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STABILITY AND NONPRODUCT FORM OF STOCHASTIC FLUID NETWORKS WITH LÉVY INPUTS[1]

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We consider a stochastic fluid network with inputs which are independent subordinators. We show that under some natural conditions the distribution of the fluid content process converges in total variation to a proper limit and that the limiting distribution has a positive mass at the origin. As a consequence of the methodology used, we obtain upper and lower bounds for the expected values of this limiting distribution. For the two-dimensional case, under certain conditions, explicit formulas for the means, variances and covariance of the steady-state fluid content are given. Further, for the two-dimensional case, it is shown that, other than for trivial setups, the limiting distribution cannot have product form.

1. Introduction. In Kaspi and Kella (1996) we have studied a feedforward fluid network with independent Lévy inputs and have shown that, under natural stability conditions, the network is stable, in the strong sense that, for every initial condition, the distribution of the fluid content vector converges in total variation to some limiting distribution which is independent of initial conditions. Furthermore, we showed that, other than for trivial setups, the limiting distribution never has product form, which contrasts the situation in certain queueing and Brownian networks. The feedforward assumption allowed us to give an inductive proof for both results. In this paper the goal is to extend these results for the case of a general network in which we allow feedback. In this case, the inductive proof developed in Kaspi and Kella (1996) does not apply and we need to seek other methods. Indeed, both results hold in the case under study and, in particular, for the two-dimensional case product form holds only in situations where either there is no flow between the stations or from some point on at least one of the stations is always empty (in particular, there is no external flow into that station). Unlike in Kaspi and Kella (1996), where for the major step in the nonproduct form proof one has an explicit representation of all but one steady-state marginal, here we do not know any of them. Also, here we are able to show an additional feature, which is that the steady-state distribution has a positive mass at the origin, which therefore implies that the process is strictly regenerative. This remained on open problem in Kaspi and Kella (1996).

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Recently, tremendous attention has been devoted to both stochastic and deterministic fluid networks as well as the stability of both queueing and Brownian networks. Regarding the stability of queueing networks, there is an immense developing literature. Since this is not the focus of this article, we will suffice by mentioning Dai (1994), Meyn and Down (1994) and Baccelli and Foss (1994). Each of these three papers has an extensive reference list to which the reader is referred for earlier and current (some yet unpublished) works on the subject. In particular, Dai (1994) shows that, for a multiclass queueing network with i.i.d. interarrival and service times within each class, the network is positive Harris recurrent if it has deterministic fluid limits which ultimately reach 0 from any initial condition. Meyn and Down (1994) consider the more restrictive single-class queueing network, but with different, in some respects more general assumptions. Baccelli and Foss (1994) also consider the single-class Jackson-type queueing network, but with weaker assumptions, more precisely, weakening the i.i.d. assumptions to stationary and ergodic. They show that, under the natural conditions discussed in previous papers, the network is stable (in the ergodic sense). As for Brownian networks, the most recent work is by Dupuis and Williams (1994), who show that semimartingale (multivariate) reflecting Brownian motions are stable if any solution of a related Skorohod problem is attracted to the origin for any initial condition. In particular, for reflected Brownian motions which are weak limits of single-class Jackson-type queueing networks, there is a unique solution for the related Skorohod problem which is a deterministic fluid network. This particular model is studied in Harrison and Williams (1987) and, as it turns out (embarrassingly discovered after the fact), their idea of bounding the network by a simpler network [see the proof of Lemma (12) there] can be used here together with some simple monotonicity results (Lemma 3.1 here) and other facts in order to provide the total variation convergence to a limiting distribution which has a positive mass at the origin (as well as some other results) for the Lévy model considered in the current paper. Their idea was used in order to obtain an exponential tail bound for (hence to show nondegeneracy of) the limiting distribution. Here this bounding idea leads to other facts, in particular, strict regeneration and positive mass at the origin, which do not hold in the Brownian case. It should also be pointed out here that recently Chang, Thomas and Kiang (1994) also used a related (but different) method of bounding a single-class Jackson queueing network by a simpler network in order to achieve stability.

The main tool in Dai (1994) and related papers is in establishing that if a Markov process, after an appropriate rescaling, obeys a functional strong law to a deterministic fluid network which is attracted to the origin from any initial condition, and if, for the original Markov process, compact sets which contain the origin are *small*, then stability (in the sense of positive Harris recurrence) is implied. Therefore, it might be argued that the only remaining problem for a given process obeying such a functional strong law is in showing that indeed such compact sets are small. In order to obtain this, an assumption which has become standard in queueing literature is that interarrival times (for the various classes) have spread out distributions with unbounded support. This ensures the accessibility of the origin and the nonarithmetic nature of the process, which is a sufficient condition for compact sets containing the origin being small. We note that the model we are considering here does indeed satisfy the desired functional strong law. Also, for the special case of compound Poisson inputs, the origin is clearly accessible and arrivals have an exponential (hence spread out) distribution. Therefore, for this case, existing results can be used to establish stability. However, we are not restricting ourselves to this setup and we emphasize that we also include inputs that may have an infinite number of jumps in any finite interval. In this case, it is impossible to talk about *interarrival distribution*, and accessibility of the origin is far from being obvious. In fact, we do show that not only is the origin accessible, but actually the limiting distribution has a positive atom there (which is a stronger statement). However, the method which is used here in showing this immediately implies stability in the strong sense of convergence in total variation for every initial condition. Hence, the method referred to above is of no direct use to the setup here and does not imply any extra results which are not obtained anyway.

In Meyn and Down (1994), which deals with a single-class queueing network, some of these difficulties are circumvented by assuming that service times have finite second moments (as well as some linearity condition for the arrival process). Also an inductive proof is given despite the fact that the network is not of feedforward type. We emphasize that for the purpose of stability we do not make any second moment assumptions here. The only places where such assumptions are introduced are in Theorem 3.2 and Section 4 where we consider moments of the stationary distribution of the workloads.

A stochastic fluid network, defined here (see Section 2) as a network in which the net input process is the difference between a nondecreasing process (input) and a deterministic linear fluid flow (output), is a much younger and less developed area than its queueing and Brownian counterparts. Nevertheless, the model itself is of significant importance, as it serves as a direct model of phenomena previously studied via queueing models or Brownian network approximations. In general, this model approximates any situation in which the material flowing in the network can be approximated as fluid, the only source of randomness is in the input, and the servers or machines at the stations are reliable to the extent that output can be considered linear and deterministic. In particular, these networks approximate any model which can be described by a queueing network with batch arrivals and deterministic service times with lengths which are "small" compared to the arrival rates times the batch size. Two obvious examples are the flow of (bits of) information in high-speed communications networks, and production networks in which the output of material from machines can be approximated (or directly modeled) as a continuous flow, such as in the oil, food and chemical industries. More motivation is discussed in earlier papers. We emphasize here that, unlike models in the recent queueing literature which concentrate on multiclass networks, the model in this paper is strictly single class. Multiclass analogs are yet to be studied.

As for earlier work, the tandem case under certain conditions was considered in Kella and Whitt (1992b) for the case where there is an external compound Poisson input only to the first station, while Kella (1993) studied both parallel and tandem networks, where, in the tandem case, the various stations (not only the first) have independent general subordinator inputs (rather than the more restrictive compound Poisson). Recently, Kella and Whitt (1996) established certain structural properties and tightness for stochastic fluid (and more general) networks with inputs which have stationary, but not necessarily independent, increments. Finally, as mentioned previously, Kaspi and Kella (1996) relates most strongly in flavor to the current paper, but the methods are different.

For earlier works on fluid and related storage models, see Gaver and Miller (1962), Miller (1963), Meyer, Rothkopf and Smith (1979, 1983), Newell (1982), Anick, Mitra and Sondhi (1982), Mitra (1988), Chen and Mandelbaum (1991), Chen and Yao (1992), Kella and Whitt (1992a, c) and the references therein.

The paper is organized as follows. In Section 2 we give the main setup. In Section 3 we state and prove the main stability result and give upper and lower bounds for certain linear combinations of the means of the steady-state buffer contents. In Section 4 we consider the special case of a two-dimensional network with an added assumption which allows us to reduce the model to a tandem fluid network which has the conditions imposed in Kella (1993) and to provide explicit expressions for means, variances and covariance. Finally, in Section 5 we establish the nonproduct form result for a two-dimensional network.

2. The model. Throughout this paper we let $\mathscr{R}$ be the real line and $\mathscr{R}_+^n$ be the nonnegative orthant. For every real $x_1, \dots, x_n$ we write $x = (x_1, \dots, x_n)'$, where $'$ denotes both transposition as well as the first derivative of a function. Which case it is will be clear from the context. We use $a \vee b \equiv \max(a, b)$, $a \wedge b \equiv \min(a, b)$, $a^+ \equiv a \vee 0$ and $a^- \equiv a \wedge 0$. Also, for $x, y \in \mathscr{R}^n$ we write $x \le y$ to mean $y - x \in \mathscr{R}_+^n$ (the usual partial order on $\mathscr{R}^n$). As usual, we use a.s. for *almost surely* (where the measure will be clear from the context) and w.l.o.g. for *without loss of generality* and denote by 1_A the indicator of a set A. As customary, a process $X = \{X(t) \mid t \ge 0\}$ will be called tight if the distribution measures $\{P[X(t) \in \cdot] \mid t \ge 0\}$ are right. Finally, we use $Y(\infty)$ to denote a random vector having the steady-state distribution of a process Y whenever it exists.

We model the inputs to the network by $J_i = \{J_i(t) \mid t \ge 0\}$, $i = 1, \dots, n$, which are right-continuous subordinators (nondecreasing Lévy processes, i.e., having stationary independent increments) with exponents $\eta_i(\alpha_i) = -\log E \exp(-\alpha_i J_i(1))$, where $\alpha_i \ge 0$. It is well known that the general form of such an exponent is

$$\eta_i(\alpha_i) = c_i\alpha_i + \int_{(0,\infty)} (1 - \exp(-\alpha_i x))\nu_i(dx), \tag{2.1}$$

where ν_i is the Lévy measure known to satisfy $\nu_i(1,\infty) < \infty$ and $\int_{(0,1]} x\nu_i(dx) < \infty$. For $i = 1,\dots,n$ let us denote

$$\rho_i = \int_{(0,\infty)} x\nu_i(dx) = \eta_i'(0) = EJ_i(1). \tag{2.2}$$

With strictly positive $r_1,\dots,r_n$ and a substochastic matrix $P = (p_{ij})$ with $P^n \to 0$ as $n \to \infty$ (spectral radius < 1), the potential output from node i is r_i and every unit of fluid processed by station i is distributed to the other stations according to the matrix P. That is, when there is a positive fluid level at station i, this station sends $r_i p_{ij}$ units per unit time to station j. This means that, when the fluid level at all stations is positive, the net outflow rate from station i is $r_i - \sum_{i\neq j} p_{ji} r_j$. Therefore, in light of this description, the model considered in this paper is the càdlàg strong Markov process

$$W(t) = W(0) + J(t) - (I - P')rt + (I - P')L(t), \tag{2.3}$$

where $W(0)$ is independent of J and L is the unique continuous nondecreasing (n-dimensional) process with $L(0) = 0$ having the property that $\int_0^\infty W_i(t)\,dL_i(t) = 0$ and known to be the minimal process for which $W(t) \geq 0$ for all $t \geq 0$.

We would like to study the steady-state properties of the process specified by (2.3) and, in particular, show that the following condition implies stability, in a strong sense, which will be made precise in Theorem 3.1 in the next section.

CONDITION 2.1. $(I - P')^{-1}\rho < r$.

Note that, in particular [as $(I - P)^{-1} = I + P + P^2 + \cdots$], Condition 2.1 implies that $\rho_i < \infty$ for $i = 1,\dots,n$. For simplicity and w.l.o.g., we will assume throughout that $c = 0$, so that J is a pure jump process. If $c \neq 0$, then we can consider $\tilde{J}(t) = J(t) - ct$, with $\tilde{\rho} = \rho - c$ and $\tilde{r} = r - (I - P')^{-1}c$. Clearly, L and W from (2.3) have not changed and Condition 2.1 is satisfied for the new process if and only if it was satisfied for the old one. One of the reasons for making this assumption is that for the two-dimensional case which is considered in Sections 4 and 5 we can write $L_i(t) = \int_0^t l_i(W(s))\,ds$ with

$$\begin{aligned} l_i(x) = {} & \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} 1_{\{x_1=0,\, x_2=0\}} + \begin{pmatrix} (r_1 - p_{21}r_2)^+ \\ 0 \end{pmatrix} 1_{\{x_1=0,\, x_2>0\}} \\ & + \begin{pmatrix} 0 \\ (r_2 - p_{12}r_1)^+ \end{pmatrix} 1_{\{x_1>0,\, x_2=0\}}. \end{aligned} \tag{2.4}$$

It is useful to note that, when $r_1 > p_{21}r_2$ and $r_2 > p_{12}r_1$,

$$(I - P')(r - l(x)) = \begin{pmatrix} (1 - p_{12}p_{21})r_1 \\ 0 \end{pmatrix} 1_{\{x_1 > 0,\ x_2 = 0\}} + \begin{pmatrix} 0 \\ (1 - p_{12}p_{21})r_2 \end{pmatrix} 1_{\{x_1 = 0,\ x_2 > 0\}} + \begin{pmatrix} r_1 - p_{21}r_2 \\ r_2 - p_{21}r_1 \end{pmatrix} 1_{\{x_1 > 0,\ x_2 > 0\}}. \tag{2.5}$$

Equation (2.4) will be referred to in Section 4 while (2.5) will be cited in Section 5.

3. Stability and bounds in the multidimensional case. In this section we will establish the stability of our n-dimensional network described by (2.3) under Condition 2.1. In particular, we will prove the following theorem. In the statement of the theorem we denote by $P_t(w, \cdot)$ the Markov kernel associated with $W(t)$ (starting from w). Also let π be the joint steady-state distribution (when it exists).

THEOREM 3.1. *For every $w \in \mathscr{R}_+^n$, $P_t(w, \cdot) \to \pi$ in total variation, where π is proper (nondegenerate) with $\pi(\{0\}) > 0$ if and only if Condition 2.1 holds. Furthermore, under Condition 2.1, the origin is accessible in finite (resp. finite expected) time for every initial condition that is a.s. finite (resp. has finite expectation).*

In preparation for the proof of the "if" part and in a similar manner as in the proof of Lemma (12) of Harrison and Williams (1987), let $\tilde{\rho} > \rho$ be such that $(I - P')^{-1}\tilde{\rho} < r$. By Condition 2.1 clearly a vector like this exists. Consider the process

$$\tilde{W}(t) = \tilde{W}(0) + J(t) - \tilde{\rho}t + \tilde{L}(t), \tag{3.1}$$

where $\tilde{W}(0)$ is independent of J and

$$\tilde{L}_i(t) = -\inf_{0 \le s \le t} \left[\tilde{W}_i(0) + J_i(s) - \tilde{\rho}_i s\right]^-, \qquad 1 \le i \le n. \tag{3.2}$$

When $W(0) = 0$ or $\tilde{W}(0) = 0$ we will emphasize this with the notation W^0, L^0, $\tilde{W}^0$ and $\tilde{L}^0$.

LEMMA 3.1. *If $W(0) \le \tilde{W}(0)$, then $(I - P')^{-1}W(t) \le (I - P')^{-1}\tilde{W}(t)$; hence $e'W(t) \le e'\tilde{W}(t)$ for every $t \ge 0$.*

PROOF. First assume that $\tilde{W}(0) = W(0)$. Then it is easy to see that $\tilde{W}(t) = W(0) + J(t) - (I - P')rt + (I - P')L^*(t)$, where

$$L^*(t) = (I - P')^{-1}\tilde{L}(t) + \left[r - (I - P')^{-1}\tilde{\rho}\right]t. \tag{3.3}$$

Note that L^* is clearly a nondecreasing process with $L^*(0) = 0$. Since L is the minimal such process for which $W(0) + J(t) - (I - P')rt + (I - P')L(t)$ is nonnegative for all $t \geq 0$, it follows that $L(t) \leq L^*(t)$ and hence $(I - P')^{-1}W(t) \leq (I - P')^{-1}\tilde{W}(t)$ for all $t \geq 0$. Then $e'W(t) \leq e'\tilde{W}(t)$ follows upon premultiplying both sides by $e'(I - P')$, which is nonnegative as P is substochastic. To see that this result holds when $\tilde{W}(0) \geq W(0)$, it is easy to check [see (3.2)] that $\tilde{W}_i(t)$ is monotonically increasing in $\tilde{W}_i(0)$ for each $i = 1, \dots, n$. $\square$

PROOF OF THEOREM 3.1. It suffices to assume (and we do) that $EW(0) < \infty$, since the case $W(0) < \infty$ a.s. can be considered as a special case, by conditioning on the value of $W(0)$ and therefore assuming that it is a finite constant. Now observe that, for each $1 \leq i \leq n$, $\tilde{W}_i(t)$ is a one-dimensional reflected Lévy process having no negative jumps. If we let

$$T_i = \inf\{t \mid \tilde{W}_i(0) + J_i(t) - \tilde{\rho}_i(t) = 0\}, \tag{3.4}$$

then it is well known [e.g., Kella and Whitt (1992a)] that $ET_i = E\tilde{W}_i(0)/(\tilde{\rho}_i - \rho_i) < \infty$ whenever $E\tilde{W}_i(0) < \infty$. It is easy to check that $\tilde{W}_i(t) = \tilde{W}_i^0(t)$ for $t \geq T_i$. Therefore, $\tilde{W}(t) = \tilde{W}^0(t)$ for $t \geq T \equiv \max(T_1, \dots, T_n)$, where clearly $ET \leq \sum_{i=1}^n ET_i < \infty$. Denote

$$\begin{aligned} \tau_0 &= \inf\{t \mid t \geq T, \tilde{W}^0(t) = 0\} \\ &= \inf\{t \mid t \geq T, \tilde{W}(t) = 0\} = \inf\{t \mid \tilde{W}(t) = 0\}. \end{aligned} \tag{3.5}$$

From standard regenerative arguments we have (since $ET < \infty$)

$$\begin{aligned} &\lim_{t \to \infty} \frac{1}{t} \int_0^t P\big[\tilde{W}^0(s) = 0\big]\, ds \\ &\quad = \begin{cases} \dfrac{1}{E\tau_0} E\displaystyle\int_0^T 1_{\{\tilde{W}^0(s) = 0\}}\, ds, & \text{if } E\tau_0 < \infty, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.6}$$

Therefore, if we show that the left-hand side of (3.6) is strictly positive, this will immediately imply that $E\tau_0 < \infty$. To see that this is indeed the case, note that since $J_1, \dots, J_n$ are independent by assumption, then so are $\tilde{W}_1^0, \dots, \tilde{W}_n^0$. From Kella and Whitt (1992a) (and many more references), $P[\tilde{W}_i^0(t) = 0] \to 1 - \rho_i/\tilde{\rho}_i > 0$ as $t \to \infty$ so that, in particular,

$$P\big[\tilde{W}^0(t) = 0\big] \to \prod_{i=1}^n \left(1 - \frac{\rho_i}{\tilde{\rho}_i}\right) > 0, \tag{3.7}$$

where by coupling (at time T) the same limit is valid for $\tilde{W}(t)$. From Lemma 3.1, when taking $\tilde{W}(0) = W(0)$ for example, $W(t)$ is 0 whenever $\tilde{W}(t)$ is, so that the above also implies that our original process reaches 0 in finite expected time, which is one of the stated results. In particular, we clearly have that

$$\liminf_{t\to\infty} P[W(t) = 0] \ge \prod_{i=1}^{n}\left(1 - \frac{\rho_i}{\tilde{\rho}_i}\right) > 0, \tag{3.8}$$

which implies that once we argue the existence of a proper limiting distribution, then this distribution must have a positive atom at the origin.

In order to complete the proof we now need to argue that the limiting distribution for our process indeed exists, is independent of initial conditions and holds in total variation. To see this, it is well known that, for any a.s. finite initial condition, $\tilde{W}_i$ is tight for every $1 \le i \le n$ as a one-dimensional reflected Lévy process with driving process having a negative mean. Therefore, the vector-valued process $\tilde{W}$ is tight as well and by Lemma 3.1 so is W. In particular, it is a standard observation that in such an instance a proper stationary distribution for W exits. Now let $W(0)$ be arbitrary (finite a.s.) and $W^*(0)$ has such a proper stationary distribution [so that $W^*(0)$ is finite a.s.] where $(W(0), W^*(0))$ is independent of J. Set $\tilde{W}_i(0) = W_i(0) \vee W_i^*(0)$ for every $i = 1, \dots, n$ (also finite a.s.). By Lemma 3.1, $e'\tilde{W}(t) \ge e'W(t) \vee e'W^*(t)$ and since $T < \infty$ a.s. we have $\tilde{W}(T) = W(T) = W^*(T) = 0$, so that $W(t) = W^*(t)$ for every $t \ge T$. By coupling we immediately obtain both total variation convergence and independence of initial conditions. In particular, this, of course, also implies that the stationary distribution we chose is unique.

In order to complete the proof we need to show the "only if" part of the theorem. For this we observe that with $Q = (I - P')^{-1}$ we have $QW(t) = Q(W(0) + J(t)) - rt + L(t) \ge 0$, so that L is greater than or equal to the minimal nondecreasing process L' having this property, which is given by

$$L_i'(t) = -\inf_{0\le s\le t}\left(\sum_{j=1}^{n} q_{ij}\big(W_j(0) + J_j(s)\big) - r_i s\right)^{-}. \tag{3.9}$$

For every i such that $\sum_{j=1}^{n} q_{ij}\rho_j \ge r_i$ we have, with $W'(t) = Q(W(0) + J(t)) - rt + L'(t)$, $P[W_i'(t) \le x] \to 0$ for every $x > 0$ as a one-dimensional reflected Lévy process. Since $W' \le QW$ (as $L' \le L$) we immediately have $P[\sum_{j=1}^{n} q_{ij} W_j(t) \le x] \to 0$ for every $x > 0$ as $t \to \infty$, which clearly implies that W is not tight and hence cannot even converge in distribution, let alone in total variation. □

REMARK 3.1. Note that as a consequence of Theorem 3.1 we immediately have that, when Condition 2.1 is satisfied, W is necessarily strictly regenerative where, for every $\delta > 0$, we can choose regeneration epochs as $T_m = \inf\{t \mid t \ge T_{m-1} + \delta, W(t) = 0\}$ (with $T_0 = 0$). Since $\pi(0) > 0$ it clearly follows that $E(T_m - T_{m-1}) < \infty$ for every $m \ge 2$ [for $m = 1$ it also holds provided $W(0)$ has a finite mean]. In fact, as argued, one may couple the process starting from an arbitrary initial condition (in particular, 0) with the process starting with the stationary distribution. Hence the regeneration epochs themselves admit coupling, and thus they have a spread-out distribution. See Theorem 2.3 on page 146 of Asmussen (1987).

We conclude this section with the following theorem which gives lower and upper bounds for the steady-state means.

THEOREM 3.2. *Assume that* $\sigma_i^2 = \mathrm{Var}(J_i(1)) < \infty$ *for* $1 \le i \le n$ *and denote* $w_i = EW_i(\infty)$ *as well as* $Q = (q_{ij}) \equiv (1 - P')^{-1}$. *Then, for every* $1 \le i \le n$,

$$\frac{\sum_{j=1}^n q_{ij}^2 \sigma_j^2}{2\left(r_i - \sum_{j=1}^n q_{ij}\rho_j\right)} \le \sum_{j=1}^n q_{ij} w_j \le \sum_{j=1}^n q_{ij} \frac{\sigma_j^2}{2\left(\tilde{\rho}_j - \rho_j\right)} \tag{3.10}$$

for every $\tilde{\rho} > \rho$ *for which* $Q\tilde{\rho} < r$.

PROOF. The right inequality in (3.10) follows from Lemma 3.1 together with the fact that $E\tilde{W}_i(\infty) = \sigma_i^2/[2(\tilde{\rho}_i - \rho_i)]$ [see Kella and Whitt (1992a)]. To obtain the left inequality, we take $W' = Q(W(0) + J(t)) - rt + L'(t)$ from the proof of Theorem 3.1 [see (3.9)]. Even though the $QJ(t)$ are not independent subordinators, since all we are interested in is the mean, it suffices to consider each i separately. However, for each i, $W_i'(t)$ is a one-dimensional reflected Lévy process having no negative jumps and satisfying the appropriate stability conditions (since $Q\rho - r < 0$), so that, once again, the same results from Kella and Whitt (1992a) imply that

$$EW'(\infty) = \frac{\mathrm{Var}\left(\sum_{j=1}^n q_{ij} J_j(1)\right)}{2\left(r_i - E\sum_{j=1}^n q_{ij} J_j(1)\right)}, \tag{3.11}$$

which is precisely the left-hand side of (3.10). □

4. A two-dimensional network and reduction to the tandem case. Consider this model under the added assumption that $p_{12} r_1 \ge r_2$. The case $p_{21} r_2 \ge r_1$ is, of course, treated in the same way. From (2.4) we have

$$\begin{aligned}(I - P')(r - l(x)) &= \begin{pmatrix} r_1 - p_{21} r_2 \\ r_2 - p_{12} r_1 \end{pmatrix} 1_{\{x > 0\}} \\ &\quad + \begin{pmatrix} 0 \\ (1 - p_{12} p_{21}) r_2 \end{pmatrix} 1_{\{x_1 = 0,\ x_2 > 0\}}.\end{aligned} \tag{4.1}$$

Therefore, if we take $r_1' = r_1 - p_{21} r_2$, $r_2' = (1 - p_{12} p_{21}) r_2$, $p_{12}' = p_{12}$ and $p_{21}' = 0$, it is easy to see that the right-hand side of (4.1) will remain unchanged. This means that the network is identical to a tandem network with the same inputs and initial conditions, with r' and P' replacing r and P. The reason this makes sense intuitively is that, under the stated condition, the second queue is never idle, so that the boundary $\{(x_1, 0) \mid x_1 > 0\}$ is never hit.

The following observation is obvious for the case of a tandem network (when $p_{21} = 0$), which is the case resulting from the previous paragraph. The fact that it holds for any two-dimensional network was discovered by Glickman (1993). It is obtained by multiplying the first equation in (2.3) by p_{12} and inspecting the parameters of both equations, noting that, trivially, $(p_{12} L_1, L_2)$

is the minimal nondecreasing process, starting at 0, which makes the appropriate right-hand sides nonnegative.

LEMMA 4.1. *For any two-dimensional network (not necessarily satisfying the conditions of this section)* $(\tilde W_1, \tilde W_2) = (p_{12}W_1, W_2)$ *is the content process of a fluid process with initial condition* $(p_{12}W_1(0), W_2(0))$, *inputs* $(\tilde J_1, \tilde J_2) = (p_{12}J_1, J_2)$, *maximal output rates* $(p_{12}r_1, r_2)$ *and routing matrix*

$$\begin{pmatrix} 0 & 1 \\ p_{12}p_{21} & 0 \end{pmatrix}.$$

The first paragraph of this section together with Lemma 4.1 immediately leads to the following result.

THEOREM 4.1. *When* $p_{12}r_1 \ge r_2$, $(\tilde W_1, \tilde W_2) = (p_{12}W_1, W_2)$ *is the content process of a tandem fluid network with* $\tilde p_{12} = 1$ *and* $\tilde p_{21} = 0$, *initial condition* $(p_{12}W_1(0), W_2(0))$, *input processes* $(\tilde J_1, \tilde J_2) = (p_{12}J_1, J_2)$ *and output rates* $(\tilde r_1, \tilde r_2) = (p_{12}(r_1 - p_{21}r_2), (1 - p_{12}p_{21})r_2)$.

Finally, with the notation of Theorem 4.1, *denoting* $\tilde\rho_i = E\tilde J_i(1)$ *for* $i = 1, 2$, *simple algebra leads to the following result.*

LEMMA 4.2. *If* $p_{12}r_1 \ge r_2$, *then* $\tilde r_1 \ge \tilde r_2$ *and Condition* 2.1 *holds if and only if* $\tilde\rho_1 < \tilde r_1$ *and* $\tilde\rho_1 + \tilde\rho_2 < \tilde r_2$.

Theorem 4.1 and Lemma 4.2 imply that $(p_{12}W_1, W_2)$ is a tandem network satisfying the conditions in Kella (1993). Hence explicit joint steady-state characteristics are readily available via substitution in the formulas of that article.

From Kella (1993), for every a.s. finite initial condition $W(0)$, the time to reach 0 is a.s. finite, and the expected time to reach 0 is explicitly given by

$$\frac{E\tilde W_1(0) + E\tilde W_2(0)}{\tilde r_2 - \tilde\rho_1 - \tilde\rho_2} = \frac{p_{12}EW_1(0) + EW_2(0)}{(1 - p_{12}p_{21})r_2 - p_{12}r_1 - r_2}. \tag{4.2}$$

Also, in this case, the steady-state atom at the origin is

$$\pi(0) = 1 - \frac{\tilde\rho_1 + \tilde\rho_2}{\tilde r_2} = 1 - \frac{p_{12}\rho_1 + \rho_2}{(1 - p_{12}p_{21})r_2} > 0. \tag{4.3}$$

For future reference we give the explicit formulas for the steady-state means, variances and covariances. We recall that $\rho_i = EJ_i(1) = \eta_i'(0)$, and denote $\sigma_i^2 = \mathrm{Var}(J_i(1)) = -\eta_i''(0)$. For the purpose of (4.4) below, we assume that $\eta_i'''(0) < \infty$, which implies that $\sigma_i < \infty$ as well, for $i = 1, 2$:

$$
\begin{aligned}
EW_1(\infty) &= \frac{\sigma_1^2}{2(r_1 - p_{21}r_2 - \rho_1)}, \\
EW_2(\infty) &= \frac{p_{12}^2\sigma_1^2 + \sigma_2^2}{2((1-p_{12}p_{21})r_2 - p_{12}\rho_1 - \rho_2)} - p_{12}EW_1(\infty), \\
\mathrm{Var}(W_1(\infty)) &= \frac{\eta_1'''(0)}{3(r_1 - p_{21}r_2 - \rho_1)} + (EW_1(\infty))^2, \\
\mathrm{Cov}(W_1(\infty), W_2(\infty)) &= \frac{p_{12}r_1 - r_2 + \rho_2}{2(r_1 - p_{21}r_2 - \rho_1)}\left[\mathrm{Var}(W_1(\infty)) + (EW_1(\infty))^2\right], \\
\mathrm{Var}(W_2(\infty)) &= \frac{p_{12}^3\eta_1'''(0) + \eta_2'''(0)}{3((1-p_{12}p_{21})r_2 - p_{12}\rho_1 - \rho_2)} \\
&\quad + (p_{12}EW_1(\infty) + EW_2(\infty))^2 \\
&\quad - p_{12}^2\mathrm{Var}(W_1(\infty)) - 2p_{12}\mathrm{Cov}(W_1(\infty), W_2(\infty)).
\end{aligned} \tag{4.4}
$$

In Corollary 4.2 of Kella and Whitt (1992b) and Corollary 4.1 of Kella (1993), it was shown that, other than for trivial setups [where either $W_1(\infty) \equiv 0$ or $W_2(\infty) \equiv 0$], the steady-state correlation coefficient in a tandem fluid network, in which the maximal output rate from the first station is at least as large as that of the second station, is strictly between 0 and $1/\sqrt{3}$. Hence, by Theorem 4.1, this also holds for the case considered in this section whenever neither of the following situations takes place: (1) $\eta_1 \equiv 0$ [where $W_1(\infty) \equiv 0$] and (2) $\eta_2 \equiv 0$ and $p_{12}r_1 = r_2$ [so that $\tilde{r}_1 = \tilde{r}_2$, hence $W_2(\infty) = 0$].

5. Nonproduct form for a two-dimensional network. In this section we will show that, other than for trivial setups, when Condition 2.1 is satisfied, the limiting distribution associated with our two-dimensional network never has a product form. More precisely, we prove the following result.

THEOREM 5.1. *The limiting distribution associated with our two-dimensional network has product form only in the following obvious cases*:

(i) $p_{12} = p_{21} = 0$ (*parallel network*), *in which case* W_1 *and* W_2 *are clearly independent processes*;
(ii) $J_1 \equiv J_2 \equiv 0$, *in which case* $W_1(\infty) = W_2(\infty) = 0$;
(iii) $J_i \equiv 0$ *and* $r_i > p_{ji}r_j$, *in which case* $W_i(\infty) = 0$, *where* $(i, j) \in \{(1, 2), (2, 1)\}$.

PROOF. For the case considered in Section 4, that is, when either $p_{12}r_1 \ge r_2$ or $p_{21}r_2 \ge r_1$, since we can reduce the network to a tandem network satisfying the conditions considered in Kella and Whitt (1992b) and Kella (1993), the result follows from the fact that the correlation coefficient is positive whenever either both J_1 and J_2 are not identically 0 or $J_i \not\equiv 0$ and

$p_{ij}r_i > r_j$ (the strict inequality is important). This, however, assumes the existence of second and third moments for J_1 and J_2. Without this assumption, this, as well as when $p_{12}p_{21} = 0$, which includes the tandem (not necessarily satisfying the conditions of Section 4) and parallel cases, follows as a special case of Lemma 4.1 and, more generally, of Theorem 4.1 in Kaspi and Kella (1996). Therefore, it suffices to consider the case where both $r_1 > p_{21}r_2$ and $r_2 > p_{12}r_1$ where $0 < p_{12}p_{21} < 1$.

Applying Theorem 2 of Kella and Whitt (1992a) to the one-dimensional process $\alpha' W(t)$, we have that

$$\begin{aligned}&\exp(-\alpha' W(t)) - \exp(-\alpha' W(0))\\ &\quad + \int_0^t [\eta_1(\alpha_1) + \eta_2(\alpha_1) - \alpha'(I-P')(r - l(W(s)))]\exp(-\alpha' W(s))\,ds\end{aligned} \tag{5.1}$$

is a zero-mean martingale, where $(I - P')(r - l(x))$ is given by (2.5). Hence, when taking a stationary version of the Markov process W, we immediately have

$$\begin{aligned}E&[\eta_1(\alpha_1) + \eta_2(\alpha_1) - \alpha'(I-P')(r - l(W(\infty)))]\exp(-\alpha' W(\infty))\\ &= \frac{1}{t}\int_0^t E[\eta_1(\alpha_1) + \eta_2(\alpha_1) - \alpha'(I-P')(r - l(W(s)))]\\ &\qquad \times \exp(-\alpha' W(s))\,ds\\ &= 0.\end{aligned} \tag{5.2}$$

Let us denote $w(\alpha) = E\exp(-\alpha' W(\infty))$ and $w_i(\alpha_i) = E\exp(-\alpha_i W_i(\infty))$ for $i = 1, 2$. In particular, $w(\alpha_1, \infty) = E\exp(-\alpha_1 W_1(\infty))1_{\{W_2(\infty)=0\}}$, $w(\infty, \infty) = P[W_1(\infty) = W_2(\infty) = 0]$ and $w_i(\infty) = P[W_i(\infty) = 0]$. From (5.2) and (2.5) we have

$$\begin{aligned}&[\eta_1(\alpha_1) + \eta_2(\alpha_2)]w(\alpha_1, \alpha_2)\\ &\quad = \alpha_1(1 - p_{12}p_{21})r_1[w(\alpha_1, \infty) - w(\infty, \infty)]\\ &\qquad + \alpha_2(1 - p_{12}p_{21})r_2[w(\infty, \alpha_2) - w(\infty, \infty)]\\ &\qquad + [\alpha_1(r_1 - p_{21}r_2) + \alpha_2(r_2 - p_{12}r_1)]\\ &\qquad\quad \times [w(\alpha_1, \alpha_2) - w(\alpha_1, \infty) - w(\infty, \alpha_2) + w(\infty, \infty)].\end{aligned} \tag{5.3}$$

The limiting distribution has product form if $w(\alpha_1, \alpha_2) = w_1(\alpha_1)w_2(\alpha_2)$. Hence substituting in (5.3) and denoting $\pi_i = w_i(\infty)$ gives

$$\begin{aligned}&[\eta_1(\alpha_1) + \eta_2(\alpha_2)]w_1(\alpha_1)w_2(\alpha_2)\\ &\quad = \alpha_1(1 - p_{12}p_{21})r_1[w_1(\alpha_1) - \pi_1]\pi_2\\ &\qquad + \alpha_2(1 - p_{12}p_{21})r_2\pi_1[w_2(\alpha_2) - \pi_2]\\ &\qquad + [\alpha_1(r_1 - p_{21}r_2) + \alpha_2(r_2 - p_{12}r_1)]\\ &\qquad\quad \times [w_1(\alpha_1) - \pi_1][w_2(\alpha_2) - \pi_2].\end{aligned} \tag{5.4}$$

Dividing by α_1 or by α_2 and letting $\alpha_1, \alpha_2 \downarrow 0$ yields

$$\begin{aligned}\rho_1 &= (1-p_{12}p_{21})r_1(1-\pi_1)\pi_2 + (r_1 - p_{21}r_2)(1-\pi_1)(1-\pi_2),\\ \rho_2 &= (1-p_{12}p_{21})r_2\pi_1(1-\pi_2) + (r_2 - p_{12}r_1)(1-\pi_1)(1-\pi_2),\end{aligned}\tag{5.5}$$

respectively. Now setting $\alpha_2 = 0$ in (5.4) results in

$$\begin{aligned}&\eta_1(\alpha_1)w_1(\alpha_1)\\ &\quad= \alpha_1[(1-p_{12}p_{21})r_1\pi_2 + (r_1 - p_{21}r_2)(1-\pi_2)](w_1(\alpha_1) - \pi_1)\\ &\quad= \frac{\alpha_1\rho_1(w_1(\alpha_1)-\pi_1)}{1-\pi_1},\end{aligned}\tag{5.6}$$

where for the bottom equality we apply (5.5). Therefore,

$$w_i(\alpha_i) = \frac{\alpha_i\pi_i\rho_i}{\alpha_i\rho_i - (1-\pi_i)\eta_i(\alpha_i)},\tag{5.7}$$

so that

$$w_i(\alpha_i) - \pi_i = \frac{\pi_i(1-\pi_i)\eta_i(\alpha_i)}{\alpha_i\rho_i - (1-\pi_i)\eta_i(\alpha_i)}\tag{5.8}$$

for $i = 1, 2$.

Substituting (5.7) and (5.8) in (5.4) and performing some cumbersome but straightforward computations, which among others involve the substitution of ρ_i by the right-hand side of (5.5) in certain (but not all) places, leads to the following equation which holds for all $\alpha_1, \alpha_2 > 0$:

$$\begin{aligned}&\pi_1\pi_2(1-\pi_1)(1-\pi_2)\\ &\quad\times\left[\frac{(\rho_1 - \eta_1(\alpha_1)/\alpha_1)\eta_2(\alpha_2)}{p_{21}(r_2 - p_{12}r_1)} + \frac{(\rho_2 - \eta_2(\alpha_2)/\alpha_2)\eta_1(\alpha_1)}{p_{12}(r_1 - p_{21}r_2)}\right] = 0.\end{aligned}\tag{5.9}$$

It is well known that $\eta_i(\alpha_i) \le \alpha_i\rho_i$ for all $\alpha_i > 0$ with equality for some $\alpha_i > 0$ if and only if it holds for all $\alpha_i > 0$. This follows from the concavity of η_i, which is strict whenever J_i is not a linear deterministic process. Hence, since we assumed that J_i, $i = 1, 2$, are pure jump subordinators, $\eta_i(\alpha_i) = \alpha_i\rho_i$ for some $\alpha_i > 0$ if and only if $J_i \equiv 0$. Next we observe that since $W_i(\infty)$ is proper, then, from (5.7), if $J_i \not\equiv 0$, then necessarily $\pi_i > 0$. Finally, we note that, from (5.5), $\pi_i = 1$ if and only if $\rho_i = 0$, which is equivalent to $J_i \equiv 0$. With these observations in mind it is now clear from (5.9) that either $J_1 \equiv 0$ or $J_2 \equiv 0$ (or both) which completes the proof. □

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