+1}^{(\nu)}$ satisfies

$$E\big(Z_{t+1}^{(\nu)} \mid \mathscr{F}_t\big) = Z_t^{(\nu)}.$$

Hence $Z_0^{(\nu)}, Z_1^{(\nu)}, Z_2^{(\nu)}, \dots,$ in a martingale. Since $0 \leq Z_t^{(\nu)} \leq 1$, it is uniformly integrable. The limit $Z^{(\nu)} = \lim_{t\to\infty} Z_t^{(\nu)}$ exists almost surely, since $\xi_t(1,\nu) = \cdots = \xi_t(n,\nu)$ a.s. for sufficiently large t (or by the martingale convergence theorem). $Z^{(\nu)}$ only takes the values 0 and 1, and $Z^{(\nu)} = 1$ iff $\xi_t(i,\nu) = 1$ for $i = 1\dots,n$ and all sufficiently large t. As $(Z_t^{(\nu)})_{t\in\mathbb{Z}_+}$ is a uniformly integrable martingale, we obtain

$$\begin{aligned}\pi_\nu(\xi) &= P(Z^{(\nu)} = 1) = E(Z^{(\nu)}) \\ &= E(Z_0^{(\nu)}) = \sum_{i=1}^{n} \sigma_i \xi_0(i,\nu) \\ &= \sum_{i:\, \xi(i,\nu)=1} \sigma_i.\end{aligned}$$

In this paper we consider two extensions of the above model.

Model 1. If (1.1) holds, each individual is completely subjected to the previous frequency distribution of the population, only influencing it by assigning possibly different weights to different individuals. A more realistic model should also include for every participant the possibility of taking decisions based on individual sources and not on the past. One way to incorporate this feature is to introduce for any individual i a probability distribution $p^{(i)} = (p_{i\nu})_{\nu=1,\dots,m}$ on the set of groups and then to replace (1.1) in the above model by

$$P\big(\xi_{t+1}^{(i)} = e_\nu \mid \mathscr{F}_t\big) = \lambda_i \sum_{j=1}^{n} \beta_{i,j} \xi_t(j,\nu) + (1-\lambda_i) p_{i\nu}, \tag{1.2}$$

where $\lambda_i \in [0,1]$. Thus, with probability λ_i the next transition of individual i is governed by the mechanism (1.1) and with probability $1-\lambda_i$, his next group affiliation is determined by using $p^{(i)}$. Individuals i, for which $\lambda_i = 1$, may be called "conformists," while those with $\lambda_i = 0$ act independently of the rest of the population. The individuals for which $\lambda_i = 1$ and $p^{(i)}$ is a unit row even have a permanently fixed affiliation. In Sections 3–5 we will study this model in detail in the case $\beta_{i,j} = 1/n$, in which for each individual all previous affiliations of other individuals carry the same weight.

Model 2. As another extension, one can allow the conditional probabilities $P(\xi_{t+1}^{(i)} = e_\nu \mid \mathscr{F}_t)$ to depend not only on ξ_t, but on $\xi_t, \xi_{t-1}, \dots, \xi_{t-c+1}$ for some fixed $c \in \mathbb{N}$, so that the next affiliation of any individual depends on the last c distributions of the population. Specifically, we will consider the model

$$P\big(\xi_{t+1}^{(i)} = e_\nu \mid \mathscr{F}_t\big) = \sum_{j=1}^{n} \beta_{i,j} \sum_{l=1}^{c} \gamma_l \; \xi_{t-l+1}(j,\nu), \tag{1.3}$$

where the additional coefficients γ_l satisfy $\gamma_l \geq 0$, $\sum_{l=1}^{c} \gamma_l = 1$. The γ_l serve to weight the l past time instants according to their impact on the next stage, while $\beta_{i,j}$ measures the importance of individual j to individual i.

As in the basic model (1.1), also under (1.3) all individuals will eventually be "absorbed" in one of the groups. As a generalization of Proposition 1, the probability $\pi_\nu(\xi_0, \dots, \xi_{c-1})$ that all individuals will eventually belong to group ν, given that the first c allocations are $\xi_0, \dots, \xi_{c-1}$, is determined in Section 6. Section 7, the final section, is devoted to various examples.

2. Description of results for the first model.

We describe the flow between groups in model (1.2) in terms of *population profiles*, as suggested by Brumelle and Gerchak [4]. The set of possible population profiles is given by

$$\mathscr{S} = \{x = (x_1, \dots, x_m)^{\mathsf{T}} \in \mathbb{Z}_+^m \mid \| x \| = n\},$$

where we set $\| x \| = \sum_{\nu=1}^{n} |x_\nu|$. For the *profile process*, being in state $x \in \mathscr{S}$ means that x_ν individuals belong to group ν for $\nu = 1, \dots, m$. The number of profiles is $\binom{m+n-1}{n}$.

The transition from one profile x to another, say $y \in \mathscr{S}$, proceeds as follows. Let $\eta_{x,1}, \dots, \eta_{x,n}$ be independent random column vectors such that

$$P(\eta_{x,i} = e_\nu^{\mathsf{T}}) = \lambda_i \frac{x_\nu}{n} + (1-\lambda_i) p_{i\nu}, \;\; x \in \mathscr{S}, \;\; i = 1, \dots, n, \;\; \nu = 1, \dots, m. \tag{2.1}$$

Then the probability of a (one-step) transition from x to y is given by

$$q(x, y) = P(\eta_{x,1} + \cdots + \eta_{x,n} = y), \quad x, y \in \mathscr{S}. \tag{2.2}$$

As explained in the Introduction, individual i determines his probability to go to group ν in the next step according to his individual convex combination of the current frequency of that group and his individual fixed preference for ν, expressed by the probability $p_{i\nu}$. For (2.1) we assume that $\lambda_i \in [0, 1]$ $p_{i\nu} \geq 0$, $\sum_{\nu=1}^{m} p_{i\nu} = 1$. Note that there may be individuals under the total influence of the collective opinion (those with $\lambda_i = 1$), while others may take completely independent decisions ($\lambda_i = 0$).

Clearly, $Q = (q(x, y))_{x, y \in \mathscr{S}}$ is a $(\operatorname{card} \mathscr{S} \times \operatorname{card} \mathscr{S})$-transition matrix which defines, together with an arbitrary initial distribution on $\mathscr{S}$, a time-homogeneous Markov chain with state space $\mathscr{S}$. Somewhat surprisingly, it turns out that rather explicit algebraic results on Q can be obtained.

We need some additional notations. For $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_+$, let $(a)_b = a(a-1)\cdots(a-b+1)$. For $x \in \mathbb{Z}^m$ and $z \in \mathbb{Z}_+^m$, we set $(x)_z = \prod_{\nu=1}^{m} (x_\nu)_{z_\nu}$ and $x^z =$

$\prod_{\nu=1}^{m} x_\nu^{z_\nu}$. Furthermore, let $\mathscr{V} = \mathbb{R}^{\mathscr{S}}$ and denote the elements of $\mathscr{V}$ by $v = (v_x)_{x\in\mathscr{S}}$. Next, we set $\mathscr{S}_j = \{z \in \mathbb{Z}_+^m \mid \| z \| = j\}$, $j = 0, \ldots, n$, so that $\mathscr{S} = \mathscr{S}_n$. For $z \in \mathscr{S}_j$ we denote by $v^{(z)}$ the column vector $v^{(z)} = (v_x^{(z)})_{x\in\mathscr{S}} \in \mathscr{V}$ whose components are given by

$$v_x^{(z)} = (x)_z = \prod_{\nu=1}^{m} (x_\nu)_{z_\nu}. \tag{2.3}$$

The vectors $v^{(z)}$, $z \in \mathscr{S}_j$, will be seen to be linearly independent for every $j \in \{0, \ldots, n\}$. These sets are an important tool in the derivation of our main result.

THEOREM 1. *The matrix Q is diagonalizable. Its eigenvalues are*

$$\mu_0 = 1,$$
$$\mu_j = n^{-j} j! \sum_{1\le i_1<\cdots<i_j\le n} \lambda_{i_1}\cdots\lambda_{i_j}, \qquad j = 1, \ldots, n.$$

The multiplicity of μ_j is $\binom{m+j-2}{j}$; if some μ_j's are equal, these multiplicities have to be added. It follows that $Q = ADA^{-1}$, where D is the diagonal matrix with diagonal entries μ_j, each one repeated according to its multiplicity, and A is a matrix of corresponding linearly independent eigenvectors.

The next two sections are devoted to a proof of these statements.

3. Preliminaries. In this section we collect some auxiliary results for the proof of Theorem 1. Let $\zeta_1, \zeta_2, \ldots, \zeta_n$ be independent, $\{0, 1\}$-valued random variables satisfying $P(\zeta_i = 1) = 1 - P(\zeta_i = 0) = \lambda_i$, $i = 1, \ldots, n$. We have to consider the factorial moments $m_j = E((\zeta)_j)$ of their sum $\zeta = \sum_{i=1}^n \zeta_i$.

LEMMA 1. *For $\mathscr{A} \subset \{1, \ldots, n\}$ let $P_{\mathscr{A}} = (\prod_{i\in\mathscr{A}} \lambda_i)(\prod_{i\in\{1,\ldots,n\}\setminus\mathscr{A}}(1 - \lambda_i))$. Then for $j = 1, \ldots, n$,*

$$m_j = \sum_{\mathscr{A}\subset\{1,\ldots,n\}} (\mathrm{card}\,\mathscr{A})_j P_{\mathscr{A}} = j! \sum_{1\le i_1<\cdots<i_j\le n} \lambda_{i_1}\cdots\lambda_{i_j} = n^j \mu_j. \tag{3.1}$$

PROOF. The first equation in (3.1) is obvious. To see the second, note that $m_j = (d^j f/du^j)(1)$, where $f(u) = E(u^\zeta)$, $u \in \mathbb{R}$, is the generating function of ζ. If we set

$$g(u) = \prod_{i=1}^{n} (1 + \lambda_i u), \tag{3.2}$$

then $g(u) = f(1+u)$; therefore $(d^j f/du^j)(1) = (d^j g/du^j)(0)$ is the coefficient of u^j in the expansion of the product in (3.2) in ascending powers of u. □

LEMMA 2. *Let k_0 be the number of indices $i \in \{1, \dots, n\}$ satisfying $\lambda_i \neq 0$. Then*

$$1 = m_0 \geq n^{-1} m_1 > n^{-2} m_2 > \cdots > n^{-k_0} m_{k_0} > 0 = m_j, \qquad j > k_0. \tag{3.3}$$

Furthermore, $n^{-1} m_1 = 1$ if and only if $\lambda_i = 1$ for all $i \in \{1, \dots, n\}$.

PROOF. By (3.1),

$$\begin{aligned} n^{-j-1} m_{j+1} &= n^{-j-1} \sum_{\mathscr{A}} (\operatorname{card} \mathscr{A})_{j+1} P_{\mathscr{A}} \\ &= n^{-j} \sum_{\mathscr{A}} (\operatorname{card} \mathscr{A})_j \frac{\operatorname{card} \mathscr{A} - j}{n} P_{\mathscr{A}} \leq n^{-j} m_j \end{aligned}$$

and, clearly, strict inequality holds if $m_{j+1} > 0$. As $P(\zeta = k_0) > 0 = P(\zeta > k_0)$, it is obvious that $m_{k_0+1} = E((\zeta)_{k_0+1}) = 0 < E((\zeta)_{k_0}) = m_{k_0}$. Finally, note that $1 = n^{-1} m_1 = n^{-1} \sum_{i=1}^{n} \lambda_i$ is equivalent to $\lambda_1 = \cdots = \lambda_n = 1$. $\square$

LEMMA 3. *For every $j \in \{0, \dots, n\}$ the set*

$$B_j = \{v^{(z)} \mid z \in \mathscr{S}_j\}$$

is linearly independent in $\mathscr{V}$.

PROOF. We carry out a backward induction on j. First, let $j = n$. If $z \in \mathscr{S}_j = \mathscr{S}$ and $x \in \mathscr{S}$,

$$(x)_z = \begin{cases} (x)_x \neq 0, & \text{if } z = x, \\ 0, & \text{if } z \neq x. \end{cases}$$

Thus, exactly the component $v_z^{(z)}$ of $v^{(z)}$ is not equal to zero. It follows that B_n is a basis of $\mathscr{V}$.

For the induction step we need the following relation. If $x = (x_1, \dots, x_m)^{\mathsf{T}} \in \mathscr{S}$ and $z = (z_1, \dots, z_m)^{\mathsf{T}} \in \mathscr{S}_{j-1}$, then

$$(n - j + 1)(x)_z = \sum_{\nu=1}^{m} (x)_z (x_\nu - z_\nu) = \sum_{\nu=1}^{m} (x)_{z + e_\nu^{\mathsf{T}}}. \tag{3.4}$$

Now suppose that B_j is linearly independent for some $j \in \{1, \dots, n\}$. For every $z \in \mathscr{S}_{j-1}$ we have, by (3.4),

$$v^{(z)} = (n - j + 1)^{-1} \sum_{\nu=1}^{m} v^{(z + e_\nu^{\mathsf{T}})}.$$

Therefore, a linear relationship of the form $\sum_{z \in \mathscr{S}_{j-1}} c_z v^{(z)} = 0$ for certain coefficients $c_z \in \mathbb{R}$ implies that

$$\sum_{\nu=1}^{m} \sum_{z \in \mathscr{S}_{j-1}} \frac{c_z}{n - j + 1} v^{(z + e_\nu^{\mathsf{T}})} = 0,$$

so that all c_z must be zero by the induction hypothesis. Hence B_{j-1} is also linearly independent. □

4. Proof of Theorem 1. The transition from a state x to a state y according to (2.1) and (2.2) can be considered as carried out in two stages. First, the set of individuals is randomly split into two disjoint sets,

$$\{1, \ldots, n\} = \mathscr{A} \cup \mathscr{B}, \qquad \mathscr{A} \cap \mathscr{B} = \varnothing,$$

where $\mathscr{A}$ denotes the set of "conformists" and $\mathscr{B}$ the set of "nonconformists." The random set $\mathscr{A}$ is chosen with probability $P_{\mathscr{A}} = (\prod_{i\in\mathscr{A}} \lambda_i)(\prod_{i\in\mathscr{B}}(1-\lambda_i))$. In the second stage every individual selects his group. Conformists follow the collective preference structure displayed in the preceding time period, while nonconformists do not take into account the decisions of other people. This means that for $1 = 1, \ldots, n$ and $\nu = 1 \ldots, m$,

$$P(\eta_{x,i} = e_\nu^\mathsf{T} \mid i \in \mathscr{A}) = x_\nu/n,$$
$$P(\eta_{x,i} = e_\nu^\mathsf{T} \mid i \in \mathscr{B}) = p_{i\nu}.$$

Let $\eta_{x,\mathscr{A}} = \sum_{i\in\mathscr{A}} \eta_{x,i}$, $\eta_{x,\mathscr{B}} = \sum_{i\in\mathscr{B}} \eta_{x,i}$. For any $z \in \mathbb{Z}_+^m$ it is clear that the conditional probability that $\eta_{x,\mathscr{A}} = z$, given that $\mathscr{A}$ is the current set of conformists, is of the multinomial form

$$P(\eta_{x,\mathscr{A}} = z \mid \mathscr{A}) = \binom{\operatorname{card}\mathscr{A}}{z_1, \ldots, z_m} \frac{x^z}{n^{\|z\|}}.$$

(Of course this probability is 0 if $\operatorname{card}\mathscr{A} \neq \|z\|$.) Let $\pi(\mathscr{B}, z) = P(\eta_{x,\mathscr{B}} = z \mid \mathscr{B})$ be the corresponding probability for the nonconformists. Also we define, for $x \in \mathbb{Z}_+^m$, the factorial product $\varphi(x) = x_1! \ldots x_m!$; if $x \in \mathbb{Z}^m$ has at least one negative component, we set $1/\varphi(x) = 0$. Then we obtain

$$\begin{aligned} q(x, y) &= \sum_{\substack{\mathscr{A}+\mathscr{B}=\{1,\ldots,n\} \\ z\in\mathbb{Z}_+^m}} P_{\mathscr{A}} \pi(\mathscr{B}, z) P(\eta_{x,\mathscr{A}} = y - z \mid \mathscr{A}) \\ &= \sum_{z\in\mathbb{Z}_+^m} \alpha(z) \frac{x^{y-z}}{\varphi(y-z)}, \end{aligned}$$

where

$$\alpha(z) = \frac{(n - \|z\|)!}{n^{n-\|z\|}} \sum_{\mathscr{A}+\mathscr{B}=\{1,\ldots,n\}} P_{\mathscr{A}} \pi(\mathscr{B}, z).$$

We want to derive an identity of the form

$$\sum_{y\in\mathscr{S}} q(x, y)(y)_z = \sum_{z'\in\mathbb{Z}_+^m} \gamma(z, z')(x)_{z'}, \quad x \in \mathscr{S}, \ z \in \mathbb{Z}_+^m$$

for certain coefficients $\gamma(z, z')$. For this some more notations and a few combinatorial equations are required. Let $b(z, z') = (z)_{z'}/\varphi(z') = \prod_{i=1}^{m}\binom{z_i}{z'_i}$ for $z, z' \in \mathbb{Z}_+^m$. We need the easily established Vandermonde type convolution

$$(y)_z = \sum_{z''\in\mathbb{R}_+^m} b(z, z'')(y-z')_{z''}(z')_{z-z''} \tag{4.1}$$

for $y \in \mathscr{S}$, $z, z' \in \mathbb{Z}_+^m$; the summand in (4.1) is zero unless $y \geq z' + z'' \geq z \geq z''$. Furthermore, the following three simple identities have to be applied:

$$(y-z')_{z''}/\varphi(y-z') = \varphi(y-z'-z''), \tag{4.2}$$

$$\sum_{y\in\mathscr{S}} \frac{x^{y-z'-z''}}{\varphi(y-z'-z'')} = \frac{n^{n-||z'||-||z''||}}{(n-||z'||-||z''||)!}, \tag{4.3}$$

$$x^{z''} = \sum_{z'\in\mathbb{Z}_+^m} s(z'', z')(x)_{z'}. \tag{4.4}$$

In (4.4) we use the Stirling numbers $s(l, k)$, $l, k \in \mathbb{Z}_+$, of the second kind, given by their generating function

$$\sum_{k=0}^{\infty} s(l, k)(u)_k = u^l, \qquad u \in \mathbb{R}.$$

The term $s(z'', z')$ is defined by

$$s(z'', z') = \prod_{i=1}^{m} s(z''_i, z'_i).$$

Using (4.1)–(4.3) we find that

$$\begin{aligned}
\sum_{y\in\mathscr{S}} q(x, y)(y)_z &= \sum_{y\in\mathscr{S}} \sum_{z'\in\mathbb{Z}_+^m} \alpha(z')\frac{x^{y-z'}}{\varphi(y-z')}(y)_z \\
&= \sum_{y\in\mathscr{S}} \sum_{z'\in\mathbb{Z}_+^m} \sum_{z''\in\mathbb{Z}_+^m} \alpha(z')\frac{x^{y-z'}}{\varphi(y-z')} b(z, z'')(y-z')_{z''}(z')_{z-z''} \\
&= \sum_{z'\in\mathbb{Z}_+^m} \sum_{z''\in\mathbb{Z}_+^m} \sum_{y\in\mathscr{S}} \alpha(z')b(z, z'')\frac{x^{y-z'-z''}}{\varphi(y-z'-z'')} x^{z''}(z')_{z-z''} \\
&= \sum_{z', z''} \alpha(z')b(z, z'')\frac{n^{n-||z'||-||z''||}}{(n-||z'||-||z''||)!} x^{z''}(z')_{z-z''}.
\end{aligned}$$

Thus, setting

$$\beta(z, z'') = b(z, z'') \sum_{z'\in\mathbb{Z}_+^m} \alpha(z')(z')_{z-z''} \frac{n^{n-||z'||-||z''||}}{(n-||z'||-||z''||)!} \tag{4.5}$$

we obtain

$$\sum_{y\in\mathscr{S}} q(x,y)(y)_z = \sum_{z''\in\mathbb{Z}_+^m} \beta(z,z'')x^{z''}. \tag{4.6}$$

Note that $\beta(z,z'')\neq 0$ implies that $b(z,z'')\neq 0$ and thus $z\geq z''$. Using (4.1) we can also write

$$\beta(z,z'') = b(z,z'')n^{-||z''||} \sum_{\substack{\mathscr{A}+\mathscr{B}=\{1,\dots,n\}\\ z'\in\mathbb{Z}_+^m}} (n-||z'||)_{||z''||}(||z'||)_{z-z''} P_{\mathscr{A}}\pi(\mathscr{B},z'). \tag{4.7}$$

In particular, for $z=z''$ this yields

$$\begin{aligned}\beta(z,z) &= n^{-||z||} \sum_{\substack{\mathscr{A}+\mathscr{B}=\{1,\dots,n\}\\ z'\in\mathbb{Z}_+^m}} (n-||z'||)_{||z||} P_{\mathscr{A}}\pi(\mathscr{B},z')\\ &= n^{-||z||} \sum_{\mathscr{A}+\mathscr{B}=\{1,\dots,n\}} (\text{card}\,\mathscr{A})_{||z||} P_{\mathscr{A}} \sum_{z'} \pi(\mathscr{B},z')\\ &= n^{-||z||} m_{||z||}.\end{aligned} \tag{4.8}$$

In (4.8) we used that $b(z,z)=1$, that $\pi(\mathscr{B},z')\neq 0$ entails $||z'|| = \text{card}\,\mathscr{B}$ and that $\sum_{z'}\pi(\mathscr{B},z')=1$. Now define $\gamma(z,z')$ by

$$\gamma(z,z') = \sum_{z''\in\mathbb{Z}_+^m} \beta(z,z'')s(z'',z'). \tag{4.9}$$

The summands in (4.9) are not zero only for values of z'' satisfying $z'\leq z''\leq z$. In particular, $\gamma(z,z')\neq 0$ implies that $z'\leq z$. For $z'=z$ we find that

$$\gamma(z,z) = \beta(z,z)s(z,z) = \beta(z,z) = n^{-||z||}m_{||z||}.$$

By (4.4), it can now be seen that

$$\begin{aligned}\sum_{z''\in\mathbb{Z}_+^m} \beta(z,z'')x^{z''} &= \sum_{z''\in\mathbb{Z}_+^m} \beta(z,z'') \sum_{z'\in\mathbb{Z}_+^m} s(z'',z')(x)_{z'}\\ &= \sum_{z'\in\mathbb{Z}_+^m} \gamma(z,z')(x)_{z'}.\end{aligned} \tag{4.10}$$

It follows from (4.6) and (4.10) that

$$\sum_{y\in\mathscr{S}} q(x,y)(y)_z = \sum_{z'\in\mathbb{Z}_+^m} \gamma(z,z')(x)_{z'}, \quad x\in\mathscr{S},\ z\in\mathbb{Z}_+^m. \tag{4.11}$$

On the right-hand side of (4.11) we need only sum over $z'\leq z$, as $\gamma(z,z')=0$ otherwise.

Relation (4.11) can also be written in vector form as

$$Qv^{(z)} = n^{-||z||}m_{||z||}v^{(z)} + \sum_{||z'||<||z||} \gamma(z,z')v^{(z')}, \qquad z\in\mathbb{Z}_+^m. \tag{4.12}$$

Thus, if we consider Q as a linear mapping from $\mathscr{V}$ to itself and denote by Id the identity mapping on $\mathscr{V}$ (and also the corresponding matrix), we obtain

$$(Q - n^{-||z||} m_{||z||} \mathrm{Id}) v^{(z)} = \sum_{||z'|| < ||z||} \gamma(z, z') v^{(z')}. \tag{4.13}$$

For $j = 0, \ldots, n$, let $\mathscr{V}_j \subset \mathscr{V}$ be the linear subspace having B_j as its basis. Using Lemma 3 and (4.13) we have the following:

1. $\mathscr{V}_0 \subset \mathscr{V}_1 \subset \cdots \subset \mathscr{V}_n = \mathscr{V}$;
2. $\dim \mathscr{V}_j = \binom{m+j-1}{j}$, $j = 0, \ldots, n$;
3. the linear mapping $g_j = (Q - n^{-j} m_j \,\mathrm{Id})\,|_{\mathscr{V}_j}$ satisfies $\mathrm{Im}\, g_j \subset \mathscr{V}_{j-1}$, $j = 1, \ldots, n$.

From the dimension formula

$$\dim \ker g_j + \dim \mathrm{Im}\, g_j = \dim \mathscr{V}_j = \binom{m+j-1}{j}$$

and the inequality

$$\dim \mathrm{Im}\, g_j \le \dim \mathscr{V}_{j-1} = \binom{m+j-2}{j-1},$$

we find that

$$\dim \ker g_j \ge \binom{m+j-1}{j} - \binom{m+j-2}{j-1} = \binom{m+j-2}{j}.$$

Set $\mathscr{W}_j = \ker((Q - n^{-j} m_j \,\mathrm{Id})|_{\mathscr{V}})$. Let us first suppose that all values $n^{-j} m_j$, $j = 0, \ldots, n$, are distinct. Then the $\mathscr{W}_j$ are eigenspaces corresponding to different eigenvalues of Q and thus, since $\dim \mathscr{W}_j \ge \dim \ker g_j$,

$$\binom{m+n-1}{n} = \dim \mathscr{V} \ge \sum_{j=0}^{n} \dim \mathscr{W}_j \ge \sum_{j=0}^{n} \binom{m+j-2}{j} = \binom{m+n-1}{n}.$$

It follows that $\dim \mathscr{W}_j = \binom{m+j-2}{j}$, $j = 0, \ldots, n$, and that these dimensions add up to $\dim \mathscr{V}$. The theorem is proved, provided that all $n^{-j} m_j$ are distinct.

It thus remains to consider the case that some $n^{-j} m_j$ are equal. We have seen in Lemma 2 that this can only happen if either all λ_j are 1 or some of them are 0.

If $\lambda_j = 1$ for $j = 0, \ldots, n$, then $\zeta \equiv n$, $m_j = (n)_j$ and

$$1 = m_0 = n^{-1} m_1 > n^{-2} m_2 > \cdots > n^{-n} m_n.$$

It can easily be checked that in this case

$$Q v^{(e_\nu)} = v^{(e_\nu)}, \qquad \nu = 1, \ldots, m.$$

Since $B_1 = \{ v^{(e_\nu)} \mid \nu = 1, \ldots, m \}$ is a basis of $\mathscr{V}_1$, $Q|\mathscr{V}_1$ is the identity mapping and $1 = m_0 = n^{-1} m_1$ is an eigenvalue of Q having multiplicity $m = \binom{m+0-2}{0} + \binom{m+1-2}{1}$, as claimed.

Finally, assume that some λ_i are equal to 0. Let $\tilde{\mathscr{A}} = \{i \in \{1, \dots, n\} \mid \lambda_i \neq 0\}$, $k = \operatorname{card} \tilde{\mathscr{A}}$. Then $m_k \neq 0 = m_j$ for $j > k$.

We have $\beta(z, z'') = 0$ for $\|z''\| > k$. To see this, consider the sum on the right-hand side of (4.7) and assume $\|z''\| > k$. For a summand corresponding to z' not to be zero, one must have $\operatorname{card} \mathscr{B} = \|z'\|$ (for otherwise $\pi(\mathscr{B}, z') = 0$) and $n - \|z'\| \geq \|z''\|$ (for otherwise $(n - \|z'\|)_{\|z''\|} = 0$). If $\|z''\| > k$, it follows from these relations that

$$\operatorname{card} \mathscr{A} = n - \operatorname{card} \mathscr{B} = n - \|z'\| \geq \|z''\| > k;$$

but obviously $\mathscr{A} \subset \tilde{\mathscr{A}}$, so that $\operatorname{card} \mathscr{A} \leq k$. Hence $\beta(z, z'') = 0$ for $\|z''\| > k$.

By (4.6),

$$\sum_{y \in \mathscr{S}} q(x, y)(y)_z = \sum_{\|z'\| \leq k} \beta(z, z'') x^{z''}.$$

Thus $Q(\mathscr{V}) \subset \mathscr{V}_k$. One can now argue as before, replacing the chain $\mathscr{V}_0 \subset \cdots \subset \mathscr{V}_n = \mathscr{V}$ of vector spaces by $\mathscr{V}_0 \subset \cdots \subset \mathscr{V}_k \subset \mathscr{V}$. The theorem is proved. □

5. The limiting distribution. As usual, diagonalization is a useful tool for computing the t-step transition probabilities, that is, the elements of Q^t, because the representation $Q = ADA^{-1}$ derived above implies that

$$Q^t = AD^tA^{-1}, \qquad t \in \mathbb{N}.$$

Next note that, except in the case that all λ_i are equal to 1, Q has 1 as an eigenvalue of multiplicity 1 so that the Markov chain is ergodic. Its stationary distribution can be calculated by carrying out some simple matrix multiplications (without any matrix inversions).

PROPOSITION 2. *Let $\lambda_i < 1$ for some $i \in \{1, \dots, n\}$. Then the matrix*

$$\prod_{j=1}^{n} ((Q - \mu_j \operatorname{Id})/(1 - \mu_j))$$

has identical rows which are all equal to the uniquely determined stationary distribution $\pi = (\pi_x)_{x \in \mathscr{S}}$ associated with Q.

PROOF. From $Q = ADA^{-1}$ we conclude that

$$Q - \mu_j \operatorname{Id} = A(D - \mu_j \operatorname{Id})A^{-1}, \qquad j = 0, \dots, n. \tag{5.1}$$

Define $R = \prod_{j=1}^{n}(Q - \mu_j \operatorname{Id})$. Multiplying equations (5.1), we obtain

$$\begin{aligned} R(Q - \operatorname{Id}) &= \prod_{j=0}^{n} (Q - \mu_j \operatorname{Id}) \\ &= A\left(\prod_{j=0}^{n} (D - \mu_j \operatorname{Id})\right)A^{-1} \\ &= 0, \end{aligned} \tag{5.2}$$

where 0 denotes the zero matrix. (Note that $\prod_{j=0}^{n}(D-\mu_j \,\mathrm{Id})$ is a product of diagonal matrices in which for every diagonal entry at least one of the matrices contributes the factor zero, so that the product is the zero matrix). By (5.2), if $\rho = (\rho_x)_{x\in\mathscr{S}}$ is an arbitrary row of R, then

$$\rho(Q-\mathrm{Id}) = 0.$$

No row of R is zero. Indeed, the definition of R yields

$$R = Q^n - Q^{n-1}\sum_{j=1}^{n}\mu_j + Q^{n-2}\sum_{j_1<j_2}\mu_{j_1}\mu_{j_2} - \cdots + (-1)^n\mu_1\cdots\mu_n \,\mathrm{Id} \tag{5.3}$$

and by (5.3) each row of R has the sum

$$1-\sum_{j=1}^{n}\mu_j + \sum_{j_1<j_2}\mu_{j_1}\mu_{j_2} - \cdots + (-1)^n\mu_1\cdots\mu_n = \prod_{j=1}^{n}(1-\mu_j) \neq 0;$$

here we of course use that Q is a stochastic matrix. Hence, for any row ρ of R the row vector $\tilde{\rho} = (\prod_{j=1}^{n}(1-\mu_j))^{-1}\rho$ satisfies $\tilde{\rho}Q = \tilde{\rho}$ and $\|\tilde{\rho}\| = 1$. These equations uniquely determine the stationary distribution belonging to Q.

Next we compute the expected number φ_ν of individuals in group ν under a stationary regime. Clearly, φ_ν is given by

$$\varphi_\nu = \sum_{x\in\mathscr{S}} x_\nu \pi_x, \qquad \nu = 1,\dots,m$$

and is also the limit of this expected number under any initial distribution on $\mathscr{S}$. We will now show that φ_ν/n is equal to the ratio of the expected number of nonconformists in group ν and the expected total number of nonconformists. □

PROPOSITION 3. *If at least one λ_i is smaller than* 1, *then*

$$\varphi_\nu = n\sum_{i=1}^{n}(1-\lambda_i)p_{i\nu}\Big/\left(n-\sum_{i=1}^{n}\lambda_i\right). \tag{5.4}$$

PROOF. Using stationarity and the relations (4.6) and (4.8) we find that

$$\begin{aligned}
\varphi_\nu &= \sum_{x\in\mathscr{S}} x_\nu \sum_{y\in\mathscr{S}} \pi_y q(y,x)\\
&= \sum_{x\in\mathscr{S}} (x)_{e_\nu^{\mathsf{T}}} \sum_{y\in\mathscr{S}} \pi_y q(y,x)\\
&= \sum_{y\in\mathscr{S}} \pi_y \sum_{x\in\mathscr{S}} (x)_{e_\nu^{\mathsf{T}}} q(y,x)\\
&= \sum_{y\in\mathscr{S}} \pi_y \sum_{\substack{z\in\mathbb{Z}_+^m\\ z\le e_\nu^{\mathsf{T}}}} \beta(e_\nu^{\mathsf{T}}, z) y^z
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{y \in \mathscr{S}} \pi_y (\beta(e_\nu^{\mathsf{T}}, 0) y^0 + \beta(e_\nu^{\mathsf{T}}, e_\nu^{\mathsf{T}}) y^{e_\nu^{\mathsf{T}}}) \\
&= \sum_{y \in \mathscr{S}} \pi_y (\beta(e_\nu^{\mathsf{T}}, 0) + \beta(e_\nu^{\mathsf{T}}, e_\nu^{\mathsf{T}}) y_\nu) \\
&= n^{-1} m_1 \varphi_\nu + \beta(e_\nu^{\mathsf{T}}, 0),
\end{aligned}
$$

where 0 denotes the zero vector in $\mathscr{S}$. Thus, $\varphi_\nu(1 - n^{-1} m_1) = \beta(e_\nu^{\mathsf{T}}, 0)$ or

$$
\varphi_\nu = \frac{n}{n - \sum_{i=1}^n \lambda_i} \beta(e_\nu^{\mathsf{T}}, 0). \tag{5.5}
$$

It remains to determine $\beta(e_\nu^{\mathsf{T}}, 0)$. By (4.7), we can write

$$
\begin{aligned}
\beta(e_\nu^{\mathsf{T}}, 0) &= \sum_{\substack{\mathscr{A} + \mathscr{B} = \{1, \ldots, n\} \\ z \in \mathbb{Z}_+^m}} (z)_{e_\nu^{\mathsf{T}}} P_{\mathscr{A}} \pi(\mathscr{B}, z) \\
&= \sum_{\mathscr{A} + \mathscr{B} = \{1, \ldots, n\}} P_{\mathscr{A}} \left(\sum_{z \in \mathbb{Z}_+^m} z_\nu \pi(\mathscr{B}, z) \right).
\end{aligned}
$$

The inner sum on the right-hand side is the conditional expected number of nonconformists going to group ν, given that $\mathscr{A}$ is the set of conformists at that stage. Therefore, $\beta(e_\nu^{\mathsf{T}}, 0)$ is the (unconditional) expected number of nonconformists going to group ν at an arbitrarily chosen stage. Hence, $\beta(e_\nu^{\mathsf{T}}, 0)$ is also equal to the sum over $(1 - \lambda_i) p_{i\nu}$, $i = 1, \ldots, n$, since $(1 - \lambda_i) p_{i\nu}$ is the probability that individual i becomes a nonconformist and joins group ν. It follows that

$$
\beta(e_\nu^{\mathsf{T}}, 0) = \sum_{i=1}^n (1 - \lambda_i) p_{i\nu}. \tag{5.6}
$$

Equation (5.4) follows from (5.5) and (5.6). □

6. Absorption in the finite memory model. Let us now consider the second model, which is characterized by the equation

$$
P(\xi_{t+1}^{(i)} = e_\nu \mid \mathscr{F}_t) = \sum_{j=1}^n \beta_{i,j} \sum_{l=1}^c \gamma_l \xi_{t-l+1}(j, \nu), \qquad i = 1, \ldots, n, \; \nu = 1, \ldots, m, \; t = 0, 1, 2, \ldots. \tag{6.1}
$$

According to (6.1), the decision of individual i at time $t + 1$ depends on the previous c decisions of all members of the population. The weight of individual j is given by $\beta_{i,j}$, where $\beta_{i,j} > 0$ and $\sum_{j=1}^n \beta_{i,j} = 1$, and the influence of the last time units is represented by the γ_l, where $\gamma_l \geq 0$, $\gamma_c > 0$, $\sum_{l=1}^c \gamma_l = 1$. Since all $\beta_{i,j}$ are positive, it is clear that all individuals will eventually belong to the same group. For fixed initial allocations $\xi_0, \dots, \xi_{c-1}$ let $\pi_\nu(\xi_0, \dots, \xi_{c-1})$ be the probability that $\xi_t^{(i)} = e_\nu$ for $i = 1, \dots, n$ and all sufficiently large t. The aim of this section is to derive a simple formula for this absorption probability.

THEOREM 2. *Let* $\Gamma_l = \sum_{h=l}^{c} \gamma_h$. *If* $(\sigma_i)_{i=1,\dots,n}$ *denotes the stationary distribution of the stochastic matrix* $(\beta_{i,j})_{i,j=1,\dots,n}$, *then*

$$\pi_\nu(\xi_0, \dots, \xi_{c-1}) = \sum_{l=1}^{c} \left(\Gamma_l \Big/ \sum_{h'=1}^{c} \Gamma_{h'} \right) \sum_{i:\xi_{c-l}^{(i)} = e_\nu} \sigma_i. \tag{6.2}$$

Thus, to compute $\pi_\nu(\xi_0, \dots, \xi_{c-1})$ we have to determine, for any $l \in \{1, \dots, c\}$, those individuals i who belong to group ν at time $c - l$ and then add the stationary probabilities σ_i corresponding to these individuals; the absorption probability for group ν is then a special weighted average of these c values.

PROOF OF THEOREM 2. We fix the values $\xi_0, \dots, \xi_{c-1}$ and ν. As in the proposition proved in the Introduction, it is easily concluded from (6.1) that the sequence

$$Z_t = Z_t^{(\nu)} = \sum_{i=1}^{n} \sigma_i 1_{\{\xi_t^{(i)} = e_\nu\}}, \qquad t \in \mathbb{Z}_+ \tag{6.3}$$

satisfies

$$E(Z_{t+1} \mid \mathcal{F}_t) = \sum_{l=1}^{c} \gamma_l Z_{t-l+1}. \tag{6.4}$$

Since eventual absorption is almost certain, $Z = \lim_{t\to\infty} Z_t$ almost surely exists and takes values in $\{0, 1\}$. Obviously,

$$\pi_\nu(\xi_0, \dots, \xi_{c-1}) = E(Z). \tag{6.5}$$

Define

$$Y_t = \sum_{l=1}^{c} \Gamma_l Z_{t-l+1}, \qquad t \ge c - 1. \tag{6.6}$$

Then for every $t \ge c - 1$, (6.4) implies that

$$\begin{aligned} E(Y_{t+1} \mid Y_{c-1}, \dots, Y_t) &= E(E(Y_{t+1} \mid \mathcal{F}_t) \mid Y_{c-1}, \dots, Y_t) \\ &= E\left(E(\Gamma_1 Z_{t+1} \mid \mathcal{F}_t) + \sum_{l=2}^{c} \Gamma_l Z_{t-l+2} \mid Y_{c-1}, \dots, Y_t \right) \\ &= E\left(\sum_{l=1}^{c} \gamma_l Z_{t-l+1} + \sum_{l=2}^{c} \Gamma_l Z_{t-l+2} \mid Y_{c-1}, \dots, Y_t \right) \\ &= E\left(\sum_{l=1}^{c} \Gamma_l Z_{t-l+1} \mid Y_{c-1}, \dots, Y_t \right) \\ &= Y_t. \end{aligned} \tag{6.7}$$

By (6.7), Y_t is a martingale. As $0 \le Y_t \le c$, the sequence Y_t is uniformly integrable; in particular, its limit $Y = \lim_{t\to\infty} Y_t$ [which exists almost surely by the martingale convergence theorem or simply by definition (6.6), since Z_t is a.s. convergent] satisfies $E(Y) = E(Y_{c-1})$. If we set $\Gamma = \sum_{l=1}^{c} \Gamma_l$, then passing to the limit in (6.6) shows that

$$Y = \Gamma Z,$$

so that

$$E(Z) = \Gamma^{-1}E(Y) = \Gamma^{-1}E(Y_{c-1}) = \Gamma^{-1}\sum_{l=1}^{c}\Gamma_l Z_{c-l}. \tag{6.8}$$

Inserting (6.8) in (6.5) and using the definition (6.3) of Z_{c-l} yields (6.2).

7. Examples.

Example 1. To illustrate Proposition 1, consider an ordered population $1, \ldots, n$ in which every individual i, in order to make his next move, takes into account only his own current group and the current decisions of his immediate neighbors. Thus, the coefficients $\beta_{i,j}$ in (1.1) form a stochastic Jacobi matrix:

$$(\beta_{i,j})_{i,j=1,\ldots,n} = \begin{pmatrix} q_1 & r_1 & 0 & & \cdots & & 0 \\ p_2 & q_2 & r_2 & 0 & & & 0 \\ 0 & & & & & & \vdots \\ \vdots & & & & & & 0 \\ 0 & \ldots & & & p_{n-1} & q_{n-1} & r_{n-1} \\ 0 & \ldots & & & 0 & p_n & q_n \end{pmatrix}.$$

Assume that all r_i are positive. Then the stationary distribution corresponding to $(\beta_{i,j})$ is given by

$$\sigma_i = \prod_{l=i}^{n-1}(p_{l+1}/r_l)\Big/\Big(1 + \sum_{k=1}^{n-1}\prod_{h=k}^{n-1}(p_{h+1}/r_h)\Big), \qquad i = 1, \ldots, n$$

(with the empty product defined to be 1). The probability of eventual absorption of the entire population in group ν is the sum of those σ_i for which individual i is initially in group ν.

Example 2. Next, let the group affiliations be governed by (1.2) and assume that the population entirely consists of (pure) nonconformists and conformists; that is, there is a $k \in \{0, \ldots, n\}$ such that $\lambda_i = 0$ for $i \le k$ and $\lambda_i = 1$ for $i > k$. The eigenvalues are then given by $\mu_0 = 1$ and $\mu_j = n^{-j}(n-k)_j$, $j = 1, \ldots, n$.

CASE 1 ($k \geq 1$). Then there is at least one nonconformist. The profile process is asymptotically stationary, since the matrix Q has 1 as a simple eigenvalue and all its other eigenvalues are in the interval $(0, 1)$. The component π_x of the stationary distribution π gives the long-run relative frequency of visits of the profile x. Let G be the set of groups that are occasionally chosen by nonconformists, that is, $G = \{\nu \mid p_{i\nu} > 0$ for some $i \leq k\}$, and set $\mathcal{T} = \{x \in \mathcal{S} \mid x_\nu = 0$ for all $\nu \notin G\}$. Once a profile $x \in \mathcal{T}$ is reached, the set $\mathcal{T}$ is never left again, as nonconformists will never go to any $\nu \notin G$ and conformists will not do so when starting from a profile in $\mathcal{T}$. By Proposition 3, the limiting (or stationary) expected fraction of individuals in group ν is given by

$$n^{-1}\varphi_\nu = k^{-1}\sum_{i=1}^{k} p_{i\nu}, \qquad \nu \in \{1, \dots, m\}.$$

Note that $\sum_{i=1}^{k} p_{i\nu}$ is the expected number of nonconformists going to group ν at an arbitrary fixed time instant.

As a very special case, suppose that $k = 1$ so that individual 1 is the only nonconformist. Then the set of groups ν with $p_{1\nu} > 0$ is absorbing. In particular, if $p_{1\nu} = 1$ for some group ν_1, all individuals will eventually end up in ν_1; the one nonconformist will have his way.

CASE 2 ($k = 0$). Then all individuals are conformists. We saw in the proof of Theorem 1 that $B_1 = \{v^{(e_\nu)} \mid \nu = 1, \dots, m\}$ is a basis of the eigenspace of Q corresponding to the eigenvalue 1. Recalling (2.3) we note that $v^{(e_\nu)} = (v_x^{(e_\nu)})_{x\in\mathcal{S}} = (x_\nu)_{x\in\mathcal{S}}$.

In this case $\mathcal{S}$ contains m absorbing "unanimous" profiles $x^{(1)} = (n, 0, \dots, 0)^{\mathsf{T}}, \dots, x^{(m)} = (0, \dots, 0, n)^{\mathsf{T}}$ [$x^{(\nu)}$ denotes the profile in which all individuals have chosen group ν], while the other $\binom{n+m-1}{m-1} - m$ profiles are transient. The limiting distribution on $\{x^{(1)}, \dots, x^{(m)}\}$ depends of course on the initial profile, say $x^{(0)} = (x_1^{(0)}, \dots, x_m^{(0)})^{\mathsf{T}} \in \mathcal{S}$. By Proposition 1 (in the special case $\beta_{i,j} = 1/n$), the probability of ultimate absorption in group ν, that is, in $x^{(\nu)}$, is given by $x_\nu^{(0)}/n$. This rather intuitive result can also be proved algebraically as follows. First order $\mathcal{S}$ such that the m "unanimous" profiles $x^{(1)}, \dots, x^{(m)}$ come first. Then extend the set $w^{(1)} = n^{-1}v^{(e_1)}, \dots, w^{(m)} = n^{-1}v^{(e_m)}$, which is a basis of the eigenspace for the eigenvalue 1, to a basis $w^{(1)}, \dots, w^{(s)}$ of eigenvectors of Q (here $s = \operatorname{card}\mathcal{S}$). Let A be the matrix $(w^{(1)}, \dots, w^{(s)})$. Then $Q = ADA^{-1}$, where D is the diagonal matrix of eigenvalues (suitably ordered). Note that A is of the form

$$A = (A_{x,y})_{x,y\in\mathcal{S}} = \begin{pmatrix} \mathrm{Id}_m & A' \\ A'' & A''' \end{pmatrix}$$

with Id_m denoting the $(m \times m)$-identity matrix. By the definition of the $w^{(\nu)}$, the submatrix A'' has the entries $A_{x,x^{(\nu)}} = x_\nu/n,\ x = (x_1, \dots, x_m)^{\mathsf{T}} \in \mathcal{S}\backslash\{x^{1)}, \dots, x^{(m)}\}$; that is, $A_{x,x^{(\nu)}}$ is the fraction of individuals in group ν for the profile x. Clearly, $\lim_{t\to\infty} Q^t = \begin{pmatrix} \mathrm{Id}_m & 0 \\ A'' & 0 \end{pmatrix}$. Thus, if the process starts from the profile $x^{(0)}$, the probability of ending unanimously in group ν equals $x_\nu^{(0)}/n$.

EXAMPLE 3. Let us finally consider a numerical example. Take $n = 10$ individuals and $m = 3$ groups and assume that there are four conformists and six nonconformists of whom one always stays in group 1, two stay in group 2 and three in group 3. There are 15 possible profiles:

(5,2,3), (4,3,3), (3,4,3), (2,5,3), (1,6,3), (4,2,4), (3,3,4), (2,4,4), (1,5,4), (1,2,5), (2,3,5), (1,4,5), (2,2,6), (1,3,6), (1,2,7).

The remaining $\binom{10+3-1}{3-1} - 15 = 51$ profiles in $\mathscr{S}$ cannot be attained because of the individuals who do not change groups.

The exact stationary probabilities of the attainable profiles (in the above order) are easily computed using *Mathematica* for the matrix multiplications in Proposition 2:

$$\frac{31167}{13387792}, \frac{48139}{5020422}, \frac{136343}{6693896}, \frac{277661}{10040844}, \frac{49985}{2510211}, \frac{48139}{3346948}, \frac{181383}{3346948}, \frac{261}{2684}, \frac{277661}{3346948},$$
$$\frac{147603}{3346948}, \frac{469551}{3346948}, \frac{1059963}{6693896}, \frac{2275}{27434}, \frac{2275}{13717}, \frac{17525}{219472}.$$

Note that the three profiles (1,3,6), (1,4,5) and (2,3,5) are by far the most likely, having probabilities 0.1659, 0.1583 and 0.1403, respectively. Every other profile has a stationary probability of less than 0.0973.

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FACHBEREICH MATHEMATIK/INFORMATIK
UNIVERSITÄT OSNABRÜCK
FB 6, D-49069 OSNABRÜCK
GERMANY
E-MAIL: wolfgang@mathematik.uni-osnabrueck.de